PERSISTENCE OF TRAVELING WAVES TO THE TIME FRACTIONAL KELLER-SEGEL SYSTEM WITH A SMALL PARAMETER

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ABSTRACT. This paper aims to investigate the time fractional Keller-Segel system with a small parameter. After the fractional order traveling wave transformation, the heteroclinic orbit to the degenerate time fractional Keller-Segel system is demonstrated through the method of constructing a suitable invariant region. Moreover, the persistence of traveling waves in the system with a small parameter can be further illustrated. The results are mainly reliance on the application of geometric singular perturbation theory and Fredholm theorem, which are fundamental theoretical frameworks for dealing with problems of complexity and high dimensionality. Eventually, the asymptotic behavior is depicted by the asymptotic theory to illustrate the rate of decay for traveling waves.

Keywords: chemotaxis, time fractional Keller-Segel system, geometric singular perturbation, traveling waves

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1. INTRODUCTION.

Chemotaxis [1, 11, 21] is one of the most broadly used mechanisms to describe the aggregation of biological species, which has been widely considered to explain the biological phenomena. Indeed, chemotaxis is known as an organism's directed movement in response to the chemical stimulus, which can be secreted by organism itself or from an external source. Dating back to the pioneering works of theoretical and mathematical modeling, Keller and Segel [27, 28, 29] introduced a basic model in chemotaxis of partial differential equations to explain the collection of specific categories, which is called Keller-Segel system and given as

$$\begin{cases} U_t = (DU_x - U\chi(U,V)V_x)_x, \\ V_t = \varepsilon V_{xx} + g(U,V), \end{cases} \tag{1.1}$$

where the function $U(x,t)$ indicates cell (or organism) bacteria density in the position x at time t, function $V(x,t)$ shows the chemical signal substance concentration. The function $\chi(U,V)$ can both be related to U and V, which describes the chemotactic

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sensitivity linked to chemotaxis. It is noteworthy that $g(U,V)$ is the function describing the production of chemical signals. The constant $\varepsilon > 0$ describes the diffusion speed for the chemical and the constant D denotes the diffusivity of cell. The Keller-Segel system for chemotaxis characterizes the cell collective motion, especially bacteria and amoebae. Actually they are released by a chemical which is attractive to them.

Inspired by the work based on Keller and Segel, the diverse dynamics for the original Keller-Segel system have become critical issues in the last decades [4, 10, 11, 21, 33, 37, 38, 39, 40, 43]. It is seen that various versions to the Keller-Segel system are available, depending on different phenomena and scales. In the case, spatial pattern formation and self-aggregation phenomenon are important properties for system (1.1). Based on review papers [1, 22, 23], it is shown that the dynamical results involving the boundedness [10], global existence [42] and blow-up [37] for solutions in the Keller-Segel systems have been detailed established.

There is no doubt that Keller-Segel systems with integer order derivatives attracted a lot of attention in previous studies, which was a fascinating research topic. Nonetheless, integer order derivative models have not been adequately applied to describe chemotaxis in complex and nonhomogeneous media. Conversely, it can be better modeled by fractional derivatives whose order is a scalar value between zero and one. Fractional partial differential system has been widely applied in biological modeling, fluid dynamics, electromagnetics, signal processing, optics, and many other fields [16, 17, 19, 32, 34]. It motivated researchers to use fractional derivatives due to the natural and complicated phenomena. Fractional derivatives [25, 32, 34, 35, 41] can better identify the consistency between the solution and the real data, which can be referred as the ideal modelling tool for mathematical biological models. Recently, along with the progression in the area of fractional mathematics, many researchers are interested in fractional partial differential equations [3, 6, 8, 9, 17, 31, 35, 41, 42] according to their potential applications. This would imply a major breakthrough in the development of keller-Segel system, which is applicable to the study of more complex situations. For instance, Cheng et al. [8] took the time fractional Keller-Segel system with diffusion term into consideration

$$\begin{cases} D_t^\alpha U = aU_{xx} - (UV_x)_x\,, \\ D_t^\alpha V = bV_{xx} + cU, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \mathcal{D}_t^\alpha U = aU_{xx} - (UV_x)_x\,, \\ \mathcal{D}_t^\alpha V = bV_{xx} + cU - dV, \end{cases} \tag{1.3}$$

where $0 < \alpha < 1$, the operator D_t^α and $\mathcal{D}_t^\alpha$ respectively represent Riemann-Liouville and Caputo fractional derivative of order α. The positive constants a and b concern the effect of cell and chemical diffusion, while the constants c and d are arbitrary. In order to deal with dynamics of nonlinear fractional partial differential system, multitudinous efficient methods have emerged, for instance invariant subspace method [8], Q-homotopy analysis method [8], homotopy perturbation transform technique [31], Laplace Adomian decomposition method [31], semigroup method [42], Duhamel's principle [9], the fixed point argument [9] and so on. Few works have been presented to address traveling waves for various Keller-Segel systems, although which have been discussed by a large number of researchers. To name a few, Du et al. [14] focused on a kind of generalized Keller-Segel system, taking the form of

$$\begin{cases} U_t = \left(\varepsilon U_x - U\phi(V)V_x\right)_x, \\ \varepsilon V_t = V_{xx} + f(U) - g(V), \end{cases} \tag{1.4}$$

where the parameter $0 < \varepsilon \ll 1$ is sufficiently small. They discussed traveling pulse solutions relying on Poincaŕe-Bendixson theorem and geometric singular perturbation theory. Similarly, Chang et al. [7] paid attention to another form of Keller-Segel system (1.4), and established the existence and linear instability of traveling pulses for the case that the chemical signal production $f(U)$ equals U and the nonlinear interaction term is $\chi U\phi(V_x)$ instead of $U\phi(V)V_x$.

In the aforementioned discussion, it can be found that the classical geometric singular perturbation theory plays a fundamental role in establishing the existence of traveling waves in the system with a small parameter. It should be pointed out that geometric singular perturbation theory [18, 20, 24] is a powerful tool to handle singular perturbation problems. In the case, the singular perturbed system can be reduced to a regular perturbed system on the invariant manifold, and the existence of invariant manifolds can also be ensured. Notably, geometric singular perturbation theory has a successfully application in some aspects, such as Camassa-Holm equations [12, 13], Belousov-Zhabotinskii systems [15], FitzHugh-Nagumo equation [36] and so on. Nonetheless, it has been applied less frequently to address the Keller-Segel system.

Motivated by the above analysis, we consider the time fractional Keller-Segel system with a small parameter in the present study, which is given as

$$\begin{cases} D_t^\alpha U = dU_{xx} - \left(UV_x\right)_x + \mu U(1-U), \\ D_t^\alpha V = \varepsilon V_{xx} - \delta_1 U + \delta_2 V, \end{cases} \tag{1.5}$$

where $t \geq 0$ and $x \in \mathbb{R}$ denote temporal and spatial variables, respectively. The functions $U(x,t) \geq 0$ and $V(x,t) \geq 0$ represent the cell density and the chemical signal concentration. The operator D_t^α is a time fractional derivative with $0 < \alpha \leq 1$. The parameter μ is positive and related to the rate of logistic cell growth, and the constants $\delta_1, \delta_2 > 0$ describe the chemical growth and the death rate. The diffusion coefficient $0 < \varepsilon \ll 1$ concerns the speed of the chemical diffusion and the constant $d > 0$ illustrates the cell diffusivity. The main contributions of this brief are as follows.

(1) An appropriate transformation is selected to transform the fractional partial differential equations into ordinary differential equations. The time fractional derivatives can model the chemotaxis in the complicated and nonhomogeneous media and mathematical biological models, which results in a challenge to analyze the complicated impact on the dynamics of Keller-Segel systems.
(2) The suitable invariant region is designed for time fractional Keller-Segel system with the small parameter $\varepsilon = 0$, which corresponds to traveling waves.
(3) Geometric singular perturbation theory plays an essential role in handling the persistence of time fractional Keller-Segel system with sufficiently small parameter $\varepsilon > 0$. Noteworthy, the asymptotic theory is a functional tool to explore the asymptotic behavior to traveling waves.

The rest is outlined as follows. Section 2 presents some basic definitions and notations concerning fractional and fractional-like derivatives. Section 3 gives the main existence results for system (1.5) with the small parameter $\varepsilon = 0$, which is based on the method of constructing an appropriate invariant region. Section 4 reveals the persistence of traveling waves of the system with the small parameter $\varepsilon > 0$ by means of the search for heteroclinic orbits under the perturbation. Using the asymptotic theory, Section 5 explores the asymptotic behavior to traveling waves.

2. Preliminaries.

Many researchers [25, 32, 34] tried to put forward the definition for different versions of fractional derivatives, which include Riemann-Liouville, Caputo, Riesz, Grunwald-Letnikov, Weyl and Marchaud fractional derivatives. Two of them were among the most popular.

Definition 2.1. *([32, 34]) For $n-1 \le \alpha < n$, the Riemann-Liouville derivative of function $u(x)$ of order α is defined by*

$$D_x^\alpha u(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{u(\xi)}{(x-\xi)^{\alpha-n+1}} d\xi, \tag{2.1}$$

and the Caputo derivative of function $u(x)$ of order α is expresseded by

$$D_x^\alpha u(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{u^{(n)}(\xi)}{(x-\xi)^{\alpha-n+1}} d\xi, \tag{2.2}$$

where $\Gamma(\alpha)$ is the Euler Gamma function and takes the form of

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} dx, \quad \alpha > 0. \tag{2.3}$$

However, both definitions may have some limitations in the proof. Other fractional and fractional-like definitions have been proposed successively.

Definition 2.2. *([25]) The Jumarie's modified Riemann-Liouville derivative of function $u(x)$ of order α is stated by the following expression*

$$D_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-\xi)^{-\alpha}(u(\xi)-u(0))\mathrm{d}\xi, & 0 < \alpha < 1, \\ \left(u^{(n)}(x)\right)^{(\alpha-n)}, & n \le \alpha < n+1, n \ge 1, \end{cases} \tag{2.4}$$

where $\Gamma(\alpha)$ is the Euler Gamma function given in (2.3).

Atangana [2] proposed beta fractional-like derivative as blew.

Definition 2.3. *([2]) The beta fractional-like derivative of function $u(x)$ of order β is given by*

$$D_x^\beta(u(x)) = \lim_{\varpi \to 0} \frac{u\left(x + \varpi\left(x + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) - u(x)}{\varpi}, \; 0 < \beta \le 1, \tag{2.5}$$

where $\Gamma(\beta)$ is the Euler Gamma function.

Similarly Khalil et al. [26] proposed a type of conformable fractional-like derivative.

Definition 2.4. *([26]) The conformable fractional-like derivative of function $u(x)$ of order α is defined by*

$$D_x^\alpha(u(x)) = \lim_{\varpi \to 0} \frac{u\left(x + \varpi x^{1-\alpha}\right) - u(x)}{\varpi}, \tag{2.6}$$

for all $x > 0$, $\alpha \in (0,1)$. If $u(x)$ is α-differentiable and $\lim_{x\to 0^+} u^{(\alpha)}(x)$ exists, then $f^{(\alpha)}(0) = \lim_{t\to 0^+} u^{(\alpha)}(x)$.

3. The time fractional Keller-Segel system with $\varepsilon = 0$.

This section draws attention to the time fractional Keller-Segel system (1.5) with the small parameter $\varepsilon = 0$, which is given as

$$\begin{cases} D_t^\alpha U = dU_{xx} - (UV_x)_x + \mu U(1-U), \\ D_t^\alpha V = -\delta_1 U + \delta_2 V. \end{cases} \tag{3.1}$$

It promotes us to seek for the heteroclinic orbit, corresponding to the existence of traveling waves, by virtue of constructing a suitable invariant region.

3.1. **Phase plane analysis.**

The first thing to do is to choose the appropriate transformation to change fractional partial differential equations into ordinary differential equations. If the fractional derivative is defined in Definition 2.4, we now take it as an example and conduct the following proof. Taking the following fractional traveling wave transformation

$$U(x,t) = U\left(kx - \frac{c}{\alpha}t^\alpha\right) = U(\xi), \quad V(x,t) = V\left(kx - \frac{c}{\alpha}t^\alpha\right) = V(\xi), \tag{3.2}$$

where $k > 0$ and $c > 0$. The parameter c represents the traveling wave speed. By formula (3.2), we obtain

$$\begin{aligned} D_t^\alpha U(x,t) &= t^{1-\alpha}\frac{dU(x,t)}{dt} = t^{1-\alpha}\frac{dU(\xi)}{dt} \\ &= t^{1-\alpha}U_\xi \frac{d\xi}{dt} = t^{1-\alpha}U_\xi\left(-\frac{c}{\alpha}\right)\alpha t^{\alpha-1} \\ &= -cU_\xi. \end{aligned}$$

Substituting the transformation (3.2) into system (3.1), then resulting in the following system

$$\begin{cases} -cU' = dk^2U'' - k^2\left(UV'\right)' + \mu U(1-U), \\ -cV' = -\delta_1 U + \delta_2 V, \end{cases} \tag{3.3}$$

where $' = \frac{d}{d\xi}$. Introducing

$$W = dk^2U' + cU - \frac{k^2}{c}U\left(\delta_1 U - \delta_2 V\right),$$

then the system (3.3) can be rewritten as the following three-dimensional ordinary differential equations

$$\begin{cases} U' = \dfrac{1}{dk^2}\left(W - cU + \dfrac{k^2}{c}U\left(\delta_1 U - \delta_2 V\right)\right), \\ V' = \dfrac{1}{c}\left(\delta_1 U - \delta_2 V\right), \\ W' = \mu U(U-1). \end{cases} \tag{3.4}$$

It is obvious that $E_1\left(0,0,0\right)$ and $E_2\left(1,\dfrac{\delta_1}{\delta_2},c\right)$ are two equilibrium points. In the system (3.4), the traveling wave solutions of interest are heteroclinic orbits connecting equilibrium point $E_2\left(1,\dfrac{\delta_1}{\delta_2},c\right)$ to the origin $E_1\left(0,0,0\right)$. Then we have the following lemma concerning the equilibrium points. Figure 1 describes the phase portrait of system (3.4) with $c=4,\ k=2,\ d=1,\ \delta_1=0.5,\ \delta_2=3,\ \mu=1$. Moreover, Figure 2 is presented to illustrate the projection portraits of the system (3.4).

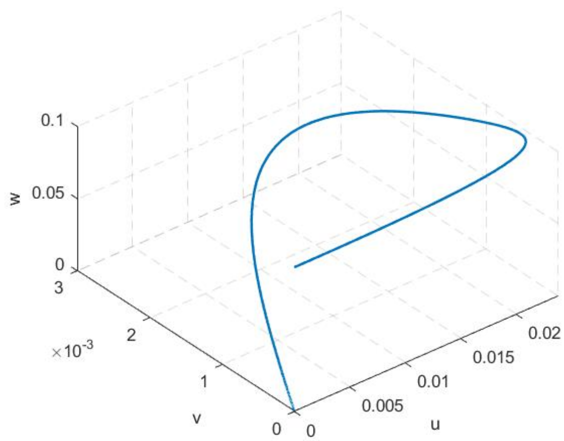

FIGURE 1. Phase portrait of system (3.4) with $c=4,\ k=2,\ d=1,\ \delta_1=0.5,\ \delta_2=3,\ \mu=1$.

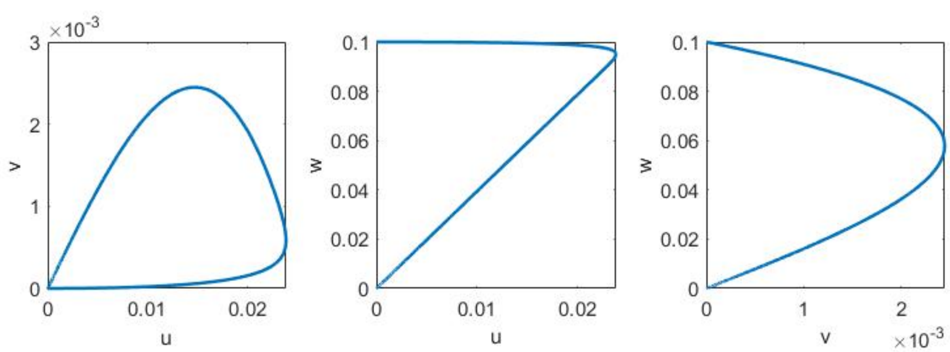

FIGURE 2. Projection portraits of system (3.4) with$c=4,\ k=2,\ d=1,\ \delta_1=0.5,\ \delta_2=3,\ \mu=1$.

Lemma 3.1. *For each $c > 0$, equilibrium point $E_1\,(0,0,0)$ is locally asymptotically stable. Specifically, if $0 < c < 2k\sqrt{d\mu}$, the linearization of system (3.4) at the equilibrium point E_1 exists at least one pair complex-conjugate eigenvalues whose real part is negative. While if $c \geq 2k\sqrt{d\mu}$, there exist all real and negative eigenvalues, i.e., the local stable manifold $W^s_{loc}(E_1)$ of system (3.4) is three-dimensional.*

Proof. It is evident that the linearization matrix for the system (3.4), which is restricted on $E_1\,(0,0,0)$, taking the form of

$$M_1 = \begin{pmatrix} -\dfrac{c}{dk^2} & 0 & \dfrac{1}{dk^2} \\ \dfrac{\delta_1}{c} & -\dfrac{\delta_2}{c} & 0 \\ -\mu & 0 & 0 \end{pmatrix}. \tag{3.5}$$

The characteristic equation is given as

$$\left(\tilde{\lambda} + \frac{\delta_2}{c}\right)\left(\tilde{\lambda}^2 + \frac{c}{dk^2}\tilde{\lambda} + \frac{\mu}{dk^2}\right) = 0. \tag{3.6}$$

Thus if $0 < c < 2k\sqrt{d\mu}$, there exists at least one pair complex-conjugate eigenvalues whose real part is negative. Obviously if $c \geq 2k\sqrt{d\mu}$, the eigenvalues of the equation (3.6) are all real and negative. That is to say, the local stable manifold $W^s_{loc}(E_1)$ of system (3.4) is three-dimensional when $c \geq 2k\sqrt{d\mu}$. $\square$

Lemma 3.2. *For each $c > 0$ and $\delta_1 < d\mu$, equilibrium point $E_2\left(1, \dfrac{\delta_1}{\delta_2}, c\right)$ is hyperbolic. Furthermore, the local unstable manifold $W^u_{loc}(E_2)$ for system (3.4) is one-dimensional.*

Proof. The linearized matrix of the system (3.4) at the equilibrium point $E_2\left(1, \dfrac{\delta_1}{\delta_2}, c\right)$, is given as

$$M_2 = \begin{pmatrix} \dfrac{1}{dk^2}\left(-c + \dfrac{\delta_1}{c}k^2\right) & -\dfrac{\delta_2}{cd} & \dfrac{1}{dk^2} \\ \dfrac{\delta_1}{c} & -\dfrac{\delta_2}{c} & 0 \\ \mu & 0 & 0 \end{pmatrix}. \tag{3.7}$$

The corresponding characteristic polynomial is

$$\mathcal{P}(\lambda) = -\lambda^3 - \frac{\delta_2 dk^2 + c^2 - \delta_1 k^2}{cdk^2}\lambda^2 - \frac{\delta_2 - \mu}{dk^2}\lambda + \frac{\mu\delta_2}{cdk^2}. \tag{3.8}$$

Let $\lambda_1, \lambda_2, \lambda_3$ be roots of $\mathcal{P}(\lambda) = 0$. Note that

$$\lambda_1\lambda_2\lambda_3 = \frac{\mu\delta_2}{cdk^2} > 0,$$

then the roots $\lambda_1, \lambda_2, \lambda_3$ may satisfy the following cases:

- **Case 1:** $\operatorname{Re}\lambda_1 > 0, \quad \operatorname{Re}\lambda_2 > 0, \quad \lambda_3 > 0,$
- **Case 2:** $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 = 0, \quad \operatorname{Im}\lambda_1 = -\operatorname{Im}\lambda_2 \neq 0, \quad \lambda_3 > 0,$
- **Case 3:** $\operatorname{Re}\lambda_1 < 0, \quad \operatorname{Re}\lambda_2 < 0, \quad \lambda_3 > 0.$

In order to rule out Case 1 and Case 2 above, we make the assumption that Case 1 and Case 2 are true.

At first, for $0 < \delta_2 \leq \mu$, if Case 1 and Case 2 are true, $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$ must be positive. However, we deduce that

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \frac{\delta_2 - \mu}{dk^2} \leq 0,$$

which produces a contradiction. Thus Case 1 and Case 2 are excluded when $0 < \delta_2 \leq \mu$.

Then, for $0 < \mu < \delta_2$, if Case 1 is true, we have

$$\mathcal{P}(-\lambda) = \lambda^3 - \frac{\delta_2 dk^2 + c^2 - \delta_1 k^2}{cdk^2}\lambda^2 + \frac{\delta_2 - \mu}{dk^2}\lambda + \frac{\mu\delta_2}{cdk^2}, \tag{3.9}$$

which have negative real parts. Subsequently, according to the Hurwitz algorithm, we obtain

$$\delta_2 dk^2 + c^2 - \delta_1 k^2 < 0, \tag{3.10}$$

and

$$-(\delta_2 dk^2 + c^2 - \delta_1 k^2) > \frac{\delta_2 - \mu}{dk^2}\mu\delta_2.$$

Then based on $\delta_1 < d\mu$, we have

$$\delta_1 k^2 - d\mu k^2 \leq c^2 < \delta_1 k^2 - d\delta_2 k^2 \implies \mu > \delta_2.$$

This is a contradiction to the condition of $0 < \mu < \delta_2$. Thereby Case 1 is excluded.

Again, assuming that Case 2 holds, it follows from polynomial (3.8) that

$$\lambda_1 + \lambda_2 + \lambda_3 = \lambda_3 = -\frac{\delta_2 dk^2 + c^2 - \delta_1}{cdk^2} > 0,$$

which implies (3.10). Similarly, we derive a contradiction to the condition of $0 < \mu < \delta_2$. From the analysis above, we accordingly get the conclusion that for each $c > 0$, $\mathcal{P}(\lambda)$ has the unique positive root. □

3.2. **Construction of the invariant region.**

The following discussion is focused on the case of $c \geq 2k\sqrt{d\mu}$. Defining the region

$$\Omega = \{(U, V, W) : 0 \leq U \leq \frac{\delta_2}{\delta_1}V \leq 1,\ 0 \leq W \leq cU\},$$

which is surrounded by curves S_i $(i = 1, \cdots, 4)$, respectively

$$S_1 = \{(U, V, W) : W = 0,\ 0 \leq U \leq \frac{\delta_2}{\delta_1}V \leq 1\},$$
$$S_2 = \{(U, V, W) : W = cU,\ 0 \leq U \leq \frac{\delta_2}{\delta_1}V \leq 1\},$$
$$S_3 = \{(U, V, W) : U = \frac{\delta_2}{\delta_1}V,\ 0 \leq U \leq 1,\ 0 \leq W \leq cU\},$$
$$S_4 = \{(U, V, W) : V = \frac{\delta_1}{\delta_2},\ 0 \leq U \leq 1,\ 0 \leq W \leq cU\},$$

which is shown in Figure 3. Notice that in order to establish a heteroclinic orbit remains in Ω, and naturally obtain solutions of system (3.4). This leads to the following lemma.

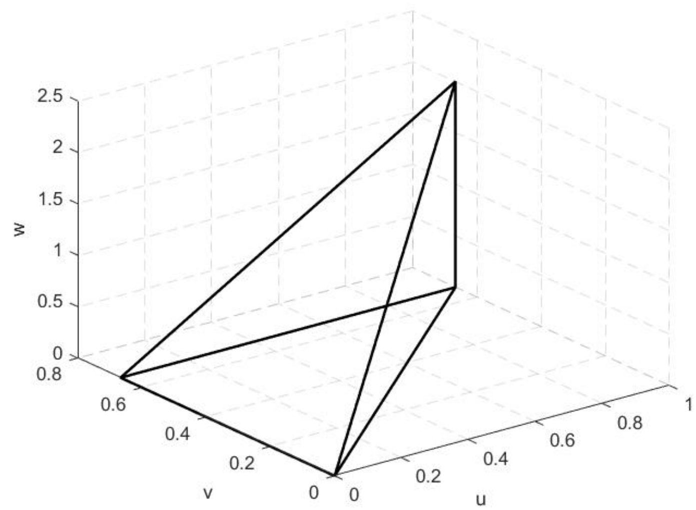

FIGURE 3. The region Ω surrounded by curves S_i $(i = 1, \cdots, 4)$.

Lemma 3.3. *For each $c \geq 2k\sqrt{d\mu}$ and $\delta_1 < d\mu$, the tangent vector of local unstable manifold $W^u_{loc}(E_2)$ points to the interior of the region Ω. Additionally, the unstable manifold can only go away the region Ω by crossing $w = 0$.*

Proof. The face S_1 has an outer normal vector $\overrightarrow{n_1} = (0, 0, -1)$. For any point on the face S_1, one has

$$f \cdot \overrightarrow{n_1} = \mu U(1 - U) \geq 0.$$

The aim is to explore the dynamics on four faces S_i $(i = 1, \cdots, 4)$ making up the boundary of the region Ω, and the results imply that the flow entering the region Ω can only leave via crossing the face S_1.

In fact, the face S_2 has an outer normal vector $\overrightarrow{n_2} = (-c, 0, 1)$. For any point on the face S_2, it leads to

$$\begin{aligned}
f \cdot \overrightarrow{n_2} &= \mu U(U-1) - \frac{c}{dk^2}\left(W - cU + \frac{k^2}{c}U\left(\delta_1 U - \delta_2 V\right)\right) \\
&= \mu U(U-1) - \frac{c}{dk^2}\left(\frac{k^2}{c}U\left(\delta_1 U - \delta_2 V\right)\right) \\
&= \mu U(U-1) + \frac{1}{d}U(\delta_2 V - \delta_1 U) \\
&\leq \mu U(U-1) + \frac{1}{d}U(\delta_1 - \delta_1 U) \\
&= U(U-1)\left(\frac{\delta_1}{d} - \mu\right) \\
&\leq 0.
\end{aligned}$$

The inequalities hold when $\delta_1 < d\mu$. Thus the unstable manifold of $E_2\left(1, \frac{\delta_1}{\delta_2}, c\right)$ can not leave the region Ω through the face S_2.

Similarly, the face S_3 has an outer normal vector $\overrightarrow{n_3} = (1, -\dfrac{\delta_2}{\delta_1}, 0)$ and the face S_4 has an outer normal vector $\overrightarrow{n_4} = (0, 1, 0)$. For any point on the face S_3 or S_4, it results in, respectively

$$\begin{aligned} f \cdot \overrightarrow{n_3} &= \frac{1}{dk^2}\left(W - cU + \frac{k^2}{c}U(\delta_1 U - \delta_2 V)\right) - \frac{\delta_2}{c\delta_1}(\delta_1 U - \delta_2 V) \\ &= \frac{1}{dk^2}(W - cU) \le 0, \end{aligned}$$

and

$$f \cdot \overrightarrow{n_4} = \frac{1}{c}(\delta_1 U - \delta_2 V) = \frac{\delta_1}{c}(U - 1) \le 0.$$

Thus the unstable manifold of $E_2\left(1, \dfrac{\delta_1}{\delta_2}, c\right)$ can not leave the region Ω through the faces S_3 and S_4. This concludes the proof. □

According to Lemma 3.3, in order to guarantee that the unstable manifold of equilibrium point E_2 never leaves the region Ω, it promotes us to seek a lower boundary which cannot be crossed by solutions of system (3.4). Setting $w = \varrho(U, V)$, which satisfies

- **Condition (a):** on the surface $w = \varrho(U, V)$, $\frac{d}{dt}(W - \varrho(U, V)) \ge 0$ along trajectories of system (3.4)
- **Condition (b):** for all $0 \le U \le \dfrac{\delta_2}{\delta_1}V \le 1$, $0 \le \varrho(U, V) \le cU$
- **Condition (c):** $\varrho(0, 0) = 0$.

Based on the conditions above, the following lemma reveals the design of $\varrho(U, V)$.

Lemma 3.4. *For each $c \ge 2k\sqrt{d\mu}$, the surface $w = \varrho(U, V)$ can be constructed to guarantee that the solutions of system (3.4) cannot leave through the bottom of the region Ω.*

Proof. Taking $\varrho(U, V) = \eta U$, where $\eta > 0$ has not be specified. It is obvious that Condition (c) is satisfied immediately. In order to verify Condition (a),

$$W' - \varrho_U U' - \varrho_V V' = \mu U(U - 1) - \eta \frac{1}{dk^2}\left(W - cU + \frac{k^2}{c}U(\delta_1 U - \delta_2 V)\right) \ge 0. \quad (3.11)$$

Rearranging (3.11), one has

$$\eta^2 + \left(\frac{k^2}{c}(\delta_1 U - \delta_2 V) - c\right)\eta + dk^2\mu(1 - U) \le 0. \quad (3.12)$$

This is a quadratic expression. Consequently, to satisfy expression (3.12), the nonnegative discriminant is required, i.e.,

$$\left(\frac{k^2}{c}(\delta_1 U - \delta_2 V) - c\right)^2 - 4dk^2\mu(1 - U) \ge 0. \quad (3.13)$$

For all $0 \leq U \leq \dfrac{\delta_2}{\delta_1}V \leq 1$, it directly leads to $\delta_1 U - \delta_2 V \leq 0$. Hence, $\dfrac{k^2}{c}(\delta_1 U - \delta_2 V) - c$ can be minimized over all relevant $0 \leq U \leq \dfrac{\delta_2}{\delta_1}V \leq 1$ to reach that

$$c^2 \geq 4dk^2\mu(1-U) \Longrightarrow c^2 \geq 4dk^2\mu,$$

which results in $c \geq 2k\sqrt{d\mu}$. Hence, we can choose an η that satisfies the expression (3.12) and $0 < \eta \leq c$. It is obvious that for all $c \geq 2k\sqrt{d\mu}$, the discriminant (3.13) is nonnegative, then the quadratic expression of η has two real roots for all $0 \leq U \leq \dfrac{\delta_2}{\delta_1}V \leq 1$. Taking $\eta = \eta^*$ to be the critical point of expression (3.12). The location of η^* is given by

$$0 \leq \eta^* = \frac{c}{2} - \frac{k^2}{2c}(\delta_1 U - \delta_2 V) \leq \frac{c}{2} + \frac{k^2}{2c}\delta_2 v.$$

In fact, due to $\delta_2 V < \delta_1 < d\mu < \dfrac{c^2}{4k^2}$, then

$$\eta^* \leq \frac{c}{2} + \frac{k^2}{2c} \cdot \frac{c^2}{4k^2} = \frac{5c}{8} < c,$$

making the surface $\varrho(U) < cU$. This completes the proof. $\square$

As a consequence, we can choose an $\eta^* = \dfrac{3}{8}c$ satisfying Condition (a),(b),(c), i.e.,

$$S_1' = \{(U, V, W) : W = \frac{3}{8}cU,\ 0 \leq U \leq \frac{\delta_2}{\delta_1}V \leq 1\}.$$

Then based on the above lemmas, we can define a new invariant region Ω^* which is surrounded by curves S_1' and S_i $(i = 2, 3, 4)$, which is directly presented in Figure 4. It directly leads to the major result as follows.

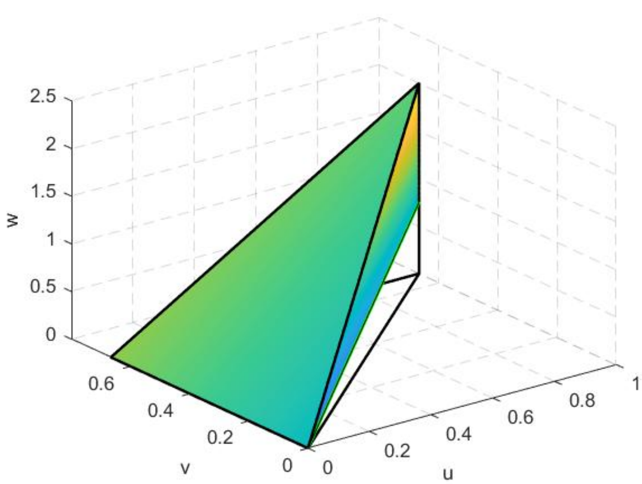

FIGURE 4. The region Ω^* surrounded by curves S_1' and S_i $(i = 2, 3, 4)$.

Theorem 3.1. *For all $c \geq 2k\sqrt{d\mu}$, the degenerate time fractional Keller-Segel system has a heteroclinic orbit from $E_2\left(1, \frac{\delta_1}{\delta_2}, c\right)$ to $E_1(0,0,0)$, which remains in the region Ω for all $\xi \in \mathbb{R}$, that is to say, the time fractional Keller-Segel system (1.5) with a small parameter $\varepsilon = 0$ exists traveling wave solutions.*

Remark 3.1. *For Keller-Segel system with integer order derivative, we can choose the transformation*

$$U(x,t) = U\left(kx - ct\right) = U(\xi).$$

If the time fractional derivative is defined by Definition 2.2, the transformation

$$U(x,t) = U\left(\frac{kx}{\Gamma(1+\alpha)} - \frac{ct}{\Gamma(1+\alpha)}\right) = U(\xi),$$

will be involved. If the time fractional derivative is defined by Definition 2.3, the transformation

$$U(x,t) = U\left(kx - \frac{\lambda}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta}\right) = U(\xi),$$

will be preferred. Consequently, the partial differential equations can equivalently be converted into ordinary differential equations. The existence of traveling waves in these Keller-Segel systems can be similarly demonstrated.

4. Traveling wave solutions for time fractional Keller-Segel system with a small parameter.

In this part, we concentrate on the time fractional Keller-Segel system (1.5) with the small parameter $\varepsilon > 0$. Undoubtedly, geometric singular perturbation theory plays a critical role in elucidating the existence of traveling wave solutions.

4.1. **Perturbation Analysis.**

Substituting the transformation (3.2) into system (1.5), then we obtain equations

$$\begin{cases} -cU' = dk^2 U'' - k^2\left(UV'\right)' + \mu U(1-U), \\ -cV' = \varepsilon k^2 V'' - \delta_1 U + \delta_2 V, \end{cases} \tag{4.1}$$

where $' = \frac{d}{d\xi}$. Introducing $W = dk^2U' - k^2UV' + cU$, the system (4.1) is equivalent to the following system

$$\begin{cases} U' = \frac{1}{dk^2}\left(W + k^2UY - cU\right), \\ V' = Y, \\ W' = \mu U(U-1), \\ \varepsilon Y' = \frac{1}{k^2}\left(\delta_1 U - \delta_2 V - cY\right), \end{cases} \tag{4.2}$$

where $' = \frac{d}{d\xi}$ and system (4.2) is regarded as the singularly perturbed system. Obviously, when $\varepsilon \to 0$, system (4.2) will reduce to system (3.4). Actually, when $\varepsilon > 0$, system (4.2) can be turned into a parallel problem through making the time scale transformation. Choosing $\xi{=}\varepsilon z$, then slow system (4.2) can be converted into fast system

$$\begin{cases} \dot{U}{=}\dfrac{\varepsilon}{dk^2}\left(W+k^2UY-cU\right), \\ \dot{V}{=}\varepsilon Y, \\ \dot{W}{=}\varepsilon\mu U(U-1), \\ \dot{Y}{=}\dfrac{1}{k^2}\left(\delta_1 U-\delta_2 V-cY\right), \end{cases} \tag{4.3}$$

where $\dot{} = \frac{d}{dz}$. When $\varepsilon > 0$, the the fast system (4.3) and slow system (4.2) are equivalent. However, different time scales generate different limiting systems. Setting $\varepsilon \to 0$ in slow system (4.2), the reduced system is given as

$$\begin{cases} U'{=}\dfrac{1}{dk^2}\left(W+k^2UY-cU\right), \\ V'{=}Y, \\ W'{=}\mu U(U-1), \\ 0{=}\dfrac{1}{k^2}\left(\delta_1 U-\delta_2 V-cY\right). \end{cases} \tag{4.4}$$

Setting $\varepsilon \to 0$ in system (4.3), we obtain the layer system

$$\begin{cases} \dot{U}{=}0, \\ \dot{V}{=}0, \\ \dot{W}{=}0, \\ \dot{Y}{=}\dfrac{1}{k^2}\left(\delta_1 U-\delta_2 V-cY\right). \end{cases} \tag{4.5}$$

Then solutions in system (4.4) are constrained to the following set

$$M_0 = \left\{(U,V,W,Y) \in \mathbb{R}^4 : Y = \frac{\delta_1}{c}U - \frac{\delta_2}{c}V\right\}.$$

Based on geometric singular perturbation theory of Fenichel [18, 24], if M_0 is normally hyperbolic, a invariant manifold M_ε of three-dimensional when $0 < \varepsilon \ll 1$ can be obtained, so that the existence of the slow manifold will also be established.

By a direct calculation, the linearization matrix of fast system (4.3) restrained to M_0 is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{\delta_1}{k^2} & -\dfrac{\delta_2}{k^2} & 0 & -\dfrac{c}{k^2} \end{pmatrix}.$$

The matrix has four eigenvalues: $0, 0, 0, -\dfrac{c}{k^2}$. There are three zero eigenvalues, and M_0 is a three-dimensional manifold. Therefore, M_0 is a normally hyperbolic manifold .

Afterwards, the theorem proposed by Fenichel [18] due to Jones [24] can be applied.

Lemma 4.1. ([18, 24]) *For the system with a small real parameter* $0 < \varepsilon \ll 1$,

$$\begin{cases} \dfrac{dx}{dt} = \phi(x, y, \varepsilon), \\ \dfrac{dy}{dt} = \varepsilon\psi(x, y, \varepsilon), \end{cases} \tag{4.6}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^l$, $n,\ l \geq 1$. *The functions* ϕ *and* ψ *are both assumed to be* C^∞. *If* $\varepsilon > 0$, *define* M_0 *to be a compact (possibly with boundary), normally hyperbolic critical manifold and given as a graph* $\{(x, y) : x = h^0(y)\}$. *If* $\varepsilon > 0$ *is sufficiently small, then*

(I) there exists a manifold M_ε, *which is* C^r *and locally invariant under the flow for system (4.6);*

(II) for some C^r *function* $h^\varepsilon(y)$, M_ε *is given as graph:*

$$M_\varepsilon = \{(x, y) : x = h^\varepsilon(y)\};$$

(III) there exist locally invariant stable manifold $W^s(M_\varepsilon)$ *and unstable manifold* $W^u(M_\varepsilon)$ *of the slow manifold* M_ε, *which are lying* $O(\varepsilon)$ *and* C^r *diffeomorphic to the stable manifold* $W^s(M_0)$ *and unstable manifold* $W^u(M_0)$ *of the critical manifold* M_0, *respectively.*

According to Lemma 4.1, for $\varepsilon > 0$, the three-dimensional manifold M_ε is $O(\varepsilon)$ close and diffeomorphic to M_0, which is given as

$$M_\varepsilon = \left\{ (U, V, W, Y) \in \mathbb{R}^4 : Y = \frac{\delta_1}{c}U - \frac{\delta_2}{c}V + g(U, V, W, \varepsilon) \right\}.$$

The function g is smooth and determined on a compact domain, moreover,

$$g(U, V, W, 0) = 0.$$

The function g can be expanded in the formation of the Taylor series relative to the small parameter ε, i.e.,

$$g(U, V, W, \varepsilon) = \varepsilon g_1(U, V, W) + \varepsilon^2 g_2(U, V, W) + \cdots. \tag{4.7}$$

Substituting

$$Y = \frac{\delta_1}{c}U - \frac{\delta_2}{c}V + g(U, V, W, \varepsilon),$$

into slow system (4.2), one has

$$\begin{aligned} \varepsilon \Bigg\{ \left(\frac{\delta_1}{c} + \frac{\partial g}{\partial U} \right) \frac{1}{dk^2} \left[W - cU + k^2 U \left(\frac{\delta_1}{c}U - \frac{\delta_2}{c}V + g \right) \right] + \left(-\frac{\delta_2}{c} + \frac{\partial g}{\partial V} \right) \\ \left(\frac{\delta_1}{c}U - \frac{\delta_2}{c}V + g \right) + \frac{\partial g}{\partial W} \mu U(U-1) \Bigg\} = -\frac{c}{k^2} g. \end{aligned}$$

Comparing the coefficients of ε, we have

$$\begin{aligned} g_1 &= -\frac{\delta_1}{c^2 d} \left[W - cU + k^2 U \left(\frac{\delta_1}{c}U - \frac{\delta_2}{c}V \right) \right] + \frac{\delta_2 k^2}{c^2} \left(\frac{\delta_1}{c}U - \frac{\delta_2}{c}V \right) \\ &= -\frac{\delta_1}{c^2 d} (W - cU) + \frac{k^2}{c^3} \left(\delta_2 - \frac{\delta_1}{d}U \right) (\delta_1 U - \delta_2 V). \end{aligned} \tag{4.8}$$

Hence the slow system limited to M_ε, taking the form of

$$\begin{cases} U'=\dfrac{1}{dk^2}\left(W+k^2U\left(\dfrac{\delta_1}{c}U-\dfrac{\delta_2}{c}V+g\right)-cU\right), \\ V'=\dfrac{\delta_1}{c}U-\dfrac{\delta_2}{c}V+g, \\ W'=\mu U(U-1), \end{cases} \tag{4.9}$$

where g is presented in (4.7) and (4.8). It is evident that system (4.9) is simplified to system (3.4) when $\varepsilon \to 0$.

4.2. **Traveling wave solutions.**

The following aims to verify the existence of a heteroclinic orbit connecting equilibrium points E_1 and E_2. Accordingly, the time fractional Keller-Segel system (1.5) with sufficiently small parameter $\varepsilon > 0$ has traveling wave solutions.

Set

$$U=U_0+\varepsilon\phi_1+\cdots, V=V_0+\varepsilon\phi_2+\cdots, W=W_0+\varepsilon\phi_3+\cdots, \tag{4.10}$$

where (u_0, v_0, w_0) is the flow of system (3.4). Substituting transformation (4.10) into system (4.9), and comparing the coefficients of ε, then one has the differential system determining ϕ_1, ϕ_2 and ϕ_3

$$\begin{cases} \phi_1'=\dfrac{1}{dk^2}\left[\phi_3+k^2\phi_1\left(\dfrac{\delta_1}{c}U_0-\dfrac{\delta_2}{c}V_0\right)+k^2U_0\left(\dfrac{\delta_1}{c}\phi_1-\dfrac{\delta_2}{c}\phi_2+g_{10}\right)-c\phi_1\right], \\ \phi_2'=\dfrac{\delta_1}{c}\phi_1-\dfrac{\delta_2}{c}\phi_2+g_{10}, \\ \phi_3'=\mu(2U_0-1)\phi_1, \end{cases} \tag{4.11}$$

where

$$g_{10}=-\frac{\delta_1}{c^2d}\left(W_0-cU_0\right)+\frac{k^2}{c^3}\left(\delta_2-\frac{\delta_1}{d}U_0\right)\left(\delta_1U_0-\delta_2V_0\right).$$

It leads system (4.11) to transform into

$$\frac{d\Psi(\xi)}{d\xi}+\Gamma(\xi)\Psi(\xi)=\mathcal{Q}(\xi), \tag{4.12}$$

where

$$\Psi(\xi)=\begin{pmatrix}\phi_1(\xi)\\ \phi_2(\xi)\\ \phi_3(\xi)\end{pmatrix},\ \Gamma(\xi)=\begin{pmatrix} \dfrac{1}{dk^2}\left(c-\dfrac{2\delta_1}{c}U_0k^2+\dfrac{\delta_2}{c}V_0k^2\right) & \dfrac{\delta_2}{dc}U_0 & -\dfrac{1}{dk^2} \\ -\dfrac{\delta_1}{c} & \dfrac{\delta_2}{c} & 0 \\ \mu\left(1-2U_0\right) & 0 & 0 \end{pmatrix},$$

and

$$\mathcal{Q}(\xi)=\left(\frac{1}{d}U_0g_{10}, g_{10}, 0\right)^T.$$

In the following, we will find that system (4.12) exists a solution meeting

$$\phi_1(\pm\infty)=0,\ \phi_2(\pm\infty)=0,\ \phi_3(\pm\infty)=0.$$

Define

$$\mathcal{L} = \frac{d}{d\xi} + \Gamma(\xi),$$

and Euclidean inner production as

$$\int_{-\infty}^{+\infty} (\varphi_1(\xi),\ \varphi_2(\xi)) d\xi.$$

Base on the Fredholm theory, for all functions $\varphi_1(\xi) \in \mathbb{R}^3$, system (4.12) exists a solution if and only if

$$\int_{-\infty}^{+\infty} (\varphi_1(\xi),\ \mathcal{Q}(\xi)) d\xi = 0,$$

in the kernel of the adjoint for operator $\mathcal{L}$, i.e., $\mathcal{L}^*$, is given as

$$\mathcal{L}^* = -\frac{d}{d\xi} + \Gamma^T(\xi),$$

where

$$\Gamma^T(\xi) = \begin{pmatrix} \frac{1}{dk^2}\left(c - \frac{2\delta_1}{c}U_0 k^2 + \frac{\delta_2}{c}V_0 k^2\right) & -\frac{\delta_1}{c} & \mu\,(1-2U_0) \\ \frac{\delta_2}{dc}U_0 & \frac{\delta_2}{c} & 0 \\ -\frac{1}{dk^2} & 0 & 0 \end{pmatrix}.$$

Next we compute Ker$\mathcal{L}^*$, all $\varphi_1(\xi)$ satisfying $\mathcal{L}^*\varphi_1(\xi) = 0$ can be calculated by

$$\frac{d\varphi_1(\xi)}{d\xi} = \Gamma^T(\xi)\varphi_1(\xi). \tag{4.13}$$

As a result, the persistence question can be regarded as the solvability of equation (4.13). However, it is difficult in finding the general solution of equation (4.13) due to the matrix $\Gamma^T(\xi)$ is nonconstant. Nevertheless, it promotes us only to look for solutions satisfying $\varphi_1(\pm\infty) = 0$. As a matter of fact, the zero solution is the unique solution. There is no doubt that $U_0(\xi)$ is the solution for the unperturbed problem. Let $\xi \to +\infty$, the matrix $\Gamma^T(\xi)$ finally becomes the following constant matrix

$$\begin{pmatrix} \frac{c}{dk^2} & -\frac{\delta_1}{c} & \mu \\ 0 & \frac{\delta_2}{c} & 0 \\ -\frac{1}{dk^2} & 0 & 0 \end{pmatrix},$$

and all eigenvalues are real and negative for $c \geq 2k\sqrt{d\mu}$.

Similarly, through computing the eigenvalues of the matrix in (4.13) when $\xi \to -\infty$, thus we find that, the zero solution is the only solution satisfying $\varphi_1(\pm\infty) = 0$. It implies that the Fredholm orthogonality theorem naturally holds

$$\int_{-\infty}^{+\infty} (\varphi_1(\xi),\ \mathcal{Q}(\xi)) d\xi = \int_{-\infty}^{+\infty} (0,\ \mathcal{Q}(\xi)) d\xi = 0,$$

and solutions of system (4.12) exist, which satisfy

$$\phi_1(\pm\infty) = 0,\ \phi_2(\pm\infty) = 0,\ \phi_3(\pm\infty) = 0.$$

Thus for $\varepsilon > 0$ sufficiently small, there exists a heteroclinic orbit connecting equilibrium points E_1 and E_2. Additionally, the time fractional Keller-Segel system (1.5) with a small parameter exists traveling waves. Hence the following main result can be concluded.

Theorem 4.1. *For each $c \geq 2k\sqrt{d\mu}$ and $\varepsilon > 0$ sufficiently small, the time fractional Keller-Segel system (1.5) with a small parameter exists traveling wave solutions.*

5. Asymptotic behavior.

Using the asymptotic theory, the asymptotic behavior to traveling waves can be detailed described as follows.

Let $\Phi(\xi) = (U(\xi), V(\xi), W(\xi))^T$ be traveling wave solutions in the degenerate time fractional Keller-Segel system (3.4), which satisfies the boundary conditions

$$\begin{cases} U(+\infty) = 0,\ V(+\infty) = 0,\ W(+\infty) = 0, \\ U(-\infty) = 1,\ V(-\infty) = \dfrac{\delta_1}{\delta_2},\ W(-\infty) = c. \end{cases}$$

Differentiate the system (3.4) with respect to ξ and denote

$$\Phi'(\xi) = (U'(\xi), V'(\xi), W'(\xi))^T = \left(\tilde{U}(\xi), \tilde{V}(\xi), \tilde{W}(\xi)\right)^T,$$

we obtain

$$\begin{cases} \tilde{U}' = \dfrac{1}{dk^2}\left(\tilde{W} - c\tilde{U} + \dfrac{k^2}{c}\tilde{U}\left(\delta_1 U - \delta_2 V\right) + \dfrac{k^2}{c}u\left(\delta_1\tilde{U} - \delta_2\tilde{V}\right)\right), \\ \tilde{V}' = \dfrac{1}{c}\left(\delta_1\tilde{U} - \delta_2\tilde{V}\right), \\ \tilde{W}' = \mu\tilde{U}(U-1) + \mu U\tilde{U}. \end{cases} \tag{5.1}$$

Then let $\xi \to +\infty$ in system (5.1), one has

$$\begin{cases} \tilde{U}'_+ = \dfrac{1}{dk^2}\left(\tilde{W} - c\tilde{U}\right), \\ \tilde{V}'_+ = \dfrac{1}{c}\left(\delta_1\tilde{U} - \delta_2\tilde{V}\right), \\ \tilde{W}'_+ = -\mu\tilde{U}U, \end{cases}$$

which can be rewritten as

$$Z_1' = M_1 Z_1, \tag{5.2}$$

where M_1 is given in (3.5), and $Z_1 = \left(\tilde{U}_+(\xi), \tilde{V}_+(\xi), \tilde{W}_+(\xi)\right)^T$. For all $c \geq 2k\sqrt{d\mu}$, the eigenvalues $\tilde{\lambda}_i$ of M_1 is given as

$$\tilde{\lambda}_1 = -\frac{\delta_2}{c} < 0, \quad \tilde{\lambda}_2 = \frac{\sqrt{c^2 - 4dk^2\mu} - c}{2dk^2} < 0, \quad \tilde{\lambda}_3 = \frac{-\sqrt{c^2 - 4dk^2\mu} - c}{2dk^2} < 0. \tag{5.3}$$

The general solution in system (5.2) satisfies

$$\left(\tilde{U}_+(\xi),\tilde{V}_+(\xi),\tilde{W}_+(\xi)\right)^T=\sum_{i=1}^{3}\alpha_i p_i e^{\tilde{\lambda}_i\xi},\ \ i=1,2,3,$$

where $\tilde{\lambda}_i$ are eigenvalues of the matrix M_1, p_i are corresponding eigenvectors, and α_i are arbitrary constants.

Since $\xi\to+\infty$, $\left(\tilde{U}_+(\xi),\tilde{V}_+(\xi),\tilde{W}_+(\xi)\right)^T\to(0,0,0)^T$, the asymptotic behavior at this case can be deduced

$$\begin{pmatrix}\tilde{U}_+(\xi)\\ \tilde{V}_+(\xi)\\ \tilde{W}_+(\xi)\end{pmatrix}=\begin{pmatrix}\sum_{i=1}^3\sigma_i(\bar{\gamma}_i+o(1))e^{\tilde{\lambda}_i\xi}\\ \sum_{i=1}^3\sigma_i(\tilde{\gamma}_i+o(1))e^{\tilde{\lambda}_i\xi}\\ \sum_{i=1}^3\sigma_i(\hat{\gamma}_i+o(1))e^{\tilde{\lambda}_i\xi}\end{pmatrix},$$

where $\bar{\gamma}_i,\tilde{\gamma}_i,\hat{\gamma}_i$ are constants and σ_i cannot be zero simultaneously. If the first component of eigenvector p_i is zero, the matrix M_1 implies that the other components are zero, which implies the $\bar{\gamma}_i,\tilde{\gamma}_i,\hat{\gamma}_i\neq 0$.

Again letting $\xi\to-\infty$ in system (5.1), i.e.,

$$\begin{cases}\tilde{U}'_-=\dfrac{1}{dk^2}\left(\tilde{W}-c\tilde{U}+\dfrac{k^2}{c}\left(\delta_1\tilde{U}-\delta_2\tilde{V}\right)\right),\\ \tilde{V}'_-=\dfrac{1}{c}\left(\delta_1\tilde{U}-\delta_2\tilde{V}\right),\\ \tilde{W}'_-=\mu\tilde{U},\end{cases}$$

which can be rewritten as

$$Z_2'=M_2Z_2,\tag{5.4}$$

where M_2 is given in (3.7), and $Z_2=\left(\tilde{U}_-(\xi),\tilde{V}_-(\xi),\tilde{W}_-(\xi)\right)^T$. The general solution in system (5.4) satisfies

$$\left(\tilde{U}_-(\xi),\tilde{V}_-(\xi),\tilde{W}_-(\xi)\right)^T=\sum_{i=1}^{3}\tilde{\alpha}_i\tilde{p}_i e^{\lambda_i\xi},\ \ i=1,2,3,\tag{5.5}$$

where λ_i are eigenvalues of the matrix M_2, $\tilde{p}_i$ are corresponding eigenvectors, and $\tilde{\alpha}_i$ are arbitrary constants. According to Lemma 3.2, we find that, the eigenvalues λ_i of M_2 satisfy $\operatorname{Re}\lambda_1<0,\ \operatorname{Re}\lambda_2<0,\ \lambda_3>0$.

Since $\xi\to-\infty$, $\left(\tilde{U}_-(\xi),\tilde{V}_-(\xi),\tilde{W}_-(\xi)\right)^T\to(0,0,0)^T$. Based on (5.5), we have

$$\left(\tilde{U}_-(\xi),\tilde{V}_-(\xi),\tilde{W}_-(\xi)\right)^T=\tilde{\alpha}_3\tilde{p}_3e^{\lambda_3\xi},$$

thus the asymptotic behavior as $\xi\to-\infty$ can be deduced

$$\begin{pmatrix}\tilde{U}_-(\xi)\\ \tilde{V}_-(\xi)\\ \tilde{W}_-(\xi)\end{pmatrix}=\begin{pmatrix}\bar{\sigma}_3(\bar{\gamma}_3+o(1))e^{\lambda_3\xi}\\ \bar{\sigma}_3(\tilde{\gamma}_3+o(1))e^{\lambda_3\xi}\\ \bar{\sigma}_3(\hat{\gamma}_3+o(1))e^{\lambda_3\xi}\end{pmatrix},$$

where $\bar{\gamma}_3,\tilde{\gamma}_3,\hat{\gamma}_3\neq 0$ are constants and $\bar{\sigma}_3$ cannot be zero simultaneously.

Theorem 5.1. *For all $c \geq 2k\sqrt{d\mu}$, there exist positive constants $K_i (i = 1, 2, 3)$ and $N_i(1, 2, 3)$ such that the degenerate time fractional Keller-Segel system (1.5) exists a traveling wave solution $\Phi(\xi)$, whose asymptotic properties are respectively presented as*

$$\Phi(\xi) = \begin{pmatrix} (-K_1 + o(1))\, e^{\tilde{\lambda}_i \xi} \\ (-K_2 + o(1))\, e^{\tilde{\lambda}_i \xi} \\ (-K_3 + o(1))\, e^{\tilde{\lambda}_i \xi} \end{pmatrix}, \ \text{as } \xi \to +\infty,$$

and

$$\Phi(\xi) = \begin{pmatrix} 1 - (-N_1 + o(1))\, e^{\lambda_3 \xi} \\ \dfrac{\delta_1}{\delta_2} - (-N_2 + o(1))\, e^{\lambda_3 \xi} \\ c - (-N_3 + o(1))\, e^{\lambda_3 \xi} \end{pmatrix}, \ \text{as } \xi \to -\infty,$$

where $\tilde{\lambda}_i$ may be $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3$ given in (5.3).

Remark 5.1. *The asymptotic properties to traveling waves for the Keller-Segel system (1.5) with parameter $\varepsilon > 0$ sufficiently small can be explored in a similar discussion. It motivates us to focus on more complex characteristic equations and the details are omitted here.*

Remark 5.2. *Essentially, for Keller-Segel system with integer order derivative or other formations of fractional order derivative, the existence and asymptotic properties of traveling waves for Keller-Segel systems can be explored in a similar manner.*

Remark 5.3. *Compared with the research in* [5, 7, 14, 30]*, the asymptotic behavior to traveling waves are detailed established in the present study. In addition, although we pursue the heteroclinic orbits for the degenerate time fractional Keller-Segel system by constructing a suitable invariant region, which is slightly different. Noteworthy, the invariant region is three-dimensional and explicitly depicted, which is more complex, interesting and challenging.*

Remark 5.4. *Distinguished from* [5, 7, 14, 30]*, the time fractional derivative is introduced in this paper, which makes it a challenge to explore the complicated impact on the dynamics of Keller-Segel systems. Fractional derivatives play an important role in modeling chemotaxis in complex and nonhomogeneous media and mathematical biological models, and are more challenging to study compared to integer order derivatives.*

6. Conclusion.

To conclude, we systematically studied traveling waves for time fractional Keller-Segel system (1.5) without and with the small parameter ε. Such a problem is of great theoretical and practical importance, and has not been well studied yet in the literature. The results of this paper are new.

There are two major technical issues: the construction of the invariant region and the search for the heteroclinic orbit for the perturbed system. Obviously, geometric singular perturbation theory plays an essential role in seeking for the invariant manifold and establishing the existence of traveling waves. It is worth mentioning that the additional time fractional derivatives shall be applicable to reveal inherently more realistic and general phenomena than integer order derivatives.

The stability and non-existence of traveling waves for time fractional Keller-Segel systems are our future efforts. Furthermore, time-space fractional Keller-Segel systems also need further investigation.

References

[1] G. Arumugam, J. Tyagi, Keller-Segel Chemotaxis Models: A Review, *Acta Appl. Math.,* **171(1)** (2021).

[2] A. Atangana, Derivative with a New Parameter: Theory, Methods and Applications, *Academic Press, Elsevier,* (2015).

[3] A. Atangana, R. T. Alqahtani, New numerical method and application to Keller-Segel model with fractional order derivative, *Chaos Solitons Fractals,* **116** (2018) 14-21.

[4] N. Bellomo, A. Bellouquid, Y. Tao, M. Winkler, Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, *Math. Models Methods Appl. Sci.*, **25** (2015) 1663-1763.

[5] J.J. Bramburger, Exact minimum speed of traveling waves in a Keller-Segel model, *Appl. Math. Lett.,* **111** (2021) 106594.

[6] J. Burczak, R.G. Belinchón, Suppression of blow up by a logistic source in 2D Keller-Segel system with fractional dissipation, *J. Differential Equations*, **263(9)** (2017) 6115-6142.

[7] C. Chang, Y. Chen, J.M. Hong, B.C. Huang, Existence and instability of traveling pulses of Keller-Segel system with nonlinear chemical gradients and small diffusions, *Nonlinearity,* **32** (2019) 143-167.

[8] X. Cheng, L. Wang, J. Hou, Solving time fractional Keller-Segel type diffusion equations with symmetry analysis, power series method, invariant subspace method and q-homotopy analysis method, *Chinese J. Phys.,* **77** (2022) 1639-1653.

[9] C. Cuevas, C. Silva, H. Soto, On the time-fractional Keller-Segel model for chemotaxis, *Math. Meth. Appl. Sci.,* **43** (2020) 769-798.

[10] F. Dai, B. Liu, Boundedness and asymptotic behavior in a Keller-Segel(-Navier)-Stokes system with indirect signal production, *J. Differential Equations*, **314** (2022) 201-250.

[11] J.I. Diaz, T. Nagai, J.M. Rakotoson, Symmetrization techniques on unbounded domains: Application to a chemotaxis system on $\mathbf{R}^N$, *J. Differential Equations*, **145** (1998) 156-183.

[12] Z. Du, J. Li, Geometric singular perturbation analysis to Camassa-Holm Kuramoto-Sivashinsky equation, *J. Differential Equations*, **306** (2022) 418-438.

[13] Z. Du, J. Li, X. Li, The existence of solitary wave solutions of delayed Camassa-Holm equation via a geometric approach, *J. Funct. Anal.*, **275** (2018) 988-1007.

[14] Z. Du, J. Liu, L. Ren, Traveling pulse solutions of a generalized Keller-Segel system with small cell diffusion via a geometric approach, *J. Differential Equations*, **270** (2021) 1019-1042.

[15] Z. Du, Q. Qiao, The Dynamics of traveling waves for a nonlinear Belousov-Zhabotinskii system, *J. Differential Equations*, **269** (2020) 7214-7230.

[16] C. Escudero, Chemotactic collapse and mesenchymal morphogenesis, *Phys. Rev. E,* **72(2)** (2005) 022903.

[17] C. Escudero, The fractional Keller-Segel model, *Nonlinearity,* **19(12)** (2006) 2909-2918.

[18] N. Fenichel, Geometric singular perturbation theory for ordinary differential equations, *J. Differential Equations*, **31** (1979) 53-98.

[19] A. Garfinkel, Y. Tintut, D. Petrasek, K. Bostrom, L.L. Demer, Pattern formation by vascular mesenchymal cells, *Proc. Natl. Acad. Sci.,* **101(25)** (2004) 9247-9250.

[20] G. Hek, Geometric singular perturbation theory in biological practice, *J. Math. Biol.*, **60** (2010) 347-386.

[21] T. Hillen, K.J. Painter, A user's guide to PDE models for chemotaxis, *J. Math. Biol.*, **58(1-2)** (2009) 183-217.

[22] D. Horstmann, From 1970 until present: The Keller-Segel model in chemotaxis and its consequences I, *Jahresber. Deutsch. Math.-Verein.,* **105** (2003) 103-165.

[23] D. Horstmann, From 1970 until present: The Keller-Segel model in chemotaxis and its consequences II, *Jahresber. Deutsch. Math.-Verein.,* **106** (2004) 51-69.

[24] C.K.R.T. Jones, Geometric singular perturbation theory, *In: Dynamical Systems. Lecture Notes in Mathematics,* **1609** (1995) 44-118.

[25] G. Jumarie, Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results, *Comput. Math. Appl.*, **51(9-10)** (2006) 1367-1376.

[26] R. Khalil, M. Al Horani, A. Yousef, M. Sababhehb, A new definition of fractional derivative, *J. Comput. Appl. Math.*, **264** (2014) 65-70.

[27] E.F. Keller, L.A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.*, **26** (1970) 399-415.

[28] E.F. Keller, L.A. Segel, Traveling bands of chemotactic bacteria: a theoretical analysis, *J. Theoret. Biol.*, **30** (1971) 377-380.

[29] E.F. Keller, L.A. Segel, Model for chemotaxis, *J. Theoret. Biol.*, **30** (1971) 225-234.

[30] T. Li, J. Park, Traveling waves in a chemotaxis model with logistic growth, *Discrete Contin. Dyn. Syst. Ser. B,* **24(12)** (2019) 6465-6480.

[31] H. Liu, H. Khan, R. Shah , A.A. Alderremy, S. Aly, D. Baleanu, On the fractional view analysis of Keller-Segel equations with sensitivity functions, *Complexity,* **2020** (2020).

[32] K.S. Miller, B. Ross, An introduction to the fractional calculus and fractional differential equations, *Wiley, New York,* 1993.

[33] M. Negreanu, J.I. Tello, On a parabolic-elliptic system with gradient dependent chemotactic coefficient, *J. Differential Equations*, **265** (2018) 733-751.

[34] K.B. Oldham, J. Spanier, The Fractional Calculus, Theory and Applications of Differentiation and Integration of Arbitrary Order, *Academic Press, USA,* (1974).

[35] I.M. Stamova, G.T. Stamov, Functional and impulsive differential equations of fractional order: qualitative analysis and applications, *CRC Press, Taylor and Francis Group, Boca Raton, USA,* (2017).

[36] J. Shen, X. Zhang, Travelling pulses in a coupled FitzHugh-Nagumo equation, *Phys. D,* **418** (2021) 132848.

[37] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, *J. Math. Pures Appl.,* **100** (2013) 748-767.

[38] M. Winkler, Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening, *J. Differential Equations*, **257(4)** (2014) 1056-1077.

[39] M. Winkler, A three-dimensional Keller-Segel-Navier-Stokes system with logistic source: global weak solutions and asymptotic stabilization, *J. Funct. Anal.*, **276(5)** (2019) 1339-1402.

[40] M. Winkler, Blow-up profiles and life beyond blow-up in the fully parabolic Keller-Segel system, *J. Anal. Math.,* **141** (2020) 585-624.

[41] X.J. Yang, J.A.T. Machado, A new fractional operator of variable order: application in the description of anomalous diffusion, *Phys. A: Stat. Mech. Appl.,* **481** (2017) 276-283.

[42] W. Zhang, Z. Liu, L. Zhou, Global existence and asymptotic behavior of classical solutions to a fractional logistic Keller-Segel system, *Nonlinear Anal.,* **189** (2019) 111624.

[43] L.H. Zhang, F.S. Xu, Conservation laws, symmetry reductions, and exact solutions of some Keller-Segel models, *Adv. Differential Equations,* **327** (2018).