# The analytic solution of the fractional Rosenau–Hyman model in Liouville-Caputo sense

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**Abstract**

The purpose of this work is to achieve the analytic solution of non-linear Liouville-Caputo fractional initial value problem including time fractional Rosenau–Hyman equation by means of ARA transform and iterative method. First of all, ARA transform is employed to reduce the problem into a simpler one. Then the series solution of the non-linear fractional initial value problem is acquired by applying iterative method which leads to the analytic solution of the Rosenau–Hyman equation (RHE) with initial condition. The obtained results verify that the combination of ARA transform with iterative method produces an effective method to tackle with non-linear fractional problems.

**Keywords**: ARA transform, Rosenau–Hyman equation, Fractional derivative, iterative method
**2010 Mathematics Subject Classification**: 26A33, 35R11

## 1 Introduction

Since nonlinear differential equations are employed frequently in the modelling of physical processes, they have a significant importance in mathematical biology, electro-chemistry, viscoelasticity, physics and fluid mechanics. Therefore solving them get attention of many researchers and diverse methods such as Homotopy Perturbation Method, Dissipative perturbation methods and Variational Iteration Method see [19, 4, 2] have been developed to solve them. However, the big challenge is to construct their solution analytically [9, 5]. As a result, their approximate solutions are established by various mathematical methods [10].

Modelling of various processes with fractional differential equations attracts a rising attention of a great number of scientist from various areas of science since fractional derivative is a non-local operator which allows us to obtain better mathematical models of processes [7, 13, 12, 14, 15]. Obviously, fractional non-linear problems become more important in modelling of diverse scientific events. In general, we tackle with non-linear differential equations by means of various mathematical methods like variational iteration method (VIM), Adomian decomposition method (ADM) and homotopy perturbation method (HPM), etc. [6, 18, 1, 3].

Recently, a new integral transform, called ARA transform, have been established to tackle with any type of differential equations. This method allow us to reduce intricate differential equations into simplier differential or algebraic equations. The advantage of ARA transform compare to other integral transforms is that the applicability of the ARA transform is grater than applicability of the other integral transforms [16]. ARA transform with other numerical methods generate influential numerical methods for the solution of differential equations. In the modelling of non-linear dispersion process which has various applications in applied sciences, the RHE, a generalization of the KdV equation and introduced by P. Rosenau and J.M. Hyman [11], is employed to analyze the behaviour of the process. Even though, the RHE is a fractional non-linear equation, its solutions have been established by making use of diverse analytical methods.

The motivation of this research is analytically constructing the solution of the time-fractional RHE in Liouville-Caputo sense:

$$ {}_{0}^{C}D_{\tau}^{\varepsilon}\varphi = \varphi\frac{\partial^3\varphi}{\partial\zeta^3} + \varphi\frac{\partial\varphi}{\partial\zeta} + 3\frac{\partial\varphi}{\partial\zeta}\frac{\partial^2\varphi}{\partial\zeta^2}, 0 < \varepsilon \leq 1, \zeta \in \mathbb{R}, \tau \in (0, T] \tag{1} $$

subject to the initial condition

$$\varphi(\zeta,0) = -\frac{8c}{3}cos^2\left(\frac{\zeta}{4}\right), c,\zeta \in \mathbb{R} \tag{2}$$

by means of IAM, formed by ARA transform and iterative method. The innovation of this research is analytically establishing the solution for the time-fractional RHE with the initial condition. There has been no attempt to achieve the analytic solution of this problem to the best our knowledge. Moreover, ARA transform method is employed first time to tackle with this fractional initial value problem. The established results prove that IAM is an effective and versatile method.

# 2 Preliminary Results

This section is devoted to basic definitions, notations and characteristics of the fractional calculus [8].

**Definition 1** *The integration of a real valued function $f(\tau)$ in Riemann-Liouville integral sense is given as*

$$I_\tau^\varepsilon f(\tau) = \frac{1}{\Gamma(\varepsilon)}\int_0^\tau (\tau-\kappa)^{\varepsilon-1} f(\kappa)d\kappa,$$

*where $\varepsilon > 0$ represents the order of the integral.*

**Definition 2** *The fractional derivative of a real valued function $f(\tau)$ of order $\varepsilon$ in Liouville-Caputo sense is given as*

$$D_\tau^\varepsilon f(\tau) = \frac{1}{\Gamma(n-\varepsilon)}\int_{\tau_0}^\tau (\tau-\kappa)^{n-\varepsilon-1} f^{(n)}(\kappa)d\kappa, \tau \in [\tau_0, \tau_0+T],$$

*where $n-1<\varepsilon<n$ and $f^{(n)}(\tau) = \frac{d^n f}{d\tau^n}$. Liouville-Caputo derivative is equal to integer-order derivative when $\varepsilon$ is an natural number.*

**Definition 3** *The Mittag-Leffler function of two parameters is defined as*

$$E_{\varepsilon,\gamma}\left(\lambda(\tau-\tau_0)^\varepsilon\right) = \sum_{k=0}^{\infty}\frac{\left(\lambda(\tau-\tau_0)^\varepsilon\right)^k}{\Gamma(\varepsilon k+\gamma)}, \varepsilon,\gamma>0, \lambda\in\mathbb{R}.$$

Taking $\tau_0 = 0, \varepsilon = \gamma = q$ leads to the following

$$E_{q,q}(\lambda\tau^q) = \sum_{k=0}^{\infty}\frac{(\lambda\tau^q)^k}{\Gamma(qk+q)}, q>0. \tag{3}$$

Moreover, $E_{1,1}(\lambda\tau) = e^{\lambda\tau}$ when $q=1$. In the construction of the solution the following functions are utilized:

**Definition 4** *The $\rho^{th}$ order ARA integral transformation of the continuous function $g(\tau)$ on $(0,\infty)$ is defined as [16]*

$$A_\rho[g(\tau)](\nu) = A(\rho,\nu) = \nu\int_0^\infty \tau^{\rho-1}e^{-\nu\tau}g(\tau)d\tau, \nu>0.$$

**Definition 5** *Inverse transformation of ARA transform is introduced as*

$$\begin{aligned} g(\tau) &= A_{\rho+1}^{-1}\left[A_{\rho+1}[g(\tau)]\right] \\ &= \frac{(-1)^\rho}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\nu\tau}\left((-1)^\rho\left(\frac{1}{\nu\Gamma(\rho-1)}\int_0^\nu(\nu-x)^{\rho-1}A(\rho+1,x)\,dx + \sum_{k=0}^{\rho-1}\frac{\nu^k}{k!}\frac{\partial^k G(0)}{\partial\nu^k}\right)\right)d\nu \end{aligned}$$

*where $G(\nu) = \int_0^\infty e^{-\nu\tau}g(\tau)d\tau$ is $(\rho-1)$ times differentiable [16].*

**Property 1** *Taking $\rho = 1$ in the ARA transformation of Mittag-Leffler function leads to the following:*

$$\begin{aligned}
A_\rho\left[\tau^{\gamma-1}E_{\varepsilon,\gamma}\left(\lambda\tau^{\varepsilon}\right)\right](\nu) &= \nu\int_0^\infty \tau^{\rho-1}e^{-\nu\tau}\tau^{\gamma-1}\sum_k^\infty \frac{\left(\lambda\tau^{\varepsilon}\right)^k}{\Gamma\left(\varepsilon k+\gamma\right)}d\tau \\
&= \sum_k^\infty \frac{\lambda^k}{\Gamma\left(\varepsilon k+\gamma\right)}\nu\int_0^\infty e^{-\nu\tau}\tau^{\rho+\gamma-2+\varepsilon k}d\tau \\
&= \sum_k^\infty \frac{\lambda^k}{\Gamma\left(\varepsilon k+\gamma\right)}\nu\int_0^\infty \tau^{\rho-1}e^{-\nu\tau}\tau^{\gamma-1+\varepsilon k}d\tau \\
&= \sum_k^\infty \frac{\lambda^k}{\Gamma\left(\varepsilon k+\gamma\right)}\frac{\Gamma\left(\gamma-1+\varepsilon k+\rho\right)}{\nu^{\gamma-1+\varepsilon k+\rho-1}} \\
&= \frac{1}{\nu^{\gamma+\rho-2}}\sum_k^\infty \frac{\lambda^k}{\Gamma\left(\varepsilon k+\gamma\right)}\frac{\Gamma\left(\gamma+\varepsilon k+\rho-1\right)}{\nu^{\varepsilon k}}.
\end{aligned}$$

*For $\rho = 1$,*

$$A_1\left[\tau^{\gamma-1}E_{\varepsilon,\gamma}\left(\lambda\tau^{\varepsilon}\right)\right](\nu) = \frac{1}{\nu^{\gamma-1}}\sum_k^\infty \frac{\lambda^k}{\Gamma\left(\varepsilon k+\gamma\right)}\frac{\Gamma\left(\gamma+\varepsilon k\right)}{\nu^{\varepsilon k}} = \frac{1}{\nu^{\gamma-1}}\left(\frac{1}{1-\frac{\lambda}{\nu^{\varepsilon}}}\right) = \frac{\nu^{\varepsilon-\gamma+1}}{\nu^{\varepsilon}-\lambda}.$$

**Property 2** *The ARA transformation of $\tau^{p\varepsilon}$ for $p \in \mathbb{N}$ is given by*

$$\begin{aligned}
A_\rho\left[\tau^{p\varepsilon}\right](\nu) &= \nu\int_0^\infty \tau^{\rho-1}e^{-\nu\tau}\tau^{p\varepsilon}d\tau = \nu\int_0^\infty \tau^{p\varepsilon+\rho-1}e^{-\nu\tau}d\tau \\
&= \Gamma\left(p\varepsilon+n\right)\left(\frac{1}{\nu}\right)^{p\varepsilon+\rho}\nu\int_0^\infty \frac{\tau^{p\varepsilon+\rho-1}e^{-\nu\tau}}{\Gamma\left(p\varepsilon+n\right)\left(\frac{1}{\nu}\right)^{p\varepsilon+n}}d\tau = \Gamma\left(p\varepsilon+n\right)\left(\frac{1}{\nu}\right)^{p\varepsilon+n}\nu \\
&= \frac{\Gamma\left(p\varepsilon+\rho\right)}{\nu^{p\varepsilon+\rho-1}}.
\end{aligned}$$

**Property 3** *Convolution property of the ARA transformation is given by [16]*

$$A_\rho\left[f(\tau)*g(\tau)\right](\nu) = (-1)^{\rho-1}\nu\sum_{j=0}^{\rho-1}c_j^{\rho-1}F^{(j)}(\nu)\,G^{(\rho-1-j)}(\nu)\,.$$

*For $\rho = 1$ it becomes*

$$A_1\left[f(\tau)*g(\tau)\right](\nu) = \nu F(\nu)\,G(\nu)\,,$$

*where $G(\nu) = \int_0^\infty e^{-\nu\tau}g(\tau)d\tau$ and $F(\nu) = \int_0^\infty e^{-\nu\tau}f(\tau)d\tau$.*

The existence proof of the ARA transformation for Liouville-Caputo derivative and Riemann-Liouville integral is introduced in [17].

# 3 Main Results

The algorithm of IAM and its implementation for the time fractional RHE are presented in this section.

## 3.1 Algorithm of IAM for time fractional problems

This subsection is devoted to the algorithm of IAM for initial non-linear problem with Liouville-Caputo derivative is presented. Take the non-linear problem below into consideration:

$$\begin{aligned}
&{}_0^C D_\tau^\varepsilon\left(\varphi(\zeta,\tau)\right) + R\varphi(\zeta,\tau) + N\varphi(\zeta,\tau) = g(\zeta,\tau)\,, m-1<\varepsilon\leq m, m\in\mathbb{N} \qquad (4)\\
&\frac{\partial^k}{\partial\tau^k}\varphi(\zeta,0) = h_k(\zeta)\,, k=0,1,2,\ldots,m-1
\end{aligned}$$

where ${}_{0}^{C}D_{\tau}^{\varepsilon}\left(\varphi\left(\zeta,\tau\right)\right), R, N$ and $g\left(\zeta,\tau\right)$ denote fractional derivative, the linear operator, the general nonlinear operator and the right hand side function, respectively. By means of the ARA transform on Eq. (4), we get

$$\nu^{\varepsilon}A_{1}\left[\varphi\right]\left(\nu\right)-\sum_{k=0}^{m-1}\frac{\partial^{k}}{\partial\tau^{k}}\varphi\left(\zeta,0\right)\nu^{\varepsilon-k}+A_{1}\left[R\varphi+N\varphi\right]\left(\nu\right)=A_{1}\left[g\left(\zeta,\tau\right)\right]\left(\nu\right).$$
$$A_{1}\left[\varphi\right]\left(\nu\right)=\frac{1}{\nu^{\varepsilon}}\sum_{k=0}^{m-1}\frac{\partial^{k}}{\partial\tau^{k}}\varphi\left(\zeta,0\right)\nu^{\varepsilon-k}-\frac{1}{\nu^{\varepsilon}}A_{1}\left[R\varphi+N\varphi\right]\left(\nu\right)+\frac{1}{\nu^{\varepsilon}}A_{1}\left[g\left(\zeta,\tau\right)\right]\left(\nu\right).$$

where $\varphi=\varphi\left(\zeta,\tau\right)$. Application of its inverse transformation produces

$$\varphi=A_{1}^{-1}\left[\frac{1}{\nu^{\varepsilon}}\left[\sum_{k=0}^{m-1}\frac{\partial^{k}}{\partial\tau^{k}}\varphi\left(\zeta,0\right)\nu^{\varepsilon-k}+A_{1}\left[g\left(\zeta,\tau\right)\right]\left(\nu\right)\right]\right]-A_{1}^{-1}\left[\frac{1}{\nu^{\varepsilon}}\left[A_{1}\left[R\varphi+N\varphi\right]\left(\nu\right)\right]\right].\tag{5}$$

Employing the Iterative method leads to

$$\varphi=\sum_{i=0}^{\infty}\varphi_{i}.\tag{6}$$

Since the operator $R$ is linear, we have

$$R\left(\sum_{i=0}^{\infty}\varphi_{i}\right)=\sum_{i=0}^{\infty}R\left(\varphi_{i}\right),\tag{7}$$

Decomposing of the operator $N$ leads to

$$N\left(\sum_{i=0}^{\infty}\varphi_{i}\right)=N\left(\varphi_{0}\right)+\sum_{i=0}^{\infty}\left\{N\left(\sum_{k=0}^{i}\varphi_{k}\right)-N\left(\sum_{k=0}^{i-1}\varphi_{k}\right)\right\}.\tag{8}$$

Substituting (6), (7) and (8) in (5) produces the following

$$\begin{aligned}\sum_{i=0}^{\infty}\varphi_{i} &= A_{1}^{-1}\left[\frac{1}{\nu^{\varepsilon}}\left[\sum_{k=0}^{m-1}\frac{\partial^{k}}{\partial\tau^{k}}\varphi\left(\zeta,0\right)\nu^{\varepsilon-k}+A_{1}\left[g\left(\zeta,\tau\right)\right]\left(\nu\right)\right]\right]\\ &- A_{1}^{-1}\left[\frac{1}{\nu^{\varepsilon}}\left[A_{1}\left[R\left(\varphi_{i}\right)+N\left(\varphi_{0}\right)+\sum_{i=1}^{\infty}\left\{N\left(\sum_{k=0}^{i}\varphi_{k}\right)-N\left(\sum_{k=0}^{i-1}\varphi_{k}\right)\right\}\right]\left(\nu\right)\right]\right].\end{aligned}$$

As a result, the recurrence relation is constructed in the following form:

$$\begin{aligned}&\varphi_{0}=A_{1}^{-1}\left[\frac{1}{\nu^{\varepsilon}}\left[\sum_{k=0}^{m-1}\frac{\partial^{k}}{\partial\tau^{k}}\varphi\left(\zeta,0\right)\nu^{\varepsilon-k}+A_{1}\left[g\left(\zeta,\tau\right)\right]\left(\nu\right)\right]\right],\\ &\varphi_{1}=-A_{1}^{-1}\left[\frac{1}{\nu^{\varepsilon}}\left[A_{1}\left[R\left(\varphi_{0}\right)+N\left(\varphi_{0}\right)\right]\left(\nu\right)\right]\right],\\ &\varphi_{m+1}=A_{1}^{-1}\left[\frac{1}{\nu^{\varepsilon}}\left[A_{1}\left[R\left(\varphi_{m}\right)-\left\{N\left(\sum_{k=0}^{m}\varphi_{k}\right)-N\left(\sum_{k=0}^{m-1}\varphi_{k}\right)\right\}\right]\left(\nu\right)\right]\right],m\geq1\end{aligned}\tag{9}$$

Eventually, the $m$-term approximate solution is established as

$$\varphi\left(\zeta,\tau\right)\cong\varphi_{0}+\varphi_{1}+\varphi_{2}+...+\varphi_{m},m=1,2,...$$

### 3.2 Implementation of IAM for the time fractional Rosenau–Hyman problem

This section is dedicated to presentation of a nonlinear time fractional RHE with initial condition (1)-(2) in order to present how to establish analytic solutions of it by IAM. Making use of the ARA transform on Eq. (1) with (2) produces the following

$$A_1\left[\varphi\right](\nu)=-\frac{8c}{3}cos^2\left(\frac{\zeta}{4}\right)+\frac{1}{\nu^{\varepsilon}}A_1\left[\varphi\frac{\partial^3\varphi}{\partial\zeta^3}+\varphi\frac{\partial\varphi}{\partial\zeta}+3\frac{\partial\varphi}{\partial\zeta}\frac{\partial^2\varphi}{\partial\zeta^2}\right](\nu). \tag{10}$$

Applying inverse ARA transform to Eq. (10) produces

$$\varphi=-\frac{8c}{3}cos^2\left(\frac{\zeta}{4}\right)+A_1^{-1}\left[\frac{1}{\nu^{\varepsilon}}\left[A_1\left[\varphi\frac{\partial^3\varphi}{\partial\zeta^3}+\varphi\frac{\partial\varphi}{\partial\zeta}+3\frac{\partial\varphi}{\partial\zeta}\frac{\partial^2\varphi}{\partial\zeta^2}\right](\nu)\right]\right]. \tag{11}$$

Plugging (6)-(8) into (11) and using (9) leads to the following portions of the solution:

$$\begin{aligned}
&\varphi_0=\varphi(\zeta,0)=-\frac{8c}{3}cos^2\left(\frac{\zeta}{4}\right),\\
&\varphi_1=A_1^{-1}\left[\frac{1}{\nu^{\varepsilon}}\left[A_1\left[\varphi_0\frac{\partial^3\varphi_0}{\partial\zeta^3}+\varphi_0\frac{\partial\varphi_0}{\partial\zeta}+3\frac{\partial\varphi_0}{\partial\zeta}\frac{\partial^2\varphi_0}{\partial\zeta^2}\right](\nu)\right]\right]\\
&=A_1^{-1}\left[\frac{1}{\nu^{\varepsilon}}\left[A_1\left[-\frac{2c^2}{3}sin\left(\frac{\zeta}{2}\right)\right](\nu)\right]\right]=-\frac{2c^2}{3}sin\left(\frac{\zeta}{2}\right)\frac{\tau^{\varepsilon}}{\Gamma(\varepsilon+1)},\\
&\varphi_2=A_1^{-1}\left[\frac{1}{\nu^{\varepsilon}}\left[A_1\left[(\varphi_0+\varphi_1)\frac{\partial^3(\varphi_0+\varphi_1)}{\partial\zeta^3}+(\varphi_0+\varphi_1)\frac{\partial(\varphi_0+\varphi_1)}{\partial\zeta}+3\frac{\partial(\varphi_0+\varphi_1)}{\partial\zeta}\frac{\partial^2(\varphi_0+\varphi_1)}{\partial\zeta^2}\right](\nu)\right]\right]\\
&-A_1^{-1}\left[\frac{1}{\nu^{\varepsilon}}\left[A_1\left[\varphi_0\frac{\partial^3\varphi_0}{\partial\zeta^3}+\varphi_0\frac{\partial\varphi_0}{\partial\zeta}+3\frac{\partial\varphi_0}{\partial\zeta}\frac{\partial^2\varphi_0}{\partial\zeta^2}\right](\nu)\right]\right]\\
&=A_1^{-1}\left[\frac{1}{\nu^{\varepsilon}}\left[A_1\left[-\frac{2c^2}{3}sin\left(\frac{\zeta}{2}\right)+\frac{c^3}{3}cos\left(\frac{\zeta}{2}\right)\frac{\tau^{\varepsilon}}{\Gamma(\varepsilon+1)}\right](\nu)\right]\right]\\
&-\left(-\frac{2c^2}{3}sin\left(\frac{\zeta}{2}\right)\frac{\tau^{\varepsilon}}{\Gamma(\varepsilon+1)}\right)=\frac{c^3}{3}cos\left(\frac{\zeta}{2}\right)\frac{\tau^{2\varepsilon}}{\Gamma(2\varepsilon+1)},\\
&\varphi_3=A_1^{-1}\left[\frac{1}{\nu^{\varepsilon}}\left[A_1\left[(\varphi_0+\varphi_1+\varphi_2)\frac{\partial^3(\varphi_0+\varphi_1+\varphi_2)}{\partial\zeta^3}+(\varphi_0+\varphi_1+\varphi_2)\frac{\partial(\varphi_0+\varphi_1+\varphi_2)}{\partial\zeta}\right.\right.\right.\\
&\left.\left.\left.+3\frac{\partial(\varphi_0+\varphi_1+\varphi_2)}{\partial\zeta}\frac{\partial^2(\varphi_0+\varphi_1+\varphi_2)}{\partial\zeta^2}\right](\nu)\right]\right]-A_1^{-1}\left[\frac{1}{\nu^{\varepsilon}}\left[A_1\left[(\varphi_0+\varphi_1)\frac{\partial^3(\varphi_0+\varphi_1)}{\partial\zeta^3}\right.\right.\right.\\
&\left.\left.\left.+(\varphi_0+\varphi_1)\frac{\partial(\varphi_0+\varphi_1)}{\partial\zeta}+3\frac{\partial(\varphi_0+\varphi_1)}{\partial\zeta}\frac{\partial^2(\varphi_0+\varphi_1)}{\partial\zeta^2}\right](\nu)\right]\right]=A_1^{-1}\left[\frac{1}{\nu^{\varepsilon}}\left[A_1\left[\frac{c^4}{6}sin\left(\frac{\zeta}{2}\right)\frac{\tau^{2\varepsilon}}{\Gamma(2\varepsilon+1)}\right.\right.\right.\\
&\left.\left.\left.+\frac{c^3}{3}cos\left(\frac{\zeta}{2}\right)\frac{\tau^{\varepsilon}}{\Gamma(\varepsilon+1)}-\frac{2c^2}{3}sin\left(\frac{\zeta}{2}\right)\right](\nu)\right]\right]-\frac{c^3}{3}cos\left(\frac{\zeta}{2}\right)\frac{\tau^{2\varepsilon}}{\Gamma(2\varepsilon+1)}+\frac{2c^2}{3}sin\left(\frac{\zeta}{2}\right)\frac{\tau^{\varepsilon}}{\Gamma(\varepsilon+1)}\\
&=\frac{c^4}{6}sin\left(\frac{\zeta}{2}\right)\frac{\tau^{3\varepsilon}}{\Gamma(3\varepsilon+1)},\\
&\ldots
\end{aligned}$$

Eventually, the truncated solution is obtained as:

$$\varphi(\zeta,\tau;\varepsilon,c)=-\frac{8c}{3}cos^2\left(\frac{\zeta}{4}\right)-\frac{2c^2}{3}sin\left(\frac{\zeta}{2}\right)\frac{\tau^{\varepsilon}}{\Gamma(\varepsilon+1)}+\frac{c^3}{3}cos\left(\frac{\zeta}{2}\right)\frac{\tau^{2\varepsilon}}{\Gamma(2\varepsilon+1)}+\frac{c^4}{6}sin\left(\frac{\zeta}{2}\right)\frac{\tau^{3\varepsilon}}{\Gamma(3\varepsilon+1)}+\ldots.$$

Therefore, the series solution can be established as:

$$\varphi(\zeta,\tau;\varepsilon,c)=\frac{8}{3}\left(-c\ cos^2\left(\frac{\zeta}{4}\right)+\sum_{n=1}^{\infty}(-1)^n\left(\frac{c}{2}\right)^{2n}\left[sin\left(\frac{\zeta}{2}\right)\frac{\tau^{(2n-1)\varepsilon}}{\Gamma((2n-1)\varepsilon+1)}-\frac{c}{2}cos\left(\frac{\zeta}{2}\right)\frac{\tau^{2n\varepsilon}}{\Gamma(2n\varepsilon+1)}\right]\right), \tag{12}$$

where $\zeta \in \mathbb{R}, \tau \in [0, T]\, and 0 < \varepsilon \leq 1$. Rearranging (12) leads to the following

$$\begin{aligned}\varphi(\zeta, \tau; \varepsilon, c) &= -\frac{8}{3}c\, cos^2\left(\frac{\zeta}{4}\right) - \frac{8}{3}\frac{c}{2} sin\left(\frac{\zeta}{2}\right)\sum_{n=0}^{\infty}(-1)^n\left(\frac{c}{2}\right)^{2n+1}\frac{\tau^{(2n+1)\varepsilon}}{\Gamma((2n+1)\varepsilon+1)} \\ &- \frac{c}{2}\frac{8}{3}\left[cos_{\varepsilon}\left(\frac{c}{2}\tau^{\varepsilon}\right) - 1\right] cos\left(\frac{\zeta}{2}\right).\end{aligned}$$

In terms of $sin_{\varepsilon}\left(\frac{c}{2}\tau^{\varepsilon}\right)$ and $cos_{\varepsilon}\left(\frac{c}{2}\tau^{\varepsilon}\right)$, the series solution can be formed as:

$$\varphi(\zeta, \tau; \varepsilon, c) = -\frac{8}{3}c\, cos^2\left(\frac{\zeta}{4}\right) - \frac{8}{3}\frac{c}{2} sin\left(\frac{\zeta}{2}\right) sin_{\varepsilon}\left(\frac{c}{2}\tau^{\varepsilon}\right) - \frac{c}{2}\frac{8}{3} cos\left(\frac{\zeta}{2}\right)\left[cos_{\varepsilon}\left(\frac{c}{2}\tau^{\varepsilon}\right) - 1\right] \tag{13}$$

where $sin_{\varepsilon}(\mu\tau^{\varepsilon}) = \sum_{k=0}^{\infty}\frac{(-1)^k(\mu\tau^{\varepsilon})^{2k+1}}{\Gamma((2k+1)\varepsilon+1)}$ and $cos_{\varepsilon}(\mu\tau^{\varepsilon}) = \sum_{k=0}^{\infty}\frac{(-1)^k(\mu\tau^{\varepsilon})^{2k}}{\Gamma(2k\varepsilon+1)}$.
(13) is equal to the exact solution [8]

$$\varphi(\zeta, \tau) = -\frac{8c}{3} cos^2\left(\frac{\zeta - c\tau}{4}\right), |\zeta - c\tau| \leq 2\pi,$$

when $\varepsilon = 1$.

Solving non-linear fractional initial value problem including the time fractional RHE leads us to the following theorem:

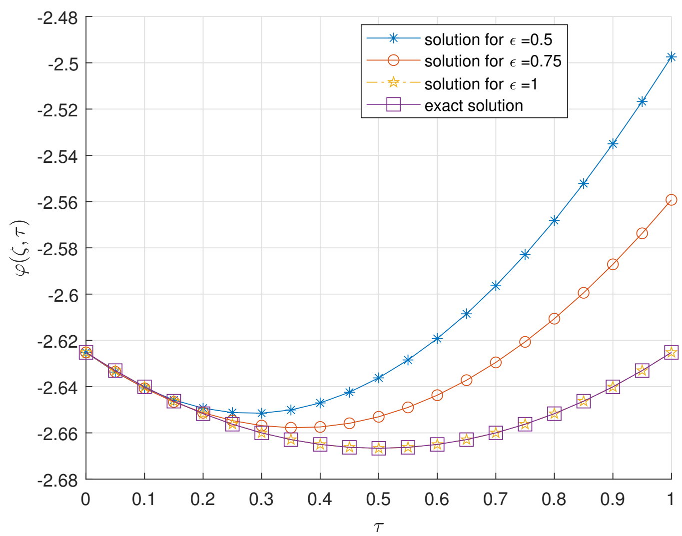


Figure 1: The analytic solutions of the initial value problem (1)-(2) for $c = 1$, $\zeta = 0.5$ and various values of $\varepsilon$.

Figures 1-3 demonstrates the analytical establishment of the solution to the time-fractional RHE in the Liouville-Caputo sense.

# 4 Conclusion

Innovation of this research is the analytic establishment of the solution for the time fractional RHE with initial condition by means of IAM. The analytic solution of this problem has not been established before. The implementation and outputs of IAM confirm the effectiveness and accuracy of this method which is supported by the outcomes.

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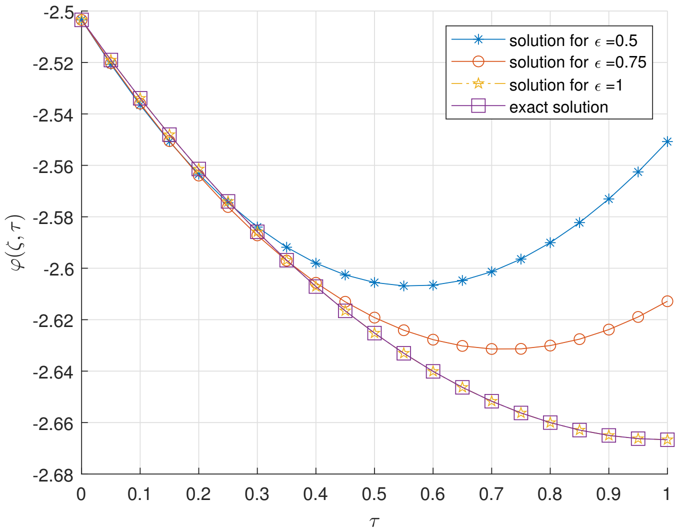


Figure 2: The analytic solutions of the initial value problem (1)-(2) for $c = 1$, $\zeta = 1$ and various values of $\varepsilon$.


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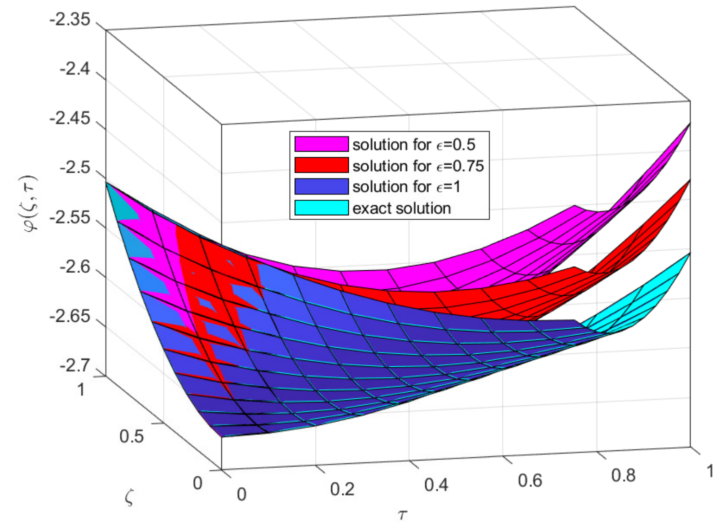


Figure 3: $3D$ analytic solutions of the initial value problem (1)-(2) for $c = 1$ and various values of $\varepsilon$.


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