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Equation of State of White Dwarfs and Mass-Radius Estimation in the Newtonian Limit

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ABSTRACT

White Dwarf stars are one of the densest form of matter following neutron star and black holes. A typical White Dwarf is as massive as our Sun has radius comparable to the Earth. This paper reviewed the Fermi gas model Equation of State of White Dwarf and numerical computation of Mass-radius and pressure density profile. A section in brief has been included for the calculation of average speed of electrons in the White Dwarf environment.

Keywords: Equation of State, White Dwarfs, Polytrope, TOV, Fermi Gas.

1. INTRODUCTION

[1]. White dwarfs are form of compact objects that stable against gravitational collapse by support of their internal pressure from degenerate matter. For the first time White Dwarf was seen in a telescope in 1862. The theoretical explanation for the properties of the White Dwarf came many years later, when famous Dirac formulated the Fermi–Dirac statistics, that was based on ideas of degenerate matter, and later Fowler in 1926 applied it to describe White Dwarfs. Ultimately, Chandrasekhar included the concept of special relativity and finally he derived a maximum mass for White Dwarfs known as Chandrasekhar mass limit. In the following section we will talk about motivation and logistics for White Dwarf in a beginner friendly way.

2. MOTIVATION AND LOGISTICS

The motivation of this paper is to get hand on experience with White Dwarf star. Constructing Equation of State(EoS) considering the Fermi electron gas in non-relativistic and relativistic regime. Using the theoretical principles of quantum mechanics and thermodynamics to numerically solve for the mass, radius and density of white dwarfs with Tolman-Opeenheimer-Volkoff (TOV) equations in the Newtonian limit. The essential ingredients required by this algorithm is the equation of state, which is developed in Section 3 and 4. To test correctness of the results, author has compared his work with the existing literature https://arxiv.org/abs/astro-ph/0506417

3. STELLAR EQUATIONS OF STATE

The main aim in astrophysics is to study the stellar structure. To do that we require the equations that could possibly describe the behavior of stellar matter in gravitational fields. Usually Equations of state draw the information about the microscopic physics of the star and that mainly correlate thermodynamic variables like Pressure P, Temperature T, density ρ. Stellar equations of state (EoS) also reflect microscopic properties of the gas in stars. A low-density gas shows classical behaviour while at a enormous high-density gas behaves quantum mechanically. In the next section we will develop the EoS in both Non-relativistic and Relativistic regime for the White Dwarf.

3.1. *Physics input to construct EoS*

The physics required to construct Equation of State can be understood in terms of fundamental physics concepts.

- Debroglie hypothesis and Wave Particle Duality: quantum description of matter is understood interns of particle–wave duality waves are characterized by a de-Broglie wavelength $\lambda = \frac{h}{p}$, where p is the momentum and h is Planck constant. .
- Heisenberg Uncertainity: Principle The Heisenberg uncertainty principle quantifies the fuzziness of particle–wave duality: $\Delta x \Delta p \geq h$

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[1] https://github.com/mr-tousif/

- Degeneracy: In the Degenerate matter, all the lowest energy levels are completely filled and all the higher-energy states are empty.
- Quantum Statistics: electrons obey Fermi–Dirac statistics as well as follow the Pauli exclusion principle: no two fermion can occupy same quantum state.

4. FERMI GAS OF ELECTRON

In astrophysics, the most common form of a equation of state is polytrope,

$$P(r) = K\rho^{\gamma} = K\rho^{\frac{n+1}{n}}(r) \tag{1}$$

where, P is the gas pressure and ρ is the density in arbitrary unit. γ is the polytropic index related to n. A polytropic approximation implies physically that the pressure is independent of temperature, depending only on density and composition. However in our study we will develop the prescription that involoves Fermi gas of electrons. With equation of state we will attempt to find the solutions for the equations of stellar structure "White Dwarf".

4.1. *White Dwarf: Nonrelativstic EoS*

Degenerate equations of state play an crucial role in a variety of astrophysical applications. In White Dwarf the electrons are particularly degenerate.

In a complete degenerate gas, the density is high enough so that all the electron states having energy less than some maximum energy are filled. Since, the total number density of electrons is to be finite, the density of states can be filled up to limiting value of electron's momentum

$$n_e(p)dp = \begin{cases} \frac{g_e}{h^3}4\pi r^2 dp & p \leq p_F \\ 0 & p \geq p_F \end{cases}$$

$g_e = 2$ is the electron degeneracy factor. It is evident that the complete degeneracy is the state of minimum kinetic energy. The electron number density in degenerate gas is related to the maximum momentum

$$n_e = \int_0^{p_F} n_e(p)dp = \frac{8\pi}{h^3}\frac{p_F^3}{3} \tag{2}$$

The energy associated with the Fermi momentum p_F is called the Fermi Energy and the $p_F = \left(\frac{3h^3 n_e}{8\pi}\right)^{1/3}$. The pressure $P_e|_{NR}$ of a completely degenerate electron gas in can be computed as

$$P_e|_{NR} = \frac{1}{3}\int_0^{p_F} p v_e n_e(p)dp = \frac{8\pi}{15h^3}\frac{p_F^5}{m_e} \tag{3}$$

where, $v_e = \frac{p_F}{m_e}$ is the non-relativistic electron velocity and m_e is the electron rest mass. It is clear $p_F \ll m_e c^2$ for the degenerate distribution and electron degeneracy pressure is completely determined by the electron number density n_e. Finally we write

$$P_e|_{NR} = \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{2/3} n_e^{5/3} \tag{4}$$

The number density of electron may be expressed in terms of mass density

$$n_e = \frac{\rho N_A}{\mu_e} \tag{5}$$

where N_A is the avogadro number and μ_e is the mean molecular weight, not the electron chemical potential, one should not get confused. Ideally for complete ionization

$$\mu_e = \frac{2}{1+x_H} = 1.25$$

with the hydrogen ionization fraction $x_H \approx 0.6$

Finally with the help of eqn.(4),(5) and (6) we can conclude,

$$P_e|_{NR} = \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{N_A}{\mu_e}\right)^{5/3}\rho^{5/3} = K_{NR}\rho^{5/3} \tag{6}$$

Where we assigned,

$$K_{NR} = \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{N_A}{\mu_e}\right)^{5/3} \tag{7}$$

At this moment, it is easy understand that the eqn. (2) and eqn.(7) are identical to each other and drawing the same physics information about pressure and density of the non-relativistic degenerate electron gas.

4.2. *White Dwarf: Relativistic EoS*

With the increase in electron density, the maximum momentum p_F also increases and certianly a density is reached where the electron distribution in white dwarfs become relativistic means at this point $p_F c \approx 2m_e c^2 \approx MeV$. The value of p_F then becomes

$$p_F = \frac{E_e}{c} \tag{8}$$

but still $p_F \ll m_N$, so nucleons are treated as being nonrelativistic. Before calculating the relativistic electron pressure, let us try to compute those densities for which the relativistic nature appears.

We simply write the relativistic electron energy as

$$E_e = p_F c = hc\left(\frac{3n_e}{8\pi}\right)^{1/3} = 5.15\times 10^{-3}\left(\frac{\rho}{\mu_e}\right)^{1/3} \tag{9}$$

With $E_e \equiv 1MeV$ we find,

$$\rho = \left(7.3 \times 10^6 gm/cm^3\right) \mu_e \tag{10}$$

The above equation tells us as the density approaches the value ρ, the relativistic kinematics must be used to compute the pressure integral inside a white-dwarf environment. From now it is straightforward to calculate the electron degeneracy pressure as

$$P_e|_R = \frac{1}{3}\int_0^{p_F} \epsilon n_e(p)dp = c\left(\frac{3}{\pi h}\right)^{1/3}\left(\frac{N_A}{\mu_e}\right)^{4/3}\rho^{4/3} \tag{11}$$

$$P_e|_R = K_R \rho^{4/3} \tag{12}$$

Where we assigned,

$$K_R = ch\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{N_A}{\mu_e}\right)^{4/3} \tag{13}$$

At this point we have constructed the equation of state under a appropriate conditions, now our goal is to compute the Mass-radius and pressure density profile using Tolman-Oppenheimer-Volkoff equations in the next section.

5. TOV EQUATION

Tolman-Oppenheimer-Volkoff(TOV) Equation for compact objects generally written as https://en.wikipedia.org/wiki/Tolman%E2%80%93Oppenheimer%E2%80%93Volkoff_equation

$$\frac{dm(r)}{dr} = \frac{4\pi r^2 \epsilon(r)}{c^2} = 4\pi r^2 \rho(r) \tag{14}$$

$$\frac{dP(r)}{dr} = -\frac{G\epsilon(r)m(r)}{c^2 r^2} \cdot X \tag{15}$$

$$X = \left[1 + \frac{P(r)}{\epsilon(r)}\right]\left[1 + \frac{4\pi r^3 P(r)}{m(r)c^2}\right]\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1} \tag{16}$$

The factor that we have in front is the classical equation for hydrostatic equilibrium and it is perceptible that The first two factors in square brackets represent special relativistic corrections of order $\frac{v^2}{c^2}$ that come from the mass–energy relation so that the denominators ϵ and mc^2 .The last term that we included in brackets is a general relativistic correction based on the physical significance of the Schwarzschild form of the metric in General relativity and the $m(r)$ is the total integrated mass out to a radial distance r. These corrections each act to strengthen the gravitational interaction. The Newtonian limit can be easily recovered by setting $\epsilon \to \rho$, $P = 0$, and ignoring the Schwarzschild factor in the denominator. However we solve the TOV equation in the Newtonian limit means (1),

$$\left[1 + \frac{P(r)}{\epsilon(r)}\right]\left[1 + \frac{4\pi r^3 P(r)}{m(r)c^2}\right]\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1} = 1 \tag{17}$$

6. SCALING OF THE TOV EQUATION

$$\frac{dP}{dr} = -\frac{G\epsilon(r)m(r)}{c^2 r^2}$$

$$\frac{dm}{dr} = \frac{4\pi r^2 \epsilon(r)}{c^2}$$

Introducing $R_o = \frac{2GM}{c^2}$ and $\bar{m}(r) = \frac{m(r)}{M_\odot}$ and rewriting the above two odes we get,

$$\frac{dP(r)}{dr} = -\frac{R_O}{2r^2}\left(\frac{P(r)}{k}\right)^{1/\gamma}\bar{m(r)} \tag{18}$$

$$\frac{d\bar{m}(r)}{dr} = \frac{4\pi r^2}{M_\odot c^2}\left(\frac{P(r)}{k}\right)^{1/\gamma} \tag{19}$$

For the given central density we can solve these coupled odes for mass and radius estimate. The boundary conditions are that the pressure at the center is given by $P(0) = P_c$ and that the pressure reached the lowest values at the surface usually . Note, that the pressure has to be vanished at the surface to ensure hydrostatic equilibrium. One can use the standard python library. Scipy integrate routine eventually use the defualt Runge-Kutta Method to solve coupled differential equation. However Euler method can also be used for this kind of problem.

Table 1. A Typical White Dwarf Profile [a]

Model	Central Pressure [dyne/cm^2]	Central Density [gm/cm^3]
(1)	(2)	(3)
P=K$\rho^{5/3}$	$\approx 10^{21}$	$\approx 10^{5.4}$
P=K$\rho^{4/3}$	$\approx 10^{25}$	$\approx 10^{6}$

[a] I. sagert et all. paper see reference (2).

Note—Here K represents the constant factor that is given by eqn.(7) and eqn.(13) for Nonrelativistic and relativistic EoS separately.

7. ELECTRON SPEED IN THE WHITE DWARF ENVIRONMENT

Since we already calculated the pressure and electron number density for both relativistic and nonrelativisc equation of state in **section3.1** and **section 3.2**, now we can easily compute the electron speed in the white dwarf environment. Calling eqn.(3) and eqn.(4) we get the electron speed nonrelativistic white dwarf turns out to be

$$v_e|_{NR} = \left[\frac{15h^3}{8\pi m_e^4} P_e|_{NR}\right]^{1/5} \tag{20}$$

However in the relativistic EoS, the Fermi momentum of electron is given by eqn. (7), introduncing relativistic definition of momentum we write,

$$\gamma m_e v_e|_R = \frac{E_e}{c} \tag{21}$$

where the $\gamma = \frac{1}{\sqrt{1-(v_e/c)^2}}$

Putting the gamma factor and re-arranging the term we get the speed of relativistic electron in the white dwarf as

$$v_e|_R = \left[\frac{E_e^2}{c^2}\frac{1}{m_e^2c^4 + E_e^2}\right]^{1/2} \tag{22}$$

Where E_e is given by eqn. (8). Computation of $v_e|_{NR}$ and $v_e|_R$ requires proper units. Here the subscript NR and R represents nonrelativtic and relativistic nature. Author has used these notation to make them clear to the reader.

8. RESULT AND DISCUSSION

In this section we report the result of the numerical computation of the TOV equation for the relativistic and non-relativistic Fermi gas Equation of state.

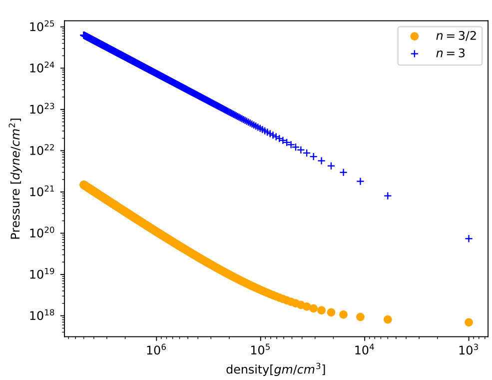

Figure 1. Pressure- density profile of White Dwarf with relativistic EoS (n=3) and nonrelativistic EoS(n=3/2) . See Table 1.

It is expected from the equation of state that as density goes on decreasing, we expect less amount of pressure and the surface of the White Dwarf usually we get lowest values of pressure incomparison to it's core.

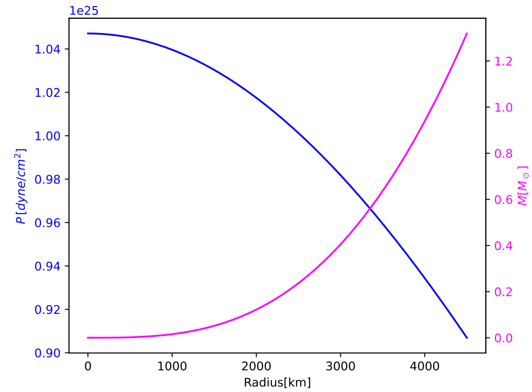

Figure 2. Pressure-Mass and Radius profile of White Dwarf with Relativistic EoS . See Table 1.

Normal gas are dilute and non-degenerate in nature, But in a white dwarf, the density which much higher, hence all the electrons are distributed much closer to each other in a compact form. This is often referred to as a degenerate Fermi gas, meaning that all the energy levels in its atoms are filled up with electrons. Once a star is in degenerate state, the gravity force no longer compress it as quantum mechanically no more available space to be taken up by the electrons. So the White Dwarf is stable because of the quantum mechanics that prevent its complete collapse of the star against the gravity.

Degenerate matter has interesting properties. For instance, the more massive a white dwarf is, the smaller the radius it will have. Reason is the more mass a white dwarf has, the more its electrons squeezed together to

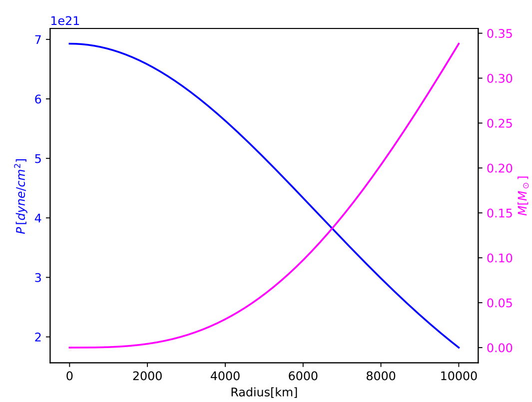

Figure 3. Pressure-Mass and Radius profile of White Dwarf with Nonrelativistic EoS . See Table 1.

maintain enough outward gravity pressure that eventually hold up the additional mass. However, there is a limit on the amount of mass a white dwarf can have. It is Chandrasekhar who discovered White Dwarf mass limit to be 1.4 times the mass of the Sun. This is popularly known as the Chandrasekhar limit in the honor of inventor. In fig 2. we see the mass of the White Dwarf is 1.2 solar mass which is still less than the Chandrasekhar mass has the radius around 4000 km. On the other hand we see in fig 3. White Dwarf has the radius around 10000 Km with lower almost 1/3 of solar mass and both these structure are stable against the gravitational collapse.

9. CONCLUSION

Numerically solving for large-scale properties of a white dwarf requires usage of the physics of a quantum scale . Investigating the pressure-density relationship for degenerate electrons are necessary for the standard stellar structure routine. It is observed that non-relativistic white dwarfs have smaller masses and larger radii where as relativistic white dwarfs have a larger mass, maximum mass known as the Chandrasekhar Limit alongside a comparatively shrinking radius. The results reported in this paper are in good agreement with the existing literature.

10. SCOPE OF IMPROVEMENTS

White dwarfs are interesting objects to study due to their known equation of state and many of them has already been discovered, One can easily compute the mass-radius and pressure density profile. The presented work in this paper assuming non-magnetic and non-rotating White Dwarf, a simplified model compared to a realistic white dwarf. For complete study of the White Dwarfs, all these effects are significant and need to be taken into the account(3).

11. ACKNOWLEDGENMENTS

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APPENDIX

A. UNIT CONVERSION

Based on our subject of interest, the author has used several unit conversion factors which are non-trivial for the numerical analysis.

$$1eV = 1.6 \times 10^{-19} J = 1.6 \times 10^{-12} dyne \cdot cm \tag{A1}$$

$$1\,MeV = \frac{1.6}{9 \times 10^{26}}\,gm \tag{A2}$$

$$\frac{dyne}{cm^2} = \frac{1}{1.6 \times 10^{33}} \frac{MeV}{fm^3} \tag{A3}$$

$$\frac{gm}{cm^3} = \frac{1}{1.777 \times 10^{12}} \frac{MeV}{fm^3} \tag{A4}$$

B. CODE LISTING

Computation of differential equation is a often problem for Physics and Astrophysics major. Keeping in mind the interest of reader of Undergrad or Graduate level, the author has decided to make this work publicly available in his github account. Following the link, https://github.com/mr-tousif/Computational_Physics.

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