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Qualitative analysis of the Prabhakar-Caputo type fractional delayed equations†

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Summary
The representation of an explicit solution to the Prabhakar fractional differential delayed system is studied employing the far-famed Laplace transform technique. Second, the existence uniqueness of the solution is debated together with the Ulam-Hyers stability of a semilinear Prabhakar fractional differential delayed system. Thirdly, the necessary and sufficient circumstances for the controllability of linear Prabhakar fractional differential delayed system are determined by describing the Gramian matrix. A sufficient circumstance for the relative controllability of a semilinear Prabhakar fractional differential delayed system is studied via the Krasnoselskii's fixed point theorem. Numerical examples are offered to verify the theoretical findings.

KEYWORDS:
Prabhakar fractional derivative, differential delayed equation, existence uniqueness, stability and controllability, Mittag-Leffler function

1 | INTRODUCTION

If an equation consists of the state and its rate of changes, it is known as either an ordinary differential equation or a partial differential equation. If it also includes the past state, it is called as a differential delayed equation. Although it did not make sense that the future state depends on the past state, at the beginning of the 1900s, Volterra modelled certain differential delayed equations in his studies [1 2] such as predator-prey and viscoelasticity. In the same years, Minorskii [3] formulated the differential delayed equations whose delays were used in the feed back mechanism in his different kinds of studies such as ship stabilization and automatic steering. These kinds of works have revealed the importance of the past state(delay parameters) in the theory of both control and differential equations. When having look at studies about (fractional) differential delayed equations in the literature, we have observed that they have been investigated by many of researchers in many aspects such as existence and uniqueness of their solutions, controllability, stability, etc. For more details, the readers can check the references [11 - 28]. As can be seen in the cited works, it is more difficult to model and solve a differential delayed equation and check whether it is controllable or stable, etc according to an ordinary differential equation which does not includes delay parameters. Even if one gets the delay parameter equal to zero, the available results are valuable again.

Fractional calculus, in brief, can be seen as an extension of integer calculus. Although its birthdate dates back to 1600 years such as the traditional calculus, it has been the focus of attention by the most of researchers for the last 30 years. One of the main reasons for this is the fact that almost all of scientific improvements for fractional calculus are valid for traditional calculus. Another main reason is the fact that it is noticed that it models many of real-life social problems or real-world systems more appropriate than the traditional calculus, etc. Today, fractional calculus is employed in many sorts of areas such as biophysics, control theory, engineering, electrochemistry, signal, mathematical physics, etc; see [4 - 10].

†Qualitative analysis of the Prabhakar fractional delayed system.

In the quite recent years, Prabhakar fractional calculus have been revealed in the scientific world. The fractional integral operator which was firstly described in the reference[29] causes the onset of the Prabhakar fractional calculus. It is profoundly investigated and studied in the reference[30] and expanded to the notion(concept) of fractional derivatives in the reference[31]. It has been applied to pure and applied mathematics[32 33] and miscellaneous applications[34 35]. On the other hands, the Prabhakar fractional derivatives can be made into different types of fractional operators like the Lorenzo-Hartly, the Miller-Ros, Riemann-Liouville, Gorenflo-Minerdi, Caputo fractional operators, etc.

To the best of our knowledge, there is no study about the Prabhakar Caputo-type fractional differential delayed equations. The above-counted explanations and the above-cited works have inspired us to take into consideration the following semilinear Prabhakar Caputo-type fractional delayed equations

$$\begin{cases} {}^{PC}_{0^+}\mathcal{D}^{H,\delta}_{\alpha,\beta} w(t) = Zw(t-r) + \daleth(t, w(t)), \quad t \in (0,T], \quad r > 0, \\ \qquad\qquad w(t) = \phi(t), \;\; t \in [-r, 0] \end{cases} \tag{1}$$

where ${}^{C}_{0^+}\mathcal{D}^{H,\delta}_{\alpha,\beta}$ represents the Prabhakar Caputo-type derivative of fractional order $0 < \beta < 1$, $H, Z \in \mathbb{R}^{n\times n}$, $T = lr$ for a fixed natural number l, and r is a retardation(delay), the disturbing function $\daleth : [-r, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $\phi : [-r, 0] \to \mathbb{R}$ is an absolutely continuously differentiable function.

This paper is arranged as noted below. In Section 1, the brief history about the differential delayed equation, Fractional calculus and Prabhakar fractional calculus is stated and the Prabhakar Caputo-type differential delayed system is introduced. In Section 2, the available tools in the literature we will use are presented. In Section 3, an analytical solution to the linear Prabhakar fractional differential delayed system is obtained from the Laplace technique and a global solution to the semilinear system is offered. In Section 4, the existence and uniqueness of the solutions to the semilinear system is proved and the Ulam-Hyers stability of the semilinear system is debated. In Section 5, the Gramian matrix is defined, the necessary and sufficient circumstances for controllability of the linear version are presented and the sufficient circumstance for relative controllability of the semilinear version is offered. In Section 6, our theoretical findings are supported by means of numerical examples.

2 | PRELIMINARIES

In this section, the most fundamental tools are presented to make the coming findings easily understandable.

$\mathbb{R}^n$ is an Euclidean space with dimension $n \in \mathbb{N}$(Natural numbers). $C([0,T], \mathbb{R}^n)$ consisting of all continuous functions is the Banach space with the norm $\|\daleth\|_C := \sup_{t\in[0,T]} \|\daleth(t)\|$ for an arbitrary norm $\|.\|$ on $\mathbb{R}^n$. The set $AC^n([0,T])$ consists of real-valued functions f such that it has derivatives up to order $n-1$ on $[0,T]$, and $f^{(n-1)}$ is absolutely continuous.

For $\alpha, \beta, \delta \in \mathbb{C}$ with $Re(\beta) > 0$, $Re(\alpha) > 0$, and $H \in \mathbb{R}^{n\times n}$, the Prabhakar (fractional) integral operator[39] is defined as follows

$$\left({}_{0^+}\mathcal{I}^{H,\delta}_{\alpha,\beta}\daleth\right)(t) = \int_0^t (t-s)^{\beta-1}\, \mathcal{E}^{\delta}_{\alpha,\beta}\left(H(t-s)^{\alpha}\right)\daleth(s)\,ds \tag{2}$$

where the well-known three-parameter Mittag-Leffler function[33]

$$\mathcal{E}^{\delta}_{\alpha,\beta}(x) = \sum_{m=0}^{\infty} \frac{(\delta)_m}{\Gamma(m\alpha+\beta)} \frac{x^m}{m!}.$$

which is a generalisation of the Mittag-Leffler function and defined by Prabhakar in 1971, here $\Gamma(.)$ is the famous Gamma function and $(\delta)_m$ is the Pochhammer symbol, that is, $(\delta)_m = \frac{\Gamma(\delta+m)}{\Gamma(\delta)}$ or

$$(\delta)_0 = 1, \quad (\delta)_m = \delta(\delta+1)...(\delta)(\delta+m-1), \quad m = 0, 1, 2, ...$$

In the study[?], the Prabhakar Riemann-Liouville-type and Caputo-type derivatives are defined respectively, as follows

$$\left({}^{PR}_{0^+}\mathcal{D}^{H,\delta}_{\alpha,\beta}\daleth\right)(t) = \frac{d^m}{dx^m}\int_0^t (t-s)^{m-\beta-1}\, \mathcal{E}^{-\delta}_{\alpha,m-\beta}\left(H(t-s)^{\alpha}\right)\daleth(s)\,ds, \tag{3}$$

and

$$\left({}^{PC}_{0^+}\mathcal{D}^{H,\delta}_{\alpha,\beta}\daleth\right)(t) = \int_0^t (t-s)^{m-\beta-1}\, \mathcal{E}^{-\delta}_{\alpha,m-\beta}\left(H(t-s)^{\alpha}\right)\frac{d^m}{ds^m}\daleth(s)\,ds, \tag{4}$$

where $\alpha, \beta, \delta \in \mathbb{C}$ with $Re(\beta) \geq 0$, $Re(\alpha) > 0$, $m = \lfloor Re(\beta) \rfloor + 1$ (here $\lfloor . \rfloor$ is the floor function), $H \in \mathbb{R}^{n\times n}$ and $\gimel \in AC^m([0,T])$.

Definition 1. [36] If $f : [0, -\infty) \to \mathbb{R}$ is both exponentially bounded and measurable on $[0, -\infty)$, then the Laplace transform $F(s) = \mathfrak{L}\{f(t)\}(s)$ is given by

$$F(s) = \mathfrak{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)dt, \quad s \in \mathbb{C},$$

exists and is an analytic function of s for $Re(s) > 0$.

Lemma 1. [36] The time-shift property for the Laplace integral transform is given by

$$\mathfrak{L}\{f(t-a)\mathcal{H}(t-a)\}(s) = e^{-as}F(s),$$

where the heaviside function $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ is defined as follows

$$\mathcal{H}(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Lemma 2. [36] The inverse Laplace transform of a function $F(s)$ is given by

$$f(t) = \mathfrak{L}^{-1}\{F(s)\}(t) = \lim_{\sigma\to\infty} \frac{1}{2\pi i} \int_{c-i\sigma}^{c+i\sigma} e^{st}F(s)ds, \ \ c = Re(s) > 0.$$

Lemma 3. [36] The convolution of two functions f and gb on $[0, \infty)$ under the Laplace integral transform is given by

$$\mathfrak{L}\{(f * g)(t)\}(s) = \mathfrak{L}\{f(t)\}(s)\,\mathfrak{L}\{g(t)\}(s), \quad s \in \mathbb{C}.$$

Lemma 4. [36] Assume that H is a linear and bounded operator defined on a Banach space with $\|H\| < 1$. Then, $(I - H)^{-1}$ is linear and bounded such that

$$(I - H)^{-1} = \sum_{k=0}^{\infty} H^k.$$

Lemma 5. [37] For any $\alpha, \beta, \delta > 0$, the Laplace transform of the general version of Mittag-Leffler type function of three parameters $\mathcal{E}^{\delta}_{\alpha,\beta}(Ht^\alpha)$ is

$$\mathfrak{L}\left\{t^{\beta-1}\mathcal{E}^{\delta}_{\alpha,\beta}(Ht^\alpha)\right\}(s) = s^{-\beta}\left(I - Hs^{-\alpha}\right)^{-\delta},$$

which holds for $Re(s) > \|A\|^{\frac{1}{\alpha}}$, where I is the identity operator.

Lemma 6. [37] The Laplace integral transform of Prabhakar fractional derivative of Caputo-type is represented by

$$\mathfrak{L}\{f(t)\}(s) = s^\beta\left(I - Hs^{-\alpha}\right)^\delta \mathfrak{L}\{f(t)\}(s) - \sum_{k=0}^{m-1} s^{\beta-k-1}\left(I - Hs^{-\alpha}\right)^\delta f^{(k)}(0).$$

where $m - 1 \leq Re(\beta) < m$.

Lemma 7. (Krasnoselskii's fixed point theorem) Let $\mathcal{B}$ be a convex bounded and closed subset of Banach space X and let $\mathcal{K}_1, \mathcal{K}_2$ be maps $\mathcal{D}$ into Y such that $\mathcal{K}_1 w + \mathcal{K}_2 v \in \mathcal{B}$ for every pair $w, v \in \mathcal{B}$. If $\mathcal{K}_2$ is compact and continuous and $\mathcal{K}_1$ is a contraction, then the equation $\mathcal{K}_1 w + \mathcal{K}_2 w = w$ is of a solution on $\mathcal{B}$.

From now on, all of the following sharing will be new contributions.

3 | AN ANALYTICAL SOLUTION TO PRABHAKAR CAPUTO-TYPE FRACTIONAL DELAYED SYSTEM

In this section, we research for an exact solution to linear and semilinear Prabhakar Caputo-type fractional delayed system by using the well-known famous Laplace technique which is prerequisite for differential systems.

First of all, we are interested in an exact solution to the following linear Prabhakar Caputo-type fractional delayed system

$$\begin{cases} {}^{PC}_{0^+}\mathcal{D}^{H,\delta}_{\alpha,\beta} w(t) = Zw(t-r) + \gimel(t), & t \in (0,T], \ \ r > 0, \\ \qquad\qquad w(t) = \phi(t), & t \in [-r, 0] \end{cases} \tag{5}$$

where ${}^{C}_{0^+}\mathcal{D}^{H,\delta}_{\alpha,\beta}$ represents the Prabhakar Caputo-type derivative of fractional order $0 < \beta < 1$, $H, Z \in \mathbb{R}^{n\times n}$, $T = lr$ for a fixed natural number l, and r is a retardation(delay), the disturbing function $\gimel : [-r, T] \to \mathbb{R}^n$ is continuous, $\phi : [-r, 0] \to \mathbb{R}^n$ is an absolutely continuously differentiable function.

Lemma 8. Under the assumption of the commutativity of H and Z, the linear Prabhakar Caputo-type fractional delayed system (5) is equivalent to the following integral equation for $w \in AC([0,T])$

$$w(t) = \sum_{k=0}^{\infty} (t-kr)^{k\beta} \mathcal{E}^{k\delta}_{\alpha,k\beta+1}(H(t-kr)^{\alpha})\mathcal{H}(t-kr)Z^k w(0) + \int_{-r}^{\min\{t-r,0\}} \Omega^{H,Z}(t,s+r)Z\phi(s)ds + \int_0^t \Omega^{H,Z}(t,s)\daleth(s)ds,$$

where

$$\Omega^{H,Z}(t,s) = \sum_{k=0}^{\infty} (t-kr-s)^{(k+1)\beta-1} \mathcal{E}^{(k+1)\delta}_{\alpha,(k+1)\beta}(H(t-kr-s)^{\alpha})\mathcal{H}(t-kr-s)Z^k.$$

Before proving this lemma, we make a little preparation. We especially deal with the Laplace transform of the retarded term $w(t-r)$. In the light of the substitution $\eta = t-r$, we get

$$\begin{aligned}\mathfrak{L}\{w(t-r)\}(s) &= \int_0^{\infty} e^{-st} w(t-r)\,dt \\ &= e^{-sr}\int_{-r}^{\infty} e^{-s\eta} w(\eta)\,d\eta \\ &= e^{-sr}\left(\int_{-r}^{0} e^{-s\eta} w(\eta)\,d\eta + \int_0^{\infty} e^{-s\eta} w(\eta)\,d\eta\right) \\ &= e^{-sr}\mathfrak{L}\{w(t)\}(s) + \int_{-r}^{0} e^{-s(\eta+r)}\phi(\eta)\,d\eta.\end{aligned}$$

Under the substitution $\eta + r = t$, one can acquire

$$\begin{aligned}\mathfrak{L}\{w(t-r)\}(s) &= e^{-sr}\mathfrak{L}\{w(t)\}(s) + \int_0^{r} e^{-st}\phi(t-r)\,dt \\ &= e^{-sr}\mathfrak{L}\{w(t)\}(s) + \int_0^{\infty} e^{-st}\bar{\phi}(t-r)\,dt \\ &= e^{-sr}\mathfrak{L}\{w(t)\}(s) + \mathfrak{L}\left\{\bar{\phi}(t-r)\right\}(s),\end{aligned} \tag{6}$$

where the unit-step function $\bar{\phi} : \mathbb{R} \to \mathbb{R}$ is defined as follows

$$\bar{\phi}(t) = \begin{cases} \phi(t), & -r \le t \le 0, \\ 0, & t > 0. \end{cases}$$

Proof. of Lemma 8: We apply the Laplace transform to both sides of equations (5) and acquire the following equalities,

$$\mathfrak{L}\left\{{}^{PC}_{0^+}\mathcal{D}^{H,\delta}_{\alpha,\beta} w(t)\right\}(s) = Z\mathfrak{L}\{w(t-r)\}(s) + \mathfrak{L}\{\daleth(t)\}(s),$$

If Lemma 6 and equation (6) are implemented in the just-above equation, one can obtain

$$\Rightarrow s^{\beta}(I-Hs^{-\alpha})^{\delta}\mathfrak{L}\{w(t)\}(s) - s^{\beta-1}(I-Hs^{-\alpha})^{\delta}w(0) = e^{-sr}Z\mathfrak{L}\{w(t)\}(s) + Z\mathfrak{L}\left\{\bar{\phi}(t-r)\right\}(s) + \mathfrak{L}\{\daleth(t)\}(s)$$

$$\Rightarrow s^{\beta}(I-Hs^{-\alpha})^{\delta}\mathfrak{L}\{w(t)\}(s) + e^{-sr}Z\mathfrak{L}\{w(t)\}(s) = s^{\beta-1}(I-Hs^{-\alpha})^{\delta}w(0) + Z\mathfrak{L}\left\{\bar{\phi}(t-r)\right\}(s) + \mathfrak{L}\{\daleth(t)\}(s)$$

$$\Rightarrow \left(s^{\beta}(I-Hs^{-\alpha})^{\delta} + e^{-sr}Z\right)\mathfrak{L}\{w(t)\}(s) = s^{\beta-1}(I-Hs^{-\alpha})^{\delta}w(0) + Z\mathfrak{L}\left\{\bar{\phi}(t-r)\right\}(s) + \mathfrak{L}\{\daleth(t)\}(s)$$

$$\Rightarrow s^{\beta}(I-Hs^{-\alpha})^{\delta}\left(I - s^{-\beta}(I-Hs^{-\alpha})^{-\delta}e^{-sr}Z\right)\mathfrak{L}\{w(t)\}(s) = s^{\beta-1}(I-Hs^{-\alpha})^{\delta}w(0) + Z\mathfrak{L}\left\{\bar{\phi}(t-r)\right\}(s) + \mathfrak{L}\{\daleth(t)\}(s)$$

one acquires the Laplace transform of $w(t)$, $\mathfrak{L}\{w(t)\}(s)$:

$$\begin{aligned}\mathfrak{L}\{w(t)\}(s) &= s^{-\beta}\left(I-Hs^{-\alpha}\right)^{-\delta}\left(I-s^{-\beta}\left(I-Hs^{-\alpha}\right)^{-\delta}e^{-sr}Z\right)^{-1}s^{\beta-1}\left(I-Hs^{-\alpha}\right)^{\delta}w(0)\\ &+ s^{-\beta}\left(I-Hs^{-\alpha}\right)^{-\delta}\left(I-s^{-\beta}\left(I-Hs^{-\alpha}\right)^{-\delta}e^{-sr}Z\right)^{-1}Z\mathfrak{L}\left\{\bar{\phi}(t-r)\right\}(s)\\ &+ s^{-\beta}\left(I-Hs^{-\alpha}\right)^{-\delta}\left(I-s^{-\beta}\left(I-Hs^{-\alpha}\right)^{-\delta}e^{-sr}Z\right)^{-1}\mathfrak{L}\{\daleth(t)\}(s).\end{aligned}$$

If Lemma 4 is applied to the above equation with the norm

$$\left\|s^{-\beta}\left(I-Hs^{-\alpha}\right)^{-\delta}e^{-sr}Z\right\|<1,$$

one gets

$$\begin{aligned}\mathfrak{L}\{w(t)\}(s) &= \sum_{k=0}^{\infty}s^{-(k\beta+1)}\left(I-Hs^{-\alpha}\right)^{-k\delta}e^{-skr}Z^{k}w(0)\\ &+ \sum_{k=0}^{\infty}s^{-(k+1)\beta}\left(I-Hs^{-\alpha}\right)^{-(k+1)\delta}e^{-skr}Z^{k+1}\mathfrak{L}\left\{\bar{\phi}(t-r)\right\}(s)\\ &+ \sum_{k=0}^{\infty}s^{-(k+1)\beta}\left(I-Hs^{-\alpha}\right)^{-(k+1)\delta}e^{-skr}Z^{k}\mathfrak{L}\{\daleth(t)\}(s)\end{aligned}$$

Based on Lemma 5, one easily obtain the following equality $\mathfrak{L}\{w(t)\}(s)$:

$$\begin{aligned}\mathfrak{L}\{w(t)\}(s) &= \sum_{k=0}^{\infty}\left\{(t-kr)^{k\beta}\mathcal{E}^{k\delta}_{\alpha,k\beta+1}(H(t-kr)^{\alpha})\mathcal{H}(t-kr)Z^{k}\right\}w(0)\\ &+ \sum_{k=0}^{\infty}\mathfrak{L}\left\{(t-kr)^{(k+1)\beta-1}\mathcal{E}^{(k+1)\delta}_{\alpha,(k+1)\beta}(H(t-kr)^{\alpha})\mathcal{H}(t-kr)Z^{k+1}\right\}(s)\,\mathfrak{L}\left\{\bar{\phi}(t-r)\right\}(s)\\ &+ \sum_{k=0}^{\infty}\mathfrak{L}\left\{(t-kr)^{(k+1)\beta-1}\mathcal{E}^{(k+1)\delta}_{\alpha,(k+1)\beta}(H(t-kr)^{\alpha})\mathcal{H}(t-kr)Z^{k}\right\}(s)\,\mathfrak{L}\{\daleth(t)\}(s)\end{aligned}$$

If the inverse Laplace transform and convolution are applied to the just-above equation, one obtains $w(t)$:

$$\begin{aligned}w(t) &= \sum_{k=0}^{\infty}(t-kr)^{k\beta}\mathcal{E}^{k\delta}_{\alpha,k\beta+1}(H(t-kr)^{\alpha})\mathcal{H}(t-kr)Z^{k}w(0)\\ &+ \sum_{k=0}^{\infty}(t-kr)^{(k+1)\beta-1}\mathcal{E}^{(k+1)\delta}_{\alpha,(k+1)\beta}(H(t-kr)^{\alpha})\mathcal{H}(t-kr)Z^{k+1}*\bar{\phi}(t-r)\\ &+ \sum_{k=0}^{\infty}(t-kr)^{(k+1)\beta-1}\mathcal{E}^{(k+1)\delta}_{\alpha,(k+1)\beta}(H(t-kr)^{\alpha})\mathcal{H}(t-kr)Z^{k}*\daleth(t)\end{aligned}$$

and so

$$\begin{aligned}w(t) &= \sum_{k=0}^{\infty}(t-kr)^{k\beta}\mathcal{E}^{k\delta}_{\alpha,k\beta+1}(H(t-kr)^{\alpha})\mathcal{H}(t-kr)Z^{k}w(0)\\ &+ \int_{-r}^{\min\{t-r,0\}}\Omega^{H,Z}(t,s+r)Z\phi(s)ds+\int_{0}^{t}\Omega^{H,Z}(t,s)\daleth(s)ds,\end{aligned}$$

in this just-above equation, we evaluate that if $t\geq r$, then

$$\int_{-r}^{t-r}\Omega^{H,Z}(t,s+r)Z\phi(s)ds=\int_{-r}^{0}\Omega^{H,Z}(t,s+r)Z\phi(s)ds,$$

and if $t<r$, then

$$\int_{-r}^{t-r}\Omega^{H,Z}(t,s+r)Z\phi(s)ds=\int_{-r}^{t-r}\Omega^{H,Z}(t,s+r)Z\phi(s)ds,$$

which provide

$$\int_{-r}^{t-r} \Omega^{H,Z}(t, s+r) Z \phi(s) ds = \int_{-r}^{\min\{t-r,0\}} \Omega^{H,Z}(t, s+r) Z \phi(s) ds.$$

This is the last point for this lemma, which means the proof is completed. □

Corollary 1. The global solution of the semilinear Prabhakar Caputo-type fractional delayed system (1) is given by

$$\begin{aligned} w(t) &= \sum_{k=0}^{\infty} (t-kr)^{k\beta}\, \mathcal{E}^{k\delta}_{\alpha,k\beta+1}(H\,(t-kr)^{\alpha}) \mathcal{H}(t-kr) Z^k w(0) \\ &+ \int_{-r}^{\min\{t-r,0\}} \Omega^{H,Z}(t, s+r) Z \phi(s) ds + \int_{0}^{t} \Omega^{H,Z}(t,s) \daleth(s, w(s)) ds, \end{aligned}$$

where

$$\Omega^{H,Z}(t,s) = \sum_{k=0}^{\infty} (t-kr-s)^{(k+1)\beta-1}\, \mathcal{E}^{(k+1)\delta}_{\alpha,(k+1)\beta}(H\,(t-kr-s)^{\alpha}) \mathcal{H}(t-kr-s) Z^k, \tag{7}$$

whose graph is as follows under the special choices of the available parameters

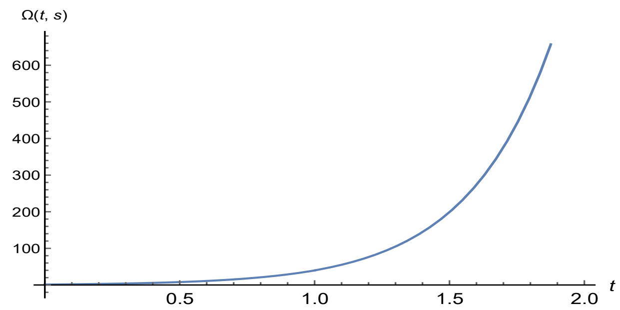

Figure 1 The graph of the function $\Omega^{H,Z}(t,s)$ given in (7) for the special choices $\alpha = 0.6$, $\beta = 0.5$, $r = 1$, $H = 2$, $Z = 1$, and $s = 0$.

4 | EXISTENCE UNIQUENESS OF THE SEMILINEAR PRABHAKAR FRACTIONAL DELAYED SYSTEM AND ITS STABILITY

In this section, we questionnaire whether there exists a solution to the system (1) and it is unique. In fact, we have just found the explicit solution to the system (1) in Lemma 8. Here, the important question is to examine if statements in the system (1) ensure that this solution is unique or not. Unfortunately, available conditions do not guarantee the uniqueness of the analytical solution, so we need to put one more condition on the disturbing function $\daleth\,(t, w\,(t))$ to assure the explicit solution is unique. This condition is the fact that disturbing function $\daleth\,(t, w\,(t))$ is the Lipschitzian in the second component with $L_{\daleth}$.

Lemma 9. Under the appropriate choices of the parameters, the following inequality holds true:

$$\int_{0}^{t} \left\| \Omega^{H,Z}(t,s) \right\| ds \le t\,\Omega^{\|H\|,\|Z\|}(t,0).$$

Proof. If the norm and the integral properties and the rule of $\Omega^{H,Z}(t,s)$ are employed, one easily gets

$$\begin{aligned}\int_0^t \left\|\Omega^{H,Z}(t,s)\right\| ds &\le \int_0^t \left\| \sum_{k=0}^{\infty} (t-kr-s)^{(k+1)\beta-1}\, \mathcal{E}^{(k+1)\delta}_{\alpha,(k+1)\beta}(H\,(t-kr-s)^{\alpha})\mathcal{H}(t-kr-s)Z^k \right\| ds \\ &\le \left\| \sum_{k=0}^{\infty} (t-kr)^{(k+1)\beta-1}\, \mathcal{E}^{(k+1)\delta}_{\alpha,(k+1)\beta}(H\,(t-kr)^{\alpha})\mathcal{H}(t-kr)Z^k \right\| \int_0^t ds \\ &\le t\sum_{k=0}^{\infty} (t-kr)^{(k+1)\beta-1}\, \mathcal{E}^{(k+1)\delta}_{\alpha,(k+1)\beta}(\|H\|\,(t-kr)^{\alpha}) \left\|Z^k\right\| := t\Omega^{\|H\|,\|Z\|}(t,0).\end{aligned}$$

□

Theorem 1. If the disturbing continuous function $\daleth\,(t,w)$ fulfills the Lipschitz condition in the second component with the Lipschitz constant $L_{\daleth} > 0$ and $TL_{\daleth}\Omega^{\|H\|,\|Z\|}(T,0) < 1$ is valid, then the integral equation in the Lemma 8 has a unique solution in $[-r,T]$.

Proof. We use the Banach fixed point theorem among fixed point theorems. Thus, we firstly define $\mathcal{S} : C([-r,T],\mathbb{R}^n) \to C([-r,T],\mathbb{R}^n)$ by

$$\begin{aligned}\mathcal{S}w(t) &= \sum_{k=0}^{\infty} (t-kr)^{k\beta}\, \mathcal{E}^{k\delta}_{\alpha,k\beta+1}(H\,(t-kr)^{\alpha})\mathcal{H}(t-kr)Z^k w(0) \\ &+ \int_{-r}^{\min\{t-r,0\}} \Omega^{H,Z}(t,s+r)Z\phi(s)ds + \int_0^t \Omega^{H,Z}(t,s)\daleth(s,w(s))ds,\end{aligned}$$

where

$$\Omega^{H,Z}(t,s) = \sum_{k=0}^{\infty} (t-kr-s)^{(k+1)\beta-1}\, \mathcal{E}^{(k+1)\delta}_{\alpha,(k+1)\beta}(H\,(t-kr-s)^{\alpha})\mathcal{H}(t-kr-s)Z^k.$$

For any $w, v \in C([-r,T],\mathbb{R}^n)$, we take into consideration

$$\begin{aligned}\|\mathcal{S}w(t) - \mathcal{S}v(t)\| &\le \int_0^t \left\|\Omega^{H,Z}(t,s)\right\| \|\daleth\,(s,w(s)) - \daleth\,(s,v(s))\|\, ds \\ &\le TL_{\daleth}\Omega^{\|H\|,\|Z\|}(T,0)\,\|w-v\|_C\,.\end{aligned}$$

which ensures that $\mathcal{G}$ is a contraction in the light of the inequality condition. It is well-known that $C([-r,T],\mathbb{R}^n)$ is the Banach space with the norm $\|.\|_C$. Then, Banach fixed point theorem provides that $\mathcal{S}$ has a unique fixed point on $[-r,T]$, i.e. $\exists! v \in C([-r,T],\mathbb{R}^n)$, $v(t) = \mathcal{S}v(t)$. □

Now, we will show that Prabhakar fractional delayed system is stable in the sense of Ulam-Hyers. Before this, We share one definition and remark related Ulam-Hyers stability.

Definition 2. Let $\varsigma > 0$. The system (1) is stable in the setting of Ulam-Hyers if for every solution $w \in C([0,T],\mathbb{R}^n)$ of inequality,

$$\left\| {}^{PC}_{0^+}\mathcal{D}^{H,\delta}_{\alpha,\beta} w(t) - Zw(t-r) - \daleth\,(t,w(t)) \right\| \le \varepsilon, \tag{8}$$

there is a solution $v \in C([0,T],\mathbb{R}^n)$ of system (1), and $\vartheta > 0$ such that

$$\|w(t) - v(t)\| \le \vartheta.\varepsilon, \quad t \in [0,T]\,.$$

Remark 1. A function $w \in C^1([0,T],\mathbb{R}^n)$ is a solution of the inequality equation (8) if and only if there is a function $g \in C([0,T],\mathbb{R}^n)$, such that

*i***.** $\|g(t)\| < \varepsilon$,

*ii***.** ${}^{PC}_{0^+}\mathcal{D}^{H,\delta}_{\alpha,\beta} w(t) = Zw(t-r) + \daleth\,(t,w(t)) + g(t)$.

Theorem 2. Under all of conditions given in Theorem 1, the system (1) is Ulam-Hyers stable.

Proof. Assume $w \in C([0,T],\mathbb{R}^n)$ which fulfills the inequality (8), and let $v \in C([0,T],\mathbb{R}^n)$ which is the unique solution of system (1) with the initial condition $v(t) = w(t)$ for all $t \in [-r, 0]$. With the aid of the definition of $\mathcal{S}$ and Remark 1 in mind, we can acquire

$$\|g(t)\| < \varsigma, \quad w(t) = \mathcal{S}w(t) + \int_0^t \Omega^{H,Z}(t,s)g(s)\,ds,$$

and also $v(t) = (\mathcal{S}v)(t)$ for each $t \in [0,T]$. One can easily get

$$\|\mathcal{S}w(t) - w(t)\| \le \int_0^t \left\|\Omega^{H,Z}(t,s)\right\| \|g(s)\|\,ds \le T\Omega^{\|H\|,\|Z\|}(T,0)\varsigma.$$

We are ready to make and estimation $\|v(t) - w(t)\|$:

$$\begin{aligned}\|v(t) - w(t)\| &\le \|\mathcal{S}v(t) - \mathcal{S}w(t)\| + \|\mathcal{S}w(t) - w(t)\| \\ &\le L_{\daleth} T\Omega^{\|H\|,\|Z\|}(T,0)\,\|v - w\|_C + T\Omega^{\|H\|,\|Z\|}(T,0)\varsigma,\end{aligned}$$

which provides

$$\left(1 - L_{\daleth} T\Omega^{\|H\|,\|Z\|}(T,0)\|\right)\|v - v\|_C \le T\Omega^{\|H\|,\|Z\|}(T,0)\varsigma,$$

from this just above inequality, we obtain the desired result

$$\|v - w\|_C \le \vartheta\varsigma, \qquad \vartheta = \frac{T\Omega^{\|H\|,\|Z\|}(T,0)}{1 - L_{\daleth} T\Omega^{\|H\|,\|Z\|}(T,0)} > 0.$$

The proof is completed. □

5 | RELATIVE CONTROLLABILITY OF THE PRABHAKAR FRACTIONAL DELAYED SYSTEM

In the current section, we have discussed about relative controllability of the Prabhakar fractional delayed system. We have determined necessary and sufficient circumstances for controllability of linear Prabhakar fractional delayed system addition to establishing sufficient circumstances for controllability of semilinear Prabhakar fractional delayed system.

Before expressing main theorems, we remind a couple of necessary tools. For $H \in \mathbb{R}^{n\times n}$, the matrix norm

$$\|A\| = \max_{1\le j\le n} \sum_{i=1}^n |a_{ij}|$$

where h_{ij} are the entries of the matrix H. Let X_1, X_2 be two Banach spaces, $B(X_1, X_2)$ is the Banach space of all bounded linear operator from X_1 to X_2. Let $J = [0, \tau]$ be a bounded closed interval. $L^2(J, Y_2)$ symbolizes the Hilbert space with $\|.\|_{L^2(J,Y_2)}$.

Definition 3. The system (1) is relatively controllable if for the final state $w_\tau \in \mathbb{R}$ and the time τ, an arbitrary continuously differentiable initial function ϕ, then there exists such a control function $u \in L^2([0,\tau],\mathbb{R}^n)$ that the system (1) has a solution $w \in C([0,\tau],\mathbb{R}^n)$ which satisfies $w(\tau) = w_\tau$ and $w(t) = \phi(t)$ $t \in [-r, 0]$.

We consider two different types of the control systems. One of them is the case $\daleth(t, w(t)) = 0 \in \mathbb{R}$, that is, we firstly investigate the linear Prabhakar fractional delayed control system:

$$\begin{cases} {}^{PC}_{0^+}\mathcal{D}^{H,\delta}_{\alpha,\beta} w(t) = Zw(t-r) + Au(t), \quad t \in (0,T], \quad r > 0, \\ w(t) = \phi(t), \quad t \in [-r, 0] \end{cases} \tag{9}$$

where ${}^{C}_{0^+}\mathcal{D}^{H,\delta}_{\alpha,\beta}$ represents the Prabhakar Caputo-type derivative of fractional order $0 < \beta < 1$, $H, Z, A \in \mathbb{R}^{n\times n}$, $T = lr$ for a fixed natural number l, and r is a retardation(delay), $\phi : [-r, 0] \to \mathbb{R}^n$ is an absolutely continuously differentiable function. Its explicit solution is given by

$$w(t) = \sum_{k=0}^{\infty} (t-kr)^{k\beta}\, \mathcal{E}_{\alpha,k\beta+1}^{k\delta}(H\,(t-kr)^{\alpha})\mathcal{H}(t-kr)Z^k w(0) + \int_{-r}^{\min\{t-r,0\}} \Omega^{H,Z}(t,s+r)Z\phi(s)ds + \int_0^t \Omega^{H,Z}(t,s)Au(s)ds,$$

where

$$\Omega^{H,Z}(t,s) = \sum_{k=0}^{\infty} (t-kr-s)^{(k+1)\beta-1}\, \mathcal{E}_{\alpha,(k+1)\beta}^{(k+1)\delta}(H\,(t-kr-s)^{\alpha})\mathcal{H}(t-kr-s)Z^k.$$

Now, it is time to describe Gramian matrix as noted below

$$\mathbb{W}\,[0,\tau] = \int_0^{\tau} \Omega^{H,Z}(t,s)AA^T\Omega^{H^T,Z^T}(t,s)ds,$$

where $\left(\Omega^{H,Z}(t,s)\right)^T =: \Omega^{H^T,Z^T}(t,s)$, and $.^T$ stands for the transpose of a matrix.

The following theorem provides necessary and sufficient circumstances to control the linear system (9).

Theorem 3. The system (9) is relatively controllable if and only if the Gramian matrix $\mathbb{W}\,[0,\tau]$ is nonsingular.

Proof. *Necessity*: Suppose the contrary, that is, $\mathbb{W}\,[0,\tau]$ is singular despite the system (9) being relatively controllable. Then there is such at least an nonzero $b \in \mathbb{R}^n$ that

$$\mathbb{W}\,[0,\tau]\,b = 0.$$

One can easily acquires that

$$b^T\mathbb{W}\,[0,\tau]\,b = \int_0^{\tau} b^T\Omega^{H,Z}(\tau,s)AA^T\Omega^{H^T,Z^T}(\tau,s)bds,$$

which gives

$$b^T\Omega^{H,Z}(\tau,s)A = 0, \quad 0 \le s \le \tau. \tag{10}$$

Because of the relatively exact controllability of the system (9), for the final state 0 and time τ, there is such a control $u_1 \in L^2\left([0,\tau],\mathbb{R}^n\right)$ that the system (9) has a solution $w \in AC^1\left([0,\tau],\mathbb{R}^n\right)$ satisfying $w(\tau) = 0$, that is,

$$w(\tau) = \sum_{k=0}^{\infty} (\tau-kr)^{k\beta}\, \mathcal{E}_{\alpha,k\beta+1}^{k\delta}(H\,(\tau-kr)^{\alpha})\mathcal{H}(\tau-kr)Z^k w(0) + \int_{-r}^{\min\{\tau-r,0\}} \Omega^{H,Z}(\tau,s+r)Z\phi(s)ds + \int_0^{\tau} \Omega^{H,Z}(\tau,s)Au_1(s)ds = 0,$$

similarly, for the final state b and time τ, there is such a control $u_2 \in L^2\left([0,\tau],\mathbb{R}^n\right)$ that the system (9) has a solution $w \in AC^1\left([0,\tau],\mathbb{R}^n\right)$ satisfying $w(\tau) = b$, that is,

$$w(\tau) = \sum_{k=0}^{\infty} (\tau-kr)^{k\beta}\, \mathcal{E}_{\alpha,k\beta+1}^{k\delta}(H\,(\tau-kr)^{\alpha})\mathcal{H}(\tau-kr)Z^k w(0) + \int_{-r}^{\min\{\tau-r,0\}} \Omega^{H,Z}(\tau,s+r)Z\phi(s)ds + \int_0^{t} \Omega^{H,Z}(\tau,s)Au_2(s)ds = b.$$

By subtracting the last two equalities and keeping zero equation (10) in mind, one acquires that

$$b = \int_0^t \Omega^{H,Z}(\tau,s)A\left[u_2(s)-u_1(s)\right]ds$$

$$b^Tb = \int_0^t b^T\Omega^{H,Z}(\tau,s)A\left[u_2(s)-u_1(s)\right]ds = 0$$

So, $b = 0$ which is a contradiction with b being nonzero.

Sufficiency: We prove that the system (9) is controllable when $\mathbb{W}[0,\tau]$ is nonsingular. Since it is nonsingular, its inverse $(\mathbb{W}[0,\tau])^{-1}$ is well-defined. For the final state $b \in \mathbb{R}$, if the following function

$$u(s) = A^T \Omega^{H^T,Z^T}(\tau,s)\,(\mathbb{W}[0,\tau])^{-1}\Bigg(b - \int_{-r}^{\min\{\tau-r,0\}} \Omega^{H,Z}(\tau,s+r)Z\phi(s)ds$$
$$- \sum_{k=0}^{\infty} (\tau-kr)^{k\beta}\,\mathcal{E}^{k\delta}_{\alpha,k\beta+1}(H\,(\tau-kr)^{\alpha})\mathcal{H}(\tau-kr)Z^k w(0)\Bigg)$$

can be chosen as a control, then one can easily verify

$$\begin{aligned}
w(\tau) &= \sum_{k=0}^{\infty} (\tau-kr)^{k\beta}\,\mathcal{E}^{k\delta}_{\alpha,k\beta+1}(H\,(\tau-kr)^{\alpha})\mathcal{H}(\tau-kr)Z^k w(0) \\
&+ \int_{-r}^{\min\{\tau-r,0\}} \Omega^{H,Z}(\tau,s+r)Z\phi(s)ds + \int_0^t \Omega^{H,Z}(\tau,s)Au(s)ds \\
&= \sum_{k=0}^{\infty} (\tau-kr)^{k\beta}\,\mathcal{E}^{k\delta}_{\alpha,k\beta+1}(H\,(\tau-kr)^{\alpha})\mathcal{H}(\tau-kr)Z^k w(0) + \int_{-r}^{\min\{\tau-r,0\}} \Omega^{H,Z}(\tau,s+r)Z\phi(s)ds \\
&+ \int_0^t \Omega^{H,Z}(\tau,s)Au(s)A^T\Omega^{H^T,Z^T}(\tau,s)ds\,(\mathbb{W}[0,\tau])^{-1}\Bigg(b - \int_{-r}^{\min\{\tau-r,0\}} \Omega^{H,Z}(\tau,s+r)Z\phi(s)ds \\
&- \sum_{k=0}^{\infty} (\tau-kr)^{k\beta}\,\mathcal{E}^{k\delta}_{\alpha,k\beta+1}(H\,(\tau-kr)^{\alpha})\mathcal{H}(\tau-kr)Z^k w(0)\Bigg) \\
&= b.
\end{aligned}$$

□

The second case $\daleth(t,w(t)) \neq 0 \in \mathbb{R}$, that is, we investigate the semilinear Prabhakar fractional delayed control system:

$$\begin{cases} {}^{PC}_{0^+}\mathcal{D}^{H,\delta}_{\alpha,\beta} w(t) = Zw(t-r) + Au(t) + \daleth(t,w(t)), \quad t \in (0,T], \quad r > 0, \\ w(t) = \phi(t), \quad t \in [-r,0] \end{cases} \tag{11}$$

where ${}^{C}_{0^+}\mathcal{D}^{H,\delta}_{\alpha,\beta}$ represents the Prabhakar Caputo-type derivative of fractional order $0 < \beta < 1$, $H, Z, A \in \mathbb{R}^{n\times n}$, $T = lr$ for a fixed natural number l, and r is a retardation(delay), $\daleth : [-r,T]\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous, $\phi : [-r,0] \to \mathbb{R}^n$ is an absolutely continuously differentiable function. Its explicit solution is given by

$$\begin{aligned}
w(t) &= \sum_{k=0}^{\infty} (t-kr)^{k\beta}\,\mathcal{E}^{k\delta}_{\alpha,k\beta+1}(H\,(t-kr)^{\alpha})\mathcal{H}(t-kr)Z^k w(0) \\
&+ \int_{-r}^{\min\{t-r,0\}} \Omega^{H,Z}(t,s+r)Z\phi(s)ds + \int_0^t \Omega^{H,Z}(t,s)Au(s)ds + \int_0^t \Omega^{H,Z}(t,s)\daleth(s,w(s))ds.
\end{aligned}$$

Unfortunately, we make impose a couple of limitations under the name of assumptions except for expressing conditions in the system (11) to guarantee the relative controllability of the semilinear Prabhakar fractional delayed system.

A_1) The operator $\mathfrak{W} : L^2(J,\mathbb{R}^n) \to \mathbb{R}^n$ defined by

$$\mathfrak{W}u = \int_0^\tau \Omega^{H,Z}(t,s)Au(s)ds$$

is of an inverse matrix operator $\mathfrak{W}^{-1}$ taking values in $L^2(J,\mathbb{R}^n)\big/ker\mathfrak{W}$.

A_2) The disturbing function $\daleth : J\times\mathbb{R}^n \to \mathbb{R}^n$ is the Lipschitzian in the second component with $L_\daleth > 0$.

For simplicity, we now introduce the following notations:

$$\Pi = \left\| \mathfrak{W}^{-1} \right\|_{B\left(\mathbb{R}^n, L^2(J,\mathbb{R}^n) \big/ ker\mathfrak{W}\right)},$$

and

$$\Pi_1 = \sum_{k=0}^{\infty} (t-kr)^{k\beta}\, \mathcal{E}^{k\delta}_{\alpha,k\beta+1}(\|H\|\,(t-kr)^{\alpha})\mathcal{H}(t-kr) \left\|Z^k\right\| \|w(0)\| + \tau \left(\|Z\|\,\|\phi\| + N_{\daleth}\right) \Omega^{\|H\|,\|Z\|}(\tau,0),$$

and

$$\Pi_2 = \tau L_{\daleth} \Omega^{\|H\|,\|Z\|}(\tau,0)$$

where $N_{\daleth} = \max_{t\in J} \|\daleth(t,0)\|$. From[38, Remark 3.3], one acquires

$$\Pi = \sqrt{\left\|(\mathbb{W}[0,\tau])^{-1}\right\|}$$

Theorem 4. Under the assumptions of (A_1) and (A_2), the system (11) is relatively controllable for $1 > \alpha \geq 0.5$ if

$$\left(1 + \tau\Omega^{\|H\|,\|Z\|}(\tau,0)\,\|A\|\,\Pi\right)\Pi_2 < 1. \tag{12}$$

Proof. In the light of (A_1), for an arbitrary $w \in C = C[J,\mathbb{R}^n]$, we describe the below control operator $u_w(t)$:

$$\begin{aligned} u_w(t) = \mathfrak{W}^{-1}\Bigg(w_\tau - \sum_{k=0}^{\infty} (t-kr)^{k\beta}\, \mathcal{E}^{k\delta}_{\alpha,k\beta+1}(H\,(t-kr)^{\alpha})\mathcal{H}(t-kr) Z^k w(0) \\ - \int_{-r}^{\min\{t-r,0\}} \Omega^{H,Z}(t,s+r)Z\phi(s)ds - \int_0^t \Omega^{H,Z}(t,s)\daleth(s,w(s))ds \Bigg). \end{aligned}$$

By employing this control function, we can define $\mathcal{K} : C \to C$ by

$$\begin{aligned} \mathcal{K}w(t) = \sum_{k=0}^{\infty} (t-kr)^{k\beta}\, \mathcal{E}^{k\delta}_{\alpha,k\beta+1}(H\,(t-kr)^{\alpha})\mathcal{H}(t-kr) Z^k w(0) \\ + \int_{-r}^{\min\{t-r,0\}} \Omega^{H,Z}(t,s+r)Z\phi(s)ds + \int_0^t \Omega^{H,Z}(t,s)Au(s)ds + \int_0^t \Omega^{H,Z}(t,s)\daleth(s,w(s))ds. \end{aligned}$$

Let's consider

$$B_\rho = \{w \in C : \|w\| \leq \rho\}$$

which is a closed, bounded and convex set in C.

Our first task is to determine the positive real number $\rho > 0$ so that $\mathcal{K}\left(B_\rho\right) \subseteq B_\rho$. Now we make an estimation to the control function u_w by using the assumptions (A_1) and (A_2) as expressed below:

$$\begin{aligned} \left\|u_w(t)\right\| &= \Bigg\| \mathfrak{W}^{-1}\Bigg(w_\tau + \sum_{k=0}^{\infty} (t-kr)^{k\beta}\, \mathcal{E}^{k\delta}_{\alpha,k\beta+1}(H\,(t-kr)^{\alpha})\mathcal{H}(t-kr) Z^k w(0) \\ &\quad + \int_{-r}^{\min\{t-r,0\}} \Omega^{H,Z}(t,s+r)Z\phi(s)ds + \int_0^t \Omega^{H,Z}(t,s)\daleth(s,w(s))ds \Bigg)\Bigg\| \\ &= \left\|\mathfrak{W}^{-1}\right\| \Bigg(\left\|w_\tau\right\| + \sum_{k=0}^{\infty} (t-kr)^{k\beta}\, \mathcal{E}^{k\delta}_{\alpha,k\beta+1}(\|H\|\,(t-kr)^{\alpha})\mathcal{H}(t-kr) \left\|Z^k\right\| \\ &\quad \times \|w(0)\| + \tau\left(\|Z\|\,\|\phi\| + N_{\daleth}\right)\Omega^{\|H\|,\|Z\|}(\tau,0) + \tau L_{\daleth}\Omega^{\|H\|,\|Z\|}(\tau,0)\,\|w\|_C \Bigg) \\ &\leq \Pi\left(\left\|w_\tau\right\| + \Pi_1 + \Pi_2\,\|w\|_C\right) = \Pi\left\|w_\tau\right\| + \Pi\Pi_1 + \Pi\Pi_2\,\|w\|_C\,. \end{aligned}$$

Now we make an estimation for $\mathcal{K}w(t)$:

$$\begin{aligned}
\|\mathcal{K}w(t)\| &\leq \left\| \sum_{k=0}^{\infty} (t-kr)^{k\beta}\, \mathcal{E}^{k\delta}_{\alpha,k\beta+1}(H\,(t-kr)^{\alpha})\mathcal{H}(t-kr)Z^k w(0) \right\| \\
&+ \left\| \int_{-r}^{\min\{t-r,0\}} \Omega^{H,Z}(t,s+r)Z\phi(s)ds + \int_0^t \Omega^{H,Z}(t,s)Au(s)ds \right\| \\
&+ \left\| \int_0^t \Omega^{H,Z}(t,s)(\daleth(s,w(s)) - \daleth(s,0))ds + \int_0^t \Omega^{H,Z}(t,s)\daleth(s,0)ds \right\| \\
&\leq \Pi_1 + \Pi_2 \|w\|_C + \tau\Omega^{\|H\|,\|Z\|}(\tau,0)\,\|A\| \left(\Pi \left\|w_\tau\right\| + \Pi\Pi_1 + \Pi\Pi_2 \|w\|_C\right) \\
&\leq \Pi_1 \left(1 + \tau\Omega^{\|H\|,\|Z\|}(\tau,0)\,\|A\|\,\Pi\right) + \left(\tau\Omega^{\|H\|,\|Z\|}(\tau,0)\,\|A\|\,\Pi\right)\left\|w_\tau\right\| \\
&+ \Pi_2 \left(1 + \tau\Omega^{\|H\|,\|Z\|}(\tau,0)\,\|A\|\,\Pi\right)\rho := \rho.
\end{aligned}$$

One can easily obtain the ratio ρ:

$$\rho := \frac{\left(1 + \tau\Omega^{\|H\|,\|Z\|}(\tau,0)\,\|A\|\,\Pi\right)\Pi_1 + \left(\tau\Omega^{\|H\|,\|Z\|}(\tau,0)\,\|A\|\,\Pi\right)\left\|w_\tau\right\|}{1 - \left(1 + \tau\Omega^{\|H\|,\|Z\|}(\tau,0)\,\|A\|\,\Pi\right)\Pi_2} > 0$$

which provides $\mathcal{K}\left(B_\rho\right) \subseteq B_\rho$. Its positiveness comes from the condition (4) given in the statements of the present theorem.

Now we divide $\mathcal{K}w(t)$ into two different operators $\mathcal{K}_1 w(t)$ and $\mathcal{K}_2 w(t)$ on B_ρ as noted below:

$$\begin{aligned}
\mathcal{K}_1 w(t) &= \sum_{k=0}^{\infty} (t-kr)^{k\beta}\, \mathcal{E}^{k\delta}_{\alpha,k\beta+1}(H\,(t-kr)^{\alpha})\mathcal{H}(t-kr)Z^k w(0) \\
&+ \int_{-r}^{\min\{t-r,0\}} \Omega^{H,Z}(t,s+r)Z\phi(s)ds + \int_0^t \Omega^{H,Z}(t,s)Au(s)ds,.
\end{aligned}$$

and

$$\mathcal{K}_2 w(t) = \int_0^t \Omega^{H,Z}(t,s)\daleth(s,w(s))ds.$$

The next task is to show $\mathcal{K}_1$ is a contraction on B_ρ. By taking any $w, v \in B_\rho$ and keeping (A_1) and (A_2) in mind, we get

$$\begin{aligned}
\left\|u_w(t) - u_v(t)\right\| &\leq \Pi \int_0^\tau \left\|\Omega^{H,Z}(\tau,s)\right\| \|\daleth(s,w(s)) - \daleth(s,v(s))\|\, ds \\
&\leq \Pi\Pi_2 \|w - v\|_C .
\end{aligned}$$

Thus

$$\begin{aligned}
\left\|\mathcal{K}_1 w(t) - \mathcal{K}_1 v(t)\right\| &\leq \int_0^\tau \left\|\Omega^{H,Z}(\tau,s)\right\| \|A\| \left\|u_w(s) - u_v(s)\right\| ds \\
&\leq \tau\Omega^{\|H\|,\|Z\|}(\tau,0)\,\|A\|\,\Pi\Pi_2 \|w - v\|_C .
\end{aligned}$$

Due to the condition (4), $\tau\Omega^{\|H\|,\|Z\|}(\tau,0)\,\|A\|\,\Pi\Pi_2 < 1$ which guarantees that $\mathcal{K}_1$ is a contraction on B_ρ.

The next task is to demonstrate that $\mathcal{K}_2$ is continuous on B_ρ. Assume that $w_n \to w \in B_\rho$. (A_2) ensures that $\daleth(t, w_n(t)) \to \daleth(t, w(t))$. Dominated convergence theorem provides

$$\left\|\mathcal{K}_2 w_n(t) - \mathcal{K}_2 w(t)\right\| \leq \int_0^t \left\|\Omega^{H,Z}(t,s)\right\| \left\|\daleth(s,w_n(s)) - \daleth(s,w(s))\right\| ds \to 0$$

as $n \to \infty$.

The last task is to check whether $\mathcal{K}_2$ is compact. For $w \in B_\rho$, $0 < t < t+h < \tau$

$$\mathcal{K}_2 w(t+h) - \mathcal{K}_2 w(t) = \int_t^{t+h} \Omega^{H,Z}(t+h,s)\daleth(s,w(s))ds + \int_0^t \left(\Omega^{H,Z}(t+h,s) - \Omega^{H,Z}(t,s)\right)\daleth(s,w(s))ds.$$

Set the following notations:

$$\lambda_1 := \int_t^{t+h} \Omega^{H,Z}(t+h,s)\daleth(s,w(s))ds,$$

$$\lambda_2 := \int_0^t \left(\Omega^{H,Z}(t+h,s) - \Omega^{H,Z}(t,s)\right)\daleth(s,w(s))ds.$$

With an easy computation, one gets

$$\left\|\lambda_1\right\| = \left(L_\daleth \left\|w\right\|_C + N_\daleth\right)\int_t^{t+h} \left\|\Omega^{H,Z}(t+h,s)\right\| ds \to 0$$

$$\left\|\lambda_2\right\| = \left(L_\daleth \left\|w\right\|_C + N_\daleth\right)\int_0^t \left\|\Omega^{H,Z}(t+h,s) - \Omega^{H,Z}(t,s)\right\| ds \to 0$$

as $h \to 0$. As a consequence, one obtains

$$\left\|\mathcal{K}_2 w(t+h) - \mathcal{K}_2 w(t)\right\| \leq \left\|\lambda_1\right\| + \left\|\lambda_2\right\| \to 0 \quad \text{as} \quad h \to 0.$$

$\mathcal{K}_2\left(B_\rho\right)$ is bounded because one easily reach to the following upper bound with the similar calculations,

$$\left\|\mathcal{K}_2\right\| \leq \left(L\daleth\rho + N_\daleth\right)\tau\Omega^{\|H\|,\|Z\|}(\tau,0).$$

Based on the equicontinuity and boundedness of $\mathcal{K}_2$, Arzela-Ascoli theorem gives it is compact. Krasnoselskii's fixed point theorem assures that $\mathcal{K}$ has a fixed point $w \in B_\rho$. This completes the proof. □

6 | EXAMPLES

In this section, we illustrate our theoretical results.

Example 1. We will take into consideration the following linear Prabhakar Caputo-type fractional differential delayed system

$$\begin{cases} {}^{PC}_{0^+}\mathcal{D}^{0.1,1}_{0.2,0.3} w(t) = 0.5w(t-0.6) + t^2, \quad t \in (0,3], \\ w(t) = t, \quad t \in [-0.6, 0] \end{cases} \tag{13}$$

whose closed-form formula of the solution is given by

$$w(t) = \sum_{k=0}^{\infty} (t-0.6k)^{0.3k}\,\mathcal{E}^k_{0.2,0.3k+1}(0.1\,(t-0.6k)^{0.2})\mathcal{H}(t-0.6k)0.5^k w(0)$$
$$+ 0.5 \int_{-0.6}^{\min\{t-0.6,0\}} \Omega^{0.1,0.5}(t,s+0.6)sds + \int_0^t \Omega^{0.1,0.5}(t,s)s^2ds,$$

where

$$\Omega^{0.1,0.5}(t,s) = \sum_{k=0}^{\infty} (t-0.6k-s)^{(k+1)0.3-1}\,\mathcal{E}^{(k+1)}_{0.2,(k+1)0.3}(H\,(t-0.6k-s)^{\alpha})\mathcal{H}(t-0.6k-s)0.5^k,$$

and the graph of the solution to the system (13) is in Figure 2.

Example 2. We will examine the following linear Prabhakar Caputo-type fractional differential delayed system

$$\begin{cases} {}^{PC}_{0^+}\mathcal{D}^{H,1}_{0.1,0.7} w(t) = Zw(t-0.2) + \frac{e^t}{1+e^t}t^2\sin(w(t)), \quad t \in (0,0.8],, \\ w(t) = \phi(t), \quad t \in [-0.2, 0] \end{cases} \tag{14}$$

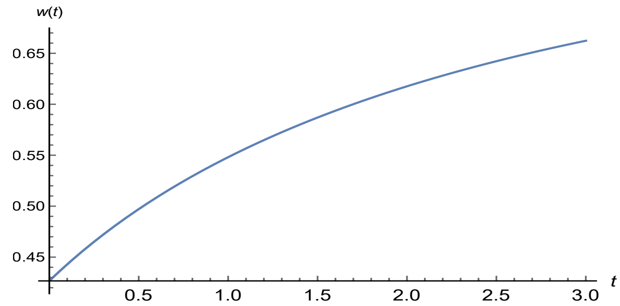

Figure 2 The graph of the exact analytical solution function $w(t)$ to the system (13).

where $H = I_{3\times3}$ and $Z = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $\phi(x) = \begin{pmatrix} t^2 \\ 3t+1 \\ t \end{pmatrix}$. One can easily calculate that $\daleth(t, w(t)) = \frac{1}{2}\frac{t^4}{1+t^2}\sin(w(t))$ is both continuous and the Lipschitzian with $L_\daleth = 0.32$ and $TL_\daleth \Omega^{\|H\|,\|Z\|}(T, 0) \cong 0.8 * 0.32 * 3.40541 = 0.871 < 1$. Hence, all of circumstances given in the statements of Theorem 1 and 2 are fulfilled, so system (14) has an unique solution as well as being Ulam-Hyers stable.

Example 3. We will consider the following homogenegous Prabhakar Caputo-type fractional differential delayed control system

$$\begin{cases} {}^{PC}_{0^+}\mathcal{D}^{H,1}_{0.6,0.5} w(t) = Zw(t-1) + Au(t) \quad t \in (0,4], \\ w(t) = \phi(t), \quad t \in [-1, 0] \end{cases} \tag{15}$$

where $H = Z = \begin{pmatrix} 0.2 & 0.4 & 0.6 \\ 0.8 & 0.1 & 0.3 \\ 0.7 & 0.9 & 0.5 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 5 & 7 \\ 4 & 9 & 6 \\ 2 & 8 & 3 \end{pmatrix}$, $\phi(t) = \begin{pmatrix} t^2+4 \\ 2t+5 \\ 3t \end{pmatrix}$. A representation of the Gramian matrix for the homogenegous Prabhakar Caputo-type fractional differential delayed system(15) as follows:

$$\begin{aligned} \mathbb{W}[0,2] &= \int_0^2 \Omega^{H,Z}(2,s)AA^T\Omega^{H^T,Z^T}(2,s)ds \\ &= \begin{bmatrix} 0.0016 & 2.048 & 2.4696 \\ 2.048 & 0.0081 & 3.4992 \\ 2.4696 & 3.4992 & 0.5625 \end{bmatrix}. \end{aligned}$$

We compute the determinant of Gramian matrix $\mathbb{W}[0,2]$ which is 32.9678. Thus, $\mathbb{W}[0,2]$ is nonsingular. Based on Theorem 3, the system (15) is relatively controllable.

Example 4. We will investigate the following inhomogeneous Prabhakar Caputo-type fractional delayed control system

$$\begin{cases} {}^{PC}_{0^+}\mathcal{D}^{H,1}_{0.2,0.9} w(t) = Zw(t-2) + Au(t) + \daleth(t, w(t)), \quad t \in (0,6], \\ w(t) = \phi(t), \quad t \in [-2, 0] \end{cases} \tag{16}$$

where $H = \begin{pmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{pmatrix}$, $Z = \begin{pmatrix} 0.5 & 0.4 \\ 0.6 & 1.1 \end{pmatrix}$, $A = \begin{pmatrix} 0.2 & 0.5 \\ 0.3 & 0.1 \end{pmatrix}$, $\phi(t) = \begin{pmatrix} t^2+4 \\ 2t+5 \end{pmatrix}$, $\daleth(t, w(t)) = \left(\frac{\sin w(t)}{\pi^6 t} \quad \frac{\tan^{-1} w(t)}{(\pi^2 t)^2} \right)^T$. A representation of the Gramian matrix for the homogenegous Prabhakar Caputo-type fractional differential delayed system(16) as follows:

$$\begin{aligned} \mathbb{W}[0,4] &= \int_0^4 \Omega^{H,Z}(4,s)AA^T\Omega^{H^T,Z^T}(4,s)ds \\ &= \begin{bmatrix} 0.250 & 8.640 \\ 8.640 & 19.360 \end{bmatrix}, \end{aligned}$$

and

$$(\mathbb{W}[0,4])^{-1} = \begin{bmatrix} 4 & 0.115741 \\ 0.115741 & 0.0516529 \end{bmatrix}$$

From[38, Remark 3.3], one acquires

$$\Pi = \left\|\mathfrak{W}^{-1}\right\|_{B\left(\mathbb{R}^2, L^2([0,4],\mathbb{R}^2)\big/ker\mathfrak{W}\right)} = \sqrt{\left\|(\mathbb{W}[0,4])^{-1}\right\|} = 2.02873$$

which guarantees that the inverse operator $\mathfrak{W}^{-1}$ exists, so the operator $\mathfrak{W} : L^2([0,4],\mathbb{R}^2) \to \mathbb{R}^2$ fulfills (A_1). The function $\daleth : [0,4] \times \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous function and for arbitrary $w, v \in \mathbb{R}^2$

$$\left\|\begin{pmatrix} \frac{\tan^{-1} v(t)}{10(\pi^2 t)^2} & \frac{\sin v(t)}{20(\pi t)^6} \end{pmatrix}^T - \begin{pmatrix} \frac{\tan^{-1} w(t)}{(\pi^2 t)^2} & \frac{\sin w(t)}{\pi^6 t} \end{pmatrix}^T\right\| \leq L_\daleth \|v(t) - w(t)\|, \quad t \in [0,4]$$

where $L_\daleth = \frac{1}{40\pi^4}$ which provides that (A_2) is satisfied for system (16). It is easy to confirm that the inequality (4) is satisfied as noted below

$$\left(1 + \tau\Omega^{\|H\|,\|Z\|}(\tau,0)\,\|A\|\,\Pi\right)\Pi_2 \cong 0.0761901 < 1.$$

As a consequence, all of the circumstances of Theorem 4 are confirmed. Theorem 4 ensures the system (16) is relatively controllable in the light of the control function $u_w(t)$:

$$u_w(t) = \mathfrak{W}^{-1}\Bigg(w_\tau - \sum_{k=0}^{\infty} (t-2k)^{0.9k}\,\mathcal{E}^k_{0.2,0.9k+1}(H\,(t-2k)^\alpha)\mathcal{H}(t-2k)Z^k(4\ \ 5)^T$$
$$- \int_{-2}^{\min\{t-2,0\}} \Omega^{H,Z}(t,s+2)Z(t^2+4\ \ 2t+5)^T ds - \int_0^t \Omega^{H,Z}(t,s)\begin{pmatrix} \frac{\sin w(s)}{\pi^6 s} & \frac{\tan^{-1} w(s)}{(\pi^2 s)^2} \end{pmatrix}^T ds\Bigg),$$

where

$$\Omega^{H,Z}(t,s) = \sum_{k=0}^{\infty} (t-2k-s)^{0.9(k+1)-1}\,\mathcal{E}^{(k+1)}_{0.2,0.9(k+1)}(H\,(t-2r-s)^{0.2})\mathcal{H}(t-2r-s)Z^k.$$

7 | CONCLUSION

This paper is devoted to introducing the Prabkahar Caputo-type fractional differential delayed system, investigating existence uniqueness of solutions to the system, discussing about its stability, and lastly demonstrating that not only the linear version but also the semilinear version are relatively controllable under some restrictions we impose.

In addition to the importance of Fractional calculus, the Prabhakar calculus, and delayed systems, the fact that the Prabhakar fractional differential delayed system is firstly introduced and studied in the setting of its qualitative properties make this paper different and give it prominence. It is clear that the obtained results are also new even if the retardation is equal to zero or is removed in the system. Because of the definition of Prabhakar fractional derivatives which include so many of different fractional operators such as the Lorenzo-Hartly, the Miller-Ros, Riemann-Liouville, Gorenflo-Minerdi, Caputo fractional operators, etc., our results are comprehensive and also valid for these fractional differential delayed systems.

As a future work, one can investigate the same system replacing the constat coefficients by variable coefficients. Another work is to consider the Langevin fractional differential equations with the Prabhakar fractional derivatives. There are lots of things to do because the Prabhakar calculus is quite novel.

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