Multiple limit cycles bifurcation and Jacobi stability for a class of segmented disc dynamo system

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Abstract. In this paper, the multiple bifurcation of limit cycles for a segmented disc dynamo system is studied. The formal series method for calculating the singular point quantities is applied to determine the highest order focus value at Hopf bifurcation point. For two cases of the segmented disc dynamo system, namely the system with or without friction coefficient (abbr. SDDF- or SDD-model), the maximum number of limit cycles is obtained at the symmetrical equilibrium points under the condition of synchronous perturbation respectively. At the same time, the parameters condition is classified for exact number of limit cycles near each weak focus. Finally, we find that all equilibrium points of the heart model are Jacobi unstable under certain parameter values.

Key words. segmented disc dynamo system; center manifold; limit cycle; singular point quantities; Jacobi stability.
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1 Introduction

Since the double disc motor system was derived in [1], its dynamic behavior has been widely concerned. And from it and based on the azimuth current distribution, the segmented disc generator model was given in [2] with the following reduced form:

$$\begin{aligned}
\dot{x}(t) &= r(y-x),\\
\dot{y}(t) &= mx-(1+m)y+xu,\\
\dot{u}(t) &= g(mx^2+1-(1+m)xy)
\end{aligned}\tag{1}$$

where $x(t)$ and $y(t)$ denote the magnetic radial and azimuthal current distributions, $u(t)$ measures the angular velocity of the disc, g represents the applied torque, moreover, m and r are dimensionless parameters. For system (1), the author of [3] investigated comprehensively its dynamic properties including Darboux integrability and existence of Hopf bifurcation. The author of [4] studied multi-stability and coexistence of three types of attractors: equilibrium points, limit cycles and hidden chaotic attractors, and found hidden chaotic solutions to occur well away from the subcritical Hopf bifurcation. Subsequently, the control of hidden chaos in system (1) was studied and applied to electronic circuit design in [5]. The authors of [6] also analyzed its mechanism of chaos from the perspective of geometry.

Furthermore, to make the segmented disc dynamo model better to reflect the practical significance in physics, especially for azimuthal currents in the rotating discs, the authors of

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[7] proposed a new description of the two-disc dynamo, yielding the segmented disc dynamo system with mechanical friction as follows:

$$\begin{aligned} \dot{x}(t) &= r(y-x), \\ \dot{y}(t) &= mx-(1+m)y+xu, \\ \dot{u}(t) &= g(mx^2+1-(1+m)xy)-fu, \end{aligned} \tag{2}$$

where f is a mechanical friction coefficient, naturally system (1) is one special case of system (2). Recently, the authors of [8] studied the complex dynamics of the equilibrium point of system (2) by using the center manifold theory, and determined the existence conditions of Hopf bifurcation, at the same time, the Darboux integrability of the system was discussed in detail. For the mechanism of chaos in system (2), some theoretical and numerical analyses were also given in [9] and [10] successively.

However, there is a little with respect to the report of investigating the multiple small-amplitude limit cycles bifurcation and local integrability restricted to one single center manifold in system (2). Then we will continue to study its dynamic properties in the two aspects, particularly, the maximal number of limit cycles in the vicinity of a Hopf singular point.

In planar polynomial systems, the maximal number of limit cycles bifurcating from an elementary focus or center is sometimes called the local version of the second part of the Hilbert's 16th problem, each reader can refer to the literatures [11, 12, 13] to learn about this famous problem and its research progress. As for the local version around an elementary focus or center, in general quadratic systems, the maximal number $M(2) = 3$ was solved by Bautin in 1952 [14]. However, in general cubic systems, the maximal number $M(3)$ is still open, many results have been obtained on its low bound [15, 16], so far, the best result is $M(3) \geq 12$ [17]. For other relevant results, one can see [18, 19] and references therein.

Compared with two-dimensional systems, the maximal number of limit cycles bifurcating from a Hopf singular point is more challenging for a three-dimensional systems [20, 21]. Such problems are solved in only a few specific models, e.g. [22, 23, 24]. For general three-dimensional systems, only some low bounds were obtained, e.g., the examples of 12 small-amplitude limit cycles in quadratic vector fields were given recently [25, 26], the readers can also refer to [21, 27] for other results.

In general, the main task of solving the above problem is to compute the focus values and determine the center conditions at the equilibrium, which is exactly a extended version from planar systems to center manifold of three dimensional systems. There exist some classical methods such as Liapunov-Schmidt method [28], the direct dimension reduction method [23] and the averaging theory [29]. And more some new research approaches were put forward, e.g., the inverse Jacobi multiplier [30, 31], the formal first integral method [32] and the simple normal form [33]. Recently, the authors of [34] also presented a useful conclusion on the bound of the cyclicity around a center on center manifold in terms of the Bautin ideal. In our research, the linear and simple algorithm proposed in [35] is applied to compute the singular point quantities corresponding to the focal values.

The rest of the paper is organized as follows. In the section 2, some preliminary methods for studying Hopf bifurcation of three-dimensional systems are given. In the section 3 and 4, the maximum number of bisymmetric limit cycles of the segmented disk generator system with friction and without friction are determined and proved strictly respectively. In the section 5, the Jacobian stability at the equilibrium point is obtained. Finally, we draw some conclusions about this work.

2 Preliminary results and method

In this section, we first introduce the method and basic results about limit cycle bifurcation at Hopf singular point on center manifold. Based on the works of Liu and Li [36], the authors of [35] extended these methods of computing singular point quantities from the planar polynomial system to three-dimensional systems, namely the following three-dimensional systems are considered:

$$\begin{aligned}
\frac{dx}{dt} &= -y + \sum_{k+j+l=2}^{\infty} A_{kjl} x^k y^j u^l = X(x, y, u), \\
\frac{dy}{dt} &= x + \sum_{k+j+l=2}^{\infty} B_{kjl} x^k y^j u^l = Y(x, y, u), \\
\frac{du}{dt} &= -du + \sum_{k+j+l=2}^{\infty} D_{kjl} x^k y^j u^l = U(x, y, u),
\end{aligned} \tag{3}$$

where $x,\ y,\ u,\ A_{kjl},\ B_{kjl},\ D_{kjl} \in\in \mathbb{R}\ (k,\ j,\ l \in \mathbb{N})$.

In general, we analyze the Hopf bifurcation of system (3) by applying the center manifold theory [37], thus system (3) has a center manifold $u(x, y)$, which can be expressed as the following series:

$$u(x, y) = u_2(x, y) + \text{h.o.t.}, \tag{4}$$

where u_2 is a quadratic homogeneous polynomial in x and y, h.o.t. stands for higher-order term. By means of transformation:

$$\begin{aligned}
&z = x + iy, \quad w = x - iy, \quad u = u, \\
&T = it, \quad i = \sqrt{-1},
\end{aligned} \tag{5}$$

system (3) can be transformed into the following complex system:

$$\begin{aligned}
\frac{dz}{dT} &= z + \sum_{k+j+l=2}^{\infty} a_{kjl} z^k w^j u^l = Z(z, w, u), \\
\frac{dw}{dT} &= -w + \sum_{k+j+l=2}^{\infty} b_{kjl} w^k z^j u^l = -W(z, w, u), \\
\frac{du}{dT} &= idu + \sum_{k+j+l=2}^{\infty} d_{kjl} z^k w^j u^l = U(z, w, u)
\end{aligned} \tag{6}$$

where $z,\ w,\ u,\ a_{kjl},\ b_{kjl},\ d_{kjl} \in \mathbb{C}\ (k,\ j,\ l \in \mathbb{N})$ and $b_{kjl} = \bar{a}_{kjl}$ with $\bar{a}_{kjl}$ denotes the conjugate complex number of a_{kjl}, we call that system (6) and (3) are concomitant. For system (6), we can obtain the following Lemma.

Lemma 2.1. *For system* (6)*, when taking* $c_{110} = 1$*,* $c_{101} = c_{011} = c_{200} = c_{020} = 0$*,* $c_{kk0} = 0$*,* $k = 2, 3 \cdots$*, we can derive successively and uniquely the following series:*

$$F(z, w, u) = zw + \sum_{\alpha+\beta+\gamma=3}^{\infty} c_{\alpha\beta\gamma} z^\alpha w^\beta u^\gamma, \tag{7}$$

such that

$$\frac{dF}{dT} = \frac{\partial F}{\partial z} Z - \frac{\partial F}{\partial w} W + \frac{\partial F}{\partial u} U = \sum_{m=1}^{\infty} \mu_m (zw)^{m+1}, \tag{8}$$

and if $\alpha \neq \beta$ or $\alpha = \beta$, $\gamma \neq 0$, $c_{\alpha\beta\gamma}$ is determined by the following recursive formula:

$$\begin{aligned} c_{\alpha\beta\gamma} = & \frac{1}{\beta - \alpha - id\gamma} \\ & \times \sum_{k+j+l=3}^{\alpha+\beta+\gamma+2} [(\alpha - k + 1)a_{k,j-1,l} - (\beta - j + 1)b_{j,k-1,l} + (\gamma - 1)d_{k-1,j-1,l+1}] \\ & \times c_{\alpha-k+1,\beta-j+1,\gamma-l}, \end{aligned} \tag{9}$$

and for any positive integer m, μ_m is determined by the following recursive formula:

$$\begin{aligned} \mu_m = & \sum_{k+j+l=3}^{2(m+1)} [(m - k + 1)a_{k,j-1,l} - (m - j + 1)b_{j,k-1,l} - ld_{k-1,j-1,l+1}] \\ & \times c_{m-k+1,m-j+1,-l}, \end{aligned} \tag{10}$$

and when $\alpha < 0$ or $\beta < 0$ or $\gamma < 0$ or $\gamma = 0$, $\alpha = \beta$, we have let $c_{\alpha\beta\gamma} = 0$. μ_m is called m-th singular quantity at the origin of system (6).

Lemma 2.2 (see [35]). *The first nonvanishing focal value of the origin of system* (3)*: v_{2m+1} and the first nonvanishing singular quantity of the origin of system* (6)*: μ_m are related as*

$$v_{2m+1} = i\pi\mu_m.$$

Definition 2.1. If the values $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0$ and $\mu_k \neq 0$, then the origin of system (3) is called the fine focus of order k, $k = 1, 2, \cdots$. If for any positive integer k, $\mu_k = 0$ hold, then the origin of system (3) is called a center.

Lemma 2.3 (see [35]). *If the origin of undisturbed system* (3) *is a fine focus with n order as its highest order, then the origin of the system* (3) *can bifurcate n at most small amplitude limit cycles under a suitable perturbation.*

In order to obtain sufficient conditions for the existence of limit cycles, we introduce the following lemma.

Lemma 2.4 (see [38]). *Suppose that the focus values depend on k parameters, expressed as $v_j = v_j(\epsilon_1, \epsilon_2, \cdots, \epsilon_k), j = 1, 3, \cdots, 2k+1$, satisfying $v_j(0, 0, \cdots, 0) = 0, j = 1, 3, \cdots, 2k-1$, $v_{2k+1}(0, 0, \cdots, 0) \neq 0$, and*

$$\det[\tfrac{\partial(v_1, v_3, \cdots, v_{2k-1})}{\partial(\epsilon_1, \epsilon_2, \cdots, \epsilon_k)}(0, 0, \cdots, 0)] \neq 0. \tag{11}$$

then the origin of the perturbed system (3) has k limit cycles.

3 Hopf bifurcation cyclicity of a segmented disk dynamo model with friction.

In this section, we analyze the limit bifurcation of system (2), namely segmented disk dynamo model with friction $f \neq 0$. It is easy to know that when $f < g$, system (2) has three equilibrium points $E_0 = (0, 0, \frac{g}{f})$ and $E_{1,2} = (\pm d_0, \pm d_0, 1)$ with writing $\sqrt{1 - \frac{f}{g}} = d_0$, when $f \geq g$, system (2) has only equilibrium point E_0.

3.1 Singular point quantities of equilibrium point E_1

Here we will study the Hopf bifurcation at the symmetric equilibrium point $E_{1,2}$ of system (2). For system (2), due to the invariance under transformation $(x, y, z) \mapsto (-x, -y, z)$, we only need to consider the equilibrium point $E_1 = (d_0, d_0, 1)$. By choosing appropriate perturbation parameters, we will obtain at most three limit cycles at the symmetric equilibrium points of system (2) via a Hopf bifurcation synchronously and respectively.

For the convenience of analysis, we translate the equilibrium point E_1 to the origin, then the system (2) changes to the following form:

$$\begin{aligned}
\dot{x}(t) &= r(y - x),\\
\dot{y}(t) &= (1 + m)(x - y) + u(x + d_0),\\
\dot{u}(t) &= g - f(1 + u) + gm(x + d_0)^2 - g(1 + m)(x + d_0)(y + d_0)
\end{aligned} \tag{12}$$

where $d_0 = \sqrt{1 - \frac{f}{g}}$.

It is easy to know that the Jacobian matrix of system (12) at the origin is

$$A_1 = \begin{pmatrix} -r & r & 0 \\ 1 + m & -1 - m & d_0 \\ (m - 1)d_0 & -(1 + m)d_0 & -f \end{pmatrix}, \tag{13}$$

and its corresponding characteristic polynomial is

$$P(\lambda) = \lambda^3 + (1 + r + m + f)\lambda^2 + (rf + g + gm)\lambda + 2r(g - f). \tag{14}$$

Then we let

$$P(\lambda) = (\lambda^2 + \omega^2)(\lambda + \lambda_0) \tag{15}$$

where $\omega > 0$ and $\lambda_0 > 0$, such that there is a pair of purely imaginary conjugate eigenvalues eigenvalues $\pm\omega i$ and one negative eigenvalue $-\lambda_0$, which implies that E_1 is of center-focus type. Further, we obtain its necessary and sufficient condition:

$$\lambda_0 = 1 + r + f + m, \;\; \omega^2 = rf + g + gm, \;\; fr(f + m + r + 3) - gd_1 = 0 \tag{16}$$

where $d_1 = r(1 - m) - (1 + m)(1 + f + m)$. And from the above third equation, if $d_1 = 0$, then $f + m + r + 3 = 0$, yielding $\lambda_0 = -2 < 0$, therefore d_1 can not vanish, we have

$$g = G(f, m, r) := \frac{fr(f+m+r+3)}{d_1}, \;\text{ and }\; \omega^2 =: \Theta(f, r, m) = \frac{2rf(1+r+m)}{d_1} > 0. \tag{17}$$

Above all, we conclude that if and only if the parameters lie in the set

$$\Omega_1 := \{(r, m, g, f) : g = G(f, r, m),\, \Theta(f, r, m) > 0,\, 1 + r + f + m > 0\}, \tag{18}$$

the equilibrium E_1, i.e., the origin of system (12) is a weak focus on the center manifold.

For system (12) in the parameter set Ω_1, there exists an invertible matrix T_2 such that A_1 becomes the following diagonal matrix:

$$T_2^{-1}A_1T_2 = \begin{pmatrix} i\omega & 0 & 0 \\ 0 & -i\omega & 0 \\ 0 & 0 & -\lambda_0 \end{pmatrix}, \tag{19}$$

where

$$T_2 = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ 1 & 1 & 1 \end{pmatrix},$$

in which T_{11}, T_{12}, T_{13}, T_{21}, T_{22} and T_{23} see Appendix (A.1).

Further, by non-degenerate linear transformation $(x, y, u)' = T_2(z, w, \tilde{u})'$, system (12) becomes the following complex symmetric system:

$$\begin{aligned} \frac{dz}{dT} &= z + a_{200}z^2 + a_{020}w^2 + a_{002}u^2 + a_{110}zw + a_{101}zu + a_{011}wu = Z(z, w, u), \\ \frac{dw}{dT} &= -(w + b_{200}z^2 + b_{020}w^2 + b_{002}u^2 + b_{110}zw + b_{101}zu + b_{011}wu) = -W(z, w, u), \\ \frac{du}{dT} &= d_{001}u + d_{200}z^2 + d_{020}w^2 + d_{002}u^2 + d_{110}zw + d_{101}zu + d_{011}wu = U(z, w, u), \end{aligned} \tag{20}$$

where $z, w, u, T \in \mathbb{C}$ and $\tilde{u}$ is still written as u, the coefficients $b_{kjl} = \bar{a}_{kjl}$, d_{kjl} are represented by the parameters of system (12), $kjl = 200, 020, 002, 110, 011, 101$.

When applying the transformation: $z = x + \mathrm{i}y$, $w = x - \mathrm{i}y$, $T = \mathrm{i}\,t$, we can get its complex conjugate system with the same form of (3):

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = -y + P_2(x, y, u) = X, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = x + Q_2(x, y, u) = Y, \\ \frac{\mathrm{d}u}{\mathrm{d}t} = -\lambda_0 u + U_2(x, y, u) = U, \end{cases} \tag{21}$$

where P_2, Q_2 and U_2 are all quadratic homogeneous polynomials in (x, y, u). In fact, there exists necessarily a nondegenerate real matrix T_1 such that system (12) ban be changed directly into (21) via the transformation: $(x, y, u)' \mapsto T_1\,(x, y, u)'$.

Next, according to Lemma 2.1, we figure out easily the first 10 singular point quantities at the origin of system (20) as follows

$$\begin{aligned} \mu_1 &= \frac{2i\omega\lambda_0^2(f-\lambda_0)^3 F_1(f, \lambda_0, \omega)}{((f-\lambda_0)^2+\omega^2)(\lambda_0^2+\omega^2)D_1}, \\ \mu_2 &= -\frac{i\omega\lambda_0^3(f-\lambda_0)^5 F_2(f, \lambda_0, \omega)}{6((f-\lambda_0)^2+\omega^2)^2(\lambda_0^2+\omega^2)^3 D_2}, \\ \mu_3 &= \frac{i\omega\lambda_0^4(f-\lambda_0)^7 F_3(f, \lambda_0, \omega)}{576((f-\lambda_0)^2+\omega^2)^3(\lambda_0^2+\omega^2)^5 D_3}, \\ \mu_4 &= \mu_5 \cdots \mu_{10} = \cdots = 0, \end{aligned} \tag{22}$$

where

$$\begin{aligned} F_1(f, \lambda_0, \omega) &= 2f^3\lambda_0 - 3f^2\lambda_0^2 - 10f^2\omega^2 + 2f\lambda_0\omega^2 + 2\omega^4, \\ D_1(f, \lambda_0, \omega) &= (\lambda_0^2 + 4\omega^2)(2f(f-\lambda_0) + (2+\lambda_0)\omega^2)^2, \\ D_2(f, \lambda_0, \omega) &= (\lambda_0^2 + 4\omega^2)^2(\lambda_0^2 + 9\omega^2)(2f(f-\lambda_0) + (2+\lambda_0)\omega^2)^4, \\ D_3(f, \lambda_0, \omega) &= (\lambda_0^2 + \omega^2)^5(4\lambda_0^2 + \omega^2)(\lambda_0^2 + 4\omega^2)^4(\lambda_0^2 + 9\omega^2)^2(\lambda_0^2 + 16\omega^2) \\ &\quad \times (2f(f-\lambda_0) + (2+\lambda_0)\omega^2)^6 \end{aligned}$$

$F_2(f, \lambda_0, \omega)$ is shown in Appendix (A.2). The polynomial $F_3(f, \lambda_0, \omega)$ of 1026 terms is too long for display, and we have let $\mu_1 = \mu_2 = \mu_3 = 0$ for each $\mu_i, i > 3$.

3.2 Center-focus problem of equilibrium points $E_{1,2}$

Here, we search possible center conditions and determine the highest order of focus at $E_{1,2}$, which is closely related to the maximum number of limit cycles via Hopf bifurcation around the equilibrium points for system (2).

By analyzing the singular point quantities in (22), one can see a common factor $(f-\lambda_0)$ in μ_1, μ_2 and μ_3. When the factor $(f-\lambda_0)$ is not zero, real zeros of μ_1, μ_2 and μ_3 are determined by $F_i(f, \lambda_0, \omega)$ $(i = 1, 2, 3)$.

Next, By computing the polynomial resultants of F_2, F_3 for F_1 with respect to f via Mathematica, we get the following results:

$$\begin{aligned}
&\text{Resultant}[F_2, F_1, f] = 7077888\lambda_0^2\omega^{18}(\lambda_0^2+\omega^2)^6(\lambda_0^2+4\omega^2)^3 G_1,\\
&\text{Resultant}[F_3, F_1, f] = 16307453952\lambda_0^3\omega^{22}(\lambda_0^2+\omega^2)^6(4\lambda_0^2+\omega^2)^3(\lambda_0^2+4\omega^2)^7 G_2,
\end{aligned}$$

where

$$\begin{aligned}
G_1 = {} & 1715\lambda_0^{14} + 26411\lambda_0^{12}\omega^2 + 133977\lambda_0^{10}\omega^4 + 126353\lambda_0^8\omega^6 - 640648\lambda_0^6\omega^8\\
& - 1518912\lambda_0^4\omega^{10} - 2592000\lambda_0^2\omega^{12} + 419904\omega^{14}
\end{aligned}$$

and G_2 is a polynomial of degree 50 with respect to ω and λ_0, see Appendix (A.3).

Furthermore, we have

$$\text{Resultant}[G_1, G_2, \lambda_0] = 779218637\cdots 0\omega^{700} \neq 0, \tag{23}$$

this shows that there is no λ_0 or ω to make $G_1 = 0$ and $G_2 = 0$ hold at the same time, that is, when $G_1 = 0$, G_2 must satisfy $G_2 \neq 0$, namely $F_1 = F_3 = 0$ can not hold. Now, set $G_1 = 0$, and calculate the following resultants

$$\begin{aligned}
&\text{Resultant}[G_1, F_1, \lambda_0] = 7077888{\lambda_0}^2\omega^{18}({\lambda_0}^2+\omega^2)^6({\lambda_0}^2+4\omega^2)^3\, H_8 H_{28},\\
&\text{Resultant}[G_1, F_2, \lambda_0] = 163\cdots 2{\lambda_0}^3\omega^{22}({\lambda_0}^2+\omega^2)^6(4{\lambda_0}^2+\omega^2)^3({\lambda_0}^2+4\omega^2)^7\, H_8 H_{84},\\
&\text{Resultant}[H_{28}, H_{84}, f] = 535890196\cdots 0\omega^{2352} \neq 0,
\end{aligned} \tag{24}$$

where $H_8 = 180f^{14} - 3953f^{12}\omega^2 - 27816f^{10}\omega^4 - 139446f^8\omega^6 + 156764f^6\omega^8 - 38553f^4\omega^{10} - 3528f^2\omega^{12} + 1372\omega^{14}$, $H_{28} = \sum_{i=0}^{14} N_i f^{2i}$ and $H_{84} = \sum_{j=0}^{42} N_j f^{2j}$ where N_i and N_j are polynomials in ω^2. From the above three resultants (24), we easily know that under the condition $G_1 = 0$, if and only if $H_8 = 0$, $F_1 = F_2 = 0$ holds, and at this time $F_3 \neq 0$.

From the above analysis, we get the following lemma.

Lemma 3.1. *For the first three singular point quantities at the origin of system* (20)*, if* $\mu_1 = \mu_2 = 0$*, then* $\mu_3 \neq 0$ *holds under the condition* $\lambda_0 \neq f$.

In fact, when $\lambda_0 = f$, from (16), we get $1 + r + m = 0$, then yielding $\omega^2 = 0$ in (17), this contradicts the fact $\omega > 0$, so $\lambda_0 \neq f$ must be satisfied. This also implies that the origin of system (20) is not a center, then we have

Theorem 3.2. *The equilibrium points* $E_{1,2}$ *can not be centers on the center manifold of system* (2).

Similarly, we can verify that when $F_1 = F_2 = 0$, the signed indeterminate factor

$$d_2 := 2f(f-\lambda_0) + (2+\lambda_0)\omega^2$$

is not equal to zero in D_1, D_2 and D_3 for (22), which shows that all the denominators in the three expressions (22) can not vanish. According to Lemma 3.1 and the above analysis, the following theorem obtained.

Theorem 3.3. *If $(r,m,g,f)\in\Omega_1$ and $\lambda_0\neq f$, the symmetrical equilibrium $E_{1,2}$ of system (2) are weak foci of order at most 3. Moreover, $E_{1,2}$ are of order i, if and only if $(r,m,g,f)\in \mathrm{C_i}$, $i=1,2,3$, where*

$$\begin{aligned}
&\mathrm{C}_1 := \Omega_1 \setminus \{(r,m,g,f) : d_2\neq 0, \lambda_0\neq f\}\setminus \mathrm{C}_2\setminus \mathrm{C}_3,\\
&\mathrm{C}_2 := \{(r,m,g,f)\in\Omega_1 : F_1(f,\lambda_0,\omega)=0, F_2(f,\lambda_0,\omega)\neq 0, d_2\neq 0, \lambda_0\neq f\},\\
&\mathrm{C}_3 := \{(r,m,g,f)\in\Omega_1 : F_1(f,\lambda_0,\omega)=F_2(f,\lambda_0,\omega)=0, \lambda_0\neq f\}.
\end{aligned}$$

3.3 Limit cycles bifurcation at equilibrium points $E_{1,2}$

Now we turn to the discussion about the maximum number of small-amplitude limit cycles from $E_{1,2}$ of system (2). According to the lemma 2.3 and the theorem 3.3, we know that there are at most three small-amplitude limit cycles around the origin of system (21) or the equilibria $E_{1,2}$ of system (20), but whether the number of limit cycles can reach three, we need further analysis and demonstration.

From the discursion in the last subsection, we know that F_1, F_2 and F_3 have no common zero, but F_1 and F_2 should have. In fact, only two classes of solution groups satisfy $F_1 = F_2 = 0$ and the other conditions in C_3 of the theorem 3.3 as follows

$$\begin{aligned}
(\lambda_0^{(1)}, f^{(1)}) &\doteq (1.58165284021\omega, 5.32357646783\omega),\\
(\lambda_0^{(2)}, f^{(2)}) &\doteq (0.38513387878\omega, 0.48551387918\omega).
\end{aligned}\tag{25}$$

Further, if g and r in the parameter conditions (17) are perturbed as follows

$$\begin{aligned}
g &= -\tfrac{\delta^2(\lambda_0+2)-4\delta\lambda_0+(\lambda_0+2)\omega^2}{2(2\delta+f-\lambda_0)} = -\tfrac{(\lambda_0+2)\omega^2}{2(f-\lambda_0)} + O(\delta),\\
r &= \tfrac{\lambda_0\left(\delta^2+\omega^2\right)(2\delta+f-\lambda_0)}{\delta^2\lambda_0+2\delta^2-4\delta\lambda_0+2f^2+4\delta f-2f\lambda_0+\lambda_0\omega^2+2\omega^2} = \tfrac{\lambda_0\omega^2(\lambda_0-f)}{2f^2-2f\lambda_0+\lambda_0\omega^2+2\omega^2} + O(\delta)
\end{aligned}\tag{26}$$

where $|\delta|\ll 1$, then we can add linear perturbations to system (21) yielding

$$\dot{x} = -y+\delta x+P_2(x,y,u),\ \dot{y} = x+\delta y+Q_2(x,y,u),\ \dot{u} = -\lambda_0 u+U_2(x,y,u).\tag{27}$$

According to the lemma 2.2, for the perturbed real system (27), each focal values v_{2i-1} of the origin is analytic at $\delta=0$ to parameter δ (the detail can be seen in [15, 35]), thus its first 4 focal values can be expressed as follows

$$\begin{aligned}
v_1 &= e^{2\pi\delta}-1 = 2\pi\delta+O(\delta), \quad & v_3 &= \mathrm{i}\pi\mu_1(\omega^2,\lambda_0,f)+O(\delta),\\
v_5 &= \mathrm{i}\pi\mu_2(\omega^2,\lambda_0,f)+O(\delta), \quad & v_7 &= \mathrm{i}\pi\mu_3(\omega^2,\lambda_0,f)+O(\delta).
\end{aligned}$$

Next, we calculate the Jacobian determinant of the function group (v_1, v_3, v_5) with respect to the variables(δ,λ_0,f). Without loss of generality, we choose freely positive real number $\omega = 1/1000$ and the first set of solutions for which the first three focal values become $v_1 = v_3 = v_5 = 0, v_7 = 0.000592461\cdots\neq 0$. Moreover, one can directly verify that the Jacobian evaluated at the critical point has

$$\left|\begin{matrix}
\frac{\partial v_1}{\partial\delta} & \frac{\partial v_1}{\partial\lambda_0} & \frac{\partial v_1}{\partial f}\\
\frac{\partial v_3}{\partial\delta} & \frac{\partial v_1}{\partial\lambda_0} & \frac{\partial v_3}{\partial f}\\
\frac{\partial v_5}{\partial\delta} & \frac{\partial v_1}{\partial\lambda_0} & \frac{\partial v_5}{\partial f}
\end{matrix}\right|_{(\delta,\lambda_0,f)=(0,\lambda_0^{(1)},f^{(1)})} = -2\pi\cdot(0.31506761\cdots)\neq 0,\tag{28}$$

implying that system (27) can have necessarily three small-amplitude limit cycles bifurcating from the origin by Lemma 2.4.

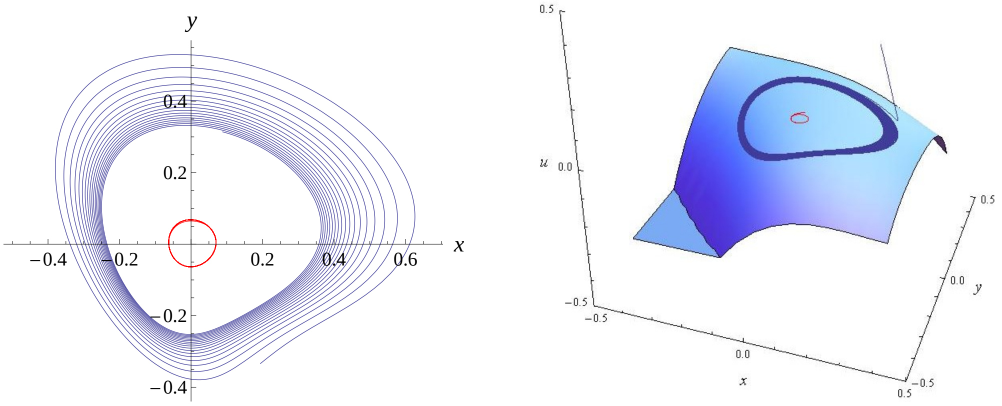
Based on the above analysis, we know that system (2) can have three small-amplitude limit cycles bifurcation from the equilibrium E_1, and the following theorem can be established.

Theorem 3.4. *At most six limit cycles can be bifurcated around the two symmetric equilibria E_1 and E_2 with a (3, 3) distribution via a Hopf bifurcation under small perturbation within the system* (2)*.*

Using numerical method, we get an example of two limit cycles bifurcated from the origin of system (27) and the symmetrical equilibria $E_{1,2}$ of system (2) respectively. First we choose $\omega = \frac{1}{1000}, a = -1, b_2 = \frac{51+\varepsilon}{115}$ with $\varepsilon > 0$, then we set

$$b_1 = \frac{32\delta}{3} - \frac{\varepsilon}{115} - \frac{1431}{115}, \quad b_4 = \frac{14\delta}{3} - \frac{\varepsilon}{115} - \frac{396}{115} \tag{29}$$

where $\delta > 0$. As a matter of fact, by comparison with the conditions (17), it is easy to see how the perturbation of coefficients occurs in (29), which just makes system (21) become (27) with perturbed linear parts. Moveover, letting $\varepsilon = 0.7, \delta = -0.001$, we have the first two focal values of the origin $v_3 = 0.000119\cdots$, $v_5 = -1.49094\cdots$.



(a) (b)

Fig.1. Phase portraits of system (27) with the initial points $(x, y, u) = (0.2, 0.2, 0.2)$ and $(0.02, 0.02, 0.02)$ respectively in the projection plane x-y for (a) and in the 3-d space x-y-u for (b).

As shown in Fig.1, these figures illustrate that the trajectories of the system (27) spiral toward one big stable limit cycle and away from another small unstable limit cycle in the neighborhood of the origin on a certain approximate center manifold.

4 Hopf bifurcation of friction free segmented disk dynamo model

In this section, we analyze the limit cycle bifurcation of system (1) at the equilibrium point. It is easy to know that system (1) has only two symmetric equilibrium points $E_\pm = (\pm 1, \pm 1, 1)$ for any parameter value. Since the system (1) is invariant under the transformation: $(x, y, z) \mapsto (-x, -y, z)$, thus we only analyze equilibrium point $E_+ = (1, 1, 1)$.

4.1 Limit cycle bifurcation at equilibrium point E_+

First, we note the Jacobian matrix of system (1) at equilibrium point E_+ as follows

$$A = \begin{pmatrix} -r & r & 0 \\ 1+m & -1-m & 1 \\ g(-1+m) & -g(1+m) & 0 \end{pmatrix}. \tag{30}$$

Moreover, the characteristic polynomial of A is

$$P(\lambda) = \lambda^3 + (1+r+m)\lambda^2 + g(1+m)\lambda + 2rg. \tag{31}$$

In order to make the equilibrium point E_+ undergo Hopf bifurcation, that is, the Jacobian matrix A has a pair of purely imaginary eigenvalues $\pm i\omega$ $(\omega > 0)$ and a real eigenvalue with negative real part $-\lambda_0$ $(\lambda_0 > 0)$, we can obtain the following critical conditions:

$$(1+r+m)(1+m)g = 2rg \Rightarrow r = \frac{(1+m)^2}{m-1}, \tag{32}$$

and yielding

$$\omega = \sqrt{g(1+m)}, \lambda_0 = \frac{2(1+m)}{1-m}; \text{ or } g = \frac{(\lambda_0+2)\,\omega^2}{2\lambda_0}, m = \frac{\lambda_0 - 2}{\lambda_0 + 2}. \tag{33}$$

where $-1 < m < 1, g(1+m) > 0$. For the convenience of calculation, we translate the equilibrium point E_+ to the origin, then system (1) becomes the following form.

$$\begin{aligned} \dot{x}(t) &= r(y-x), \\ \dot{y}(t) &= z(1+x) + (1+m)(x-y), \\ \dot{z}(t) &= g[m(x+1)^2 + 1 - (1+m)(x+1)(y+1)]. \end{aligned} \tag{34}$$

We can construct a matrix T_1, which transforms A into a diagonal matrix, that is, using the non-degenerate linear transformation $(x,y,z) = T_1(z,w,u)$, so that

$$T_1^{-1} A T_1 = \begin{pmatrix} i\omega & 0 & 0 \\ 0 & -i\omega & 0 \\ 0 & 0 & -\lambda_0 \end{pmatrix}, \tag{35}$$

where

$$T_1 = \begin{pmatrix} -\frac{\omega^2}{g(2g+2i\omega-\omega^2)} & -\frac{\omega^2}{g(2g+2i\omega-\omega^2)} & \frac{\omega^2}{-2g+\omega^2} \\ -\frac{2ig^2-ig\omega^2+\omega^3}{g\omega(2g-2i\omega-\omega^2)} & -\frac{-2ig^2+ig\omega^2+\omega^3}{g\omega(2g+2i\omega-\omega^2)} & 1 \\ 1 & 1 & 0 \end{pmatrix}, \tag{36}$$

Therefore, by nonlinear transformation $(x,y,z) = T_1(z,w,u)^T$ and setting $T = i\omega t$, system (34) becomes the following form similar to system (6):

$$\begin{aligned} \frac{dz}{dT} &= z + a_{200}z^2 + a_{020}w^2 + a_{002}u^2 + a_{110}zw + a_{101}zu + a_{011}wu = Z(z,w,u), \\ \frac{dw}{dT} &= -(w + b_{200}z^2 + b_{020}w^2 + b_{002}u^2 + b_{110}zw + b_{101}zu + b_{011}wu) = -W(z,w,u), \\ \frac{du}{dT} &= d_{001}u + d_{200}z^2 + d_{020}w^2 + d_{002}u^2 + d_{110}zw + d_{101}zu + d_{011}wu = U(z,w,u), \end{aligned} \tag{37}$$

where $T \in R$, z, w, $u \in C$, and $b_{kjl} = \bar{a}_{kjl}$, d_{kjl} $(kjl = 200, 020, 002, 110, 011, 101)$ Appendix (A.4).

Now, using the recursive formula in Lemma 2.1, we can easily calculate the first 10 singular point quantities of the origin of system (37):

If $m \neq 1$, then

$$\mu_1 = \frac{2i(-1+m)^3(1+m)^2\sqrt{g(1+m)}}{(1+g(-1+m)^2+m)(g(-1+m)^2+4(1+m))^2}. \tag{38}$$

From the condition of (32), we know that $m \neq \pm 1$, $g \neq 0$, therefore μ_1 must satisfy $\mu_1 \neq 0$ (namely the equilibrium point E_+ is never a center). This also implies that equilibrium E_+ can produce a limit cycle at most. However, we can take g as a small perturbation parameter (i.e. $g = \varepsilon \ll 1, \mu_1 \approx 0$), so that the equilibrium point E_+ becomes a second-order weak focus.

If $m \neq 1$, then

$$
\begin{aligned}
\mu_2 &= -\frac{2i(-1+m)^3(1+m)^6 f_1(\varepsilon, m)}{3M_1(\varepsilon, m)}, \\
\mu_3 &= \mu_4 \cdots = 0,
\end{aligned}
\tag{39}
$$

where $f_1(\varepsilon, m) = 80(1+m)^3 + \sum_{i=1}^{3} H_i \varepsilon^i$, $M_1(\varepsilon, m) = 4096(1+m)^8 + \sum_{j=1}^{8} H_j \varepsilon^j$ (H_i and H_j are all polynomials just in m).

In fact, when the perturbation parameter ε is small enough to approach zero, the first singular point quantity also approaches zero (i.e.$\mu_1 \approx 0$). Thus $f_1 \approx 80(1+m)^3 \neq 0$ (i.e.$\mu_2 \neq 0$). Further, we obtain the following Theorem.

Theorem 4.1. *At most two limit cycles can be bifurcated around the two symmetric equilibria E_+ and E_1 with a (2, 2) distribution via a Hopf bifurcation under small perturbation within the system* (1).

In order to verify the results of the theoretical analysis, in the next subsection, we perturb the bisymmetric limit cycle at the symmetric equilibrium point E_+ of system (1) by numerical simulation.

4.1.1 Numerical simulation of bisymmetric limit cycles at symmetric equilibrium point E_+

By satisfying conditions (32) and (38), the parameter values are set as follows:

$$r = 12.25, \quad m = 0.75, \quad g = 20.$$

Then this characteristic equation (31) has a pair of purely imaginary roots and a real root with negative real part, $\pm 5.91608i$, -14. First, we set a sufficiently small perturbation parameter δ on the coefficient m. And δ as a new independent variable, namely,

$$m = 0.75 + \delta,$$

then system (34) can be converted into the following form:

$$
\begin{aligned}
\dot{z_1} &= -5.91608 z_2 + f_1(z_1, z_2, z_3, \delta), \\
\dot{z_2} &= 5.91608 z_1 + f_2(z_1, z_2, z_3, \delta), \\
\dot{z_3} &= -14 z_3 + f_3(z_1, z_2, z_3, \delta),
\end{aligned}
\tag{40}
$$

via the variable transformation:

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -0.0530303 & -0.125493 & -7 \\ 0.00757576 & -0.151103 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix},
\tag{41}
$$

where

$$
\begin{aligned}
f_1(z_1, z_2, z_3, \delta) &= 0,004329(1.18322 \times 10^2 z_2 \delta - 280(132 z_3 + z_1)\delta) + N f_1(z_1, z_2, z_3, \delta), \\
f_2(z_1, z_2, z_3, \delta) &= 47.3286 z_3 \delta - 0.151515 z_2 \delta + 0.35855 z_1 \delta + N f_2(z_1, z_2, z_3, \delta), \\
f_3(z_1, z_2, z_3, \delta) &= 0.363636 z_3 \delta + 0.000196773(-5.91608 z_2 \delta + 14 z_1 \delta) + N f_3(z_1, z_2, z_3, \delta),
\end{aligned}
$$

and Nf_i $(i = 1, 2, 3)$ can be seen in Appendix (A.5).

Similar to the previous calculation, system (40) is restricted to the following two-dimensional center manifold.

$$\begin{aligned}\dot{z_1} &= -2.42424z_1(1 + 0.0362062\delta)\delta + 0.000070553z_2 \\ &\quad \times (83853 + 14520\delta + 1280\delta^2) + O(||(z_1, z_2)||^2), \\ \dot{z_2} &= -0.000834794z_2\delta(363 + 32\delta) + 0.179275z_1 \\ &\quad \times (-33 + 4\delta + 0.144825\delta^2) + O(||(z_1, z_2)||^2).\end{aligned} \tag{42}$$

Thus, the eigenvalues of the linear part of system (42) are $\alpha(\delta) \pm \beta(\delta)i$, where

$$\begin{aligned}\alpha(\delta) &= -0.121212\delta - 0.057243\delta^2, \\ \beta(\delta) &= 0.5\sqrt{140 + 7.27273\delta - 5.91535\delta^2 - 0.624469\delta^3 - 0.0131071\delta^4}.\end{aligned}$$

Then we have

$$\begin{aligned}\alpha(0) = 0, \quad \beta(0) = 5.91608 > 0, \\ \frac{d\alpha}{d\delta}\Big|_{\delta=0} = -0.121212 \neq 0.\end{aligned}$$

According to the Hopf bifurcation theorem, system (42) will produce a Hopf bifurcation near the origin. In other words, system (40) experiences a Hopf bifurcation near the origin. In order to verify the results of the theoretical analysis in the previous section, we can obtain the Hopf bifurcation solution and its stability via the calculation of the normal form [33]:

$$\begin{aligned}\frac{dr}{dt} &= r(-0.121212\delta - 0.0289052r^2), \\ \frac{d\theta}{dt} &= 5.91608 + 0.35855\delta - 1.06061r - 0.012499r^2,\end{aligned} \tag{43}$$

where r and θ represent the phase and amplitude of periodic orbital motion respectively.

By equation (43), we can obtain an initial equilibrium solution $r_1 = 0$ and a unique Hopf bifurcation solution $r_1^2 = -4.19343\delta$. When $\delta < 0$, the stability of the Hopf bifurcation solution is determined by $\frac{d}{dr}(\frac{dr}{dt}) = 0.24299\delta$, which means that the limit cycle bifurcation solution is stable, as shown in Fig. 1.

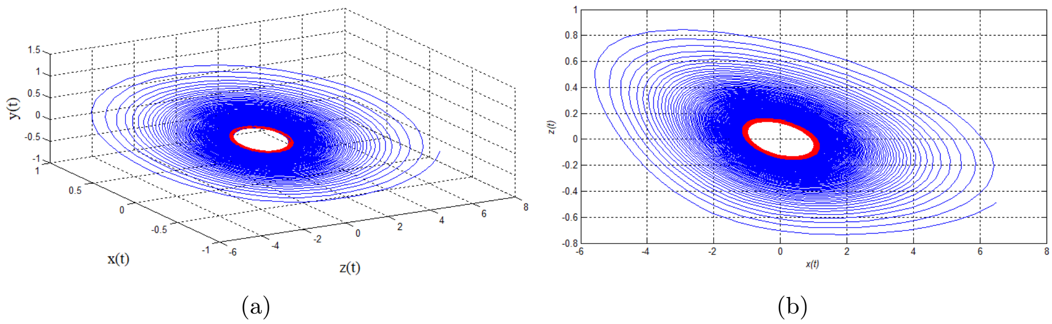


Figure 1: Trajectory projection of system (34) with initial condition (x_0, y_0, z_0) =(0.15, 0.15, 0.15) when $\delta = -0.01$. (a) $x(t)$-$y(t)$-$z(t)$-space and (b) $x(t)$-$z(t)$-plane.

5 Jacobi Stability of The Segmented Disc Dynamo System with Friction Coefficient

In this section, we calculate the KCC invariant of system (2), by changing system (2) into a second-order differential equation. From the first equation of system (2), y can be expressed as

$$y = \frac{rx + \dot{x}}{r}. \tag{44}$$

By substituting y into the second equation of system (2), we obtain

$$-\frac{1}{r}(r\dot{x} + \ddot{x} + m\dot{x} + \dot{x}) + xz - x = 0. \tag{45}$$

Taking the derivative of the third equation of system (2) with respect to time t we get

$$\ddot{z} = -f\dot{z} + 2gmx\dot{x} - gmx\dot{y} - gm\dot{x}y - gx\dot{y} - g\dot{x}y. \tag{46}$$

Substituting Eq. (44) and Eq. (45) into Eq. (46), we obtain the following equation for $\ddot{z}$,

$$\begin{aligned}&\ddot{z} - \frac{1}{r}[rzf^2 + \dot{x}xfg(m+1) + fgr\left(x^2 - 1\right) + grx\dot{x}((m-1)\\&- x(m+1)(z-1)) + g\dot{x}(m+1)(mx + x - \dot{x})] = 0.\end{aligned} \tag{47}$$

In above system of equation, let us change the notation as

$$x = x_1, \quad \frac{dx}{dt} = y_1, \quad z = x_2, \quad \frac{dz}{dt} = y_2,$$

then Eq. (47) can be changed into the following second-order differential equation

$$\frac{d^2x_i}{dt^2} - 2G^i(x_i, y_i, t) = 0, \quad i = 1, 2,$$

where

$$\begin{aligned}G^1(x_1, x_2, y_1) &= \frac{1}{4}[rx_1(-1 + x_2) - (1+m)y_1 - ry_1],\\G^2(x_1, x_2, y_1) &= \frac{1}{4r}[fgr(-1 + x_1^2) - f^2rx_2 - fgx_1y_1(1+m)\\&\quad - grx_1(x_1(-1-m)(-1+x_2) + (-1+m)y_1)\\&\quad - gy_1(1+m)(x_1 + mx_1 - y_1)].\end{aligned}$$

Therefore, we can first get the components of nonlinear connection, as shown in the following form

$$\begin{aligned}N_1^1 &= \frac{1}{2}(-1 - m - r), \quad N_2^1 = N_2^2 = 0,\\N_1^2 &= \frac{1}{2r}[-g(1 + m^2 + f(1+m) - r + m(2+r))x_1 + 2g(1+m)y_1].\end{aligned}$$

For the components of the Berwald connection we obtain

$$\begin{aligned}&G_{12}^1 = G_{21}^1 = G_{22}^1 = G_{12}^2 = G_{21}^2 = G_{22}^2 = G_{11}^1 = 0,\\&G_{11}^2 = \frac{1}{r}g(1+m).\end{aligned}$$

Meanwhile, the components of the first KCC-invariant and the deviation curvature tensor namely the second KCC-invariants can be obtained as

$$\begin{aligned}\epsilon^1 &= rx_1(-1 + x_2) - \frac{1}{2}[(1+m)y_1 - ry_1],\\\epsilon^2 &= \frac{1}{2r}[-2f^2rx_2 - fg(2r(-1 + x_1^2) + (1+m)x_1y_1)\\&\quad + gx_1(-(1+m)^2y_1 + r(2x_1(1+m)(-1+x_2) + y_1 - my_1))],\end{aligned}$$

and

$$\begin{aligned}
&P_1^1 = r + \frac{1}{4}(1+m+r)^2 - rx_2, \quad P_2^1 = -rx_1, \\
&P_2^2 = f^2 - g(1+m)x_1^2, \quad P_1^2 = \frac{1}{2r}[g((1+m)(3+f+3m)y_1 \\
&\quad + r(4fx_1 - 6x_1(1+m)(-1+x_2) + y_1 + 3my_1)))].
\end{aligned} \tag{48}$$

In fact, the matrix of the deviation tensor is given by

$$\begin{pmatrix} P_1^1 & P_2^1 \\ P_1^2 & P_2^2 \end{pmatrix},$$

and its characteristic equation is

$$\lambda^2 - (P_1^1 + P_2^2)\lambda + P_1^1 P_2^2 - P_1^2 P_2^1 = 0. \tag{49}$$

Via the definition [39] and the Routh-Hurwitz criteria, we can determine the Jacobi stability of the segmented disc dynamo system. That is, system (2) is Jacobi stable only when

$$-(P_1^1 + P_2^2) > 0, \quad P_1^1 P_2^2 - P_1^2 P_2^1 > 0, \tag{50}$$

otherwise it is Jacobi unstable. Therefore, the results of Jacobi stability at the equilibrium point can be obtained as follows.

Theorem 5.1. *(a) The equilibrium point $E_0 = (0, 0, \frac{g}{f})$ of the system (2) is Jacobi unstable for any parameter value.*
(b) The equilibrium point $E_{1,2} = (\pm d_0, \pm d_0, 1)$ are Jacobi stable if it satisfies simultaneously the constraints

$$\begin{aligned}
&r \in \mathbb{R} \quad \text{and} \quad g > \frac{r(2m+r+2)}{4(m+1)}, \quad m > -1 \\
&\frac{1}{2}\left(-\sqrt{\Delta} - m - 1\right) < f < \frac{1}{2}\left(\sqrt{\Delta} - m - 1\right),
\end{aligned} \tag{51}$$

where $\Delta = 4g(m+1) - r(2m+r+2)$, and Jacobi unstable, otherwise.

Proof. (a) Calculation the components of the deviation curvature tensor at the the equilibrium point E_0,

$$p_1^1(E_0) = -\frac{gr}{f} + \frac{1}{4}(m+r+1)^2 + r, \quad P_2^1(E_0) = P_1^2(E_0) = 0, \quad P_2^2(E_0) = f^2.$$

Then it is easy to obtain the eigenvalues of the deviation curvature tensor at at the the equilibrium point E_0 by using Eq. (53) as

$$\lambda_+(E_0) = f^2, \quad \lambda_-(E_0) = \frac{1}{4}\left(r\left(-\frac{4g}{f} + r + 6\right) + m^2 + 2m(r+1) + 1\right). \tag{52}$$

Therefore, Eq. (52) means that the eigenvalue is always positive, so the equilibrium point E_0 is Jacobi unstable.
(b)Due to the system (2) are invariant under the transformation $(x, y, z) \mapsto (-x, -y, z)$, thus we only analyze equilibrium point E_1. The Jacobi matrix at the equilibrium point E_1 is

$$p_j^i = \begin{pmatrix} \frac{1}{4}(m+r+1)^2 & -r\sqrt{1-\frac{f}{g}} \\ 2f\sqrt{g^2 - gf} & f^2 + (m+1)(f-g) \end{pmatrix},$$

□

its characteristic equation is

$$\lambda^2 - tr(p_j^i)\lambda + det(p_j^i) = 0. \tag{53}$$

By using Routh-Hurwitz criteria, we obtain the result (51).

6 Conclusion

In this work, we have studied the limit cycle bifurcation problem of two kinds of segmented disc dynamo system by applying the formal series method of computing singular point quantities. And we have strictly proved that the SDD model perturbs at most four small amplitude limit cycles simultaneously at the symmetric equilibrium point $E_{\pm} = (\pm 1, \pm 1, 1)$, two of which are stable. Furthermore, we also study in detail that SDDF model perturbs at most six limit cycles synchronously at the symmetric equilibrium point $E_{1,2} = (\pm d_0, \pm d_0, 1)$, four of which are stable. In addition, by choosing the appropriate bifurcation parameters, we have found the bifurcation path of the limit cycle and judged the bifurcation direction. Via numerical simulation, the results of theoretical analysis have been perfectly verified. At the same time, it also implies that limit cycle oscillation will occur when SDD and SDDF systems are operating on a certain parameter interval.

Acknowledgments

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Appendix A

$$\begin{aligned}
&T_{11} = \frac{iM_2\lambda_0(-f+\lambda_0)\omega}{(f-\lambda_0-i\omega)(2f(f-\lambda_0)+(2+\lambda_0)\omega^2)}, T_{12} = -\frac{iM_2\lambda_0(-f+\lambda_0)\omega}{(f-\lambda_0+i\omega)(2f(f-\lambda_0)+(2+\lambda_0)\omega^2)},\\
&T_{13} = -\frac{f-\lambda_0}{2M_2f+M_2f\lambda_0}, T_{21} = -\frac{M_2(2f(f-\lambda_0)+i(f-\lambda_0)\lambda_0\omega+(2+\lambda_0)\omega^2)}{(f-\lambda_0-i\omega)(2f(f-\lambda_0)+(2+\lambda_0)\omega^2)},\\
&T_{22} = -\frac{M_2(2f(f-\lambda_0)-i(f-\lambda_0)\lambda_0\omega+(2+\lambda_0)\omega^2)}{(f-\lambda_0+i\omega)(2f(f-\lambda_0)+(2+\lambda_0)\omega^2)}, T_{23} = \frac{M_2(2f(-f+\lambda_0)-(2+f)\omega^2)}{f(2f(f-\lambda_0)+(2+\lambda_0)\omega^2)},\\
&M_2 = \sqrt{1+2f(f-\lambda_0)/(2+\lambda_0)\omega^2}.
\end{aligned} \tag{54}$$

$$\begin{aligned}
&a_{200} = \frac{\lambda_0^2(-3i\lambda_0+4\omega)}{4(2+\lambda_0)(\lambda_0+i\omega)^2\omega}, \quad a_{110} = \frac{i\lambda_0^2(\lambda_0+2i\omega)}{2(2+\lambda_0)\omega(\lambda_0^2+\omega^2)},\\
&a_{010} = -\frac{\lambda_0(\lambda_0+2i\omega)}{4(\lambda_0+i\omega)}, \quad a_{020} = \frac{i\lambda_0^3}{4(2+\lambda_0)(\lambda_0-i\omega)^2\omega},\\
&a_{011} = -\frac{\lambda_0^2}{4\lambda_0-4i\omega}, \quad a_{002} = -\frac{i\lambda_0(2+\lambda_0)\omega}{16},\\
&d_{200} = -\frac{\lambda_0^3(2\lambda_0^2+i\lambda_0\omega+2\omega^2)}{(2+\lambda_0)^2(\lambda_0-i\omega)(\lambda_0+i\omega)^3\omega^2}, \quad d_{110} = \frac{2i\lambda_0^4}{(2+\lambda_0)^2\omega(\lambda_0^2+\omega^2)^2},\\
&d_{101} = \frac{i\lambda_0^2(\lambda_0^2+i\lambda_0\omega+\omega^2)}{(2+\lambda_0)(\lambda_0-i\omega)(\lambda_0+i\omega)^2\omega}, \quad d_{020} = \frac{\lambda_0^3(2\lambda_0^2-i\lambda_0\omega+2\omega^2)}{(2+\lambda_0)^2(\lambda_0-i\omega)^3(\lambda_0+i\omega)\omega^2},\\
&d_{011} = \frac{\lambda_0^2(i\lambda_0^2+\lambda_0\omega+i\omega^2)}{(2+\lambda_0)(\lambda_0-i\omega)^2(\lambda_0+i\omega)\omega}, \quad d_{002} = \frac{i\lambda_0^2\omega}{4(\lambda_0^2+\omega^2)}.
\end{aligned} \tag{55}$$

$$
\begin{aligned}
Nf_1 = {} & z_2^2(-0.4274568-0.0642715\delta)+z_3^2(979.99999+119.99999\delta)+z_1(0.0564426079z_1\\
& -1.2122111212\delta+0.062715518z_1\delta)+z_2(-0.04753631z_1+0.51246698\delta+0.1249443z_1\delta)\\
& +z_3(14.84848848z_1-159.999999\delta+16.969696z_1\delta+z_2(-6.27463007+16.493313333\delta)),\\
Nf_2 = {} & 41.41258z_3z_1+0.7242z_2z_1+0.31331z_1^2+47.32882z_3\delta-0.155151z_2\delta+0.3585284z_1\delta,\\
Nf_3 = {} & z_3^2(-7.42424-8.484444\delta)+z_2^2(0.00383+0.000869\delta)+z_3(-0.854124z_1+z_2(0.047535\\
& -0.124993\delta)+0.36366\delta-0.12858z_1\delta)+z_2(-0.012947z_1-0.014127\delta-0.094658z_1\delta)\\
& +z_1(-0.00605051z_1+0.002754482\delta-0.00048696z_1\delta).
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
f_2 = {} & 64f^4(f-\lambda_0)^2\lambda_0^8(-2f+3\lambda_0)+f^2\lambda_0^6(-640f^5+3536f^4\lambda_0-5688f^3\lambda_0^2\\
& +3128f^2\lambda_0^3-270f\lambda_0^4+15\lambda_0^5)\omega^2-2f\lambda_0^4(448f^6-4784f^5\lambda_0+12992f^4\lambda_0^2\\
& -12528f^3\lambda_0^3+3330f^2\lambda_0^4-378f\lambda_0^5+5\lambda_0^6)\omega^4+\lambda_0^2(-3456f^7+22032f^6\lambda_0\\
& -69400f^5\lambda_0^2+100440f^4\lambda_0^3-43966f^3\lambda_0^4+6311f^2\lambda_0^5+412f\lambda_0^6-10\lambda_0^7)\omega^6\\
& +2\lambda_0(14688f^6-55880f^5\lambda_0+96856f^4\lambda_0^2-60508f^3\lambda_0^3+10865f^2\lambda_0^4\\
& +1111f\lambda_0^5+50\lambda_0^6)\omega^8+2(-31968f^5+67296f^4\lambda_0-70904f^3\lambda_0^2+16000f^2\lambda_0^3\\
& +2724f\lambda_0^4-233\lambda_0^5)\omega^{10}-192(180f^3+46f^2\lambda_0-109f\lambda_0^2+22\lambda_0^3)\omega^{12}+8640f\omega^{14}).
\end{aligned}
\tag{57}
$$

$$
\begin{aligned}
G_2 = {} & 5126135\lambda_0^{50}+268110066\lambda_0^{48}\omega^2+1502553085\lambda-0^{46}\omega^4-520598497926\lambda_0^{44}\omega^6\\
& -34949021828178\lambda_0^{42}\omega^{14}-1260337141317916\lambda_0^{40}\omega^{10}-29914183333475478\lambda_0^{38}\omega^{12}\\
& -500020317085453044\lambda_0^{36}\omega^{14}-6094302809080027349\lambda_0^{34}\omega^{16}-5531653499158\lambda_0^{32}\omega^{18}\\
& -37894434937162191779 1\lambda_0^{30}\omega^{20}-19740103509002512\lambda_0^{28}\omega^{22}-782381173445\lambda_0^{26}\omega^{24}\\
& -232478181683428\lambda_0^{24}\omega^{26}-4853512758411730803\lambda_0^{22}\omega^{28}-50557038465137280\lambda_0^{20}\omega^{30}\\
& +9177992526648467\lambda_0^{18}\omega^{32}+6219840525317786682 9\lambda_0^{16}\omega^{34}+179615716630024\lambda_0^{14}\omega^{36}\\
& +35355356946162879894\lambda_0^{12}\omega^{38}+505917133669081\lambda_0^{10}\omega^{40}+5187437948120\lambda_0^{8}\omega^{42}\\
& +351790582998186433\lambda_0^{6}\omega^{44}+12050077796558696223\lambda_0^{4}\omega^{46}+56540934714576\lambda_0^{2}\omega^{48}\\
& -4753836506838624640 00\omega^{50}.
\end{aligned}
\tag{58}
$$

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