AGU PUBLICATIONS

[Geophysical Research Letters]

Supporting Information for

Strongly nonlinear effects on determining internal solitary wave parameters from remote sensing signatures

Tao Xu[1], Xu Chen[2*], Qun Li[3], Xiao He[2], Jing Wang[1], and Jing Meng[2]

[1] School of Physics and Optoelectronic Engineering, Ocean University of China, Qingdao, China.

[2] Key Laboratory of Physical Oceanography, Ocean University of China and Qingdao National Laboratory for Marine Science and Technology, Qingdao, China.

[3] MNR Key Laboratory for Polar Science, Polar Research Institute of China, Shanghai, China.

Corresponding author: Xu Chen (chenxu001@ouc.edu.cn)

Contents of this file

Text S1. Solutions of internal solitary wave theories

The Korteweg–de Vires (KdV) equation (Korteweg & de Vries, 1895) has the solution

$$\eta(x,t) = \eta_0 \mathrm{sech}^2(X/\lambda), \tag{S1}$$

where $X = x - ct$, η_0 is the amplitude.

$$c_0 = \sqrt{\frac{g\Delta\rho}{\rho}\frac{h_1 h_2}{h_1 + h_2}}, \tag{S2}$$

$$c = c_0 + \frac{\alpha\eta_0}{3}, \tag{S3}$$

$$\lambda^2 = \frac{12\beta}{\alpha\eta_0}, \tag{S4}$$

where g is the gravitational acceleration, $\Delta\rho$ is the density difference, ρ is the reference density, h_1 and h_2 are the thickness of the upper layer and the lower layer, respectively. The parameters

$$\alpha = \frac{3}{2} c_0 \frac{h_1 - h_2}{h_1 h_2}, \tag{S5}$$

$$\beta = \frac{c_0 h_1 h_2}{6}. \tag{S6}$$

For Joseph–Kubota–Ko–Dobbs (JKKD) equation (Joseph, 1977; Kubota et al., 1978), we consider the solution used by Zheng et al. (1993)

$$\eta(x,t) = \frac{\eta_0}{[\cosh^2 aX + (\sinh^2 aX)/a^2 b^2]}, \tag{S7}$$

$$c_0 = \sqrt{\frac{\mathrm{g}\Delta\rho h_1}{\rho_1}}, \tag{S8}$$

$$c = c_0 \left\{1 + \frac{h_1}{2H}\left[1 + \frac{H}{b}(1 - a^2 b^2)\right]\right\}, \tag{S9}$$

$$b = \frac{4h_1^2}{3\eta_0}. \tag{S10}$$

The parameter a is parameter satisfying the relationship

$$ab\tan(aH) = 1. \tag{S11}$$

The Benjamin–Ono (BO) equation (Benjamin, 1967; Ono, 1975) has the solution

$$\eta(x,t) = \frac{\eta_0}{1 + (X/\lambda)^2}. \tag{S12}$$

$$c = c_0 + \frac{\alpha\eta_0}{4}, \tag{S13}$$

where c_0 is the same as in Eq. S8.

$$\lambda = \frac{4\beta}{\alpha\eta_0}, \tag{S14}$$

$$\alpha = -\frac{3c_0}{2h_1}, \tag{S15}$$

$$\beta = \frac{c_0 h_1 \rho_2}{2\rho_1}. \tag{S16}$$

If the higher-order nonlinear terms were took into account in the KdV equation, the extended KdV (eKdV) equation (Grimshaw et al., 2004; Kakutani & Yamasaki, 1978) has the solution

$$\eta(x,t) = \frac{\eta_0}{B + (1-B)\cosh^2(X/\lambda)}. \tag{S17}$$

$$c = c_0 + \frac{1}{3}\eta_0\left(\alpha + \frac{1}{2}\gamma\eta_0\right), \tag{S18}$$

$$\lambda^2 = \frac{12\beta}{\eta_0\left(\alpha + \frac{1}{2}\gamma\eta_0\right)}, \tag{S19}$$

$$B = \frac{-\gamma\eta_0}{2\alpha + \gamma\eta_0}, \tag{S20}$$

the parameters α and β are the same as in Eq. S5 and Eq. S6, the parameter

$$\gamma = \frac{3c_0}{(h_1 h_2)^2}\left[\frac{7}{8}(h_1 - h_2)^2 - \left(\frac{h_1^3 + h_2^3}{h_1 + h_2}\right)\right]. \tag{S21}$$

The solution of Miyata–Choi–Camassa (MCC) equation (Camassa et al., 2006; Choi & Camassa, 1999; Miyata, 1988) is a nonlinear ordinary differential equation

$$\left(\frac{\partial\eta(x,t)}{\partial X}\right)^2 = \delta\,\frac{\eta^2(\eta - a_-)(\eta - a_+)}{\eta - a_*}, \tag{S22}$$

the parameter

$$\delta = \frac{3g\Delta\rho}{c^2(\rho_1 h_1^2 - \rho_2 h_2^2)}, \tag{S23}$$

$$c^2 = \frac{c_0^2(h_1 - \eta_0)(h_2 + \eta_0)}{h_1 h_2 - (c_0^2/g)\eta_0}, \tag{S24}$$

where c_0 is the same as in Eq. S2.

$$a_* = \frac{h_1 h_2}{h_2 - h_1}, \tag{S25}$$

where a_- and a_+ are the two roots of a quadratic equation

$$\eta^2 + q_1\eta + q_2 = 0, \tag{S26}$$

$$q_1 = -(c^2/g) + h_2 - h_1,\ q_2 = h_1 h_2(c^2/c_0^2 - 1). \tag{S27}$$

Without any assumption about the wavelength and amplitude, Dubreil–Jacotin–Long (DJL) (Long, 1953) is established, the equation is as follows

$$\nabla^2\eta + \frac{N^2(z-\eta)}{c^2}\eta = 0, \tag{S28}$$

where isopycnal displacement η is a function of x and z, c is the phase speed, N is the buoyancy frequency, expressed as:

$$N^2(z) = -\frac{g}{\rho_0}\frac{d\rho(z)}{dz}, \tag{S29}$$

where g is the gravitational acceleration, $\rho(z)$ is the density profile, ρ_0 is the reference density. It should be noted that the DJL equation does not have explicit solutions, which can only be solved by the numerical method (Dunphy et al., 2011; Stastna & Lamb, 2002).

Text S2. Iteration procedure of the solution of wave-induced velocity in inseparable form

When the ISW which has a waveform of $\eta(x)$ exists, the buoyancy frequency in the domain can be expressed as

$$N_{wave}^{i}\left[z+\eta^{i-1}(x,z)\right]=\begin{cases}0 & 0\le z\le -\eta^{i-1}(x,z)\\ N_b(z) & -H-\eta^{i-1}(x,z)\le z<0\end{cases}, \qquad \text{(S30)}$$

where H is the total depth of the domain, $N_b(z)$ is the background buoyancy frequency, i presents the iterations. The $\eta^{i-1}(x,z)$ is calculated by the following equation with the initial value of $\eta(x)$

$$\eta^{i}(x,z)=\eta(x){\phi_{wave}}^{i}(x,z),\quad \eta^{0}(x,z)=\eta(x), \qquad \text{(S31)}$$

where the vertical structure function $\phi^{i}(x,z)$ is calculated by

$$\left(\frac{d^2}{dz^2}+\frac{{N_{wave}^{2}}^{i}(x,z)}{c_0^2}\right){\phi_{wave}}^{i}(x,z)=0 \qquad \phi_{wave}(x,-H)=\phi_{wave}(x,0)=0, \quad \text{(S32)}$$

where c_0 is the linear phase speed. Therefore, according to the convergent iteration results, the solution of wave-induced horizontal velocity can be expressed as

$$u(x,z)=c\frac{\partial\eta(x,z)}{\partial z}=c\eta(x)\frac{\partial\phi_{wave}(x,z)}{\partial z}. \qquad \text{(S33)}$$

References

Benjamin, T. B. (1967). Internal waves of permanent form in fluids of great depth. *Journal of Fluid Mechanics*, *29*(3), 559-592. https://doi.org/https://doi.org/10.1017/S002211206700103X

Camassa, R., Choi, W., Michallet, H., RusÅS, P. O., & Sveen, J. K. (2006). On the realm of validity of strongly nonlinear asymptotic approximations for internal waves. *Journal of Fluid Mechanics*, *549*(-1). https://doi.org/10.1017/s0022112005007226

Choi, W., & Camassa, R. (1999). Fully nonlinear internal waves in a two-fluid system. *Journal of Fluid Mechanics*, *396*, 1-36. https://doi.org/10.1017/s0022112099005820

Dunphy, M., Subich, C., & Stastna, M. (2011). Spectral methods for internal waves: indistinguishable density profiles and double-humped solitary waves. *Nonlinear Processes in Geophysics*, *18*(3), 351-358. https://doi.org/10.5194/npg-18-351-2011

Grimshaw, R., Pelinovsky, E., Talipova, T., & Kurkin, A. (2004). Simulation of the Transformation of Internal Solitary Waves on Oceanic Shelves. *Journal of Physical Oceanography*, *34*(12), 2774-2791. https://doi.org/10.1175/jpo2652.1

Kakutani, T., & Yamasaki, N. (1978). Solitary Waves on a Two-Layer Fluid. *Journal of the Physical Society of Japan*, *45*(2), 674-679. https://doi.org/10.1143/jpsj.45.674

Korteweg, D. J., & de Vries, G. (1895). XLI. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, *39*(240), 422-443. https://doi.org/10.1080/14786449508620739

Long, R. R. (1953). Some Aspects of the Flow of Stratified Fluids: I. A Theoretical Investigation. *Tellus*, *5*(1), 42-58. https://doi.org/10.3402/tellusa.v5i1.8563

Miyata, M. (1988). Long internal waves of large amplitude. In *Nonlinear water waves* (pp. 399-406). Springer. https://doi.org/10.1007/978-3-642-83331-1_44

Ono, H. (1975). Algebraic Solitary Waves in Stratified Fluids. *Journal of the Physical Society of Japan*, *39*(4), 1082-1091. https://doi.org/10.1143/jpsj.39.1082

Stastna, M., & Lamb, K. G. (2002). Large fully nonlinear internal solitary waves: The effect of background current. *Physics of Fluids*, *14*(9), 2987-2999. https://doi.org/10.1063/1.1496510

Zheng, Q., Yan, X.-H., & Klemas, V. (1993). Statistical and dynamical analysis of internal waves on the continental shelf of the Middle Atlantic Bight from space shuttle photographs. *Journal of Geophysical Research: Oceans*, *98*(C5), 8495-8504. https://doi.org/10.1029/92jc02955

Table S1. Summary of experimental conditions

Case	h_1(m)	h_2(m)	h_2/h_1	η_0/h_1
1	0.04	0.12	3	0.21-0.89
2	0.04	0.16	4	0.18-1.21
3	0.04	0.20	5	0.49-1.71
3(b)	0.08	0.40	5	0.34-1.49
4	0.04	0.24	6	0.46-2.06
5	0.04	0.28	7	0.50-2.35
6	0.04	0.32	8	0.43-2.50
7	0.04	0.40	10	0.46-2.36

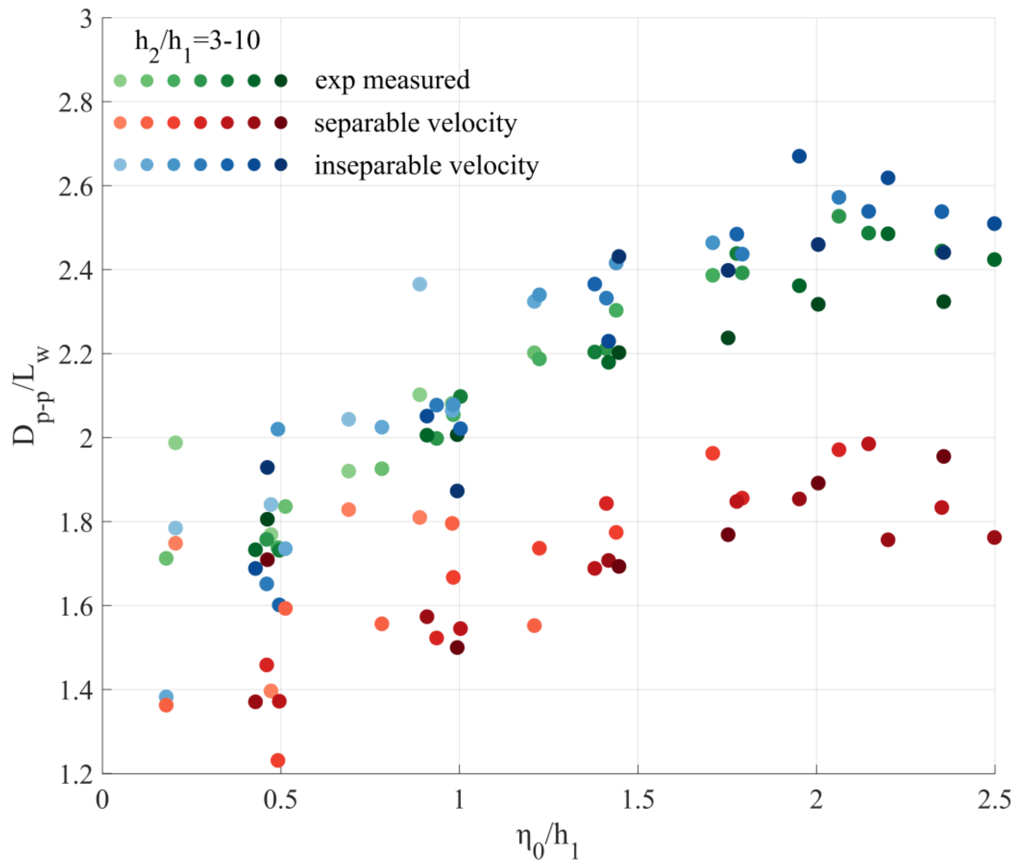

Figure S1. The variation of the ratio of D_{p-p} to wavelength with amplitude. The green dots are measured in experiments, and the red and blue dots are calculated by the solution of velocity in separable and inseparable forms, respectively. Each color from light to dark corresponds to the h_2/h_1 from 3 to 10.

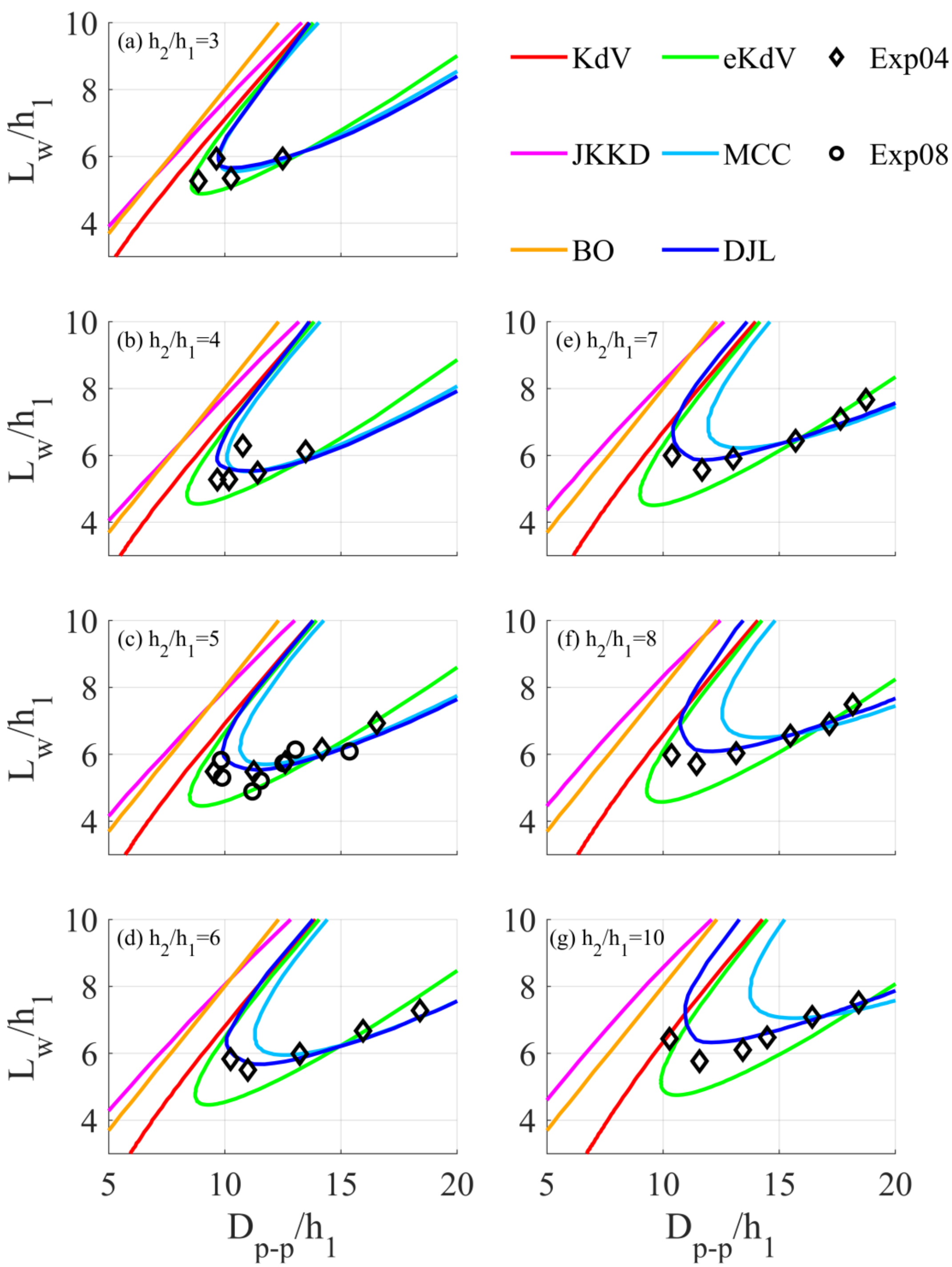

Figure S2. Theoretical and experimental results of wavelength retrievals under different stratifications, (a)-(g) are the relationships between D_{p-p} and wavelength with h_2/h_1 from 3 to 10 respectively. The red, magenta, yellow, green, light blue, and dark blue lines represent the KdV, JKKD, BO, eKdV, MCC, and DJL equations respectively, and the black diamond and circle represent the experimental results that $h_1 = 0.04$ m and 0.08 m, respectively.

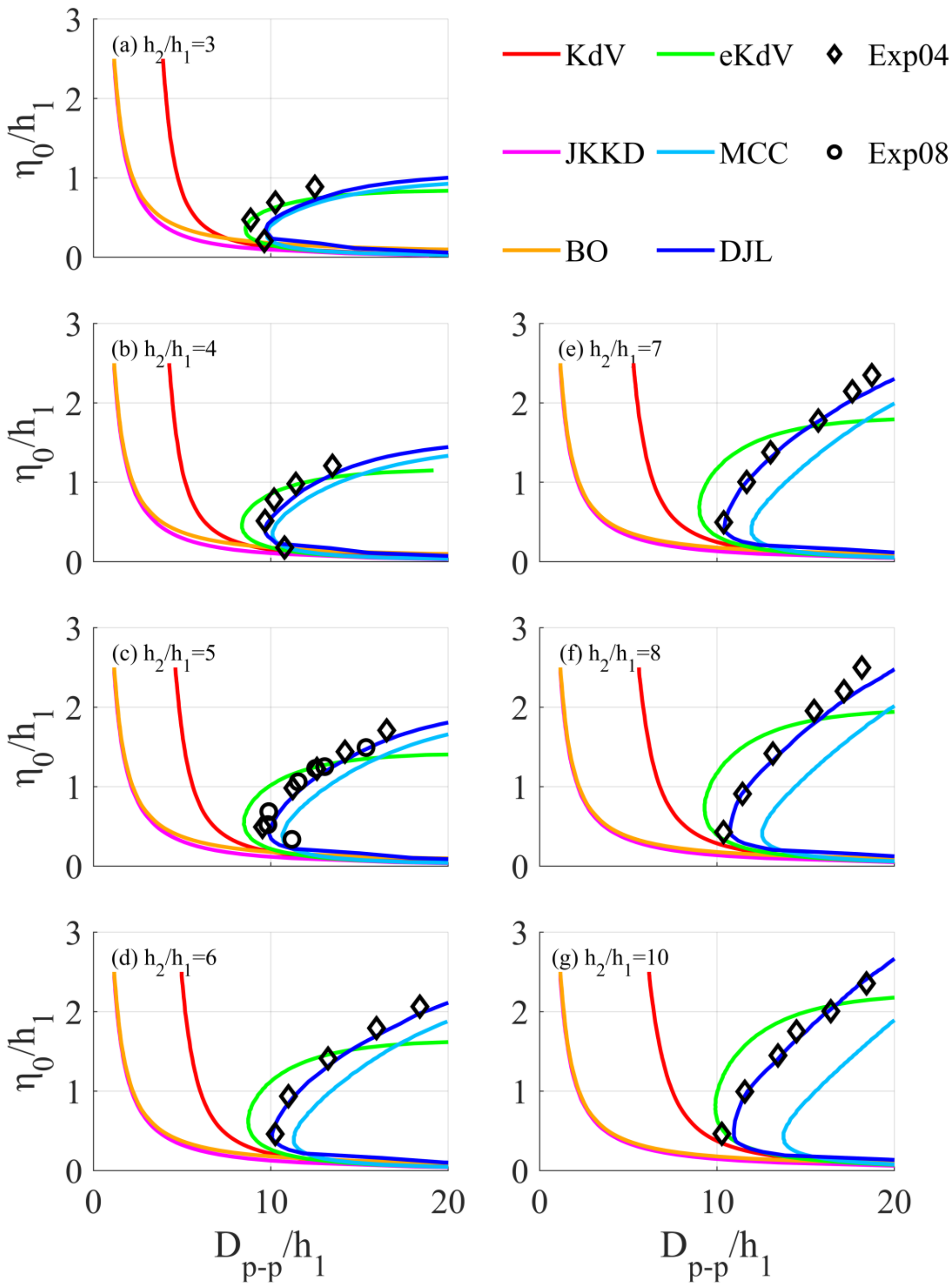

Figure S3. Theoretical and Experimental results of amplitude retrievals under different stratifications, (a)-(g) are the relationships between D_{p-p} and amplitude with h_2/h_1 from 3 to 10 respectively. The red, magenta, yellow, green, light blue, and dark blue lines represent the KdV, JKKD, BO, eKdV, MCC, and DJL equations respectively, and the black diamond and circle represent the experimental results that h_1 =0.04 m and 0.08 m, respectively.

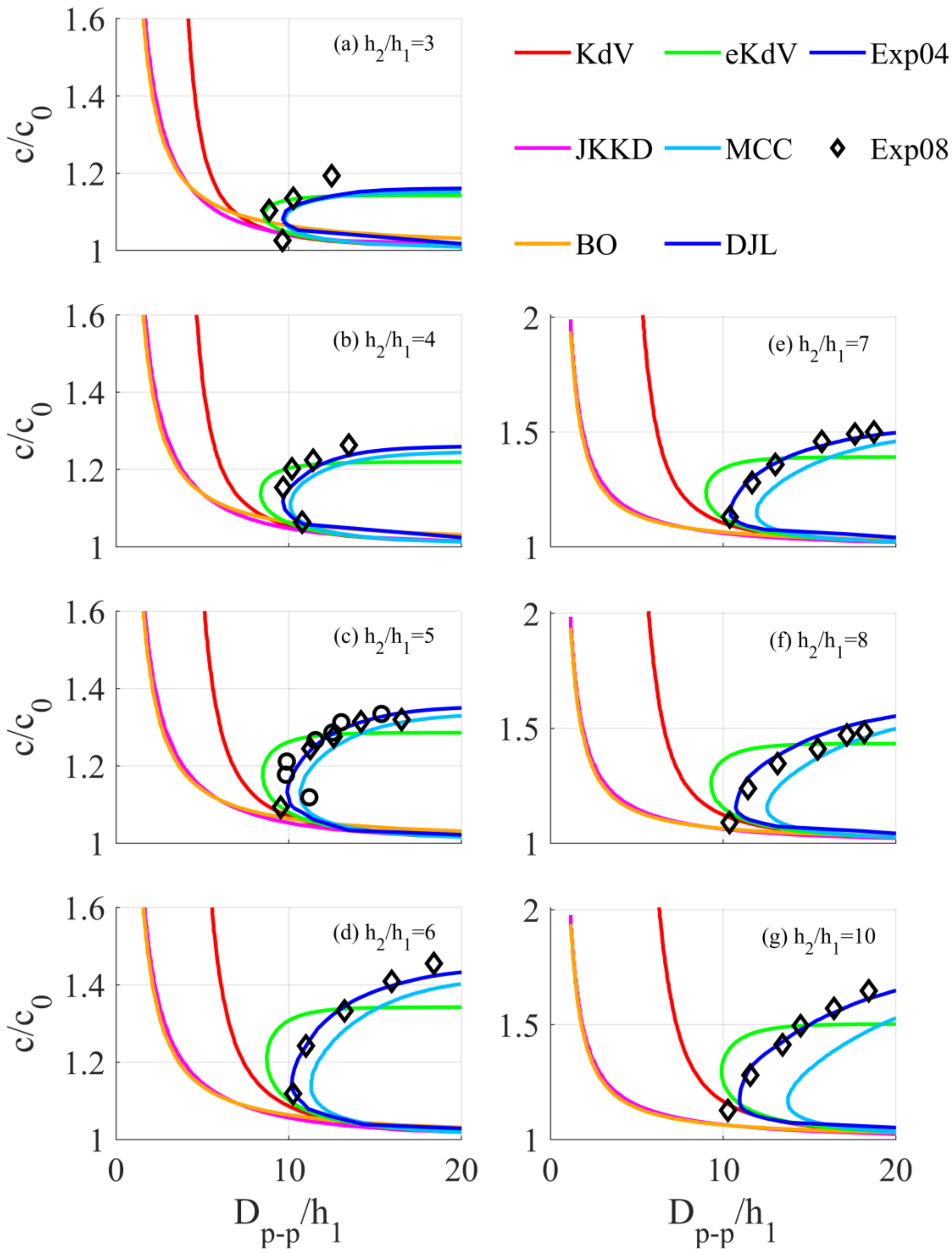

Figure S4. Theoretical and Experimental results of phase speed retrievals under different stratifications, (a)-(g) are the relationships between D_{p-p} and phase speed with h_2/h_1 from 3 to 10 respectively. The red, magenta, yellow, green, light blue, and dark blue lines represent the KdV, JKKD, BO, eKdV, MCC, and DJL equations respectively, and the black diamond and circle represent the experimental results that $h_1 = 0.04$ m and 0.08 m, respectively.