MULTIPLE NORMALIZED SOLUTIONS FOR THE FRACTIONAL SCHRÖDINGER EQUATIONS

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Abstract. In this paper, we study the existence and asymptotic properties of solutions to the following fractional Schrödinger equation

$$(-\Delta)^{\sigma}u = \lambda u + |u|^{q-2}u + \mu\left(I_{\alpha} * |u|^{p}\right)|u|^{p-2}u \text{ in } \mathbb{R}^N$$

under the normalized constraint

$$\int_{\mathbb{R}^N} u^2 = a^2,$$

where $N \geq 2,\ \sigma \in (0,1),\ \alpha \in (0,N),\ q \in (2+\frac{4\sigma}{N}, \frac{2N}{N-2\sigma}],\ p \in [2, 1+\frac{2\sigma+\alpha}{N}),\ a > 0,\ \mu > 0$, $I_{\alpha}(x)=|x|^{\alpha-N}$ and $\lambda\in\mathbb{R}$ appears as a Lagrange multiplier. In the Sobolev subcritical case $q\in(2+\frac{4\sigma}{N}, \frac{2N}{N-2\sigma})$, we show that the problem admits at least two solutions under suitable assumptions on α, a and μ. In the Sobolev critical case $q = \frac{2N}{N-2\sigma}$, we prove that the problem possesses at least one ground state solution. Furthermore, we give some stability and asymptotic properties of the solutions. We mainly extend the results in S. Bhattarai [1](J. Differ. Equ. 2017) and B. H. Feng et al. in [21](J. Math. Phys. 2019) concerning the above problem from L^2-subcritical and L^2-critical setting to L^2-supercritical setting with respect to q, involving Sobolev critical case especially.

Key words : Fractional Schrödinger equation; Normalized solutions; Variational methods; Stability.

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1. Introduction and Main Result

This paper concerns the existence of solutions $(u,\lambda) \in H^{\sigma}(\mathbb{R}^N) \times \mathbb{R}$ to the following fractional Schrödinger equation

$$(-\Delta)^{\sigma}u = \lambda u + |u|^{q-2}u + \mu\left(I_{\alpha} * |u|^{p}\right)|u|^{p-2}u \text{ in } \mathbb{R}^N \tag{1.1$_\lambda$}$$

under the constraint

$$\int_{\mathbb{R}^N} u^2 = a^2, \tag{1.2}$$

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where $N\geq 2$, $\sigma\in(0,1)$, $\alpha\in(0,N)$, $q\in(2+\frac{4\sigma}{N},2^*_\sigma]$, $2^*_\sigma=\frac{2N}{N-2\sigma}$, $p\in[2,1+\frac{2\sigma+\alpha}{N})$, $a>0$, $\mu>0$ and $I_\alpha(x)=|x|^{\alpha-N}$. Because of the nonlocality of the Choquard term $\mu\left(I_\alpha*|u|^p\right)|u|^{p-2}u$, we call $(1.1)_\lambda$ the fractional nonlinear Schrödinger equation with a focusing nonlocal perturbation if $\mu>0$. Here the fractional Laplacian $(-\Delta)^\sigma$ is defined as

$$(-\Delta)^\sigma u(x)=C_{N,\sigma}\text{P.V.}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2\sigma}}dy,\forall u\in S(\mathbb{R}^N),$$

where $S(\mathbb{R}^N)$ denotes the Schwartz space of rapidly decreasing smooth functions, P.V. stands for the principle value of the integral and $C_{N,\sigma}$ is some positive normalization constant. The Hilbert space $H^\sigma(\mathbb{R}^N)$ is defined as

$$H^\sigma(\mathbb{R}^N):=\{u\in L^2(\mathbb{R}^N):(-\Delta)^{\frac{\sigma}{2}}u\in L^2(\mathbb{R}^N)\},$$

with the inner product and norm are given respectively by

$$(u,v):=\int_{\mathbb{R}^N}(-\Delta)^{\frac{\sigma}{2}}u(-\Delta)^{\frac{\sigma}{2}}v+\int_{\mathbb{R}^N}uv,\ \|u\|:=\left(\left\|(-\Delta)^{\frac{\sigma}{2}}u\right\|_2^2+\|u\|_2^2\right)^{\frac{1}{2}},$$

where $\left\|(-\Delta)^{\frac{\sigma}{2}}u\right\|_2^2:=\frac{C_{N,\sigma}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2\sigma}}$.

Problem $(1.1)_\lambda$-(1.2) arises from seeking standing waves for the following nonlinear fractional Schrödinger equation

$$i\partial_t\psi+(-\Delta)^\sigma\psi=|\psi|^{q-2}\psi+\mu\left(I_\alpha*|\psi|^p\right)|\psi|^{p-2}\psi\text{ in }\mathbb{R}\times\mathbb{R}^N.\tag{1.3}$$

A standing wave of (1.3) is a solution having the form $\psi(t,x)=e^{-i\lambda t}u(x)$ for some $\lambda\in\mathbb{R}$ and u satisfying $(1.1)_\lambda$. So $(1.1)_\lambda$ is the stationary equation of the time-dependent equation (1.3). From Propositions 2.3-2.4 in [21], we know that the Cauchy problem for (1.3) is locally well-posed.

We say that a function $u\in H^\sigma(\mathbb{R}^N)$ is a weak solution to $(1.1)_\lambda$ if

$$\int_{\mathbb{R}^N}[(-\Delta)^{\frac{\sigma}{2}}u(-\Delta)^{\frac{\sigma}{2}}\varphi-\lambda u\varphi]-\int_{\mathbb{R}^N}|u|^{q-2}u\varphi-\mu\int_{\mathbb{R}^N}\left(I_\alpha*|u|^p\right)|u|^{p-2}u\varphi=0,\ \forall\varphi\in H^\sigma(\mathbb{R}^N).$$

For fixed $a>0$, we aim at finding a real number $\lambda\in\mathbb{R}$ and a function $u\in H^\sigma(\mathbb{R}^N)$ weakly solving $(1.1)_\lambda$ with $||u||_2=a$. Physicists call a solution u of $(1.1)_\lambda$ with $||u||_2=a$ a **normalized solution**. Normalized solutions to $(1.1)_\lambda$ can be obtained by searching critical points of the energy functional

$$E_\mu(u)=\frac{1}{2}||(-\Delta)^{\frac{\sigma}{2}}u||_2^2-\frac{\mu}{2p}\int_{\mathbb{R}^N}\left(I_\alpha*|u|^p\right)|u|^p-\frac{1}{q}||u||_q^q,\ \mu\geq 0\tag{1.4}$$

on the constraint

$$S_a:=\left\{u\in H^\sigma(\mathbb{R}^N):||u||_2^2=a^2\right\}$$

with Lagrange multipliers λ. By the L^2-norm preserving dilations $u_t(x)=t^{\frac{N}{2}}u(tx)$ with $t>0$, it is easy to know that

$$\bar{q}:=2+\frac{4\sigma}{N},\ \bar{p}:=1+\frac{2\sigma+\alpha}{N}$$

is the L^2-critical exponent with respect to q, p respectively. Indeed, for any $\mu > 0$, we have

$$\inf_{u\in S_a} E_\mu(u) = -\infty, \text{ if } \bar{q} < q \le 2^*_\sigma \text{ or } \bar{p} < p \le (N+\alpha)/(N-2\sigma)$$

and

$$\inf_{u\in S_a} E_\mu(u) > -\infty, \text{ if } 2 < q < \bar{q} \text{ and } 1+\alpha/N \le p < \bar{p}.$$

Equation (1.3) is a special case of the following equation

$$i\partial_t\psi + (-\Delta)^\sigma\psi = f(|\psi|)\psi \text{ in } \mathbb{R}\times\mathbb{R}^N, \tag{1.5}$$

which is a fundamental equation of the space-fractional quantum mechanics, see [28]. For $\sigma = 1/2$, (1.5) have been also used as models to describe Boson-stars, see [34]. In [31], S. Longhi proposed an optical realization of the fractional Schrödinger equation. One can refer to [37, 38] for more information about physical backgrounds on (1.5).

Y. Cho et al. in [9] proved existence and uniqueness of local and global solutions of (1.5) with the Hartree-type nonlinearity $f(|\psi|)\psi = (|x|^{-\alpha} * |\psi|^2)\psi$ for $\alpha \in (0, N)$. They also showed the existence of blow-up solutions in [8]. Some stable results of (1.5) with $f(|\psi|)\psi = (|x|^{-\alpha} * |\psi|^2)\psi$ for $\alpha \in (0, 2\sigma)$ are obtained by D. Wu in [48]. For the local nonlinearity $f(|\psi|)\psi = |\psi|^{q-2}\psi$ with $q \in (2, 2^*_\sigma]$, the well-posedness and ill-posedness in $H^\sigma(\mathbb{R}^N)$ have been investigated in [11, 23]. By using a sharp Gagliardo-Nirenberg-type inequality and profile decomposition in $H^\sigma(\mathbb{R}^N)$, C. M. Peng et al. in [40] proved that the standing waves of (1.5) with $f(|\psi|)\psi = |\psi|^{q-2}\psi$ are orbitally stable when $2 < q < \bar{q}$ and the ground state solitary waves are strongly unstable to blowup when $q = \bar{q}$. In [47], B. Thomas et al. obtained a general criterion for blow-up of radial solution of (1.5) with $f(|\psi|)\psi = |\psi|^{q-2}\psi$, $q \ge \bar{q}$ while $N \ge 2$, see also V. D. Dinh in [12, 13]. For (1.5) with combined power-type nonlinearities, i.e. $f(\psi) = \gamma|\psi|^{q-2}\psi + \mu|\psi|^{p-2}\psi$, one can refer to [16, 50].

In [1], S. Bhattarai considered $(1.1)_\lambda$ with $\mu \ge 0$, $q \in (2, \bar{q})$ and $p \in [2, \bar{p})$. Soon after, B. H. Feng et al. in [21] also considered $(1.1)_\lambda$ in three cases: (1) $\mu < 0$, $q = \bar{q}$, $\bar{p} < p < \frac{N+\alpha}{N-2\sigma}$; (2) $\mu > 0$, $q = \bar{q}$, $1 + \frac{\alpha}{N} < p < \bar{p}$; (3) $\mu > 0$, $2 < q < \bar{q}$, $p = \bar{p}$. Both the authors in [1, 21] studied $(1.1)_\lambda$ by considering a minimization problem

$$\inf_{u\in S_a} E_\mu(u) > -\infty, \text{ where } S_a := \Big\{u \in H^\sigma(\mathbb{R}^N) : ||u||_2^2 = a^2\Big\}. \tag{1.6}$$

They established the relative compactness of energy minimizing sequences (and hence, existence and stability of minimizers) via concentration compactness principle (see Lemma I.1 of [29]). Furthermore, a more general convolution potential was considered in [1]. S. Bhattarai also generalized the existence result and the stability of associated standing waves to a coupled system with Hartree type nonlinearities.

Recently, much attention are paid to the existence of normalized solutions to the classical Schrödinger equations (i.e. $\sigma = 1$) when the energy functional is unbounded from below on the L^2-constraint, see [25, 4, 44, 10]. In this case, the constrained minimization method used in [1, 21] does not work any more and it is also very difficult to prove the boundedness of the related Palais-Smale sequences. To overcome this difficulty, L. Jeanjean in [24] constructed a special Palais-Smale sequences which concentrates around the related Pohozaev manifold; this localization not only leads to the boundedness of the Palais-Smale sequences but also provide the information which is vital in compactness analysis. This method was also adopted in [25, 4, 44, 10].

We first point out that [25] studied normalized solutions to quasi-linear Schrödinger equations and [4, 44] studied normalized solutions to Schrödinger equations with combined nonlinearities(i.e., taking $f(\psi) = -\left(|x|^{-1} * |\psi|^2\right)\psi + |\psi|^{p-2}\psi$ or $|\psi|^{p-2}\psi + |\psi|^{q-2}\psi$). In [10], S. Cingolani et al. considered the following equation

$$-\Delta u = \lambda u + \gamma |u|^{q-2}u + \mu\left(\mathcal{I}_N * |u|^2\right) u \text{ in } \mathbb{R}^N, \tag{1.7}$$

where $q \in (2, 2_\sigma^*)$, $\gamma \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\mathcal{I}_N(x) = \frac{1}{N(N-2)\omega_N}|x|^{2-N}$ in case $N \geq 3$ and $\mathcal{I}_N(x) = \frac{1}{2\pi}\log|x|$ in case $N = 2$. Here ω_N denotes the volume of the unit ball in $\mathbb{R}^N$. In the case of $N \geq 3$, one can refer to [4, 26, 30, 32] for the existence and multiplicity of solutions with prescribed L^2-norm. S. Cingolani et al. in [10] mainly solve the case $N = 2$, i.e.

$$-\Delta u = \lambda u + \gamma |u|^{q-2}u + \mu\left(\log|\cdot| * |u|^2\right) u \text{ in } \mathbb{R}^2. \tag{1.8}$$

It is easy to see that the associated energy functional is not well-defined on $H^1(\mathbb{R}^2)$ due to the logarithmic convolution term. By choosing a suitable workspace

$$X := \left\{ u \in H^1(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} \log(1+|x|)|u(x)|^2 dx < \infty \right\},$$

they obtained several existence and multiplicity results under different assumptions on $q > 2$, $\gamma \in \mathbb{R}$ and $\mu \in \mathbb{R}$.

To our best knowledge, $(1.1)_\lambda$ with $\sigma = 1$ is quite well understood in [10] and the references therein, and the existing results on normalized solutions to $(1.1)_\lambda$ with $\sigma \in (0,1)$ and $\inf_{u \in S_a} E_\mu(u) > -\infty$ can be summarized in [1, 21]. **However, the existence of normalized solutions to $(1.1)_\lambda$ with $\sigma \in (0,1)$ and $\inf_{u \in S_a} E_\mu(u) = -\infty$ is still unknown.**

In this paper, we consider the existence and asymptotic properties of normalized solutions to $(1.1)_\lambda$ with

$$\sigma \in (0,1), \ \mu > 0, \ q \in (\bar{q}, 2_\sigma^*] \text{ and } p \in [2, \bar{p}).$$

That is, we are in the setting $\sigma \in (0,1)$ and $\inf_{u \in S_a} E_\mu(u) = -\infty$.

To state our main results, we introduce two definitions and some frequently used constants. We say that $\tilde{u}$ is a **ground state** of $E_\mu|_{S_a}$ if

$$d\, E_\mu|_{S_a}(\tilde{u}) = 0 \quad \text{and} \quad E_\mu(\tilde{u}) = \inf\left\{E_\mu(u) : \ d\, E_\mu|_{S_a}(u) = 0, \text{ and } u \in S_a\right\}.$$

The set of the ground states to $E_\mu|_{S_a}$ will be denoted by $Z_{a,\mu}$. We say that $Z_{a,\mu}$ is **stable** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $\psi_0 \in H^\sigma(\mathbb{R}^N)$ with $\inf_{v \in Z_{a,\mu}} \|\psi_0 - v\|_{H^\sigma(\mathbb{R}^N)} < \delta$, we have

$$\sup_{t \in (T_{min}, T_{max})} \inf_{v \in Z_{a,\mu}} \|\psi(t,\cdot) - v\|_{H^\sigma(\mathbb{R}^N)} < \varepsilon,$$

where $\psi(t,\cdot)\in C\big((T_{min},T_{max}),H^{\sigma}(\mathbb{R}^N)\big)$ denotes the unique solution to (1.3) with $\psi(0,x)=\psi_0(x)$ and (T_{min},T_{max}) denotes the maximal time interval on which the existence and uniqueness of the solution to the Cauchy problem is guaranteed. For $2<q\le 2_\sigma^*$ and $1+\frac{\alpha}{N}<p<\frac{N+\alpha}{N-2\sigma}$, we introduce two frequently used constants:

$$\gamma_q=\frac{N(q-2)}{2q\sigma},\ \delta_p=\frac{N(p-1)-\alpha}{2p\sigma}. \tag{1.9}$$

Notice that $0<\gamma_q\le 1$, $0<\delta_p<1$. If $\bar{q}<q\le 2_\sigma^*$ and $2\le p<\bar{p}$, we check that $2p\delta_p<2<q\gamma_q$ and

$$\bar{\mathcal{C}}(p,q):=2p(1-\delta_p)(q\gamma_q-2)+2q(1-\gamma_q)(1-p\delta_p)>0.$$

Let us assume that μ,a satisfy the following conditions:

$$(A_1^*)\qquad \mu^{q\gamma_q-2}a^{\bar{\mathcal{C}}(p,q)}<\tilde{\mathcal{C}}(p,q):=\left[\tfrac{q(1-p\delta_p)}{(q\gamma_q-2p\delta_p)\mathcal{A}_q^q}\right]^{2(1-p\delta_p)}\left[\tfrac{p(q\gamma_q-2)}{(q\gamma_q-2p\delta_p)\mathcal{C}_p^{2p}}\right]^{(q\gamma_q-2)}.$$

$$(A_2^*)\qquad \mu a^{2p(1-\delta_p)}<\tfrac{2\sigma}{N}\cdot\tfrac{2_\sigma^*}{\delta_p(2_\sigma^*-2p\delta_p)\mathcal{C}_p^{2p}}\cdot\left[\tfrac{p\delta_p\mathcal{S}^{\frac{N}{2\sigma}}}{1-p\delta_p}\right]^{1-p\delta_p}.$$

Here $\mathcal{A}_q=\mathcal{S}^{-\frac{\gamma_q}{2}}$, $\mathcal{C}_p=\mathcal{S}_p^{-1}$ while $\mathcal{S}$ and $\mathcal{S}_p$ denoting some embedding constants given by

$$\mathcal{S}=\inf_{u\in\dot{H}^{\sigma}(\mathbb{R}^N)\setminus\{0\}}\frac{\big\|(-\Delta)^{\frac{\sigma}{2}}u\big\|_2^2}{||u||_{2_\sigma^*}^2},\ \mathcal{S}_p=\inf_{u\in H^{\sigma}(\mathbb{R}^N)\setminus\{0\}}\frac{\big\|(-\Delta)^{\frac{\sigma}{2}}u\big\|_2^{\delta_p}\,\|u\|_2^{(1-\delta_p)}}{\big(\int_{\mathbb{R}^N}(I_\alpha*|u|^p)\,|u|^p\big)^{\frac{1}{2p}}}.$$

For more information about $\mathcal{S}$ and $\mathcal{S}_p$, we refer to [36, 17] (See also Section 2 for details).

Let u_0 be the ground state of $E_0|_{S_a}$ (See Lemma 5.5). Our main results are as follows.

Theorem 1.1. *Let $N\ge 2$, $\sigma\in(0,1)$, $\alpha\in(N-2\sigma,N)$, $q\in(2+\frac{4\sigma}{N},2_\sigma^*)$, $p\in[2,1+\frac{2\sigma+\alpha}{N})$ and $a,\mu>0$ satisfy condition (A_1^*). Then*
(1) *$E_\mu|_{S_a}$ has a critical point $\tilde{u}_{a,\mu}$ at some energy level $m(a,\mu)<0$, which is an interior local minimizer of E_μ on the set*

$$A_{R_0}:=\left\{u\in S_a:||(-\Delta)^{\frac{\sigma}{2}}u||_2<R_0\right\}$$

for a suitable $R_0=R_0(a,\mu)>0$. Moreover, $\tilde{u}_{a,\mu}$ is a ground state of $E_\mu|_{S_a}$, and any ground state of $E_\mu|_{S_a}$ is a local minimizer of E_μ on A_{R_0}.
(2) *$E_\mu|_{S_a}$ has a second critical point of mountain pass type $\hat{u}_{a,\mu}$ at some energy level $\sigma(a,\mu)>0$.*
(3) *There exist $\tilde{\lambda}_{a,\mu},\hat{\lambda}_{a,\mu}<0$ such that $\tilde{u}_{a,\mu}$ solves $(1.1)_{\tilde{\lambda}_{a,\mu}}$ and $\hat{u}_{a,\mu}$ solves $(1.1)_{\hat{\lambda}_{a,\mu}}$. Both $\tilde{u}_{a,\mu}$ and $\hat{u}_{a,\mu}$ are positive and radially symmetric. Moreover, $\tilde{u}_{a,\mu}$ is radially deceasing.*
(4) *$m(a,\mu)\to 0^-$, and any ground state $\tilde{u}_{a,\mu}\in S_a$ for $E_\mu|_{S_a}$ satisfies $||(-\Delta)^{\frac{\sigma}{2}}\tilde{u}_{a,\mu}||_2\to 0$ as $\mu\to 0^+$.*
(5) *$\sigma(a,\mu)\to m(a,0)$ and $\hat{u}_{a,\mu}\to u_0$ in $H^{\sigma}(\mathbb{R}^N)$ as $\mu\to 0^+$, where $m(a,0)=E_0(u_0)$ and u_0 is the ground state of $E_0|_{S_a}$.*

Theorem 1.2. *Let $N\ge 2$, $\sigma\in(0,1)$, $\alpha\in(N-2\sigma,N)$, $q=2_\sigma^*$, $p\in[2,1+\frac{2\sigma+\alpha}{N})$, $a,\mu>0$ satisfying conditions $(A_1^*)-(A_2^*)$. Then*

(1) *$E_\mu|_{S_a}$ has a critical point $\tilde{u}_{a,\mu}$ at some energy level $m(a,\mu) < 0$, which is an interior local minimizer of E_μ on the set*

$$A_{R_0} := \left\{ u \in S_a : ||(-\Delta)^{\frac{\sigma}{2}} u||_2 < R_0 \right\}$$

for a suitable $R_0 = R_0(a,\mu) > 0$. Moreover, $\tilde{u}_{a,\mu}$ is a ground state of $E_\mu|_{S_a}$, and any ground state of $E_\mu|_{S_a}$ is a local minimizer of E_μ on A_{R_0}.
(2) *There exists $\tilde{\lambda}_{a,\mu} < 0$ such that $\tilde{u}_{a,\mu}$ solves $(1.1)_{\tilde{\lambda}_{a,\mu}}$. Moreover, $\tilde{u}_{a,\mu}$ is positive and radially deceasing.*
(3) *$m(a,\mu) \to 0^-$, and any ground state $\tilde{u}_{a,\mu} \in S_a$ for $E_\mu|_{S_a}$ satisfies $||(-\Delta)^{\frac{\sigma}{2}} \tilde{u}_{a,\mu}||_2 \to 0$ as $\mu \to 0^+$.*

From Theorem 1.1, we know that the set of ground states of $E_\mu|_{S_a}$ is not empty, i.e. $Z_{a,\mu} \neq \emptyset$. For the subcritical case, we have the following stability result.

Theorem 1.3. *Let $N \geq 2$, $\sigma \in (0,1)$, $\alpha \in (N-2\sigma, N)$, $q \in (2 + \frac{4\sigma}{N}, 2_\sigma^*)$, $p \in [2, 1 + \frac{2\sigma+\alpha}{N})$ and $a > 0$. There exists $\tilde{\mu} > 0$ sufficiently small such that, if $0 < \mu < \tilde{\mu}$, then the set of ground states $Z_{a,\mu}$ is stable.*

Remark 1.4. For the Sobolev critical case $q = 2_\sigma^*$, the stability of the set of ground states is still open even for the classical case where $\sigma = 1$ (See [45]).

Remark 1.5. Theorem 1.1 indicates that there exist at least two normalized solutions to $(1.1)_\lambda$ when $q \in (\bar{q}, 2_\sigma^*)$, one ground state and one excited state, i.e. the solution's energy is strictly larger than that of the ground state. Moreover, the ground state to $(1.1)_\lambda$ vanishes and the excited state converges to a ground state of the related limiting equation $(-\Delta)^\sigma u = \hat{\lambda} u + |u|^{q-2} u$ as $\mu \to 0^+$. Theorem 1.2 is new on the existence of normalized solutions to the fractional Schrödinger equation with Sobolev critical exponents. For $q = 2_\sigma^*$ and $\mu = 0$, the related Pohozaev identity implies that $(1.1)_\lambda$ does not have any nontrivial solutions. When it comes to $\mu > 0$, we obtain at least one ground state to $(1.1)_\lambda$, which vanishes gradually as $\mu \to 0^+$. Since $E_\mu|_{S_a}$ is unbounded from below if $q = 2_\sigma^*$, it could be natural to expect that there exists a second critical point on S_a, which is an excited state as in the case of $q \in (\bar{q}, 2_\sigma^*)$. Actually, we can prove the existence of a Palais-Smale sequence for $E_\mu|_{S_a}$ at a mountain pass level $\sigma(a,\mu) > m(a,\mu)$, but it seems difficult to prove the convergence of such sequence. These results show that the nonlocal term $\mu\,(I_\alpha * |u|^p)\,|u|^{p-2}u$ has crucial effect on the structure of the energy functional E_μ and makes the solution set to $(1.1)_\lambda$ much richer.

Theorems 1.1-1.2 mainly extend the results in [1, 21], which deal with $(1.1)_\lambda$, from $\inf_{u \in S_a} E_\mu(u) > -\infty$ to the case when $\inf_{u \in S_a} E_\mu(u) = -\infty$, and also partially extend the results in [10], which deal with $(1.1)_\lambda$, from classical Schrödinger equation to fractional Schrödinger equation. Theorem 1.3 implies that the introduction of a L^2-subcritical focusing nonlocal term $\mu\,(I_\alpha * |u|^p)\,|u|^{p-2}u$ can stable the original unstable model $(1.1)_\lambda$ with $\mu = 0$ and $q \in (\bar{q}, 2_\sigma^*)$. If $\mu = 0$ and $q \in (\bar{q}, 2_\sigma^*)$, we learn from the forthcoming Lemma 5.5 that $(1.1)_\lambda$ possesses a positive radial ground state at energy level $m(a,0) > 0$ for any $a > 0$. Moreover, the associated standing wave is strongly unstable since we are in a L^2-supercritical (with respect to q) regime (see [47, 13]). From the variational point of view, the stabilization is reflected by the discontinuity of the ground state energy level $m(a, \mu)$: we have $m(a, \mu) < 0$ for every $\mu > 0$ small, while $m(a, 0) > 0$. Similar behaviors were already observed in [3, 44]. In [3], this discontinuity was created by the introduction of a trapping potential; in [44], this discontinuity was created by the introduction of a L^2-subcritical focusing ($\mu > 0$) local power dispersion term.

Remark 1.6. The condition $\alpha \in (N - 2\sigma, N)$ in the above theorems is used for getting L^∞ estimates and better regularity of solutions to $(1.1)_\lambda$, which are useful in obtaining the related Pohozaev identity (See Lemma 2.8 below). The condition (A_1^*) in Theorems 1.1-1.2 makes sure that E_μ presents a so-called convex-concave geometry. Therefore, it is reasonable to expect the existence of a local minimizer and a mountain pass critical point for $E_\mu|_{S_a}$ (See Remark 1.5 for details). The condition (A_2^*) in Theorem 1.2 is used for obtaining some energy estimates, which guarantee the compactness of the Palais-Smale sequences.

The main idea of the proof of Theorems 1.1-1.3 comes from [24, 44, 45, 10] which dealt with the classical Schrödinger equations where $\sigma = 1$. We now underline some of the difficulties that arise in the proof of our main results.

It is standard as in [24, 44, 45, 10] that the Pohozaev constraint approach can tackle the case of $\inf_{u \in S_a} E_\mu(u) = -\infty$. **But the first difficulty is that**, we need C^2 regularity of solutions to $(1.1)_\lambda$ to obtain the Pohozaev identity. For the case $\sigma = 1$, $(1.1)_\lambda$ becomes

$$-\Delta u = a(x)u := \lambda u + |u|^{q-2}u + \mu \left(I_\alpha * |u|^p\right)|u|^{p-2}u \text{ in } \mathbb{R}^N \tag{1.10}$$

where $N \geq 2$, $\alpha \in (0, N)$, $q \in (2, \frac{2N}{N-2}]$, $p \in [1 + \frac{\alpha}{N}, \frac{N+\alpha}{N-2}]$, $\mu > 0$ and $I_\alpha(x) = |x|^{\alpha - N}$. It is easy to check that $a(x) \in L^{N/2}_{loc}(\mathbb{R}^N)$ if $\max\{0, N-4\} < \alpha < N$, and the Brezis-Kato theorem in [2] implies that $u \in L^r_{loc}(\mathbb{R}^N)$ for all $1 \leq r < +\infty$ and hence $u \in W^{2,r}_{loc}(\mathbb{R}^N)$ for all $1 \leq r < +\infty$. Then, the Sobolev embedding theory implies $u \in C^{0,\beta}_{loc}(\mathbb{R}^N)$ for any $\beta \in (0, 1)$ and Schauder theory implies $u \in C^2(\mathbb{R}^N)$ (See Lemma 1.30 in [49]). But we are in the setting $\sigma \in (0, 1)$, the nonlocal term $\mu \left(I_\alpha * |u|^p\right)|u|^{p-2}u$ makes it difficult to obtain a prior L^∞ estimate and Schauder theory doesn't imply C^2 regularity of solutions to $(1.1)_\lambda$ (See Theorem 2.11 in [27]).

In this paper, we try to modify the methods in [15, 42] to prove the C^2 regularity of solutions to $(1.1)_\lambda$. In [15], P. D'avenia et al. considered the problem

$$(-\Delta)^\sigma u = \lambda u + \left(I_\alpha * |u|^p\right)|u|^{p-2}u \text{ in } \mathbb{R}^N \tag{1.11}$$

where $\sigma \in (0, 1)$, $N \geq 3$, $\lambda < 0$, $\alpha \in (0, N)$ and $1 + \frac{\alpha}{N} \leq p \leq \frac{N+\alpha}{N-2\sigma}$. They proved the C^2 regularity of the positive solutions to (1.11) provided $\sigma \in (\frac{1}{2}, 1)$ and $p \geq 2$ (See Remark 3.1 in [15]). In [42], Z. F. Shen et al. studied

$$(-\Delta)^\sigma u + u = \left(I_\alpha * F(u)\right) f(u) \text{ in } \mathbb{R}^N \tag{1.12}$$

where $\sigma \in (0, 1)$, $N \geq 3$, $\alpha \in (0, N)$ and $F(u) = \int_0^u f(\tau)d\tau \in C^1(\mathbb{R}^N, \mathbb{R})$. With the help of the Dirichlet-Neumann map, the authors transformed (1.12) into a local problem on

$\mathbb{R}_+^{N+1}$ and then they obtained a prior L^∞ estimate by using the standard Moser iteration procedure. Our proof was based on a careful analysis of the related parameters and an iteration technique. Compared with [42], we get a prior L^∞ estimate in a direct but simple way. Furthermore, we recover the loss of $\sigma \in (0, \frac{1}{2}]$ and allow $\sigma \in (0,1)$ by using an iteration technique, which relax the restriction in [15] that $\sigma \in (\frac{1}{2}, 1)$. **This part is new and important since it makes the Pohozaev constraint approach meaningful, see Lemma 2.8 for details**.

Now we give the outline of the proof for the existence results in Theorem 1.1. To be precise, for any $u \in S_a$ and $s \in \mathbb{R}$, set $(s \star u)(x) := e^{\frac{N}{2}s} u(e^s x)$, it results that $s \star u \in S_a$, and we introduce the fiber map

$$\Psi_u^\mu(s) := E_\mu(s \star u) = \frac{e^{2\sigma s}}{2}||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 - \frac{e^{q\gamma_q \sigma s}}{q}||u||_q^q - \frac{\mu e^{2p\delta_p \sigma s}}{2p} B(u,u),$$

where $B(u,u)$ was defined in (2.1). Under suitable assumptions on a and μ, we can prove that the function $\Psi_u^\mu(s)$ has exactly two critical points, one is a local mimimum and another one is a global maximum (See Lemma 2.15 below). Since $(\Psi_u^\mu)'(s) = \sigma\big[e^{2\sigma s}||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 - \gamma_q e^{q\gamma_q\sigma s}||u||_q^q - \mu\delta_p e^{2p\delta_p\sigma s} B(u,u)\big] = \sigma P_\mu(s \star u)$, we shall see that critical point of $\Psi_u^\mu(s)$ allows to project a function on the Pohozaev set

$$\mathcal{P}_{a,\mu} = \left\{ u \in S_a : 0 = P_\mu(u) =: ||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 - \gamma_q ||u||_q^q - \mu\delta_p B(u,u) \right\}.$$

Here $P_\mu(u) = 0$ denotes the Pohozaev identity. Naturally, we expect that $E_\mu|_{S_a}$ has two critical points, one is a local mimimizer and another one is of mountain pass type, which belong to the set $\mathcal{P}_{a,\mu}$. To begin with, we try to find the first critical point of $E_\mu|_{S_a}$. Define $m(a,\mu) := \inf_{u \in A_{R_0}} E_\mu(u)$ with $A_{R_0} := \left\{u \in S_a : ||(-\Delta)^{\frac{\sigma}{2}} u||_2 < R_0\right\}$ for some $R_0 > 0$, then Lemma 2.17 implies that

$$m(a,\mu) = \inf_{\mathcal{P}_{a,\mu}} E_\mu < 0.$$

The Ekeland's variational principle guarantees the existence of a Palais-Smale sequence $\{u_n\} \subset S_a$ for $E_\mu|_{S_a}$ at level $m(a,\mu) < 0$ with the additional properties

$$P_\mu(u_n) \to 0 \text{ as } n \to \infty.$$

The fact $P_\mu(u_n) \to 0$ gives the boundedness of $\{u_n\}$ in $H^\sigma(\mathbb{R}^N)$. By proving a compactness result (See Proposition 3.1 below), we have $u_n \to u$ in $H^\sigma(\mathbb{R}^N)$ and so we get the desired results. Next, we try to find the second critical point of $E_\mu|_{S_a}$. This relies heavily on a refined version of the min-max principle by N. Ghoussoub [22] (See Lemma 4.2 below). A min-max procedure guarantees the existence of a Palais-Smale sequence $\{u_n\} \subset S_a$ for $E_\mu|_{S_a}$ at the mountain pass level $\sigma(a,\mu) > 0$ with $P_\mu(u_n) \to 0$. Then, Proposition 3.1 implies that $u_n \to u$ in $H^\sigma(\mathbb{R}^N)$ and so we get the second critical point of $E_\mu|_{S_a}$. It only remains to prove the compactness result, i.e. Proposition 3.1, and this is standard as in [44] since $\bar{q} < q < 2_\sigma^*$. For the Sobolev critical case $q = 2_\sigma^*$ (See Theorem 1.2), we can only obtain a local minimizer as stated in Remark 1.5. In fact, we can similarly prove the existence of a Palais-Smale sequence $\{u_n\} \subset S_a$ for $E_\mu|_{S_a}$ at level $m(a,\mu) < 0$ with $P_\mu(u_n) \to 0$. As is well known that the energy estimate $m(a,\mu) < \frac{\sigma}{N}\mathcal{S}^{\frac{N}{2\sigma}}$ is necessary in compactness analysis when tackling Sobolev critical problems. We see immediately that $m(a,\mu) < \frac{\sigma}{N}\mathcal{S}^{\frac{N}{2\sigma}}$ since $m(a,\mu) < 0$. **But the second difficulty is that**, $m(a,\mu) < \frac{\sigma}{N}\mathcal{S}^{\frac{N}{2\sigma}}$ is not sufficient in obtaining compactness of $\{u_n\}$ in $H^\sigma(\mathbb{R}^N)$ since two alternatives may occur in Proposition 3.2. As can be seen from Section 3, the proof of Proposition 3.2 is more delicate than that of Proposition 3.1 since $q = 2_\sigma^*$. To rule out the non-compact case, we need not only the condition $m(a,\mu) < \frac{\sigma}{N}\mathcal{S}^{\frac{N}{2\sigma}}$ but also some additional energy estimates (See Section 4 for details).

Thirdly, we need to prove the relative compactness of every minimizing sequence of $m(a,\mu) = \inf_{A_{R_1}} E_\mu$ where $A_{R_1} = \left\{u \in S_a : ||(-\Delta)^{\frac{\sigma}{2}}u||_2 < R_1\right\}$ for some $R_1 > 0$ in proving Theorem 1.3. The difficulty lies in ruling out the dichotomy of every minimizing sequence of $m(a,\mu)$. However, the concentration compactness principle developed in $H^\sigma(\mathbb{R}^N)$ (see Lemma 2.4 in [20]) seems not applicable to equation $(1.1)_\lambda$ for the appearance of the focusing nonlocal term $\mu\left(I_\alpha * |u|^p\right)|u|^{p-2}u$. Consequently, we try to modify the concentration analysis in [1] to overcome this difficulty. The proof of our result is more delicate than that of [1]. Indeed, since $m(a,\mu)$ is a local minimizer value rather than a global one, we shall always keep the dichotomy of every minimizing sequence of $m(a,\mu)$ staying in the admissible set A_{R_1}(See Lemma 6.2 for details).

This paper is organized as follows, in Section 2, we give some preliminary results. In Section 3, we give the compactness analysis of Palais-Smale sequences. In Section 4, we prove the existence results, i.e. Theorem 1.1-(1)(2)(3) and Theorem 1.2-(1)(2). In Section 5, we prove the asymptotic results, i.e. Theorem 1.1-(4)(5) and Theorem 1.2-(3). In Section 6, we deal with the stability results, i.e. Theorem 1.3.

Notations: The homogeneous fractional Sobolev space of order $\sigma \in (0,1)$ is defined as $\dot{H}^\sigma(\mathbb{R}^N) := \{u \in L^{2_\sigma^*}(\mathbb{R}^N) : |\xi|^\sigma \hat{u}(\xi) \in L^2(\mathbb{R}^N)\}$, which is in fact the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm $||u||^2_{\dot{H}^\sigma(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2\sigma}|\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^N} |(-\Delta)^{\sigma/2}u|^2 dx$. The dual space of $\dot{H}^\sigma(\mathbb{R}^N)$ is denoted by $\dot{H}^\sigma(\mathbb{R}^N)'$. See [36] and references therein for the basics on the fractional Laplacian. For $\beta \in (0,1)$, $C^{0,\beta}(\mathbb{R}^N)$ denotes the standard Hölder space on $\mathbb{R}^N$. $L^p = L^p(\mathbb{R}^N)$ $(1 < p \le \infty)$ is the Lebesgue space with the standard norm $||u||_p = \left(\int_{\mathbb{R}^N} |u(x)|^p dx\right)^{\frac{1}{p}}$. We use " $\to$ " and " $\rightharpoonup$ " to denote the strong and weak convergence in the related function spaces respectively. C and C_i will denote positive constants. $\langle\cdot,\cdot\rangle$ denote the dual pair for any Banach space and its dual space. $\mathbb{N} = \{1, 2, \cdots\}$ is the set of natural numbers. $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers respectively. $\overline{\Omega}$ denotes the closure of Ω. Ω^c denotes the complement set of Ω. $X \hookrightarrow Y$ means X embeds into Y. $o_n(1)$ and $O_n(1)$ mean that $|o_n(1)| \to 0$ as $n \to +\infty$ and $|O_n(1)| \le C$ as $n \to +\infty$, respectively. $\Gamma(\cdot)$ is the Gamma function.

2. Preliminaries

In this section, we give some preliminary results. To simplify notations, we denote

$$B(u,u) := \int_{\mathbb{R}^N} (I_\alpha * |u|^p)\, |u|^p = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{N-\alpha}} dxdy. \tag{2.1}$$

The following lemma is the fractional Sobolev embedding.

Lemma 2.1. *([36], Theorem 6.5) Let $0<\sigma<1$ be such that $N>2\sigma$. Then there exists a constant $\mathcal{S}=\mathcal{S}(N,\sigma)>0$ such that*

$$\mathcal{S} = \inf_{u\in \dot{H}^\sigma(\mathbb{R}^N)\setminus\{0\}} \frac{\left\|(-\Delta)^{\frac{\sigma}{2}} u\right\|_2^2}{||u||_{2_\sigma^*}^2} \tag{2.2}$$

where $2_\sigma^* = \frac{2N}{N-2\sigma}$. Moreover, $H^\sigma(\mathbb{R}^N)$ is continuously embedded into $L^r(\mathbb{R}^N)$ for any $2\le r\le 2_\sigma^*$ and compactly embedded into $L^r_{loc}(\mathbb{R}^N)$ for every $2\le r<2_\sigma^*$.

We also require the fractional Gagliardo-Nirenberg inequality.

Lemma 2.2. *Let $0<\sigma<1$, $N>2\sigma$ and $r\in(2,2_\sigma^*)$. Then there exists a constant $C(N,\sigma,r)=\mathcal{S}^{-\frac{\gamma_r}{2}}>0$ such that*

$$||u||_r \le C(N,\sigma,r)\left\|(-\Delta)^{\frac{\sigma}{2}}u\right\|_2^{\gamma_r} \|u\|_2^{(1-\gamma_r)}, \qquad \forall u\in H^\sigma(\mathbb{R}^N), \tag{2.3}$$

where $2_\sigma^* = \frac{2N}{N-2\sigma}$ and $\gamma_r = \frac{N(r-2)}{2r\sigma}$.

Proof. By Hölder's inequality, we have $||u||_r \le ||u||_2^{(1-\gamma_r)}||u||_{2_\sigma^*}^{\gamma_r}$. Using (2.2), we have

$$||u||_r \le ||u||_2^{(1-\gamma_r)}\Big(\mathcal{S}^{-\frac{1}{2}}||(-\Delta)^{\frac{\sigma}{2}}u||_2\Big)^{\gamma_r} = \mathcal{S}^{-\frac{\gamma_r}{2}}\left\|(-\Delta)^{\frac{\sigma}{2}}u\right\|_2^{\gamma_r}\|u\|_2^{(1-\gamma_r)}.$$

□

Lemma 2.3. *(Hardy-Littlewood-Sobolev inequality, Theorem 4.3 in [33]) Let $t,r>1$ and $\alpha\in(0,N)$ with $\frac{1}{t}+\frac{1}{r}=1+\frac{\alpha}{N}$, $f\in L^t(\mathbb{R}^N)$ and $h\in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(N,\alpha,t,r)$, independent of f, h such that*

$$\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^{N-\alpha}}dxdy\Big| \le C(N,\alpha,t,r)||f||_t||h||_r. \tag{2.4}$$

If $t=r=\frac{2N}{N+\alpha}$, then

$$\mathcal{C}(N,\alpha) := C(N,\alpha,t,r) = \pi^{\frac{N-\alpha}{2}}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})}\Big\{\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\Big\}^{-\frac{\alpha}{N}}. \tag{2.5}$$

Lemma 2.4. *(Weak Young inequality, Section 4.3 in [33]) Let $N\in\mathbb{N}$, $\alpha\in(0,N)$, $\hat{p},\hat{r}>1$ and $\frac{1}{\hat{p}}=\frac{\alpha}{N}+\frac{1}{\hat{r}}$. If $v\in L^{\hat{p}}(\mathbb{R}^N)$, then $I_\alpha * v\in L^{\hat{r}}(\mathbb{R}^N)$ and*

$$\Big(\int_{\mathbb{R}^N}|I_\alpha * v|^{\hat{r}}\Big)^{\frac{1}{\hat{r}}} \le C(N,\alpha,\hat{p})\Big(\int_{\mathbb{R}^N}|v|^{\hat{p}}\Big)^{\frac{1}{\hat{p}}} \tag{2.6}$$

where $I_\alpha(x)=|x|^{\alpha-N}$. In particularly, we can set $\hat{p}=\frac{N}{\alpha}$ and $\hat{r}=+\infty$.

For any $u\in H^\sigma(\mathbb{R}^N)$, take $t=r=\frac{2N}{N+\alpha}$, $f=h=|u|^p$ in Lemma 2.3, by using (2.3), we have

$$B(u,u)\leq \mathcal{C}(N,\alpha)||u||^{2p}_{\frac{2Np}{N+\alpha}}\leq \mathcal{C}_p^{2p}\left\|(-\Delta)^{\frac{\sigma}{2}}u\right\|_2^{2p\delta_p}\|u\|_2^{2p(1-\delta_p)}, \tag{2.7}$$

where $\mathcal{C}_p=\left[\frac{\mathcal{C}(N,\alpha)}{\mathcal{S}^{p\delta_p}}\right]^{\frac{1}{2p}}=\left[\frac{\pi^{\frac{N-\alpha}{2}}}{\mathcal{S}^{p\delta_p}}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})}\left\{\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right\}^{-\frac{\alpha}{N}}\right]^{\frac{1}{2p}}>0$ and $\delta_p=\frac{N(p-1)-\alpha}{2p\sigma}$. Therefore, $B(u,u)$ is **well-defined** for any $u\in H^\sigma(\mathbb{R}^N)$ if $1+\alpha/N\leq p\leq(N+\alpha)/(N-2\sigma)$. Let $\mathcal{S}_p=\mathcal{C}_p^{-1}$, we rewrite (2.7) as $\mathcal{S}_p=\inf_{u\in H^\sigma(\mathbb{R}^N)\setminus\{0\}}\frac{\left\|(-\Delta)^{\frac{\sigma}{2}}u\right\|_2^{\delta_p}\|u\|_2^{(1-\delta_p)}}{\left(\int_{\mathbb{R}^N}(I_\alpha*|u|^p)|u|^p\right)^{\frac{1}{2p}}}$. B. H. Feng et al. in [17] proved that $\mathcal{S}_p$ is achieved.

Denote $\mathcal{A}_q=\mathcal{S}^{-\frac{\gamma_q}{2}}$. For any $u\in S_a$, (2.3) and (2.7) implies that

$$E_\mu(u)\geq\frac{1}{2}||(-\Delta)^{\frac{\sigma}{2}}u||_2^2-\frac{\mu\mathcal{C}_p^{2p}}{2p}||(-\Delta)^{\frac{\sigma}{2}}u||_2^{2p\delta_p}a^{2p(1-\delta_p)}-\frac{\mathcal{A}_q^q}{q}||(-\Delta)^{\frac{\sigma}{2}}u||_2^{q\gamma_q}a^{q(1-\gamma_q)}, \tag{2.8}$$

which indicates that $\inf_{u\in S_a}E_\mu(u)>-\infty$ for $2<q<\bar{q}$ and $2\leq p<\bar{p}$, see [1]. However, by the L^2-norm preserving dilations $u_t(x)=t^{\frac{N}{2}}u(tx)$ with $t>0$, we deduce that $\inf_{u\in S_a}E_\mu(u)=-\infty$ for for $\bar{q}<q\leq 2_\sigma^*$ or $\bar{p}<p\leq(N+\alpha)/(N-2\sigma)$. The constrained minimization method used in [1, 21] does not work any more. Naturally, we would hope to overcome this difficulty by using the Pohozaev constraint used in [44, 10]. To this end, we need the following lemmas which is related to the Pohozaev identity.

Lemma 2.5. *([14], Proposition 5.1.1) Let $u\in\dot{H}^\sigma(\mathbb{R}^N)$ be a weak solution to the problem $(-\Delta)^\sigma u=f(x,u)$ in $\mathbb{R}^N$ and assume that $|f(x,u)|\leq C(1+|u|^p)$, for some $1\leq p\leq 2_\sigma^*-1$ and $C>0$. Then $u\in L^\infty(\mathbb{R}^N)$.*

Lemma 2.6. *([41], Proposition 2.8) Let $\sigma>0$ and $(-\Delta)^\sigma u=w$. Assume that $w\in C^{0,\beta}(\mathbb{R}^N)$ and $u\in L^\infty(\mathbb{R}^N)$ for $\beta\in(0,1]$.*
(i) If $\beta+2\sigma\leq1$, then $u\in C^{0,\beta+2\sigma}(\mathbb{R}^N)$. Moreover, we have
$||u||_{C^{0,\beta+2\sigma}}\leq C\Big(||u||_{L^\infty}+||w||_{C^{0,\beta}}\Big)$ *for a constant $C=C(N,\sigma,\beta)>0$.*
(ii) If $\beta+2\sigma>1$, then $u\in C^{1,\beta+2\sigma-1}(\mathbb{R}^N)$. Moreover, we have
$||u||_{C^{1,\beta+2\sigma-1}}\leq C\Big(||u||_{L^\infty}+||w||_{C^{0,\beta}}\Big)$ *for a constant $C=C(N,\sigma,\beta)>0$.*

Lemma 2.7. *([41], Proposition 2.9) Let $\sigma>0$ and $(-\Delta)^\sigma u=h(x)$. Assume that $u\in L^\infty(\mathbb{R}^N)$ and $h\in L^\infty(\mathbb{R}^N)$.*
(i) If $2\sigma\leq1$, then $u\in C^{0,\beta}(\mathbb{R}^N)$ for any $0<\beta<2\sigma$. Moreover, we have
$||u||_{C^{0,\beta}}\leq C\Big(||u||_{L^\infty}+||h||_{L^\infty}\Big)$ *for a constant $C=C(N,\sigma,\beta)>0$.*
(ii) If $2\sigma>1$, then $u\in C^{1,\beta}(\mathbb{R}^N)$ for any $0<\beta<2\sigma-1$. Moreover, we have
$||u||_{C^{1,\beta}}\leq C\Big(||u||_{L^\infty}+||h||_{L^\infty}\Big)$ *for a constant $C=C(N,\sigma,\beta)>0$.*

Lemma 2.8. *Let $N \geq 2$, $\sigma \in (0,1)$, $\alpha \in (N-2\sigma, N)$, $q \in [2, \frac{2N}{N-2\sigma}]$, $p \in [2, \frac{2\alpha}{N-2\sigma}]$, $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. If $u \in H^\sigma(\mathbb{R}^N)$ is a nonnegative weak solution of*

$$(-\Delta)^\sigma u = \lambda u + |u|^{q-2}u + \mu\left(I_\alpha * |u|^p\right)|u|^{p-2}u, \tag{2.9}$$

then the Pohozaev identity holds true

$$0 = P_\mu(u) := ||(-\Delta)^{\frac{\sigma}{2}}u||_2^2 - \gamma_q||u||_q^q - \mu\delta_p B(u,u),$$

where $\gamma_q = \frac{N(q-2)}{2q\sigma}$ and $\delta_p = \frac{N(p-1)-\alpha}{2p\sigma}$.

Proof. Rewrite (2.9) as $(-\Delta)^\sigma u = g(x,u) := \lambda u + u^{q-1} + \mu\left(I_\alpha * u^p\right)u^{p-1}$. Sine $u \in H^\sigma(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for any $2 \leq r \leq 2_\sigma^*$, then $u^p \in L^{\frac{r}{p}}(\mathbb{R}^N)$ for any $2 \leq r \leq 2_\sigma^*$. From $p \in [2, \frac{2\alpha}{N-2\sigma}]$, we have

$$2 < \frac{N}{\alpha}\cdot p \leq 2_\sigma^* \qquad \text{and} \qquad \frac{2}{p} < \frac{N}{\alpha} \leq \frac{2_\sigma^*}{p}.$$

Therefore, $u^p \in L^{\frac{N}{\alpha}}(\mathbb{R}^N)$ and Lemma 2.4 implies that $I_\alpha * u^p \in L^\infty(\mathbb{R}^N)$(see also [42]). Considering $|u| \leq 1$ and $|u| > 1$, we deduce that there exists a constant $C > 0$ such that $|g(x,u)| \leq C(1 + |u|^{2_\sigma^*-1})$. So we have $u \in L^\infty(\mathbb{R}^N)$ from Lemma 2.5.

If $\sigma \in (\frac{1}{2}, 1)$, we know that $u \in C^{1,\beta}(\mathbb{R}^N)$ for any $0 < \beta < 2\sigma - 1$ from Lemma 2.7-(ii). By the usual properties of convolution (see page-1456 of [15]), $I_\alpha * u^p$ is C^1 with derivatives

$$\partial_{x_i}\big(I_\alpha * u^p\big) = I_\alpha * \partial_{x_i}(u^p) = I_\alpha * (pu^{p-1}\partial_{x_i}u).$$

Therefore, $\partial_{x_i}u$ satisfies

$$(-\Delta)^\sigma \partial_{x_i}u = \partial_{x_i}g(x,u) = \partial_{x_i}\Big(\lambda u + u^{q-1} + \mu\left(I_\alpha * u^p\right)u^{p-1}\Big) \in C^{0,\beta}(\mathbb{R}^N).$$

By Lemma 2.4, we can check that $|\partial_{x_i}g(x,u)| \leq C(1 + |\partial_{x_i}u|^{2_\sigma^*-1})$ for some $C > 0$. Then Lemma 2.5 implies that $\partial_{x_i}u \in L^\infty(\mathbb{R}^N)$. It follows from Lemma 2.6-(ii) that $\partial_{x_i}u \in C^{1,\beta+2\sigma-1}(\mathbb{R}^N)$ and then $u \in C^2(\mathbb{R}^N)$.

When it comes to $\sigma \in (0, \frac{1}{2}]$, we can imitate the proof of Proposition 3.7 in [42] and obtain $u \in C^2(\mathbb{R}^N)$. Indeed, for $\sigma \in (0, \frac{1}{2}]$, Lemma 2.7-(i) implies that $u \in C^{0,\beta}(\mathbb{R}^N)$ for any $0 < \beta < 2\sigma$; if $\beta + 2\sigma \leq 1$, Lemma 2.6-(i) implies that $u \in C^{0,\beta+2\sigma}(\mathbb{R}^N)$ for any $0 < \beta < 2\sigma$, repeat this step k times ($k \in \mathbb{N}$) such that

$$\beta + k\cdot 2\sigma \leq 1,\ \beta + (k+1)\cdot 2\sigma > 1,\ u \in C^{0,\beta+k\cdot 2\sigma}(\mathbb{R}^N).$$

It results to $u \in C^{1,\beta+(k+1)\cdot 2\sigma-1}(\mathbb{R}^N)$ by Lemma 2.6-(ii). Therefore, we have $\partial_{x_i}u \in C^{0,\beta+(k+1)\cdot 2\sigma-1}(\mathbb{R}^N)$. If $(\beta + (k+1)\cdot 2\sigma - 1) + 2\sigma > 1$, it follows from Lemma 2.6-(ii) that $\partial_{x_i}u \in C^{1,\beta+(k+2)\cdot 2\sigma-2}(\mathbb{R}^N)$ and we finish the proof. Otherwise, if $(\beta+(k+1)\cdot 2\sigma-1)+2\sigma \leq 1$, repeat the same step of u for j times ($j \in \mathbb{N}$) such that

$$(\beta + (k+1)\cdot 2\sigma - 1) + j\cdot 2\sigma \leq 1,\ (\beta + (k+1)\cdot 2\sigma - 1) + (j+1)\cdot 2\sigma > 1$$

and $\partial_{x_i}u \in C^{0,\beta+(k+1)\cdot 2\sigma-1+j\cdot 2\sigma}(\mathbb{R}^N)$. It results to $\partial_{x_i}u \in C^{1,\beta+(k+1)\cdot 2\sigma-2+(j+1)\cdot 2\sigma}(\mathbb{R}^N)$ by Lemma 2.6-(ii).

Finally, it is reasonable to multiply (2.9) by $x \cdot \nabla u$. We can proceed as in the proof of Proposition 2.10 in [46] and get $P_\mu(u) = 0$.

$\square$

Remark 2.9. Notice that $\max\{2, \bar{p}\} < \frac{2\alpha}{N-2\sigma} < \frac{N+\alpha}{N-2\sigma}$ since $\alpha \in (N-2\sigma, N)$.

To overcome the difficulty that $\inf_{u\in S_a} E_\mu(u) = -\infty$, we introduce the Pohozaev set:

$$\mathcal{P}_{a,\mu} = \{u \in S_a : P_\mu(u) = 0\}, \tag{2.10}$$

where

$$P_\mu(u) =: ||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 - \gamma_q ||u||_q^q - \mu\delta_p B(u,u) \tag{2.11}$$

for $\gamma_q = \frac{N(q-2)}{2q\sigma}$ and $\delta_p = \frac{N(p-1)-\alpha}{2p\sigma}$. As a consequence of Lemma 2.8, we known that any nonnegative critical point of $E_\mu|_{S_a}$ stays in $\mathcal{P}_{a,\mu}$. The properties of $\mathcal{P}_{a,\mu}$ are related to the minimax structure of $E_\mu|_{S_a}$, and in particular to the behavior of E_μ with respect to dilations preserving the L^2-norm. To be more precise, for $u \in S_a$ and $s \in \mathbb{R}$, let

$$(s \star u)(x) := e^{\frac{N}{2}s} u(e^s x), \text{ for a.e. } x \in \mathbb{R}^N. \tag{2.12}$$

It results that $s \star u \in S_a$, and hence it is natural to study the fiber map

$$\Psi_u^\mu(s) := E_\mu(s \star u) = \frac{e^{2\sigma s}}{2}||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 - \frac{e^{q\gamma_q\sigma s}}{q}||u||_q^q - \frac{\mu e^{2p\delta_p\sigma s}}{2p} B(u,u). \tag{2.13}$$

We shall see that critical point of $\Psi_u^\mu(s)$ allow to project a function on $\mathcal{P}_{a,\mu}$. Thus, monotonicity and convexity properties of $\Psi_u^\mu(s)$ strongly affects the structure of $\mathcal{P}_{a,\mu}$ (and in turn the geometry of $E_\mu|_{S_a}$), and also have a strong impact on properties of equation $(1.1)_\lambda$. In this direction, let us consider the decomposition of $\mathcal{P}_{a,\mu}$ into the disjoint union $\mathcal{P}_{a,\mu} = \mathcal{P}_+^{a,\mu} \cup \mathcal{P}_0^{a,\mu} \cup \mathcal{P}_-^{a,\mu}$, where

$$\mathcal{P}_+^{a,\mu} := \left\{u \in \mathcal{P}_{a,\mu} : 2||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 > q\gamma_q^2||u||_q^q + 2\mu p\delta_p^2 B(u,u)\right\} = \left\{u \in \mathcal{P}_{a,\mu} : (\Psi_u^\mu)''(0) > 0\right\}$$

$$\mathcal{P}_0^{a,\mu} := \left\{u \in \mathcal{P}_{a,\mu} : 2||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 = q\gamma_q^2||u||_q^q + 2\mu p\delta_p^2 B(u,u)\right\} = \left\{u \in \mathcal{P}_{a,\mu} : (\Psi_u^\mu)''(0) = 0\right\}$$

$$\mathcal{P}_-^{a,\mu} := \left\{u \in \mathcal{P}_{a,\mu} : 2||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 < q\gamma_q^2||u||_q^q + 2\mu p\delta_p^2 B(u,u)\right\} = \left\{u \in \mathcal{P}_{a,\mu} : (\Psi_u^\mu)''(0) < 0\right\}$$

For $u \in S_a$, $s \in \mathbb{R}$ and the fiber Ψ_u^μ introduced in (2.13), we have

$$(\Psi_u^\mu)'(s) = \sigma\big[e^{2\sigma s}||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 - \gamma_q e^{q\gamma_q\sigma s}||u||_q^q - \mu\delta_p e^{2p\delta_p\sigma s} B(u,u)\big] = \sigma P_\mu(s \star u) \tag{2.14}$$

where P_μ is defined by (2.11). From (2.14), we can see immediately that:

Corollary 2.10. *Let $u \in S_a$. Then $s \in \mathbb{R}$ is a critical point for Ψ_u^μ if and only if $s \star u \in \mathcal{P}_{a,\mu}$.*

In particular, $u \in \mathcal{P}_{a,\mu}$ if and only if 0 is a critical point of Ψ_u^μ. For future convenience, we also recall that the map $(s,u) \in \mathbb{R} \times H^\sigma(\mathbb{R}^N) \mapsto s \star u \in H^\sigma(\mathbb{R}^N)$ is continuous (The proof is similar to the one of Lemma 3.5 in [6]).

We also need the following result, where $T_u S_a$ denotes the tangent space to S_a in u.

Lemma 2.11. *For $u \in S_a$ and $s \in \mathbb{R}$ the map*

$$T_u S_a \to T_{s\star u} S_a, \quad \varphi \mapsto s \star \varphi$$

is a linear isomorphism with inverse $\psi \mapsto (-s) \star \psi$.

Proof. It is similar to the proof of Lemma 3.6 in [6].

□

We now study the structure of the Pohozaev manifold $\mathcal{P}_{a,\mu}$ and E_μ. Since $q \in (2+\frac{4\sigma}{N}, 2_\sigma^*]$ and $p \in [2, 1+\frac{2\sigma+\alpha}{N})$, we have $2p\delta_p < 2 < q\gamma_q$. Recalling the decomposition of $\mathcal{P}_{a,\mu} = \mathcal{P}_+^{a,\mu} \cup \mathcal{P}_0^{a,\mu} \cup \mathcal{P}_-^{a,\mu}$, we have:

Lemma 2.12. *Let $N \geq 2$, $\sigma \in (0,1)$, $\alpha \in (N-2\sigma, N)$, $q \in (2+\frac{4\sigma}{N}, 2_\sigma^*]$, $p \in [2, 1+\frac{2\sigma+\alpha}{N})$, $a, \mu > 0$ satisfying condition (A_1^*). Then $\mathcal{P}_0^{a,\mu} = \emptyset$, and $\mathcal{P}_{a,\mu}$ is a smooth manifold of codimension 2 in $H^\sigma(\mathbb{R}^N)$.*

Proof. We only prove the case $q \in (2+\frac{4\sigma}{N}, 2_\sigma^*)$. For the case $q = 2_\sigma^*$, the proof is much easier since $\gamma_{2_\sigma^*} = 1$. Firstly, we claim that $\mathcal{P}_0^{a,\mu} = \emptyset$. Otherwise, there exists $u \in \mathcal{P}_0^{a,\mu}$. From $P_\mu(u) = 0$ and $(\Psi_u^\mu)''(0) = 0$, we have

$$||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 = \gamma_q ||u||_q^q + \mu\delta_p B(u,u), \ 2||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 = q\gamma_q^2 ||u||_q^q + 2\mu p\delta_p^2 B(u,u),$$

which imply that $||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 = \frac{\gamma_q(q\gamma_q - 2p\delta_p)}{2(1-p\delta_p)} ||u||_q^q$ and $||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 = \frac{\mu\delta_p(q\gamma_q - 2p\delta_p)}{q\gamma_q - 2} B(u,u)$. By using (2.3) and (2.7), the lower and upper bounds of $||(-\Delta)^{\frac{\sigma}{2}} u||_2$ are given by

$$\Big[\frac{2(1-p\delta_p)}{\gamma_q(q\gamma_q - 2p\delta_p)\mathcal{A}_q^q a^{q(1-\gamma_q)}}\Big]^{\frac{1}{q\gamma_q - 2}} \leq ||(-\Delta)^{\frac{\sigma}{2}} u||_2 \leq \Big[\frac{\mu\delta_p(q\gamma_q - 2p\delta_p)\mathcal{C}_p^{2p} a^{2p(1-\delta_p)}}{q\gamma_q - 2}\Big]^{\frac{1}{2(1-p\delta_p)}}.$$

This leads to $\mu^{q\gamma_q - 2} a^{\bar{\mathcal{C}}(p,q)} > \tilde{\mathcal{C}}(p,q)$, which contradicts with (A_1^*). Here we used the fact that $(\frac{q\gamma_q}{2})^{2-2p\delta_p}(\frac{2p\delta_p}{2})^{q\gamma_q - 2} < 1$ and this can be proved by using the monotonicity of $\frac{\ln x}{x-1}$.

Next we check that $\mathcal{P}_{a,\mu}$ is a smooth manifold of codimension 2 in $H^\sigma(\mathbb{R}^N)$. We note that $\mathcal{P}_{a,\mu} = \{u \in H^\sigma(\mathbb{R}^N) : P_\mu(u) = 0, G(u) = 0\}$ for $G(u) = ||u||_2^2 - a^2$, with P_μ and G of class C^1 in $H^\sigma(\mathbb{R}^N)$. Thus, we have to show that the differential $(dG(u), dP_\mu(u)) : H^\sigma(\mathbb{R}^N) \to \mathbb{R}^2$ is surjective, for every $u \in \mathcal{P}_{a,\mu}$. We claim that: $\forall\ u \in \mathcal{P}_{a,\mu}$, $\exists\ \varphi \in T_u S_a$ such that $dP_\mu(u)[\varphi] \neq 0$. Otherwise, $\exists\ u \in \mathcal{P}_{a,\mu}$ such that $dP_\mu(u)[\varphi] = 0$ for any $\varphi \in T_u S_a$. Then u is a constrained critical point for P_μ on S_a, and hence by the Lagrange multipliers rule there exists $\nu \in \mathbb{R}$ such that

$$2(-\Delta)^\sigma u - \nu u - q\gamma_q |u|^{q-2} u - 2\mu p\delta_p \left(I_\alpha * |u|^p\right) |u|^{p-2} u = 0 \text{ in } \mathbb{R}^N. \tag{2.15}$$

By Lemma 2.8, we have the following Pohozaev identity

$$(2\sigma - N)||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 + \frac{\nu N}{2}||u||_2^2 + N\gamma_q ||u||_q^q + \mu(N+\alpha)\delta_p B(u,u) = 0.$$

Combined with (2.15), we derive that $(\Psi_u^\mu)''(0) = 0$, that is $u \in \mathcal{P}_0^{a,\mu}$, a contradiction. Then the claim is true. Once that the existence of φ is established, the system

$$\begin{cases} dG(u)[\alpha\varphi + \beta u] = x \\ dP_\mu(u)[\alpha\varphi + \beta u] = y \end{cases} \Leftrightarrow \begin{cases} 2\beta a^2 = x \\ \alpha dP_\mu(u)[\varphi] + \beta dP_\mu(u)[u] = y \end{cases}$$

is solvable with respect to α, β, for any $(x, y) \in \mathbb{R}^2$. Thus the surjectivity is proved. $\square$

The manifold $\mathcal{P}_{a,\mu}$ is then divided into its two components $\mathcal{P}_+^{a,\mu}$ and $\mathcal{P}_-^{a,\mu}$, having disjoint closure. We can prove that $\mathcal{P}_{a,\mu}$ is a natural constraint, in the following sense:

Lemma 2.13. *Let $N \geq 2$, $\sigma \in (0,1)$, $\alpha \in (N-2\sigma, N)$, $q \in (2+\frac{4\sigma}{N}, 2_\sigma^*]$, $p \in [2, 1+\frac{2\sigma+\alpha}{N})$, $a, \mu > 0$ satisfying condition (A_1^*). If $u \in \mathcal{P}_{a,\mu}$ is a critical point for $E_\mu|_{\mathcal{P}_{a,\mu}}$, then u is a critical point for $E_\mu|_{S_a}$.*

Proof. We only prove the case $q \in (2+\frac{4\sigma}{N}, 2_\sigma^*)$. For the case $q = 2_\sigma^*$, the proof is much easier since $\gamma_{2_\sigma^*} = 1$. We recall that by Lemma 2.12, $\mathcal{P}_{a,\mu}$ is a smooth manifold of codimension 2 in H^σ, and its subset $\mathcal{P}_0^{a,\mu}$ is empty. If $u \in \mathcal{P}_{a,\mu}$ is a critical point for $E_\mu|_{\mathcal{P}_{a,\mu}}$, then by the Lagrange multipliers rule there exists $\lambda, \nu \in \mathbb{R}$ such that

$$dE_\mu(u)[\varphi] - \lambda \int_{\mathbb{R}^N} u\varphi - \nu dP_\mu(u)[\varphi] = 0, \ \forall \varphi \in H^\sigma.$$

That is $(1-2\nu)(-\Delta)^\sigma u - \lambda u + (\nu q\gamma_q - 1)|u|^{q-2}u + \mu(2\nu p\delta_p - 1)\left(I_\alpha * |u|^p\right)|u|^{p-2}u = 0$. But, by the Pohozaev identity, this implies that

$$(1-2\nu)||(-\Delta)^{\frac{\sigma}{2}}u||_2^2 + (\nu q\gamma_q^2 - \gamma_q)||u||_q^q + \mu(2\nu p\delta_p^2 - \delta_p)B(u,u) = 0.$$

Since $u \in \mathcal{P}_{a,\mu}$, this implies that $\nu\Big(2||(-\Delta)^{\frac{\sigma}{2}}u||_2^2 - q\gamma_q^2||u||_q^q - 2\mu p\delta_p^2 B(u,u)\Big) = 0$. But the term inside the bracket cannot be 0, since $u \notin \mathcal{P}_0^{a,\mu}$, and then necessarily $\nu = 0$. $\square$

Next, we study the fiber map $\Psi_u^\mu(s)$ and determine the location and types of critical points for $E_\mu|_{S_a}$. Recall that $q \in (\bar{q}, 2_\sigma^*]$, $p \in [2, \bar{p})$ and $2p\delta_p < 2 < q\gamma_q$. Consider the constrained functional $E_\mu|_{S_a}$, by (2.8), we have

$$E_\mu(u) \geq \frac{1}{2}||(-\Delta)^{\frac{\sigma}{2}}u||_2^2 - \frac{\mu\mathcal{C}_p^{2p}}{2p}||(-\Delta)^{\frac{\sigma}{2}}u||_2^{2p\delta_p} a^{2p(1-\delta_p)} - \frac{\mathcal{A}_q^q}{q}||(-\Delta)^{\frac{\sigma}{2}}u||_2^{q\gamma_q} a^{q(1-\gamma_q)}, \ \forall u \in S_a.$$

Therefore, to understand the geometry of the functional $E_\mu|_{S_a}$ it is useful to consider the function $h : \mathbb{R}^+ \to \mathbb{R}$:

$$h(t) = \frac{1}{2}t^2 - \frac{\mu\mathcal{C}_p^{2p}}{2p}a^{2p(1-\delta_p)}t^{2p\delta_p} - \frac{\mathcal{A}_q^q}{q}a^{q(1-\gamma_q)}t^{q\gamma_q}.$$

Since $a, \mu > 0$ and $2p\delta_p < 2 < q\gamma_q$, we have that $h(0^+) = 0^-$ and $h(+\infty) = -\infty$. If $q = 2_\sigma^*$, we have $\gamma_{2_\sigma^*} = 1$, $\mathcal{A}_{2_\sigma^*} = \mathcal{S}^{-\frac{1}{2}}$ and hence $h(t) = \frac{1}{2}t^2 - \frac{\mu\mathcal{C}_p^{2p}}{2p}a^{2p(1-\delta_p)}t^{2p\delta_p} - \frac{\mathcal{S}^{-\frac{2_\sigma^*}{2}}}{2_\sigma^*}t^{2_\sigma^*}$.

Lemma 2.14. *Let $N \geq 2$, $\sigma \in (0,1)$, $\alpha \in (N-2\sigma, N)$, $q \in (2+\frac{4\sigma}{N}, 2_\sigma^*]$, $p \in [2, 1+\frac{2\sigma+\alpha}{N})$, $a, \mu > 0$ satisfying condition (A_1^*). Then the function h has a local strict minimum at negative level and a global strict maximum at positive level. Moreover, there exist $0 < R_0 < R_1$, both depending on a and μ, such that $h(R_0) = 0 = h(R_1)$ and $h(t) > 0$ if and only if $t \in (R_0, R_1)$.*

Proof. We only prove the case $q \in (2+\frac{4\sigma}{N}, 2_\sigma^*)$. For the case $q = 2_\sigma^*$, the proof is much easier since $\gamma_{2_\sigma^*} = 1$. For $t > 0$, it is easy to check that $h(t) > 0$ if and only if

$$\varphi(t) > \frac{\mu \mathcal{C}_p^{2p}}{2p} a^{2p(1-\delta_p)}, \text{ with } \varphi(t) = \frac{1}{2} t^{2-2p\delta_p} - \frac{\mathcal{A}_q^q}{q} a^{q(1-\gamma_q)} t^{q\gamma_q - 2p\delta_p}.$$

Also φ has a unique critical point on $(0, +\infty)$, which is a global maximum point at positive level, in $\bar{t} = \left[\frac{q(1-p\delta_p)}{(q\gamma_q - 2p\delta_p)\mathcal{A}_q^q a^{q(1-\gamma_q)}}\right]^{\frac{1}{q\gamma_q - 2}}$ and the maximum level is $\varphi(\bar{t}) = \frac{q\gamma_q - 2}{2(q\gamma_q - 2p\delta_p)} \bar{t}^{(2-2p\delta_p)}$.

From condition (A_1^*), we see that $\mu^{q\gamma_q - 2} a^{\bar{\mathcal{C}}(p,q)} < \tilde{\mathcal{C}}(p,q) \Longleftrightarrow \varphi(\bar{t}) > \frac{\mu \mathcal{C}_p^{2p}}{2p} a^{2p(1-\delta_p)}$. Therefore, h is positive on an open interval (R_0, R_1) if and only if $\mu^{q\gamma_q - 2} a^{\bar{\mathcal{C}}(p,q)} < \tilde{\mathcal{C}}(p,q)$. It follows immediately that h has a global maximum at positive level in (R_0, R_1). Moreover, since $h(0^+) = 0^-$, there exists a local minimum point at negative level in $(0, R_0)$. The fact that h has no other critical points can be verified observing that $h'(t) = 0$ if and only if

$$\psi(t) = \mu \delta_p \mathcal{C}_p^{2p} a^{2p(1-\delta_p)}, \text{ with } \psi(t) = t^{2-2p\delta_p} - \gamma_q \mathcal{A}_q^q a^{q(1-\gamma_q)} t^{q\gamma_q - 2p\delta_p}.$$

Clearly ψ has only one critical point at $\tilde{t} = \left[\frac{2(1-p\delta_p)}{\gamma_q(q\gamma_q - 2p\delta_p)\mathcal{A}_q^q a^{q(1-\gamma_q)}}\right]^{\frac{1}{q\gamma_q - 2}}$ which is a strict maximum and $\psi(\tilde{t}) = \frac{q\gamma_q - 2}{q\gamma_q - 2p\delta_p} \tilde{t}^{2-2p\delta_p}$. Using $(\frac{q\gamma_q}{2})^{2-2p\delta_p} (\frac{2p\delta_p}{2})^{q\gamma_q - 2} < 1$ and condition (A_1^*), we check that $\psi(\tilde{t}) > \mu \delta_p \mathcal{C}_p^{2p} a^{2p(1-\delta_p)}$. □

Lemma 2.15. *Let $N \geq 2$, $\sigma \in (0,1)$, $\alpha \in (N - 2\sigma, N)$, $q \in (2 + \frac{4\sigma}{N}, 2_\sigma^*]$, $p \in [2, 1 + \frac{2\sigma+\alpha}{N})$, $a, \mu > 0$ satisfying condition (A_1^*). For every $u \in S_a$, the function Ψ_u^μ has exactly two critical points $s_u < t_u \in \mathbb{R}$ and two zeros $c_u < d_u \in \mathbb{R}$, with $s_u < c_u < t_u < d_u$. Moreover:*
(1) $s_u \star u \in \mathcal{P}_+^{a,\mu}$ and $t_u \star u \in \mathcal{P}_-^{a,\mu}$, and if $s \star u \in \mathcal{P}_{a,\mu}$, then either $s = s_u$ or $s = t_u$;
(2) $||(-\Delta)^{\frac{\sigma}{2}}(s \star u)||_2 \leq R_0$ for every $s \leq c_u$, and

$$E_\mu(s_u \star u) = \min\left\{E_\mu(s \star u) : s \in \mathbb{R} \text{ and } ||(-\Delta)^{\frac{\sigma}{2}}(s \star u)||_2 < R_0\right\} < 0.$$

(3) We have

$$E_\mu(t_u \star u) = \max\{E_\mu(s \star u) : s \in \mathbb{R}\} > 0,$$

and Ψ_u^μ is strictly decreasing and concave on $(t_u, +\infty)$.
(4) The maps $u \in S_a \mapsto s_u \in \mathbb{R}$ and $u \in S_a \mapsto t_u \in \mathbb{R}$ are of class C^1.

Proof. We only prove the case $q \in (2+\frac{4\sigma}{N}, 2_\sigma^*)$. For the case $q = 2_\sigma^*$, the proof is much easier since $\gamma_{2_\sigma^*} = 1$. Let $u \in S_a$, as observed in Corollary 2.10, $s \star u \in \mathcal{P}_{a,\mu}$ if and only if $(\Psi_u^\mu)'(s) = 0$. Thus, we first show that Ψ_u^μ has at least two critical points. To this end, we recall that by (2.8)

$$\Psi_u^\mu(s) = E_\mu(s \star u) \geq h\left(||(-\Delta)^{\frac{\sigma}{2}}(s \star u)||_2\right) = h\left(e^{\sigma s}||(-\Delta)^{\frac{\sigma}{2}} u||_2\right).$$

Thus, the C^2 function Ψ_u^μ is positive on $\left(\frac{1}{\sigma}\log\frac{R_0}{||(-\Delta)^{\frac{\sigma}{2}}u||_2}, \frac{1}{\sigma}\log\frac{R_1}{||(-\Delta)^{\frac{\sigma}{2}}u||_2}\right)$, and clearly $\Psi_u^\mu(-\infty) = 0^-$, $\Psi_u^\mu(+\infty) = -\infty$. It follows that Ψ_u^μ has at least two critical points $s_u < t_u$, with s_u local minimum point on $(-\infty, \frac{1}{\sigma}\log\frac{R_0}{||(-\Delta)^{\frac{\sigma}{2}}u||_2})$ at negative level, and $t_u > s_u$ global maximum point at positive level. It is not difficult to check that there are no other critical points. Indeed $(\Psi_u^\mu)'(s) = 0$ reads

$$\varphi(s) = \mu\delta_p B(u,u), \text{ with } \varphi(s) = e^{\sigma(2-2p\delta_p)s}||(-\Delta)^{\frac{\sigma}{2}}u||_2^2 - \gamma_q e^{\sigma(q\gamma_q-2p\delta_p)s}||u||_q^q.$$

But φ has a unique maximum point at $\bar{s}$ with $e^{\sigma(q\gamma_q-2)\bar{s}} = \frac{(2-2p\delta_p)||(-\Delta)^{\frac{\sigma}{2}}u||_2^2}{\gamma_q(q\gamma_q-2p\delta_p)||u||_q^q}$. By the Gagliardo-Nirenberg inequality (2.3) and $\mu^{q\gamma_q-2}a^{\bar{\mathcal{C}}(p,q)} < \tilde{\mathcal{C}}(p,q)$, we deduce that

$$\varphi(\bar{s}) = \frac{q\gamma_q - 2}{q\gamma_q - 2p\delta_p} e^{2\sigma(1-p\delta_p)\bar{s}}||(-\Delta)^{\frac{\sigma}{2}}u||_2^2 > \mu\delta_p\mathcal{C}_p^{2p}a^{2p(1-\delta_p)}||(-\Delta)^{\frac{\sigma}{2}}u||_2^{2p\delta_p} \geq \mu\delta_p B(u,u).$$

That is $\varphi(\bar{s}) > \mu\delta_p B(u,u)$, so Ψ_u^μ has exactly two critical points. By Corollary 2.10, we have $s_u \star u$, $t_u \star u \in \mathcal{P}_{a,\mu}$, $s \star u \in \mathcal{P}_{a,\mu}$ implies $s \in \{s_u, t_u\}$. By minimality $(\Psi_{s_u\star u}^\mu)''(0) = (\Psi_u^\mu)''(s_u) \geq 0$, and in fact strict inequality must hold, since $\mathcal{P}_0^{a,\mu} = \emptyset$; namely $s_u \star u \in \mathcal{P}_+^{a,\mu}$. In the same way $t_u \star u \in \mathcal{P}_-^{a,\mu}$.

By monotonicity and recalling the behavior at infinity, Ψ_u^μ has moreover exactly two zeros $c_u < d_u$, with $s_u < c_u < t_u < d_u$; and, being a C^2 function, Ψ_u^μ has at least two inflection points. Arguing as before, we can easily check that actually Ψ_u^μ has exactly two inflection points. In particular, Ψ_u^μ is concave on $[t_u, +\infty)$.

It remains to show that $u \mapsto s_u$ and $u \mapsto t_u$ are of class C^1; to this end, we apply the implicit function theorem on the C^1 function $\Phi(s,u) := (\Psi_u^\mu)'(s)$. We use that $\Phi(s_u,u) = 0$, that $\partial_s\Phi(s_u,u) = (\Psi_u^\mu)''(s_u) > 0$, and the fact that it is not possible to pass with continuity from $\mathcal{P}_+^{a,\mu}$ to $\mathcal{P}_-^{a,\mu}$ (since $\mathcal{P}_0^{a,\mu} = \emptyset$). The same argument proves that $u \mapsto t_u$ is C^1. □

For $k > 0$, let us set

$$A_k := \left\{u \in S_a : ||(-\Delta)^{\frac{\sigma}{2}}u||_2 < k\right\}, \text{ and } m(a,\mu) := \inf_{u\in A_{R_0}} E_\mu(u).$$

Corollary 2.16. *Let $N \geq 2$, $\sigma \in (0,1)$, $\alpha \in (N-2\sigma, N)$, $q \in (\bar{q}, 2_\sigma^*]$, $p \in [2,\bar{p})$, $a,\mu > 0$ satisfying condition (A_1^*). Then the set $\mathcal{P}_+^{a,\mu}$ is contained in $A_{R_0} = \left\{u \in S_a : ||(-\Delta)^{\frac{\sigma}{2}}u||_2 < R_0\right\}$, and $\sup_{\mathcal{P}_+^{a,\mu}} E_\mu \leq 0 \leq \inf_{\mathcal{P}_-^{a,\mu}} E_\mu$.*

Proof. It is a direct conclusion of Lemma 2.15. Indeed, $\forall u \in \mathcal{P}_+^{a,\mu}$, Lemma 2.15 implies that $s_u = 0$, $E_\mu(u) \leq 0$ and $||(-\Delta)^{\frac{\sigma}{2}}u||_2 < R_0$. Similarly, $u \in \mathcal{P}_-^{a,\mu}$ implies that $t_u = 0$ and $E_\mu(u) \geq 0$. □

Lemma 2.17. *Let $N \geq 2$, $\sigma \in (0,1)$, $\alpha \in (N-2\sigma, N)$, $q \in (2+\frac{4\sigma}{N}, 2_\sigma^*]$, $p \in [2, 1+\frac{2\sigma+\alpha}{N})$, $a,\mu > 0$ satisfying condition (A_1^*). It results that $m(a,\mu) \in (-\infty, 0)$ and*

$$m(a,\mu) = \inf_{\mathcal{P}_{a,\mu}} E_\mu = \inf_{\mathcal{P}_+^{a,\mu}} E_\mu.$$

Moreover, there exists a constant $\rho > 0$ (independent of a and μ) small enough such that

$$m(a,\mu) < \inf_{\overline{A_{R_0}} \setminus A_{R_0-\rho}} E_\mu.$$

Proof. For $u \in A_{R_0}$, we have $E_\mu(u) \geq h\left(||(-\Delta)^{\frac{\sigma}{2}}u||_2\right) \geq \min_{t\in[0,R_0]} h(t) > -\infty$, and hence $m(a,\mu) > -\infty$. Moreover, for any $u \in S_a$ we have $||(-\Delta)^{\frac{\sigma}{2}}(s\star u)||_2 < R_0$ and $E_\mu(s\star u) < 0$ for $s \ll -1$, and hence $m(a,\mu) < 0$.

By Corollary 2.16, we have $m(a,\mu) \leq \inf_{\mathcal{P}_+^{a,\mu}} E_\mu$ since $\mathcal{P}_+^{a,\mu} \subset A_{R_0}$. On the other hand, if $u \in A_{R_0}$ then $s_u \star u \in \mathcal{P}_+^{a,\mu} \subset A_{R_0}$, and

$$E_\mu\left(s_u \star u\right) = \min\left\{E_\mu(s\star u) : s \in \mathbb{R} \text{ and } ||(-\Delta)^{\frac{\sigma}{2}}(s\star u)||_2 < R_0\right\} \leq E_\mu(u).$$

which implies that $\inf_{\mathcal{P}_+^{a,\mu}} E_\mu \leq m(a,\mu)$. To prove that $\inf_{\mathcal{P}_+^{a,\mu}} E_\mu = \inf_{\mathcal{P}_{a,\mu}} E_\mu$, it is sufficient to recall that $E_\mu \geq 0$ on $\mathcal{P}_-^{a,\mu}$, see Corollary 2.16.

Finally, by continuity of h there exists $\rho > 0$ (independent of a and μ) such that $h(t) \geq \frac{m(a,\mu)}{2}$ if $t \in [R_0-\rho, R_0]$. Therefore $E_\mu(u) \geq h\left(||(-\Delta)^{\frac{\sigma}{2}}u||_2\right) \geq \frac{m(a,\mu)}{2} > m(a,\mu)$ for every $u \in S_a$ with $||(-\Delta)^{\frac{\sigma}{2}}u||_2 \in [R_0-\rho, R_0]$. □

Lemma 2.18. *Let $N \geq 2$, $\sigma \in (0,1)$, $\alpha \in (N-2\sigma, N)$, $q \in (2+\frac{4\sigma}{N}, 2_\sigma^*]$, $p \in [2, 1+\frac{2\sigma+\alpha}{N})$, $a,\mu > 0$ satisfying condition (A_1^*). Suppose that $E_\mu(u) < m(a,\mu)$. Then the value t_u defined by Lemma 2.15 is negative.*

Proof. We consider again the function Ψ_u^μ, and we consider $s_u < c_u < t_u < d_u$ as in Lemma 2.15. If $d_u \leq 0$, then $t_u < 0$, and hence we can assume by contradiction that $d_u > 0$. If $0 \in (c_u, d_u)$, then $E_\mu(u) = \Psi_u^\mu(0) > 0$, which is not possible since $E_\mu(u) < m(a,\mu) < 0$. Therefore $c_u > 0$, and by Lemma 2.15-(2)

$$\begin{aligned} m(a,\mu) > E_\mu(u) = \Psi_u^\mu(0) &\geq \inf_{s\in(-\infty,c_u]} \Psi_u^\mu(s) \\ &\geq \inf\left\{E_\mu(s\star u) : s \in \mathbb{R} \text{ and } ||(-\Delta)^{\frac{\sigma}{2}}(s\star u)||_2 < R_0\right\} = E_\mu\left(s_u\star u\right) \geq m(a,\mu) \end{aligned}$$

which is again a contradiction. □

Lemma 2.19. *Let $N \geq 2$, $\sigma \in (0,1)$, $\alpha \in (N-2\sigma, N)$, $q \in (2+\frac{4\sigma}{N}, 2_\sigma^*]$, $p \in [2, 1+\frac{2\sigma+\alpha}{N})$, $a,\mu > 0$ satisfying condition (A_1^*). It results that*

$$\tilde{\sigma}(a,\mu) := \inf_{u\in\mathcal{P}_-^{a,\mu}} E_\mu(u) > 0.$$

Proof. Let t_{max} denote the strict maximum of the function h at positive level, see Lemma 2.14. For every $u \in \mathcal{P}_-^{a,\mu}$, there exists $\tau_u \in \mathbb{R}$ such that $||(-\Delta)^{\frac{\sigma}{2}}\left(\tau_u\star u\right)||_2 = t_{\max}$. Moreover, since $u \in \mathcal{P}_-^{a,\mu}$ we also have by Lemma 2.15 that the value 0 is the unique strict maximum of the function Ψ_u^μ. Therefore

$$E_\mu(u) = \Psi_u^\mu(0) \geq \Psi_u^\mu\left(\tau_u\right) = E_\mu\left(\tau_u\star u\right) \geq h\left(||(-\Delta)^{\frac{\sigma}{2}}\left(\tau_u\star u\right)||_2\right) = h\left(t_{\max}\right) > 0.$$

Since $u \in \mathcal{P}_-^{a,\mu}$ was arbitrarily chosen, we deduce that $\inf_{\mathcal{P}_-^{a,\mu}} E_\mu \geq \max_{\mathbb{R}} h > 0$. □

3. Compactness of Palais-Smale sequences

In this section, we give the compactness analysis of Palais-Smale sequences. Denote

$$S_{a,r} := H_{rad}^\sigma \cap S_a = \left\{u \in H_{rad}^\sigma(\mathbb{R}^N) : ||u||_2^2 = a^2\right\}. \tag{3.1}$$

Next, we will prove the following two Propositions.

Proposition 3.1. *Let $N \geq 2$, $\sigma \in (0,1)$, $\alpha \in (N-2\sigma, N)$, $q \in (2+\frac{4\sigma}{N}, 2_\sigma^*)$, $p \in [2, 1+\frac{2\sigma+\alpha}{N})$ with $a, \mu > 0$ satisfying condition (A_1^*). Let $\{u_n\} \subset S_{a,r}$ be a Palais-Smale sequence for $E_\mu|_{S_a}$ at level $c \neq 0$ with $P_\mu(u_n) \to 0$ as $n \to \infty$. Then up to a subsequence $u_n \to u$ strongly in H^σ, and $u \in S_a$ is a real-valued radial solution to $(1.1)_\lambda$ for some $\lambda < 0$.*

Proof. The proof is divided into four main steps.
(1) Boundedness of $\{u_n\}$ in H^σ.
Since $P_\mu(u_n) = o(1)$, we have $E_\mu(u_n) = \left(\frac{1}{2} - \frac{1}{q\gamma_q}\right) ||(-\Delta)^{\frac{\sigma}{2}} u_n||_2^2 - \mu\delta_p \left(\frac{1}{2p\delta_p} - \frac{1}{q\gamma_q}\right) B(u_n, u_n) + o(1)$. It results to

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{q\gamma_q}\right) ||(-\Delta)^{\frac{\sigma}{2}} u_n||_2^2 &\leq (c+1) + \mu\delta_p \left(\frac{1}{2p\delta_p} - \frac{1}{q\gamma_q}\right) B(u_n, u_n) \\ &\leq (c+1) + \mu\delta_p \left(\frac{1}{2p\delta_p} - \frac{1}{q\gamma_q}\right) \mathcal{C}_p^{2p} a^{2p(1-\delta_p)} \left\|(-\Delta)^{\frac{\sigma}{2}} u_n\right\|_2^{2p\delta_p}.\end{aligned}$$

As $2p\delta_p < 2 < q\gamma_q$, we have $||(-\Delta)^{\frac{\sigma}{2}} u_n||_2 \leq C$. So $\{u_n\}$ is bounded in H^σ since $||u_n||_2 = a$.
(2) $\exists$ Lagrange multipliers $\lambda_n \to \lambda \in \mathbb{R}$. Since $N \geq 2$, the embedding $H^\sigma_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is compact for $r \in (2, 2_\sigma^*)$, and we deduce that there exists $u \in H^\sigma_{\text{rad}}$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in H^σ, $u_n \to u$ strongly in $L^r(\mathbb{R}^N)$ for $r \in (2, 2_\sigma^*)$, and a.e. in $\mathbb{R}^N$. Now, since $\{u_n\}$ is a Palais-Smale sequence of $E_\mu|_{S_a}$, by the Lagrange multipliers rule there exists $\lambda_n \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} [(-\Delta)^{\frac{\sigma}{2}} u_n \cdot (-\Delta)^{\frac{\sigma}{2}} \varphi - \lambda_n u_n \varphi] - \int_{\mathbb{R}^N} |u_n|^{q-2} u_n \varphi - \mu \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p) |u_n|^{p-2} u_n \varphi = o(1)(||\varphi||_{H^\sigma}) \tag{3.2}$$

for every $\varphi \in H^\sigma$, where $o(1) \to 0$ as $n \to \infty$. In particular, take $\varphi = u_n$, then

$$\lambda_n a^2 = ||(-\Delta)^{\frac{\sigma}{2}} u_n||_2^2 - ||u_n||_q^q - \mu B(u_n, u_n) + o(1),$$

and the boundedness of $\{u_n\}$ in $H^\sigma \cap L^q \cap L^{\frac{2Np}{N+\alpha}}$ implies that $\{\lambda_n\}$ is bounded as well; thus, up to a subsequence $\lambda_n \to \lambda \in \mathbb{R}$.
(3) We claim that $\lambda < 0$ and $u \not\equiv 0$. Recalling that $P_\mu(u_n) \to 0$, we have

$$\lambda_n a^2 = (\gamma_q - 1)||u_n||_q^q + \mu(\delta_p - 1) B(u_n, u_n) + o(1).$$

Let $n \to +\infty$, then $\lambda a^2 = (\gamma_q - 1)||u||_q^q + \mu(\delta_p - 1)B(u, u)$. Since $\mu > 0$ and $0 < \gamma_q, \delta_p < 1$, we deduce that $\lambda \leq 0$, with equality if and only if $u \equiv 0$. If $\lambda_n \to 0$, we have $\lim\limits_{n\to\infty} ||u_n||_q^q = 0 = \lim\limits_{n\to\infty} B(u_n, u_n)$. Using again $P_\mu(u_n) \to 0$, we have $E_\mu(u_n) \to 0$. A contradiction with $E_\mu(u_n) \to c \neq 0$ and thus $\lambda_n \to \lambda < 0$ and $u \not\equiv 0$.
(4) We claim $u_n \to u$ in H^σ. Since $u_n \rightharpoonup u$ in H^σ, then (3.2) implies that

$$dE_\mu(u)\varphi - \lambda \int_{\mathbb{R}^N} u\varphi = 0, \ \forall \varphi \in H^\sigma. \tag{3.3}$$

That is u is a weak radial (and real) solution to

$$(-\Delta)^\sigma u = \lambda u + |u|^{q-2} u + \mu (I_\alpha * |u|^p) |u|^{p-2} u \text{ in } \mathbb{R}^N.$$

Choosing $\varphi = u_n - u$ in (3.2) and (3.3), and subtracting, we obtain

$$(dE_\mu(u_n) - dE_\mu(u))[u_n - u] - \lambda \int_{\mathbb{R}^N} |u_n - u|^2 = o(1).$$

Using the strong $L^{\frac{2Np}{N+\alpha}}$ convergence of $\{u_n\}$, we infer that

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p) |u_n|^{p-2} u_n (u_n - u) \leq ||u_n||^{2p-1}_{\frac{2Np}{N+\alpha}} ||u_n - u||_{\frac{2Np}{N+\alpha}} \to 0.$$

Similarly, we have $\int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^{p-2} u(u_n - u) \to 0$. Therefore, we obtain that $||(-\Delta)^{\frac{\sigma}{2}}(u_n - u)||_2^2 - \lambda ||u_n - u||_2^2 = o(1)$. □

Proposition 3.2. *Let $N \geq 2$, $\sigma \in (0,1)$, $\alpha \in (N-2\sigma, N)$, $q = 2^*_\sigma$, $p \in [2, 1 + \frac{2\sigma+\alpha}{N})$ with $a, \mu > 0$ satisfying condition (A_1^*). Let $\{u_n\} \subset S_{a,r} = S_a \cap H^\sigma_{\mathrm{rad}}$ be a Palais-Smale sequence for $E_\mu|_{S_a}$ at level $c \neq 0$, with*

$$c < \frac{\sigma}{N} \mathcal{S}^{\frac{N}{2\sigma}} \quad \text{and} \quad P_\mu(u_n) \to 0 \quad \text{as} \quad n \to \infty,$$

where $\mathcal{S}$ denotes the best constant in the Sobolev inequality (2.2). Then one of the following alternatives holds:
(i) either up to a subsequence $u_n \rightharpoonup u$ weakly in $H^\sigma(\mathbb{R}^N)$ but not strongly, where $u \not\equiv 0$ is a solution to $(1.1)_\lambda$ for some $\lambda < 0$, and

$$E_\mu(u) \leq c - \frac{\sigma}{N} \mathcal{S}^{\frac{N}{2\sigma}};$$

(ii) or up to a subsequence $u_n \to u$ strongly in $H^\sigma(\mathbb{R}^N)$, $E_\mu(u) = c$, and u solves $(1.1)_\lambda$-(1.2) for some $\lambda < 0$.

Proof. The proof is divided into four main steps. Similar to the proof of Proposition 3.1, we can easily get steps (1) and (2), that is,
(1) $\{u_n\}$ is bounded in H^σ.
(2) $\exists$ Lagrange multipliers $\lambda_n \to \lambda \in \mathbb{R}$. Moreover, we have

$$\int_{\mathbb{R}^N} [(-\Delta)^{\frac{\sigma}{2}} u_n (-\Delta)^{\frac{\sigma}{2}} \varphi - \lambda_n u_n \varphi] - \int_{\mathbb{R}^N} |u_n|^{2^*_\sigma - 2} u_n \varphi - \mu \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p) |u_n|^{p-2} u_n \varphi = o(1)(||\varphi||_{H^\sigma}) \tag{3.4}$$

for every $\varphi \in H^\sigma$, where $o(1) \to 0$ as $n \to \infty$. In particular, take $\varphi = u_n$, then

$$\lambda_n a^2 = ||(-\Delta)^{\frac{\sigma}{2}} u_n||_2^2 - ||u_n||^{2^*_\sigma}_{2^*_\sigma} - \mu B(u_n, u_n) + o(1).$$

(3) We claim that $\lambda < 0$ and $u \not\equiv 0$. Recalling that $P_\mu(u_n) \to 0$, we have

$$\lambda_n a^2 = \mu(\delta_p - 1) B(u_n, u_n) + o(1).$$

Let $n \to +\infty$, then $\lambda a^2 = \mu(\delta_p - 1) B(u, u)$. Since $\mu > 0$ and $0 < \delta_p < 1$, we deduce that $\lambda \leq 0$, with equality if and only if $u \equiv 0$. If $\lambda_n \to 0$, we have $\lim_{n\to\infty} B(u_n, u_n) = 0$.

Using again $P_\mu(u_n) \to 0$, we have $\lim_{n\to\infty} ||(-\Delta)^{\frac{\sigma}{2}} u_n||_2^2 = \lim_{n\to\infty} ||u_n||_{2_\sigma^*}^{2_\sigma^*} = \ell$. Therefore, by the Sobolev inequality $\ell \geq \mathcal{S}\ell^{\frac{2}{2_\sigma^*}}$. We have $\ell = 0$ or $\ell \geq \mathcal{S}^{\frac{N}{2\sigma}}$. Since

$$0 \neq c = \lim_{n\to+\infty} E_\mu(u_n) = \lim_{n\to+\infty} \left[\frac{1}{2}||(-\Delta)^{\frac{\sigma}{2}} u_n||_2^2 - \frac{\mu}{2p} B(u_n, u_n) - \frac{1}{2_\sigma^*}||u_n||_{2_\sigma^*}^{2_\sigma^*}\right] = \frac{\sigma}{N}\ell,$$

we have $\ell \neq 0$ and $\ell \geq \mathcal{S}^{\frac{N}{2\sigma}}$. This leads to $c = \lim_{n\to\infty} E_\mu(u_n) = \frac{\sigma}{N}\ell \geq \frac{\sigma}{N}\mathcal{S}^{\frac{N}{2\sigma}}$, and this contradicts our assumptions $c < \frac{\sigma}{N}\mathcal{S}^{\frac{N}{2\sigma}}$. Therefore, we have $\lambda < 0$ and $u \not\equiv 0$.

(4) Conclusion. Since $u_n \rightharpoonup u \not\equiv 0$ weakly in H^σ, then (3.4) implies that

$$dE_\mu(u)\varphi - \lambda \int_{\mathbb{R}^N} u\varphi = 0, \ \forall \varphi \in H^\sigma. \tag{3.5}$$

That is u is a weak radial (and real) solution to

$$(-\Delta)^\sigma u = \lambda u + |u|^{2_\sigma^*-2}u + \mu\left(I_\alpha * |u|^p\right)|u|^{p-2}u \text{ in } \mathbb{R}^N.$$

Therefore, we have $P_\mu(u) = ||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 - ||u||_{2_\sigma^*}^{2_\sigma^*} - \mu\delta_p B(u,u) = 0$.

Denote $v_n = u_n - u$, then $v_n \rightharpoonup 0$ in $H^\sigma\left(\mathbb{R}^N\right)$ and therefore

$$||(-\Delta)^{\frac{\sigma}{2}} u_n||_2^2 = ||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 + ||(-\Delta)^{\frac{\sigma}{2}} v_n||_2^2 + o(1).$$

By the Brézis-Lieb type lemmas in [5, 35], we have

$$||u_n||_{2_\sigma^*}^{2_\sigma^*} = ||u||_{2_\sigma^*}^{2_\sigma^*} + ||v_n||_{2_\sigma^*}^{2_\sigma^*} + o(1), \ B(u_n, u_n) = B(v_n, v_n) + B(u,u) + o(1).$$

Since $v_n \to 0$ strongly in $L^{\frac{2Np}{N+\alpha}}$, we have $B(u_n, u_n) = B(u,u) + o(1)$. Consequently, from $P_\mu(u_n) = o(1)$ and $P_\mu(u) = 0$, we deduce that $\lim_{n\to\infty} ||(-\Delta)^{\frac{\sigma}{2}} v_n||_2^2 = \lim_{n\to\infty} ||v_n||_{2_\sigma^*}^{2_\sigma^*} = \ell$. Therefore, by the Sobolev inequality $\ell \geq \mathcal{S}\ell^{\frac{2}{2_\sigma^*}}$. We have $\ell = 0$ or $\ell \geq \mathcal{S}^{\frac{N}{2\sigma}}$.

If $\ell \geq \mathcal{S}^{\frac{N}{2\sigma}}$, then we have

$$c = \lim_{n\to+\infty} E_\mu(u_n) = E_\mu(u) + \lim_{n\to+\infty} E_\mu(v_n) = E_\mu(u) + \frac{\sigma}{N}\ell \geq E_\mu(u) + \frac{\sigma}{N}\mathcal{S}^{\frac{N}{2\sigma}},$$

whence alternative (i) in the thesis of the proposition follows.

If instead $\ell = 0$, then $u_n \to u$ in $\dot{H}^\sigma(\mathbb{R}^N)$ and $L^{2_\sigma^*}(\mathbb{R}^N)$. In order to prove that $u_n \to u$ in $L^2\left(\mathbb{R}^N\right)$, we test (3.4) with $\varphi = u_n - u$, test (3.5) with $u_n - u$, and subtract, obtaining

$$\begin{aligned} &||(-\Delta)^{\frac{\sigma}{2}}(u_n - u)||_2^2 - \int_{\mathbb{R}^N} (\lambda_n u_n - \lambda u)(u_n - u) \\ &- \mu \int_{\mathbb{R}^N} \Big[\left(I_\alpha * |u_n|^p\right)|u_n|^{p-2}u_n - \left(I_\alpha * |u|^p\right)|u|^{p-2}u\Big](u_n - u) \\ &= \int_{\mathbb{R}^N} \left(|u_n|^{2_\sigma^*-2} u_n - |u|^{2_\sigma^*-2}u\right)(u_n - u) + o(1). \end{aligned}$$

Using the strong $L^{2_\sigma^*} \cap L^{\frac{2Np}{N+\alpha}}$ convergence of $\{u_n\}$, we infer that

$$0 = \lim_{n\to\infty} \int_{\mathbb{R}^N} (\lambda_n u_n - \lambda u)(u_n - u) = \lim_{n\to\infty} \lambda \int_{\mathbb{R}^N} (u_n - u)^2,$$

and we deduce that $u_n \to u$ strongly in $H^\sigma(\mathbb{R}^N)$. Therefore, alternative (ii) in the thesis of the proposition holds.

□

4. The existence results

In this Section, we prove the existence results, i.e. Theorem 1.1-(1)(2)(3) and Theorem 1.2-(1)(2). The proof of Theorem 1.1 is divided into two parts. Firstly, we prove the existence of a local minimizer for $E_\mu|_{S_a}$. Secondly, we construct a mountain pass type critical point for $E_\mu|_{S_a}$. The later relies heavily on a refined version of the min-max principle by N. Ghoussoub [22], the forth coming Lemma 4.2, and was already applied in [44] and [45].

Definition 4.1. Let B be a closed subset of X. We shall say that a class $\mathcal{F}$ of compact subsets of X is a homotopy-stable family with extended boundary B if for any set A in $\mathcal{F}$ and any $\eta \in C([0,1]\times X; X)$ satisfying $\eta(t,x) = x$ for all $(t,x) \in (\{0\}\times X) \cup ([0,1]\times B)$ we have that $\eta(\{1\}\times A) \in \mathcal{F}$.

Lemma 4.2. *([22], Theorem 5.2) Let φ be a C^1-functional on a complete connected C^1-Finsler manifold X and consider a homotopy-stable family $\mathcal{F}$ with an extended closed boundary B. Set $c = c(\varphi, \mathcal{F})$ and let F be a closed subset of X satisfying*

(1) $(A \cap F)\backslash B \neq \emptyset$ *for every* $A \in \mathcal{F}$,

(2) $\sup \varphi(B) \leq c \leq \inf \varphi(F)$.

Then, for any sequence of sets $(A_n)_n$ in $\mathcal{F}$ such that $\lim_n \sup_{A_n} \varphi = c$, there exists a sequence $(x_n)_n$ in X such that

$$\lim_{n\to+\infty} \varphi(x_n) = c, \quad \lim_{n\to+\infty} \|d\varphi(x_n)\| = 0, \quad \lim_{n\to+\infty} dist(x_n, F) = 0, \quad \lim_{n\to+\infty} dist(x_n, A_n) = 0.$$

Proof of Theorem 1.1-(1),(2),(3):

(i) Existence of a local minimizer.

Let us consider a minimizing sequence $\{v_n\}$ for $E_\mu|_{A_{R_0}}$. From the fractional Polya-Szegö inequality in [39] or formula (A.11) in [43], we have

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\sigma}{2}} |v_n|^*|^2 dx \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\sigma}{2}} |v_n||^2 dx \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\sigma}{2}} v_n|^2 dx, \tag{4.1}$$

where $|v_n|^*$ is the symmetric decreasing rearrangement of $|v_n|$. Furthermore, it is clear(Theorem 3.4 in [33]) that

$$||v_n||_2 = |||v_n|^*||_2, \ ||v_n||_q = |||v_n|^*||_q, \ B(|v_n|, |v_n|) \leq B(|v_n|^*, |v_n|^*). \tag{4.2}$$

Therefore, we have $E_\mu(|v_n|^*) \leq E_\mu(v_n)$. We can assume that $v_n \in S_a$ is nonnegative and radially decreasing for every n. Furthermore, for every n we can take $s_{v_n} \star v_n \in \mathcal{P}_+^{a,\mu}$, observing that then by Lemma 2.15 and Corollary 2.16 $||(-\Delta)^{\frac{\sigma}{2}}(s_{v_n} \star v_n)||_2 < R_0$ and

$$E_\mu(s_{v_n} \star v_n) = \min\left\{E_\mu(s \star v_n) : s \in \mathbb{R} \text{ and } ||(-\Delta)^{\frac{\sigma}{2}}(s \star v_n)||_2 < R_0\right\} \leq E_\mu(v_n);$$

in this way we obtain a new minimizing sequence $\{w_n = s_{v_n} \star v_n\}$, with

$$w_n \in S_{a,r} \cap \mathcal{P}_+^{a,\mu} \text{ and } P_\mu(w_n) = 0$$

for every n. By Lemma 2.17, $||(-\Delta)^{\frac{\sigma}{2}} w_n||_2 < R_0 - \rho$ for every n, and hence the Ekelands variational principle yields in a standard way the existence of a new minimizing sequence $\{u_n\} \subset A_{R_0}$ for $m(a,\mu) < 0$, with the property that $||u_n - w_n||_{H^\sigma} \to 0$ as $n \to +\infty$, which is also a Palais-Smale sequence for E_μ on $S_{a,r}$. The condition $||u_n - w_n||_{H^\sigma} \to 0$ implies

$$||(-\Delta)^{\frac{\sigma}{2}} u_n||_2 \leq R_0 - \rho \text{ and } P_\mu(u_n) \to 0 \text{ as } n \to \infty$$

and hence $\{u_n\}$ satisfies all the assumptions of Proposition 3.1. Consequently, up to a subsequence $u_n \to \tilde{u}_\mu$ strongly in H^σ, $\tilde{u}_\mu$ is an interior local minimizer for $E_\mu|_{A_{R_0}}$, and solves $(1.1)_{\tilde{\lambda}}$ for some $\tilde{\lambda} < 0$. It is easy to know that $\tilde{u}_\mu$ is nonnegative and radially deceasing. We have $\tilde{u}_\mu > 0$, otherwise, there exists $x_0 \in \mathbb{R}^N$ such that $\tilde{u}_\mu(x_0) = 0$. Then it follows from equation $(1.1)_{\tilde{\lambda}}$ that $0 = (-\Delta)^\sigma \tilde{u}_\mu(x_0) = C_{N,\sigma}\mathrm{P.V.} \int_{\mathbb{R}^N} \frac{-\tilde{u}_\mu(y)}{|x-y|^{N+2\sigma}} dy$, which implies that $\tilde{u}_\mu \equiv 0$ in $\mathbb{R}^N$. This is impossible since $\tilde{u}_\mu \in S_a$.

Since any critical point of $E_\mu|_{S_a}$ lies in $\mathcal{P}_{a,\mu}$ and $m(a,\mu) = \inf_{\mathcal{P}_{a,\mu}} E_\mu$ (see Lemma 2.17), we see that $\tilde{u}_\mu$ is a ground state for $E_\mu|_{S_a}$.

It only remains to prove that any ground state of $E_\mu|_{S_a}$ is a local minimizer of E_μ in A_{R_0}. Let then u be a critical point of $E_\mu|_{S_a}$ with $E_\mu(u) = m(a,\mu) = \inf_{\mathcal{P}_{a,\mu}} E_\mu$. Since $E_\mu(u) < 0 < \inf_{\mathcal{P}_-^{a,\mu}} E_\mu$, necessarily $u \in \mathcal{P}_+^{a,\mu}$. Then Corollary 2.16 implies that $\mathcal{P}_+^{a,\mu} \subset A_{R_0}$. It results that $||(-\Delta)^{\frac{\sigma}{2}} u||_2 < R_0$, and as a consequence u is a local minimizer for $E_\mu|_{A_{R_0}}$.

(ii) Existence of a Mountain pass type solution.

We focus now on the existence of a second critical point for $E_\mu|_{S_a}$. Denote $E_\mu^c = \{u \in S_a : E_\mu(u) \leq c\}$. Motivated by [24], we define the augmented functional $\tilde{E}_\mu : \mathbb{R} \times H^\sigma \to \mathbb{R}$

$$\tilde{E}_\mu(s,u) := E_\mu(s \star u) = \frac{e^{2\sigma s}}{2}||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 - \frac{e^{q\gamma_q \sigma s}}{q}||u||_q^q - \frac{\mu e^{2p\delta_p \sigma s}}{2p} B(u,u)$$

and consider the restriction $\tilde{E}_\mu|_{\mathbb{R}\times S_a}$. Notice that $S_{a,r} = H^\sigma_{\mathrm{rad}} \cap S_a$ and $\tilde{E}_\mu$ is of class C^1. Moreover, since $\tilde{E}_\mu$ is invariant under rotations applied to u, a Palais-Smale sequence for $\tilde{E}_\mu|_{\mathbb{R}\times S_{a,r}}$ is a Palais-Smale sequence for $\tilde{E}_\mu|_{\mathbb{R}\times S_a}$.

We introduce the minimax class

$$\Gamma := \left\{\gamma(\tau) = \big(\zeta(\tau), \beta(\tau)\big) \in C\left([0,1], \mathbb{R} \times S_{a,r}\right); \gamma(0) \in (0, \mathcal{P}_+^{a,\mu}), \gamma(1) \in (0, E_\mu^{2m(a,\mu)})\right\}.$$

The family Γ is not empty. Indeed, for every $u \in S_{a,r}$, by Lemma 2.15 we know that there exists $s_1 \gg 1$ such that

$$\gamma_u : \tau \in [0,1] \mapsto \big(0, ((1-\tau)s_u + \tau s_1) \star u\big) \in \mathbb{R} \times S_{a,r} \tag{4.3}$$

is a path in Γ (recall that $s \in \mathbb{R} \mapsto s \star u \in S_{a,r}$ is continuous, $s_u \star u \in \mathcal{P}_+^{a,\mu}$ and $E_\mu(s \star u) \to -\infty$ as $s \to +\infty$). Thus, the minimax value

$$\sigma(a,\mu) := \inf_{\gamma\in\Gamma} \max_{(s,u)\in\gamma([0,1])} \tilde{E}_\mu(s,u)$$

is a real number. We claim that

$$\forall \gamma \in \Gamma \text{ there exists } \tau_\gamma \in (0,1) \text{ such that } \zeta(\tau_\gamma) \star \beta(\tau_\gamma) \in \mathcal{P}_-^{a,\mu}. \tag{4.4}$$

Indeed, since $\gamma(0) = \big(\zeta(0), \beta(0)\big) \in (0, \mathcal{P}_+^{a,\mu})$, by Corollary 2.10 and Lemma 2.15, we have $t_{\zeta(0)\star\beta(0)} = t_{\beta(0)} > s_{\beta(0)} = 0$; since $E_\mu(\beta(1)) = \tilde{E}_\mu(\gamma(1)) \leq 2m(a,\mu)$, by Lemma 2.18, we have

$$t_{\zeta(1)\star\beta(1)} = t_{\beta(1)} < 0,$$

and moreover the map $t_{\zeta(\tau)\star\beta(\tau)}$ is continuous in τ(we refer again to Lemma 2.15 and recall that $s \in \mathbb{R} \mapsto s \star u \in S_{a,r}$ is continuous). It follows that for every $\gamma \in \Gamma$ there exists $\tau_\gamma \in (0,1)$ such that $t_{\zeta(\tau_\gamma)\star\beta(\tau_\gamma)} = 0$, and so $\zeta(\tau_\gamma) \star \beta(\tau_\gamma) \in \mathcal{P}_-^{a,\mu}$. Thus (4.4) holds.

For every $\gamma \in \Gamma$, by (4.4) we have

$$\max_{\gamma([0,1])} \tilde{E}_\mu \geq \tilde{E}_\mu\left(\gamma\left(\tau_\gamma\right)\right) = E_\mu(\zeta(\tau_\gamma) \star \beta(\tau_\gamma)) \geq \inf_{\mathcal{P}_-^{a,\mu} \cap S_{a,r}} E_\mu, \tag{4.5}$$

which gives $\sigma(a,\mu) \geq \inf_{\mathcal{P}_-^{a,\mu} \cap S_{a,r}} E_\mu$. On the other hand, if $u \in \mathcal{P}_-^{a,\mu} \cap S_{a,r}$, then γ_u defined in (4.3) is a path in Γ with

$$E_\mu(u) = \tilde{E}_\mu(0,u) = \max_{\gamma_u([0,1])} \tilde{E}_\mu \geq \sigma(a,\mu),$$

which gives $\inf_{\mathcal{P}_-^{a,\mu} \cap S_{a,r}} E_\mu \geq \sigma(a,\mu)$. This, Corollary 2.16 and Lemma 2.19 imply that

$$\sigma(a,\mu) = \inf_{\mathcal{P}_-^{a,\mu} \cap S_{a,r}} E_\mu > 0 \geq \sup_{\left(\mathcal{P}_+^{a,\mu} \cup E_\mu^{2m(a,\mu)}\right) \cap S_{a,r}} E_\mu = \sup_{\left((0,\mathcal{P}_+^{a,\mu}) \cup (0, E_\mu^{2m(a,\mu)})\right) \cap (\mathbb{R} \times S_{a,r})} \tilde{E}_\mu. \tag{4.6}$$

Let $\gamma_n(\tau) = \big(\zeta_n(\tau), \beta_n(\tau)\big)$ be any minimizing sequence for $\sigma(a,\mu)$ with the property that $\zeta_n(\tau) \equiv 0$ and $\beta_n(\tau) \geq 0$ a.e. in $\mathbb{R}^N$ for every $\tau \in [0,1]$ (Notice that, if $\{\gamma_n = \big(\zeta_n, \beta_n\big)\}$ is a minimizing sequence, then also $\{(0, \zeta_n \star |\beta_n|)\}$ has the same property). Take

$$X = \mathbb{R} \times S_{a,r},\ \mathcal{F} = \{\gamma([0,1]) :\ \gamma \in \Gamma\},\ B = (0, \mathcal{P}_+^{a,\mu}) \cup (0, E_\mu^{2m(a,\mu)}),$$

$$F = \{(s,u) \in \mathbb{R} \times S_{a,r} \mid \tilde{E}_\mu(s,u) \geq \sigma(a,\mu)\},\ A = \gamma([0,1]),\ A_n = \gamma_n([0,1])$$

in Lemma 4.2. We need to checked that $\mathcal{F}$ is a homotopy stable family of compact subsets of X with extended closed boundary B, and that F is a dual set for $\mathcal{F}$, in the sense that assumptions (1) and (2) in Lemma 4.2 are satisfied.

Indeed, since $\sigma(a,\mu) = \inf_{\mathcal{P}_-^{a,\mu} \cap S_{a,r}} E_\mu$, (4.5) $\Rightarrow \gamma\left(\tau_\gamma\right) = (\zeta(\tau_\gamma), \beta(\tau_\gamma)) \in A \cap F$, (4.6) $\Rightarrow F \cap B = \emptyset$ and (2) in Lemma 4.2, then $A \cap F \neq \emptyset$ and $F \cap B = \emptyset$ give (1) in Lemma 4.2. For every $\gamma \in \Gamma$, since $\gamma(0) \in (0, \mathcal{P}_+^{a,\mu})$ and $\gamma(1) \in (0, E_\mu^{2m(a,\mu)})$, we have $\gamma(0), \gamma(1) \in B$. Then for any set A in $\mathcal{F}$ and any $\eta \in C([0,1] \times X; X)$ satisfying $\eta(t,x) = x$ for all $(t,x) \in (\{0\} \times X) \cup ([0,1] \times B)$, it holds that $\eta(1, \gamma(0)) = \gamma(0)$, $\eta(1, \gamma(1)) = \gamma(1)$. So we have $\eta(\{1\} \times A) \in \mathcal{F}$.

Consequently, by Lemma 4.2, there exists a Palais-Smale sequence $\{(s_n, w_n)\} \subset \mathbb{R} \times S_{a,r}$ for $\tilde{E}_\mu|_{\mathbb{R} \times S_{a,r}}$ at level $\sigma(a,\mu) > 0$ such that

$$\partial_s \tilde{E}_\mu\left(s_n, w_n\right) \to 0 \quad \text{and} \quad \left\|\partial_u \tilde{E}_\mu\left(s_n, w_n\right)\right\|_{\left(T_{w_n} S_{a,r}\right)^*} \to 0 \quad \text{as } n \to \infty, \tag{4.7}$$

with the additional property that

$$|s_n| + \operatorname{dist}_{H^\sigma}(w_n, \beta_n([0,1])) \to 0 \quad \text{as } n \to \infty. \tag{4.8}$$

The first condition in (4.7) reads $P_\mu(s_n \star w_n) \to 0$, and the second condition in (4.7) gives

$$\begin{aligned} &e^{2\sigma s_n} \int_{\mathbb{R}^N} (-\Delta)^{\frac{\sigma}{2}} w_n \cdot (-\Delta)^{\frac{\sigma}{2}} \varphi - e^{q\gamma_q \sigma s_n} \int_{\mathbb{R}^N} |w_n|^{q-2} w_n \varphi \\ &-\mu e^{2p\delta_p \sigma s_n} \int_{\mathbb{R}^N} (I_\alpha * |w_n|^p) |w_n|^{p-2} w_n \varphi = o(1)(\|\varphi\|_{H^\sigma}), \ \forall \varphi \in T_{w_n} S_{a,r}. \end{aligned} \tag{4.9}$$

Since s_n is bounded from above and from below, due to (4.8), we have

$$dE_\mu(s_n \star w_n)[s_n \star \varphi] = o(1)\|\varphi\|_{H^\sigma} = o(1)\|s_n \star \varphi\|_{H^\sigma} \ \text{ as } n \to \infty, \forall \varphi \in T_{w_n} S_{a,r}. \tag{4.10}$$

By Lemma 2.11, (4.10) implies that $\{u_n := s_n \star w_n\} \subset S_{a,r}$ is a Palais-Smale sequence for $E_\mu|_{S_{a,r}}$ (thus a Palais-Smale sequence for $E_\mu|_{S_a}$, since the problem is invariant under rotations) at level $\sigma(a,\mu) > 0$, with $P_\mu(u_n) \to 0$. Therefore, all the assumptions of Proposition 3.1 are satisfied, and we deduce that up to a subsequence $u_n \to \hat{u}_\mu$ strongly in H^σ, with $\hat{u}_\mu \in S_{a,r}$ real-valued nonnegative radial solution to $(1.1)_{\hat{\lambda}}$ for some $\hat{\lambda} < 0$. We can check that $\hat{u}_\mu > 0$ as in the first part.

□

Proof of Theorem 1.2-(1),(2):
Imitate the proof of Theorem 1.1-(1), we get a Palais-Smale sequence $\{u_n\}$ for $E_\mu|_{S_a}$ with

$$\|(-\Delta)^{\frac{\sigma}{2}} u_n\|_2 \le R_0 - \rho \text{ and } P_\mu(u_n) \to 0 \text{ as } n \to \infty$$

and hence $\{u_n\}$ satisfies all the assumptions of Proposition 3.2. Hence one of the alternatives in Proposition 3.2 holds. We wish to show that necessarily the second alternative occurs. Assume then by contradiction that up to a subsequence $u_n \rightharpoonup \tilde{u}_\mu$ weakly in $H^\sigma(\mathbb{R}^N)$ but not strongly, where $\tilde{u}_\mu \not\equiv 0$ is a solution to $(1.1)_{\tilde{\lambda}}$ for some $\tilde{\lambda} < 0$, and

$$E_\mu(\tilde{u}_\mu) \le m(a,\mu) - \frac{\sigma}{N} \mathcal{S}^{\frac{N}{2\sigma}}.$$

Since $\tilde{u}_\mu$ solves $(1.1)_{\tilde{\lambda}}$, the Pohozaev identity $P_\mu(\tilde{u}_\mu) = 0$ holds. We see that $\|\tilde{u}_\mu\|_2 \le a$ and

$$\begin{aligned} m(a,\mu) &\ge \frac{\sigma}{N}\mathcal{S}^{\frac{N}{2\sigma}} + E_\mu(\tilde{u}_\mu) = \frac{\sigma}{N}\mathcal{S}^{\frac{N}{2\sigma}} + \frac{\sigma}{N}\|(-\Delta)^{\frac{\sigma}{2}}\tilde{u}_\mu\|_2^2 - \mu\delta_p\left(\frac{1}{2p\delta_p} - \frac{1}{2_\sigma^*}\right) B(\tilde{u}_\mu, \tilde{u}_\mu) \\ &\ge \frac{\sigma}{N}\mathcal{S}^{\frac{N}{2\sigma}} + \frac{\sigma}{N}\|(-\Delta)^{\frac{\sigma}{2}}\tilde{u}_\mu\|_2^2 - \mu\delta_p\left(\frac{1}{2p\delta_p} - \frac{1}{2_\sigma^*}\right)\mathcal{C}_p^{2p} a^{2p(1-\delta_p)} \left\|(-\Delta)^{\frac{\sigma}{2}}\tilde{u}_\mu\right\|_2^{2p\delta_p}. \end{aligned} \tag{4.11}$$

Denote $g(t) = \frac{\sigma}{N}t^2 - \mu\delta_p\left(\frac{1}{2p\delta_p} - \frac{1}{2_\sigma^*}\right)\mathcal{C}_p^{2p} a^{2p(1-\delta_p)} t^{2p\delta_p}$, $\forall t \ge 0$. Direct calculation gives $\min_{t\ge0} g(t) = -\frac{1-p\delta_p}{p\delta_p}\left(\frac{N}{\sigma}\right)^{\frac{p\delta_p}{1-p\delta_p}} \left[\left(\frac{1}{2p\delta_p} - \frac{1}{2_\sigma^*}\right)\mu p\delta_p^2 \mathcal{C}_p^{2p} a^{2p(1-\delta_p)}\right]^{\frac{1}{1-p\delta_p}} > -\frac{\sigma}{N}\mathcal{S}^{\frac{N}{2\sigma}}$. In the last strict inequality, we used the condition (A_2^*). Therefore, from (4.11), we deduce that

$$0 > m(a,\mu) \ge \frac{\sigma}{N}\mathcal{S}^{\frac{N}{2\sigma}} + g(\|(-\Delta)^{\frac{\sigma}{2}}\tilde{u}_\mu\|_2) \ge \frac{\sigma}{N}\mathcal{S}^{\frac{N}{2\sigma}} + \min_{t\ge0} g(t) > 0.$$

Consequently, up to a subsequence $u_n \to \tilde{u}_\mu$ strongly in H^σ, $\tilde{u}_\mu$ is an interior local minimizer for $E_\mu|_{A_{R_0}}$, and solves $(1.1)_{\tilde{\lambda}}$ for some $\tilde{\lambda} < 0$. We can also check that $\tilde{u}_\mu$ is positive and radially decreasing. Since any critical point of $E_\mu|_{S_a}$ lies in $\mathcal{P}_{a,\mu}$ and $m(a,\mu) = \inf_{\mathcal{P}_{a,\mu}} E_\mu$ (see Lemma 2.17), we see that $\tilde{u}_\mu$ is a ground state for $E_\mu|_{S_a}$.

It only remains to prove that any ground state of $E_\mu|_{S_a}$ is a local minimizer of E_μ in A_{R_0}. Let then u be a critical point of $E_\mu|_{S_a}$ with $E_\mu(u) = m(a,\mu) = \inf_{\mathcal{P}_{a,\mu}} E_\mu$. Since $E_\mu(u) < 0 < \inf_{\mathcal{P}_-^{a,\mu}} E_\mu$, necessarily $u \in \mathcal{P}_+^{a,\mu}$. Then Corollary 2.16 implies that $\mathcal{P}_+^{a,\mu} \subset A_{R_0}$. It results that $||(-\Delta)^{\frac{\sigma}{2}} u||_2 < R_0$, and as a consequence u is a local minimizer for $E_\mu|_{A_{R_0}}$. $\square$

5. The asymptotic results

In this Section, we prove the asymptotic results, i.e. Theorem 1.1-(4)(5) and Theorem 1.2-(3). To obtain the asymptotic property of $m(a,\mu)$ and $\sigma(a,\mu)$ as $\mu \to 0^+$, we need to study equation $(1.1)_\lambda$ with $\mu = 0$. Modify the arguments in Section 2, especially Lemma 2.12 and Lemma 2.15, we can derive the following Lemmas 5.1-5.2.

Lemma 5.1. *Let $N \geq 2$, $\sigma \in (0,1)$, $\bar{q} < q < 2_\sigma^*$, $a > 0$ and $\mu = 0$. Then $\mathcal{P}_0^{a,\mu} = \emptyset$, and $\mathcal{P}_{a,\mu}$ is a smooth manifold of codimension 2 in $H^\sigma(\mathbb{R}^N)$.*

Lemma 5.2. *Let $N \geq 2$, $\sigma \in (0,1)$, $\bar{q} < q < 2_\sigma^*$, $a > 0$ and $\mu = 0$. For every $u \in S_a$, there exists a unique $t_u \in \mathbb{R}$ such that $t_u \star u \in \mathcal{P}_{a,\mu}$. t_u is the unique critical point of the function Ψ_u^μ, and is a strict maximum point at positive level. Moreover:*
(1) $\mathcal{P}_{a,\mu} = \mathcal{P}_-^{a,\mu}$.
(2) Ψ_u^μ is strictly decreasing and concave on $(t_u, +\infty)$.
(3) The maps $u \in S_a \mapsto t_u \in \mathbb{R}$ are of class C^1.
(4) If $P_\mu(u) < 0$, then $t_u < 0$.

Lemma 5.3. *Let $N \geq 2$, $\sigma \in (0,1)$, $\bar{q} < q < 2_\sigma^*$, $a > 0$ and $\mu = 0$. It results that*
$$m(a,0) := \inf_{u \in \mathcal{P}_{a,0}} E_0(u) > 0.$$

Proof. Since $u \in \mathcal{P}_{a,0}$, we have $P_0(u) = 0$. By the embedding inequality (2.3)
$$||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 = \gamma_q ||u||_q^q \leq \gamma_q \mathcal{S}^{-\frac{q\gamma_q}{2}} \left\|(-\Delta)^{\frac{\sigma}{2}} u\right\|_2^{q\gamma_q} a^{q(1-\gamma_q)}.$$
Recall that $q\gamma_q > 2$, so this implies that $\inf_{u \in \mathcal{P}_{a,0}} ||(-\Delta)^{\frac{\sigma}{2}} u||_2 \geq C > 0$. As $P_0(u) = 0$, we can also deduce that
$$\inf_{u \in \mathcal{P}_{a,\mu}} \gamma_q ||u||_q^q \geq C > 0, \ \inf_{u \in \mathcal{P}_{a,0}} E_0(u) = \inf_{u \in \mathcal{P}_{a,0}} \left\{ \frac{q\gamma_q - 2}{2q} ||u||_q^q \right\} \geq C > 0.$$
$\square$

Lemma 5.4. *Let $N \geq 2$, $\sigma \in (0,1)$, $\bar{q} < q < 2_\sigma^*$, $a > 0$ and $\mu = 0$. There exists $k > 0$ sufficiently small such that*
$$0 < \sup_{\overline{A_k}} E_0 < m(a,0) \quad \textit{and} \quad u \in \overline{A_k} \Longrightarrow E_0(u) > 0, \ P_0(u) > 0,$$

where $A_k := \left\{u \in S_a : ||(-\Delta)^{\frac{\sigma}{2}} u||_2 < k\right\}$.

Proof. By the embedding inequalities (2.3), we have

$$E_0(u) \geq \frac{1}{2}||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 - \frac{\mathcal{A}_q^q}{q}||(-\Delta)^{\frac{\sigma}{2}} u||_2^{q\gamma_q} a^{q(1-\gamma_q)},$$

$$P_0(u) \geq ||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 - \gamma_q \mathcal{S}^{-\frac{q\gamma_q}{2}} \left\|(-\Delta)^{\frac{\sigma}{2}} u\right\|_2^{q\gamma_q} a^{q(1-\gamma_q)}.$$

Therefore, for any $u \in \overline{A_k}$ with k small enough, we have

$$0 < \sup_{\overline{A_k}} E_0 \quad \text{ and } u \in \overline{A_k} \Longrightarrow E_0(u) > 0, \ P_0(u) > 0.$$

If necessary replacing k with a smaller quantity, we also have

$$E_0(u) \leq \frac{1}{2}||(-\Delta)^{\frac{\sigma}{2}} u||_2^2 < m(a, 0), \ \forall u \in \overline{A_k}$$

since $m(a, 0) > 0$ by Lemma 5.3. □

Lemma 5.5. *Let $N \geq 2$, $\sigma \in (0, 1)$, $\bar{q} < q < 2_\sigma^*$, $a > 0$ and $\mu = 0$. Then, there exists a real valued positive radial critical point u_0 for $E_0|_{S_a}$ at a positive level*

$$m(a, 0) := \inf_{\mathcal{P}_{a,0}} E_0 = E_0(u_0)$$

and as a result u_0 is a ground state of $E_0|_{S_a}$.

Proof. Utilising Lemmas 5.1-5.4 and by using the same arguments in Section 7 in [44], we can drive that there exists a real valued positive radial critical point u_0 for $E_0|_{S_a}$ at a mountain pass level $\sigma(a, 0) > 0$ with $\sigma(a, 0) = \inf_{\mathcal{P}_{a,0} \cap S_{a,r}} E_0 = E_0(u_0)$. By the symmetric decreasing rearrangement technique, we have $\inf_{\mathcal{P}_{a,0}} E_0 = \inf_{\mathcal{P}_{a,0} \cap S_{a,r}} E_0$, and hence u_0 is a ground state of $E_0|_{S_a}$.

□

In the following discussion, the value $a > 0$ will always be fixed.

Lemma 5.6. *Let $N \geq 2$, $\sigma \in (0, 1)$, $\alpha \in (N - 2\sigma, N)$, $q \in (\bar{q}, 2_\sigma^*)$, $p \in [2, \bar{p})$ and $a > 0$. For any $\mu > 0$ such that μ and a satisfying condition (A_1^*), we have*

$$\sigma(a, \mu) = \inf_{u \in S_{a,r}} \max_{s \in \mathbb{R}} E_\mu(s \star u), \quad \text{ and } \quad m(a, 0) = \inf_{u \in S_{a,r}} \max_{s \in \mathbb{R}} E_0(s \star u).$$

Proof. From (4.6), we have $\sigma(a, \mu) = \inf_{\mathcal{P}_-^{a,\mu} \cap S_{a,r}} E_\mu = E_\mu(\hat{u}_\mu)$. Then, by Lemma 2.15,

$$\sigma(a, \mu) = E_\mu(\hat{u}_\mu) = \max_{s \in \mathbb{R}} E_\mu(s \star \hat{u}_\mu) \geq \inf_{u \in S_{a,r}} \max_{s \in \mathbb{R}} E_\mu(s \star u).$$

On the other hand, for any $u \in S_{a,r}$ we have $t_{u,\mu} \star u \in \mathcal{P}_-^{a,\mu}$, and hence

$$\max_{s \in \mathbb{R}} E_\mu(s \star u) = E_\mu(t_{u,\mu} \star u) \geq \sigma(a, \mu).$$

By using Lemma 5.2 and Lemma 5.5, we can similarly prove

$$m(a, 0) = \inf_{u \in S_{a,r}} \max_{s \in \mathbb{R}} E_0(s \star u).$$

□

Lemma 5.7. *Let $N \geq 2$, $\sigma \in (0,1)$, $\alpha \in (N-2\sigma, N)$, $q \in (\bar{q}, 2_\sigma^*)$, $p \in [2, \bar{p})$ and $a > 0$. For any $0 \leq \mu_1 < \mu_2$ such that μ_2 and a satisfying condition (A_1^*), it results that*

$$\sigma(a, \mu_2) \leq \sigma(a, \mu_1) \leq m(a, 0).$$

Proof. By Lemma 5.6

$$\sigma(a, \mu_2) \leq \max_{s\in\mathbb{R}} E_{\mu_2}(s \star \hat{u}_{\mu_1}) \leq \max_{s\in\mathbb{R}} E_{\mu_1}(s \star \hat{u}_{\mu_1}) = E_{\mu_1}(\hat{u}_{\mu_1}) = \sigma(a, \mu_1)$$

and

$$\sigma(a, \mu_1) \leq \max_{s\in\mathbb{R}} E_{\mu_1}(s \star u_0) \leq \max_{s\in\mathbb{R}} E_0(s \star u_0) = E_0(u_0) = m(a, 0).$$

□

Proof of Theorem 1.1-(4): convergence of $\tilde{u}_\mu$.
Let $a > 0$ fixed. From Lemma 2.14, we know that $R_0(a,\mu) \to 0$ as $\mu \to 0^+$, and hence $||(-\Delta)^{\frac{\sigma}{2}}\tilde{u}_\mu||_2 < R_0(a,\mu) \to 0$ as well. Moreover

$$\begin{aligned} 0 &> m(a,\mu) = E_\mu(\tilde{u}_\mu) \\ &\geq \frac{1}{2}||(-\Delta)^{\frac{\sigma}{2}}\tilde{u}_\mu||_2^2 - \frac{\mu\mathcal{C}_p^{2p}}{2p}||(-\Delta)^{\frac{\sigma}{2}}\tilde{u}_\mu||_2^{2p\delta_p} a^{2p(1-\delta_p)} - \frac{\mathcal{A}_q^q}{q}||(-\Delta)^{\frac{\sigma}{2}}\tilde{u}_\mu||_2^{q\gamma_q} a^{q(1-\gamma_q)} \to 0, \end{aligned}$$

which implies that $m(a,\mu) \to 0$. □

We consider now the behavior of $\hat{u}_\mu$.
Proof of Theorem 1.1-(5): convergence of $\hat{u}_\mu$.
Let us consider $\{\hat{u}_\mu : 0 < \mu < \overline{\mu}\}$, with $\overline{\mu}$ small enough. Since $\hat{u}_\mu \in \mathcal{P}_{a,\mu}$, from Lemma 5.7, we have

$$\begin{aligned} m(a,0) \geq \sigma(a,\mu) = E_\mu(\hat{u}_\mu) &= \left(\frac{1}{2} - \frac{1}{q\gamma_q}\right)||(-\Delta)^{\frac{\sigma}{2}}\hat{u}_\mu||_2^2 - \mu\delta_p\left(\frac{1}{2p\delta_p} - \frac{1}{q\gamma_q}\right)B(\hat{u}_\mu, \hat{u}_\mu) \\ &\geq \left(\frac{1}{2} - \frac{1}{q\gamma_q}\right)||(-\Delta)^{\frac{\sigma}{2}}\hat{u}_\mu||_2^2 - \mu\delta_p\left(\frac{1}{2p\delta_p} - \frac{1}{q\gamma_q}\right)\mathcal{C}_p^{2p}a^{2p(1-\delta_p)}\left\|(-\Delta)^{\frac{\sigma}{2}}\hat{u}_\mu\right\|_2^{2p\delta_p}. \end{aligned}$$

This implies that $\{\hat{u}_\mu\}$ is bounded in H^σ. Since each $\hat{u}_\mu$ is a real-valued function in $S_{a,r}$, we deduce that up to a subsequence $\hat{u}_\mu \rightharpoonup \hat{u}$ weakly in H^σ, strongly in L^r for $2 < r < 2_\sigma^*$ and a.e. in $\mathbb{R}^N$, as $\mu \to 0^+$. Using the fact that $\hat{u}_\mu$ solves

$$(-\Delta)^\sigma \hat{u}_\mu = \hat{\lambda}_\mu \hat{u}_\mu + |\hat{u}_\mu|^{q-2}\hat{u}_\mu + \mu\left(I_\alpha * |\hat{u}_\mu|^p\right)|\hat{u}_\mu|^{p-2}\hat{u}_\mu \text{ in } \mathbb{R}^N \tag{5.1}$$

for $\hat{\lambda}_\mu < 0$ and $P_\mu(\hat{u}_\mu) = 0$, we infer that $\hat{\lambda}_\mu a^2 = (\gamma_q - 1)||\hat{u}_\mu||_q^q + \mu(\delta_p - 1)B(\hat{u}_\mu, \hat{u}_\mu)$. Since $\mu > 0$ and $0 < \gamma_q, \delta_p < 1$, we deduce that $\hat{\lambda}_\mu$ converges (up to a subsequence) to some $\hat{\lambda} \leq 0$, with $\hat{\lambda} = 0$ if and only if the weak limit $\hat{u} \equiv 0$. We claim that $\hat{\lambda} < 0$. In fact, $\hat{u}_\mu \rightharpoonup \hat{u}$ weakly in H^σ implies that $\hat{u}$ is a weak radial (and real) solution to

$$(-\Delta)^\sigma \hat{u} = \hat{\lambda}\hat{u} + |\hat{u}|^{q-2}\hat{u} \text{ in } \mathbb{R}^N, \tag{5.2}$$

and in particular by the Pohozaev identity $||(-\Delta)^{\frac{\sigma}{2}}\hat{u}||_2^2 = \gamma_q||\hat{u}||_q^q$. But then, using the boundedness of $\{\hat{u}_\mu\}$ and Lemma 5.7, we deduce that

$$\begin{aligned} E_0(\hat{u}) &= \frac{q\gamma_q - 2}{2q}||\hat{u}||_q^q = \lim_{\mu\to 0^+}\left[\frac{q\gamma_q - 2}{2q}||\hat{u}_\mu||_q^q - \frac{\mu(1-p\delta_p)}{2p}B(\hat{u}_\mu,\hat{u}_\mu)\right] \\ &= \lim_{\mu\to 0^+} E_\mu(\hat{u}_\mu) = \lim_{\mu\to 0^+}\sigma(a,\mu) \geq \sigma(a,\overline{\mu}) > 0, \end{aligned}$$

which implies that $\hat{u} \not\equiv 0$, and in turn yields $\hat{\lambda} < 0$. Test (5.1) and (5.2) with $\hat{u}_\mu - \hat{u}$, and subtract, we have

$$\begin{aligned} ||(-\Delta)^{\frac{\sigma}{2}}(\hat{u}_\mu - \hat{u})||_2^2 - \mu\int_{\mathbb{R}^N}(I_\alpha * |\hat{u}_\mu|^p)|\hat{u}_\mu|^{p-2}\hat{u}_\mu(\hat{u}_\mu - \hat{u}) - \int_{\mathbb{R}^N}\left(\hat{\lambda}_\mu\hat{u}_\mu - \hat{\lambda}\hat{u}\right)(\hat{u}_\mu - \hat{u}) \\ = \int_{\mathbb{R}^N}\left(|\hat{u}_\mu|^{q-2}\hat{u}_\mu - |\hat{u}|^{q-2}\hat{u}\right)(\hat{u}_\mu - \hat{u}) = o(1), \end{aligned}$$

i.e. $||(-\Delta)^{\frac{\sigma}{2}}(\hat{u}_\mu - \hat{u})||_2^2 - \hat{\lambda}||(\hat{u}_\mu - \hat{u})||_2^2 = o(1)$, which implies that $\hat{u}_\mu \to \hat{u}$ in H^σ. Moreover, we have

$$m(a,0) \leq E_0(\hat{u}) = \lim_{\mu\to 0^+}\sigma(a,\mu) \leq m(a,0).$$

Consequently, $E_0(\hat{u}) = \lim_{\mu\to 0^+}\sigma(a,\mu) = m(a,0)$ and $\hat{u}$ is a ground state to (5.2). From [18, 19], we know that the ground state u_0 for (5.2) is unique. Thus $\hat{u} = u_0$. □

Proof of Theorem 1.2-(3):
Let $a > 0$ fixed. From Lemma 2.14, we know that $R_0(a,\mu) \to 0$ for $\mu \to 0^+$, and hence $||(-\Delta)^{\frac{\sigma}{2}}\tilde{u}_\mu||_2 < R_0(a,\mu) \to 0$ as well. Moreover

$$\begin{aligned} 0 &> m(a,\mu) = E_\mu(\tilde{u}_\mu) \\ &\geq \frac{1}{2}||(-\Delta)^{\frac{\sigma}{2}}\tilde{u}_\mu||_2^2 - \frac{\mu\mathcal{C}_p^{2p}}{2p}||(-\Delta)^{\frac{\sigma}{2}}\tilde{u}_\mu||_2^{2p\delta_p}a^{2p(1-\delta_p)} - \frac{\mathcal{S}^{-\frac{2_\sigma^*}{2}}}{2_\sigma^*}||(-\Delta)^{\frac{\sigma}{2}}\tilde{u}_\mu||_2^{2_\sigma^*} \to 0, \end{aligned}$$

which implies that $m(a,\mu) \to 0$. □

6. The stability results

In this Section, we prove the stability of the set of ground states $Z_{a,\mu}$, i.e. Theorem 1.3. Let $a > 0$ be fixed, and let $\mu > 0$ satisfy the assumptions of Theorem 1.1, then $Z_{a,\mu} \neq \emptyset$. The relative compactness (up to translations) of all the minimizing sequences for $m(a,\mu) = \inf_{A_{R_0}} E_\mu$ is vital in proving the stability of $Z_{a,\mu}$.

Notice that $m(a,\mu) = \inf_{A_{R_0}} E_\mu$ can be relaxed to

$$m(a,\mu) = \inf_{A_{R_1}} E_\mu = \inf\left\{E_\mu(u) : u \in S_a, ||(-\Delta)^{\frac{\sigma}{2}}u||_2 < R_1\right\}. \tag{6.1}$$

Indeed, if $||(-\Delta)^{\frac{\sigma}{2}}u||_2 \in [R_0, R_1]$, then $E_\mu(u) \geq h(||(-\Delta)^{\frac{\sigma}{2}}u||_2) \geq 0 > \inf_{A_{R_0}} E_\mu$, see (2.8) and Lemma 2.14. We see that R_0 and R_1 depend on a and μ by means of Lemma 2.14. In the following discussion, we stress this dependence writing $R_0(a,\mu)$ and $R_1(a,\mu)$. Similarly, the definition of A_{R_0} depends on a and μ, and hence we explicitly write $A_{a,R_0(a,\mu)}$.

Lemma 6.1. *Let $\tilde{a},\delta>0$. There exists $\tilde{\mu}=\tilde{\mu}(\tilde{a}+\delta)>0$ such that, if $0<a\leq\tilde{a}$ and $0<\mu<\tilde{\mu}$, then:*
(1) $2R_0^2(\tilde{a}+\delta,\mu)<R_1^2(\tilde{a},\mu)$.
(2) The functions $(a,\mu)\mapsto R_0(a,\mu)$ and $(a,\mu)\mapsto R_1(a,\mu)$ are of class C^1 in $(0,\tilde{a}+\delta)\times(0,\tilde{\mu})$, $R_0(a,\mu)$ is monotone increasing in a, while $R_1(a,\mu)$ is monotone decreasing in a.
(3) For any $a_1,a_2>0$ with $a_1^2+a_2^2=a^2$, we have $m(a,\mu)<m(a_1,\mu)+m(a_2,\mu)$.

Proof. The proof is motivated by [44]. We give out the details for reader's convenience. From Lemma 2.14, we see that $0<R_0=R_0(a,\mu)<R_1=R_1(a,\mu)$ are the roots of

$$0=g(t,a,\mu):=\varphi(t,a)-\frac{\mu\mathcal{C}_p^{2p}}{2p}a^{2p(1-\delta_p)}$$

where $\varphi(t,a):=\frac{1}{2}t^{2-2p\delta_p}-\frac{\mathcal{A}_q^q}{q}a^{q(1-\gamma_q)}t^{q\gamma_q-2p\delta_p}$. The condition $\mu^{q\gamma_q-2}a^{\bar{\mathcal{C}}(p,q)}<\tilde{\mathcal{C}}(p,q)$ guarantees the existence of R_0 and R_1. Let then $\tilde{a},\delta>0$, and consider the range of $\mu>0$ such that $\mu^{q\gamma_q-2}(\tilde{a}+\delta)^{\bar{\mathcal{C}}(p,q)}<\tilde{\mathcal{C}}(p,q)$. This range contains a right neighborhood of 0. By continuity we have that

$$R_0(\tilde{a}+\delta,\mu)\to 0 \text{ and } R_1(\tilde{a}+\delta,\mu)\to C_{\tilde{a}+\delta}=\left[\frac{q}{2(\tilde{a}+\delta)^{q(1-\gamma_q)}\mathcal{A}_q^q}\right]^{\frac{1}{q\gamma_q-2}} \quad \text{as } \mu\to 0^+$$

where $C_{\tilde{a}+\delta}$ the only positive root of $\varphi(t,\tilde{a}+\delta)=0$. In particular, for every $\tilde{a},\delta>0$ fixed and any $\theta>1$, we have

$$R_0(a,\mu)\to 0,\ R_1(\theta a,\mu)\to C_{\theta a}=\left[\frac{q}{2(\theta a)^{q(1-\gamma_q)}\mathcal{A}_q^q}\right]^{\frac{1}{q\gamma_q-2}} \quad \text{as } \mu\to 0^+$$

where $C_{\theta a}$ the only positive root of $\varphi(t,\theta a)=0$. Consequently, there exists $\tilde{\mu}=\tilde{\mu}(\tilde{a}+\delta)>0$ (independent of θ) sufficiently small such that

$$2R_0^2(\tilde{a}+\delta,\mu)<R_1^2(\tilde{a},\mu) \text{ and } \theta R_0(a,\mu)<R_1(\theta a,\mu) \tag{6.2}$$

if $0<a\leq\tilde{a}$ and $0<\mu<\tilde{\mu}$.

Let now $0<a\leq\tilde{a}+\delta$ and $0<\mu<\tilde{\mu}$. Under assumption $\mu^{q\gamma_q-2}a^{\bar{\mathcal{C}}(p,q)}<\tilde{\mathcal{C}}(p,q)$, we have

$$\partial_t g(t,a,\mu)=\partial_t\varphi(t,a).$$

We checked that $\varphi(\cdot,a)$ has a unique critical point on $(0,+\infty)$, which is a strict maximum point, in $\bar{t}=\bar{t}(a)$, with $0<R_0<\bar{t}<R_1$, and hence in particular

$$\partial_t g\left(R_0(a,\mu),a,\mu\right)>0, \text{ and } \partial_t g\left(R_1(a,\mu),a,\mu\right)<0.$$

Thus, the implicit function theorem implies that $R_0(a,\mu)$ is a locally unique C^1 function of (a,μ), with $\frac{\partial R_0(a,\mu)}{\partial a}=-\frac{\partial_a g(R_0(a,\mu),a,\mu)}{\partial_t g(R_0(a,\mu),a,\mu)}>0$. In a similar way, one can show that $R_1(a,\mu)$ is a locally unique C^1 function of (a,μ) with $\frac{\partial R_1(a,\mu)}{\partial_a}<0$. In particular, R_0 is monotone increasing and R_1 is monotone decreasing in a. Then we finish the proof of (1) and (2).

Next, we prove (3). Let $0 < c < \tilde{a}$ and $0<\mu<\tilde{\mu}$, let $\theta > 1$ be such that $\theta c < \tilde{a}$ and let $\{u_n\} \subset S_c$ be a minimizing sequence for $m(c, \mu)$, i.e.

$$m(c,\mu) = \lim_{n\to+\infty} E_\mu(u_n), \;\; u_n \in S_c, \;\; ||(-\Delta)^{\frac{\sigma}{2}} u_n||_2 < R_0(c,\mu). \tag{6.3}$$

From (6.2) and (6.3), we have $||(-\Delta)^{\frac{\sigma}{2}}(\theta u_n)||_2 < \theta R_0(c,\mu) < R_1(\theta c,\mu)$. Therefore, $\theta u_n \in S_{\theta c}$ and $||(-\Delta)^{\frac{\sigma}{2}}(\theta u_n)||_2 < R_1(\theta c,\mu)$. By using (6.1), it follows immediately that $m(\theta c,\mu) \le E_\mu(\theta u_n)$. Moreover, since $q \in (2+\frac{4\sigma}{N}, 2_\sigma^*)$, $p \in [2, \frac{2\alpha}{N-2\sigma}]$ and $\theta > 1$, we obtain

$$m(\theta c,\mu) \le E_\mu(\theta u_n) = \frac{\theta^2}{2}||(-\Delta)^{\frac{\sigma}{2}} u_n||_2^2 - \frac{\mu\theta^{2p}}{2p} B(u_n,u_n) - \frac{\theta^q}{q}||u_n||_q^q < \theta^2 E_\mu(u_n).$$

Letting $n \to +\infty$, it results that $m(\theta c,\mu) \le \theta^2 m(c,\mu)$, with equality if and only if $\lim_{n\to\infty} B(u_n,u_n) = 0 = \lim_{n\to\infty} ||u_n||_q^q$. But this is not possible, since otherwise we would find $0 > m(c,\mu) = \lim_{n\to\infty} E_\mu(u_n) = \frac{1}{2}\lim_{n\to\infty} ||(-\Delta)^{\frac{\sigma}{2}} u_n||_2^2 \ge 0$. Thus $m(\theta c,\mu) < \theta^2 m(c,\mu)$. Without loss of generality, let $0 < a_1 < a_2 < a$, take $\theta = \frac{a}{a_2}$ and then $\theta = \frac{a_2}{a_1}$, we have

$$\begin{aligned} m(a,\mu) = m(a_2\frac{a}{a_2},\mu) &< \frac{a^2}{a_2^2} m(a_2,\mu) = m(a_2,\mu) + \frac{a_1^2}{a_2^2} m(a_2,\mu) \\ &= m(a_2,\mu) + \frac{a_1^2}{a_2^2} m(a_1\frac{a_2}{a_1},\mu) < m(a_2,\mu) + m(a_1,\mu). \end{aligned}$$

□

Lemma 6.2. *Let $\tilde{a}, \delta>0$. There exists $\tilde{\mu}=\tilde{\mu}(\tilde{a}+\delta)>0$ such that, if $0<a\le\tilde{a}$ and $0<\mu<\tilde{\mu}$, then any sequences $\{z_n\} \subset H^\sigma(\mathbb{R}^N)$ such that*

$$E_\mu(z_n) \to m(a,\mu), \;\; ||z_n||_2 \to a, \;\; ||(-\Delta)^{\frac{\sigma}{2}} z_n||_2 < R_0(a+\delta,\mu)$$

is relatively compact in $H^\sigma(\mathbb{R}^N)$ up to translations.

Proof. Let $u_n = \frac{a z_n}{||z_n||_2}$, we can check that $E_\mu(u_n) \to m(a,\mu)$, $||u_n||_2 = a$ and $||(-\Delta)^{\frac{\sigma}{2}} u_n||_2 = \frac{a}{||z_n||_2}||(-\Delta)^{\frac{\sigma}{2}} z_n||_2 \le R_0(a+\delta,\mu)$ for n sufficiently large. Therefore, $\{u_n\}$ is bounded in $H^\sigma(\mathbb{R}^N)$ and $u_n \rightharpoonup u$ in $H^\sigma(\mathbb{R}^N)$ for some $u \in H^\sigma(\mathbb{R}^N)$. Now, let $R > 0$. If it were

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2 = 0, \quad \forall R > 0,$$

then, by the vanishing Lemma (see [15] Lemma 2.3), we have $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for $2 < r < 2_\sigma^*$. It results that $B(u_n,u_n) \to 0$ by (2.7). But this is not possible, since otherwise we would find $0 > m(a,\mu) = \lim_{n\to\infty} E_\mu(u_n) = \frac{1}{2}\lim_{n\to\infty} ||(-\Delta)^{\frac{\sigma}{2}} u_n||_2^2 \ge 0$. Then there exists an $\varepsilon_0 > 0$ such that

$$a_1^2 := \lim_{R\to\infty}\left(\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2\right) \ge \varepsilon_0.$$

Given any $\varepsilon > 0$, by definition of a_1, there exists $\bar{R} > 0$ such that if $R > \bar{R} > 0$, then

$$a_1^2 - \varepsilon < \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2 \leq a_1^2.$$

Then we can say that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$a_1^2 - \varepsilon < \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2 \leq \sup_{y\in\mathbb{R}^N} \int_{B_{2R}(y)} |u_n|^2 < a_1^2 + \varepsilon.$$

It then follows that for every $n \geq n_0$, there exists $y_n \in \mathbb{R}^N$ such that

$$a_1^2 - \varepsilon < \int_{B_R(y_n)} |u_n|^2 \leq \int_{B_{2R}(y_n)} |u_n|^2 < a_1^2 + \varepsilon. \tag{6.4}$$

Now introduce smooth cut-off functions ϕ and ψ, defined on $\mathbb{R}^N$, such that

$$\phi(x) = \begin{cases} 1, & \text{for } |x| \leq 1 \\ 0, & \text{for } |x| \geq 2, \end{cases} \quad \psi(x) = \begin{cases} 0, & \text{for } |x| \leq 1 \\ 1, & \text{for } |x| \geq 2, \end{cases} \quad \phi^2 + \psi^2 \equiv 1 \text{ on } \mathbb{R}^N.$$

Denote $\phi_R(x) = \phi(\frac{x-y_n}{R})$ and $\psi_R(x) = \psi(\frac{x-y_n}{R})$, respectively. Define $v_n(x) = \phi_R(x)u_n(x)$ and $w_n(x) = \psi_R(x)u_n(x)$. If it were $a_2 = \sqrt{a^2 - a_1^2} > 0$, then it is standard as the proof of Lemma 2.14 in [1] that

$$||(-\Delta)^{\frac{\sigma}{2}} v_n||_2^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\sigma}{2}}(\phi_R u_n)|^2 \leq \int_{\mathbb{R}^N} (\phi_R)^2 |(-\Delta)^{\frac{\sigma}{2}} u_n|^2 + C\varepsilon, \tag{6.5}$$

$$||(-\Delta)^{\frac{\sigma}{2}} w_n||_2^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\sigma}{2}}(\psi_R u_n)|^2 \leq \int_{\mathbb{R}^N} (\psi_R)^2 |(-\Delta)^{\frac{\sigma}{2}} u_n|^2 + C\varepsilon, \tag{6.6}$$

and

$$\begin{cases} \int_{\mathbb{R}^N} |v_n(x)|^2 \, dx \in (a_1^2 - \varepsilon, a_1^2 + \varepsilon) \\ \int_{\mathbb{R}^N} |w_n(x)|^2 \, dx \in (a_2^2 - \varepsilon, a_2^2 + \varepsilon) \\ E_\mu(u_n) \geq E_\mu(v_n) + E_\mu(w_n) - \varepsilon. \end{cases} \tag{6.7}$$

for sufficiently large R and some constant $C > 0$. By (6.5) and (6.6), we have

$$||(-\Delta)^{\frac{\sigma}{2}} v_n||_2^2 + ||(-\Delta)^{\frac{\sigma}{2}} w_n||_2^2 \leq ||(-\Delta)^{\frac{\sigma}{2}} u_n||_2^2 + 2C\varepsilon \leq R_0^2(a+\delta,\mu) + 2C\varepsilon.$$

Similar to (6.2), we deduce that

$$||(-\Delta)^{\frac{\sigma}{2}} v_n||_2 < R_1(a_1,\mu) \text{ and } ||(-\Delta)^{\frac{\sigma}{2}} w_n||_2 < R_1(a_2,\mu)$$

if we take $\tilde{\mu}$ sufficiently small. Let $\varepsilon \to 0$ and then $n \to +\infty$ in (6.7), we have

$$m(a,\mu) \geq m(a_1,\mu) + m(a_2,\mu),$$

which contradicts with Lemma 6.1-(3). As a result, we deduce that $a^2 = a_1^2$. From (6.4), we have

$$a^2 - \varepsilon < \int_{B_R(0)} |u_n(\cdot + y_n)|^2 = \int_{B_R(y_n)} |u_n|^2 < a^2 + \varepsilon. \tag{6.8}$$

Consequently, $\tilde{u}_n = u_n(\cdot + y_n)$ converges strongly (up to a subsequence) in $L^2(\mathbb{R}^N)$ and weakly in $H^\sigma(\mathbb{R}^N)$ to some $\tilde{u} \in S_a$. If $2 < r < 2_\sigma^*$, by Hölder and Sobolev inequality (2.2)

$$||\tilde{u}_n - \tilde{u}||_r \le ||\tilde{u}_n - \tilde{u}||_2^{(1-\gamma_r)} ||\tilde{u}_n - \tilde{u}||_{2_\sigma^*}^{\gamma_r} \le C||\tilde{u}_n - \tilde{u}||_2^{\gamma_r} \to 0$$

for some constant $C > 0$ and $\gamma_r = \frac{N(r-2)}{2r\sigma}$. Therefore, we have

$$||(-\Delta)^{\frac{\sigma}{2}}\tilde{u}||_2 \le \lim_{n\to\infty} ||(-\Delta)^{\frac{\sigma}{2}}\tilde{u}_n||_2, \ \lim_{n\to\infty} ||\tilde{u}_n||_q = ||\tilde{u}||_q, \ \lim_{n\to\infty} B(\tilde{u}_n, \tilde{u}_n) = B(\tilde{u}, \tilde{u}).$$

These facts leads to

$$m(a,\mu) \le E_\mu(\tilde{u}) \le \lim_{n\to\infty} E_\mu(\tilde{u}_n) = \lim_{n\to\infty} E_\mu(u_n) = m(a,\mu).$$

Finally, we deduce that the previous inequalities are equalities and $||\tilde{u}_n||_{H^\sigma} \to ||\tilde{u}||_{H^\sigma}$.

□

Proof of Theorem 1.3

Recall that we fixed $a > 0$, and for any small δ we considered $\tilde{\mu} = \mu(a+\delta)$ and $0 < \mu < \tilde{\mu}$. Suppose that there exists $\varepsilon > 0$, a sequence of initial data $\{\psi_{n,0}\} \subset H^\sigma$ and a sequence $\{t_n\} \subset (0, +\infty)$ such that the maximal solution $\psi_n(t, x)$ with $\psi_n(0, x) = \psi_{n,0}(x)$ satisfies

$$\lim_{n\to\infty} \inf_{v\in Z_{a,\mu}} ||\psi_{n,0} - v||_{H^\sigma} = 0, \text{ and } \inf_{v\in Z_{a,\mu}} ||\psi_n(t_n, \cdot) - v||_{H^\sigma} \ge \varepsilon. \tag{6.9}$$

(we refer to Propositions 2.3-2.4 in [21] for the local well-posedness for $(1.1)_\lambda$. Similar to the proof of Lemma 3.3 in [44], we can check that $m(a,\mu)$ is continuous in a. Clearly $||\psi_{n,0}||_2 =: a_n \to a$ and $E_\mu(\psi_{n,0}) \to m(a,\mu)$, by continuity. Furthermore, always by continuity and using point (1) of Lemma 6.1, we deduce that $||(-\Delta)^{\frac{\sigma}{2}}\psi_{n,0}||_2 < R_0(a+\delta,\mu) < R_1(a_n,\mu)$ for every n sufficiently large. Since $||(-\Delta)^{\frac{\sigma}{2}}\psi_{n,0}||_2 \in [R_0(a_n,\mu), R_1(a_n,\mu)]$ implies that $E_\mu(\psi_{n,0}) \ge 0$, we deduce that in fact $||(-\Delta)^{\frac{\sigma}{2}}\psi_{n,0}||_2 < R_0(a_n,\mu) < R_0(a+\delta,\mu)$.

Let us consider now the solution $\psi_n(t,\cdot)$. Since $\psi_{n,0} \in A_{a_n,R_0(a_n,\mu)}$, if $\psi_n(t,\cdot)$ exits from $A_{a_n,R_0(a_n,\mu)}$ there exists $t \in (0, T_{max})$ such that $||(-\Delta)^{\frac{\sigma}{2}}\psi_n(t,\cdot)||_2 = R_0(a_n,\mu)$; but then $E_\mu(\psi_n(t,\cdot)) \ge h(R_0) = 0$, against the conservation of energy. This shows that solutions starting in $A_{a_n,R_0(a_n,\mu)}$ are globally defined in time and satisfy $||(-\Delta)^{\frac{\sigma}{2}}\psi_n(t,\cdot)||_2 < R_0(a_n,\mu) < R_0(a+\delta,\mu)$ for every $t \in (0,+\infty)$. Moreover, by conservation of mass and of energy $||\psi_n(t_n,\cdot)||_2 \to a$, and $E_\mu(\psi_n(t_n,\cdot)) \to m(a,\mu)$ as $n \to +\infty$. It follows that $\{\psi_n(t_n,\cdot)\}$ is relatively compact up to translations in H^σ, and hence it converges, up to a translation, to a ground state in $Z_{a,\mu}$, in contradiction with (6.9). □

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