Anomalous Diffusion Equation Modeled by the Joint Use of Domain Boundary Element Method and Analytical Derived Solution Based on Green Equation

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Abstract

Mathematical formulation of the diffusion phenomenon might be described through a differential equation, which takes into account complementary and different effects with respect to the physical processes simulated with the support of the Fick´s equation, which is usually adopted to represent the diffusion process. In particular, diffusion applied to spatio-temporal retention problems with bimodal mass transmission are highlighted. To better understand this physical phenomenon, the proper use of the analytical Green function (GF) or the steady-state fundamental solution was investigated. In this case, we use the Boundary Element Method formulation is presented for the solution of the anomalous diffusion equation for one-dimensional problems. The formulation employs the steady-state fundamental solution. Besides the basic integral equation, another one is required, due to the fourth-order differential operator in the differential equation of the problem. The domain discretization employs linear cells. The first order time derivative is approximated by a backward finite difference scheme. Two examples are presented. Numerical results are compared with analytical solutions, showing good agreement between them.

Key-words: Anomalous diffusion, boundary element method, steady-state fundamental solution, Green function, fourth-order differential operator and finite difference scheme.

1. Introduction

The main motivation towards the solution of the problem of diffusion with retention, as reported by Jiang (2017), was the analysis of population dynamics and its impact on ecological systems (Simas, 2012). In order to make the concepts clearer, one could think of a given population marching to occupy a certain territory.

If such invasion progresses along an aisle, just in one direction, the process is equivalent to the propagation of a wave front. However, if the invading population finds a weak breaking point in the middle of the native population, the invasion could be modelled like a diffusion process. This is the case also of tumors spreading into a living organism. The main difference in the analytical expression for the wave propagation in contrast to the diffusion problem appears if retention modelling intends to take into account the colonizers moving slowly to settle down on the new territory.

On the other hand, several mathematical formulations have been proposed to study different processes of mass transport considering time delay phenomena retention effects (D'Angelo et al., 2003; Deleersnijder et al., 2006; Huang &. Madey, 1982; Kindler et al., 2010; Ferreira et al., 2010; Jianhong & Xingfu, 2001). Diffusion phenomena in fractal media also has been receiving attention, as can be seen in Mainardi (1996) and Scherer et al. (2008).

New physical approaches have been adopted to represent the transport of contaminants in groundwater, as shown in LaBolle et al. (2006). They adopted fractional diffusion equation to represent small isotopic effects on aqueous diffusion coefficients. The authors showed that diffusion can result in similar degrees of depletion and enrichment of isotopically heavy solutes during transport in heterogeneous systems with significant diffusion rate implying limited mass transfer between fast and slow-flow zones. Other works showed that the diffusion process can dominate transport within low-permeability materials, e.g., silts, clays, and rock, wherein the movement of groundwater is relatively slow see LaBolle et al. (2006); Maloszewski & Zuber (1990) (1991); Zhang et al., (2007); Haggerty et al., (2004).

The anomalous diffusion equation, which is sometimes referred to as the so-called fourth order diffusion equation was presented previously by Bevilacqua et al. (2011), (2012) and Bevilacqua et al. 2013.

In this work, the Domain Boundary Element Method (D-BEM) is explored to model that anomalous diffusion process taking into consideration that we were also able to originally develop the Green analytical solutions for the fourth order diffusion equation (see Saito, 2018; Saito et al., 2019). Such combination of approaches proves to establish a new conceptual reference in this area.

In fact, under the knowledge of the authors, this paradigmatic step, as we believe to be, is the first attempt ever made to solve the problem by means of the BEM implemented based on our original derived analytical solutions. In this paper, we address 1-D problems for showing the successful results achieved.

Given this framework, it can be said that, once a fundamental solution corresponding to the steady state problem was obtained, a D-BEM type formulation was then developed. As it is well-known, such kind of formulations present a domain integral whose integrand is, for the problem at hand, the fundamental solution multiplied by the first order time derivative of the variable of interest, or variable of the problem e.g., Carrer et al. 2009.

As the problem presents two natural boundary conditions, which are made upof the derivatives of order two and three of the problem variable, and two essential boundary conditions, namely the problem variable and its first order derivative, the basic BEM integral equation alone is not sufficient for providing the solution of the problem. In this way, similarly to what has been done to the problem of flexural analysis of beams (see Scuciato, et al., 2016), another BEM equation turns to be necessary. Such equation is that related to the first order derivative of the problem variable, and it is obtained by taking the derivative of the basic BEM equation with respect to the source point coordinate. Thus, a set of two integral equations is obtained and the problem can be solved appropriately. The domain integrals that remain in the system of equations are computed through domain discretization. Such a discretization employs linear cells, over which the first order time derivative of the variable of interest is assumed to vary linearly. The time-marching, by its turn, is carried out by simply employing a backward finite-difference scheme (see Smith, 1985).

One can assume that the domain of the problem is within the interval [0,L]. Consequently, its boundary is represented by the nodes at x = 0 and at x = L. Two examples are presented in this work, in which DBEM results are compared to available analytical solutions, showing an excellent accuracy and adherence.

2. The Anomalous Diffusion Equation

The anomalous diffusion equation, as presented by Bevilacqua et al. (2011), (2012) and Bevilacqua et al. 2013, reads:

$$\beta D \frac{\partial^2 v(x,t)}{\partial x^2} - (1-\beta)\beta R \frac{\partial^4 v(x,t)}{\partial x^4} = \frac{\partial v(x,t)}{\partial t} \qquad (1)$$

Equation (1) was obtained by considering a bi-modal flux distribution for the diffusion process associated with two energy states. The parameter β indicates the fraction of the particles in the main energy state, and the parameter R controls the effect of the secondary flux. Complementarily, D is the usual diffusion coefficient. The fourth order term with negative sign introduces the effect of retention. When, in Equation (1), β equals 1, one obtains the classical diffusion equation for isotropic media. It should be noticed that similar equations could be obtained by introducing non-linear effects on the Fick's law see Simas (2012) and D'Angelo et al. (2003).

The boundary conditions, at x = 0 or at x = L, are:

i) Dirichlet type

$$v(x,t) = \underline{v}(x,t) \qquad (2)$$

$$\frac{\partial v(x,t)}{\partial x} = \underline{v}'(x,t) \qquad (3)$$

ii) Neumann type

$$\frac{\partial^2 v(x,t)}{\partial x^2} = \underline{v}''(x,t) \qquad (4)$$

$$\frac{\partial^3 v(x,t)}{\partial x^3} = \underline{v}'''(x,t) \qquad (5)$$

The initial condition for the interval $0 \le x \le L$ is:

$$v(x,0) = v_0 \qquad (6)$$

3. BEM formulation

A residual statement (see Smith, 1985 and Brebbia et al., 1984), can be applied to Equation (1), with the fundamental solution of the steady-state problem playing the role of the weighting function. The following equation arises:

$$
\begin{aligned}
v(\varepsilon) = &-\sqrt{(1-\beta)R\beta}\left[v^{*}(x|\varepsilon)\underline{v}'''(x)\right]_{x=0}^{x=L} \\
&+\sqrt{(1-\beta)R\beta}\left[\frac{\partial\, v^{*}(x|\varepsilon)}{\partial x}\underline{v}''(x)\right]_{x=0}^{x=L} \\
&-\left[(\sqrt{(1-\beta)R\beta}\,\frac{\partial^{2}\, v^{*}(x|\varepsilon)}{\partial x^{2}}-\beta D v^{*}(x|\varepsilon)\underline{v}'(x)\right]_{x=0}^{x=L} \\
&+\left[(\sqrt{(1-\beta)R\beta}\,\frac{\partial^{3}\, v^{*}(x|\varepsilon)}{\partial x^{3}}-\beta D\,\frac{\partial\, v^{*}(x|\varepsilon)}{\partial x})\underline{v}(x)\right]_{x=0}^{x=L} \\
&-\int_{0}^{\varepsilon}\dot{v}(x,t)\, v^{*}(x|\varepsilon)dx-\int_{\varepsilon}^{L}\dot{v}(x,t)\, v^{*}(x|\varepsilon)dx
\end{aligned}
\tag{7}
$$

The fundamental solution of the steady-state problem, $v^{*} = v^{*}(x|\varepsilon)$, is the solution of Equation (14) (see Saito, 2018 & Saito et al., 2019) that can be written as

$$
v^{*}(x|\varepsilon) = \left[\frac{\sqrt{(1-\beta)R\beta}\left(sinh\left[\sqrt{\frac{\beta D}{(1-\beta)R\beta}}\; r\right]\right)+\sqrt{\beta D}\; r}{2(\beta D)^{\frac{3}{2}}}\right]
\tag{8}
$$

where $r = |x-\varepsilon|$ is the distance between field, x, and source, ε, points. As previously mentioned in the introductory section of this work, Equation (15) alone is not sufficient to provide the solution of the problem. Another equation becomes necessary. This equation is obtained by taking the derivative of Equation (15) with respect to the source point coordinate, and reads:

$$
\begin{aligned}
\frac{\partial v(x|\varepsilon)}{\partial \varepsilon} = & -\sqrt{(1-\beta)R\beta}\left[\frac{\partial v^*(x|\varepsilon)}{\partial \varepsilon}\underline{v}'''(x)\right]_{x=0}^{x=L} \\
& +\sqrt{(1-\beta)R\beta}\left[\frac{\partial^2 v^*(x|\varepsilon)}{\partial \varepsilon \partial x}\underline{v}''(x)\right]_{x=0}^{x=L} \\
& -\left[(\sqrt{(1-\beta)R\beta}\,\frac{\partial^3 v^*(x|\varepsilon)}{\partial \varepsilon \partial x^2} - \beta D\frac{\partial v^*(x|\varepsilon)}{\partial \varepsilon})\underline{v}'(x)\right]_{x=0}^{x=L} \\
& +\left[(\sqrt{(1-\beta)R\beta}\,\frac{\partial^4 v^*(x|\varepsilon)}{\partial \varepsilon \partial x^3} - \beta D\frac{\partial^2 v^*(x|\varepsilon)}{\partial x \partial \varepsilon})\underline{v}(x)\right]_{x=0}^{x=L} \\
& -\int_0^{\varepsilon} \dot{v}(x,t)\frac{\partial v^*(x|\varepsilon)}{\partial \varepsilon}dx - \int_{\varepsilon}^{L} \dot{v}(x,t)\frac{\partial v^*(x|\varepsilon)}{\partial \varepsilon}\,dx
\end{aligned}
\tag{9}
$$

From Equation (8), the derivatives that appear in Equations (7) and (9) are computed. One has, in a simplified notation, from Equation (7), the following:

$$\frac{\partial v^*(x|\varepsilon)}{\partial x} = \frac{-1}{2\beta D}\left(cosh\sqrt{\frac{\beta D}{(1-\beta)R\beta}}\,r + 1\right)\frac{\partial r}{\partial x} \tag{10}$$

$$\frac{\partial^2 v^*(x|\varepsilon)}{\partial x^2} = \frac{-1}{2\sqrt{\beta^2 D(1-\beta)R}}\left(sinh\sqrt{\frac{\beta D}{(1-\beta)R\beta}}\,r\right)\left(\frac{\partial r}{\partial x}\right)^2 \tag{11}$$

$$\frac{\partial^3 v^*(x|\varepsilon)}{\partial x^3} = \frac{-1}{2\sqrt{(1-\beta)R\beta}}\left(cosh\sqrt{\frac{\beta D}{(1-\beta)R\beta}}\,r\right)\left(\frac{\partial r}{\partial x}\right)^3 \tag{12}$$

And from equation (9), one has:

$$\frac{\partial v^*(x|\varepsilon)}{\partial \varepsilon} = \frac{-1}{2\beta D}\left(cosh\sqrt{\frac{\beta D}{(1-\beta)R\beta}}\,r + 1\right)\frac{\partial r}{\partial \varepsilon} \tag{13}$$

$$\frac{\partial^2 v^*(x|\varepsilon)}{\partial \varepsilon \partial x} = \frac{-1}{2\sqrt{\beta^2 D(1-\beta)R}}\left(sinh\sqrt{\frac{\beta D}{(1-\beta)R\beta}}r\right)\frac{\partial r}{\partial x}\,\frac{\partial r}{\partial \varepsilon} \tag{14}$$

$$\frac{\partial^3 v^*(x|\varepsilon)}{\partial \varepsilon \partial x^2} = \frac{-1}{2\sqrt{\beta(1-\beta)R}}\left(cosh\sqrt{\frac{\beta D}{(1-\beta)R\beta}}r\right)\left(\frac{\partial r}{\partial x}\right)^2\frac{\partial r}{\partial \varepsilon} \tag{15}$$

$$\frac{\partial^4 v^*(x|\varepsilon)}{\partial \varepsilon \partial x^3} = \frac{-\sqrt{\beta D}}{2(\beta(1-\beta)R)^{\frac{3}{2}}}\left(sinh\sqrt{\frac{\beta D}{(1-\beta)R\beta}}r\right)\left(\frac{\partial r}{\partial x}\right)^3\frac{\partial r}{\partial \varepsilon} \tag{16}$$

The domain discretization is necessary due to the domain integrals indicated in Equations (7) and (9). In this work, linear cells were adopted, i.e. $\dot{v}(x,t)$ varies linearly inside each cell. Analytical integration is easily carried out. For this reason, further details are omitted here. Finally, the last required approximation is related to $\dot{v}(x,t)$.

The time derivative is approximated by adopting a backward finite difference. For a given time, say $t_{n+1} = (n+1)\Delta t$, where Δt is the time-step, one has:

$$\dot{v}(x, t_{n+1}) = \dot{v}_{n+1} = \frac{v_{n+1} - v_n}{\Delta t} \tag{17}$$

$n = 0, 1, 2, 3 \dots$

To solve the problem, Equations (7) and (9) are written, or particularized, for $\varepsilon = 0$ and for $\varepsilon = L$, and the domain integrals are computed. The next step is to replace, in Equation (7), the derivatives given in Equations (10), (11) and (12). By doing some algebraic manipulations, we obtain the following expression:

$$
\begin{aligned}
& v(\varepsilon,t) + \int_0^L v^*(x|\varepsilon)\dot{v}\,(x,t)d\varepsilon \\
&= \left[\frac{(1-\beta)R\beta\left(\sqrt{(1-\beta)R\beta}\,sinh\left[\sqrt{\frac{D}{(1-\beta)R}}\ (L-x')\right]\right) + \sqrt{\beta D}(L-x')}{2(\beta D)^{\frac{3}{2}}}\right]\underline{v}'''(L,t) \\
&- \left[\frac{(1-\beta)R\beta\left(\sqrt{(1-\beta)R\beta}sinh\left[\sqrt{\frac{D}{(1-\beta)R}}\ (x'-0)\right]\right) + \sqrt{\beta D}(x'-0)}{2(\beta D)^{\frac{3}{2}}}\right]\underline{v}'''(0,t) \\
&- \left[\frac{(1-\beta)R}{2D}\left(cosh\sqrt{\frac{D}{(1-\beta)R}}(L-x') + 1\right)\right]\underline{v}''(L,t) \\
&- \left[\frac{(1-\beta)R}{2D}\left(cosh\sqrt{\frac{D}{(1-\beta)R}}(x'-0) + 1\right)\right]\underline{v}''(0,t) \\
&+ \left[\frac{(1-\beta)R\beta}{2\sqrt{(1-\beta)R\beta^2 D}}sinh\left[\sqrt{\frac{D}{(1-\beta)R}}\ (L-x')\right] + \frac{\beta D}{2(\beta D)^{\frac{3}{2}}}\left[\sqrt{\frac{D}{(1-\beta)R}}\ (L-x')\right]\right. \qquad (18) \\
&\left. + \sqrt{\beta D}\ (L-x')\right]\underline{v}'(L,t) \\
&- \left[\frac{(1-\beta)R\beta}{2\sqrt{(1-\beta)R\beta^2 D}}sinh\left[\sqrt{\frac{D}{(1-\beta)R}}\ (L-x')\right] + \frac{\beta D}{2(\beta D)^{\frac{3}{2}}}\left[\sqrt{\frac{D}{(1-\beta)R}}\ \ (x'-0)\right]\right. \\
&\left. + \sqrt{\beta D}\ (x'-0)\right]\underline{v}'(0,t) \\
&- \left[\frac{1}{2}cosh\sqrt{\frac{D}{(1-\beta)R}}(L-x') + \frac{1}{2}\left(cosh\sqrt{\frac{D}{(1-\beta)R}}(L-x') + 1\right)\right]\underline{v}(L,t) \\
&- \left[\frac{(-1)}{2}cosh\sqrt{\frac{D}{(1-\beta)R}}(x'-0) + \frac{1}{2}\left(cosh\sqrt{\frac{D}{(1-\beta)R}}(x'-0) + 1\right)\right]\underline{v}(0,t)
\end{aligned}
$$

we replace the value of ε by L

$$v(L,t)+\int_0^L v^*(x|\varepsilon)\dot{v}\,(x,t)d\varepsilon$$
$$=-\left[\frac{\beta(1-\beta)R\left(\sqrt{\beta(1-\beta)R}\,sinh\,\left[\sqrt{\frac{D}{(1-\beta)R}}\,L\right]\right)+\beta(1-\beta)R\sqrt{\beta D}L}{2(\beta D)^{\frac{3}{2}}}\right]\underline{v}'''(0,t)$$
$$-\frac{(1-\beta)R}{2D}\underline{v}''(L,t)-\left[\frac{(1-\beta)R}{2D}\left(cosh\sqrt{\frac{D}{(1-\beta)R}}\,L+1\right)\right]\underline{v}''(0,t)+\frac{1}{2}L\underline{v}'(0,t) \tag{19}$$
$$+\frac{1}{2}\underline{v}(L,t)+\frac{1}{2}\underline{v}(0,t)$$

And $\varepsilon = L$

$$v(0,t)+\int_0^L v^*(x|\varepsilon)\dot{v}(x,t)\,dx$$
$$=\left[\frac{\beta(1-\beta)R\left(\sqrt{\beta(1-\beta)R}\,sinh\,\left[\sqrt{\frac{D}{(1-\beta)R}}\,L\right]\right)+\beta(1-\beta)R\sqrt{B}L}{2(\beta D)^{\frac{3}{2}}}\right]\underline{v}'''(L,t) \tag{20}$$
$$-\left[\frac{(1-\beta)R}{2D}\left(cosh\sqrt{\frac{D}{(1-\beta)R}}\,L+1\right)\right]\underline{v}''(L,t)-\frac{(1-\beta)R}{D}\underline{v}''(0,t)-\frac{1}{2}L\underline{v}'(L,t)$$
$$+\frac{1}{2}\underline{v}(L,t)+\frac{1}{2}\underline{v}(0,t)$$

And now, we replace in Equation (17) the derivatives given in Equations (21), (22), (23) and (24) obtaining:

$$\underline{v}'(L,t)+\int_0^L \frac{\partial\, v^*(x|\varepsilon)}{\partial\varepsilon}\dot{v}(x,t)dx$$
$$=\frac{(1-\beta)R}{D}\underline{v}'''(L,t)-\left[\frac{(1-\beta)R}{2D}\left(cosh\sqrt{\frac{(1-\beta)R}{D}}\,L+1\right)\right]\underline{v}'''(0,t) \tag{21}$$
$$-\frac{1}{2}\sqrt{\frac{(1-\beta)R}{D}}\,sinh\,\left[\sqrt{\frac{(1-\beta)R}{D}}\,L\right]\underline{v}''(0,t)\;+\frac{1}{2}\underline{v}'(L,t)+\frac{1}{2}\underline{v}'(0,t)$$

$$\underline{v}'(0,t) + \int_0^L \frac{\partial\, v^*(x|\varepsilon)}{\partial x'}\dot{v}(x,t)dx = \left[-\frac{(1-\beta)R}{2D}\left(\cosh\sqrt{\frac{(1-\beta)R}{D}}\;L + 1\right)\right]\underline{v}'''(L,t) - \frac{(1-\beta)R}{D}\underline{v}'''(0,t) + \frac{1}{2}\sqrt{\frac{(1-\beta)R}{D}}\sinh\left[\sqrt{\frac{(1-\beta)R}{D}}\;L\right]\underline{v}''(L,t) + \frac{1}{2}\underline{v}'(L,t) + \frac{1}{2}\underline{v}'(0,t) \tag{22}$$

In the absence of studies concerning the choice of the time-step length, it was chosen empirically. For some recommendations concerning its choice, the reader is referred to Carrer et al. (2009). In matrix form, the system of algebraic equations (19, 20, 21 and 22) can be written as:

$$\begin{bmatrix} (\Delta t\mathbf{H}^{bb} + \mathbf{W}^{bb}) & -\Delta t\boldsymbol{v}^{bb} & (0 + \mathbf{W}^{bd}) \\ (0 + \overline{\mathbf{W}}^{bb}) & \Delta t\boldsymbol{v}^{bb} & (0 + \overline{\mathbf{W}}^{bd}) \\ (-\Delta t\mathbf{H}^{db} + \mathbf{W}^{db}) & -\Delta t\boldsymbol{v}^{db} & (\Delta t I + \mathbf{W}^{dd}) \end{bmatrix} \begin{Bmatrix} \underline{v}^{b}_{n+1} \\ \underline{v}'^{b}_{n+1} \\ \underline{v}^{d}_{n+1} \end{Bmatrix} = \begin{bmatrix} \Delta t\mathbf{G}^{bb} & \Delta t\boldsymbol{B}^{bb} \\ \Delta t\overline{\mathbf{G}}^{bb} & \Delta t\overline{\mathbf{B}}^{bb} \\ \Delta t\mathbf{G}^{db} & \Delta t\mathbf{B}^{db} \end{bmatrix} \begin{Bmatrix} v'''^{b}_{n+1} \\ v''^{b}_{n+1} \end{Bmatrix} - \begin{bmatrix} \mathbf{D}^{bb} & 0 & \mathbf{W}^{bd} \\ \overline{\mathbf{D}}^{bb} & 0 & \overline{\mathbf{W}}^{bd} \\ \mathbf{D}^{db} & 0 & \mathbf{W}^{dd} \end{bmatrix} \begin{Bmatrix} 5\underline{v}^{b}_{n} \\ 0 \\ \underline{v}^{d}_{n} \end{Bmatrix} \tag{23}$$

The superscripts b and d, concerning the vectors in Equation (23), correspond to the boundary nodes and to the domain internal points, respectively. Then, vectors $\underline{v}^{b}_{n+1}$, $\underline{v}'^{b}_{n+1}$, $\underline{v}''^{b}_{n+1}$ and $\underline{v}'''^{b}_{n+1}$have dimension (2x1), whereas vector $\underline{v}^{d}_{n+1}$ has dimension (n_i), with n_i being the number of internal points. Note that the number of cells is equal to $(n_i + 1)$. The identity matrix I is related to the internal points. In the sub-matrices, the first superscript corresponds to the position of the source point and the second to the position of the field point. Concisely, Equation (23) can be written as presented in Saito [2018]:

$$\overline{\overline{H}}\, d_{n+1} = \overline{\overline{G}}\, n_{n+1} + \overline{\overline{W}}\, u_n \tag{24}$$

In Equation (32), the vector d_{n+1} contains the values of v and $\underline{v}'$, related to the essential boundary conditions, and the vector n_{n+1} contains the values of $\underline{v}''$ and $\underline{v}'''$, related to the natural boundary conditions. Matrices $\overline{\overline{H}}$and $\overline{\overline{G}}$ come from the expressions (9) – (7) and matrix $\overline{\overline{W}}$comes from the domain integrations (see Saito, 2018).

4. Numerical results DBEM

In the examples presented in this section, the following parameters were adopted:

$D = 1$
$R = 0.05$
$R = 0.5$
$\beta = 1$

All the analyses were carried out with the domain discretized into 16 cells. The time-step length was:

$\Delta t = 0.05s$

4.1 Domain under initial condition: cosine distribution – Example 1

This example consists of a domain of unity length, with all the boundary conditions null and with an initial condition field given by:

$$v(x,t) = v_0 cos\left(\frac{\pi}{2}x\right) \tag{25}$$

The analytical solution to this problem is given by (see Bevilacqua et al., 2011) as

$$v(x,t) = v_0 exp\left(\frac{\pi^2}{4} D\rho t\right) cos\ cos\ \left(\frac{\pi x}{2}\right) \tag{26}$$

with

$$\rho = -\beta\left(1 + \frac{\pi^2}{4}\frac{R}{D}(1-\beta)\right) \tag{27}$$

Figure 1 depicts the results at various values of t for the first analysis, carried out with $R = 0.05$, whereas Figure 2 depicts the results for the second analysis, for which $R = 0.5$.

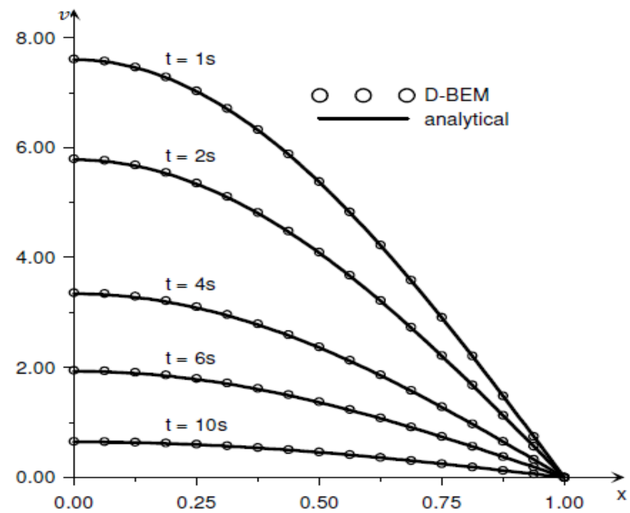

Figure 1. Example 1: results at different instants of time for $R = 0.05$.

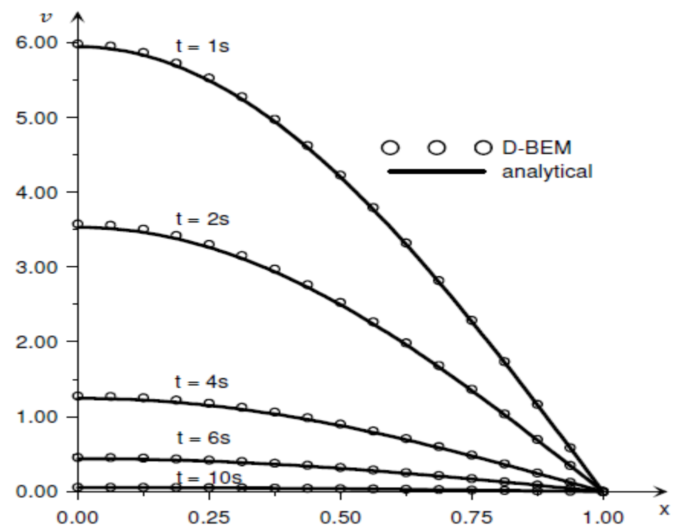

Figure 2. Example 1: results at different instants of time for $R = 0.5$.

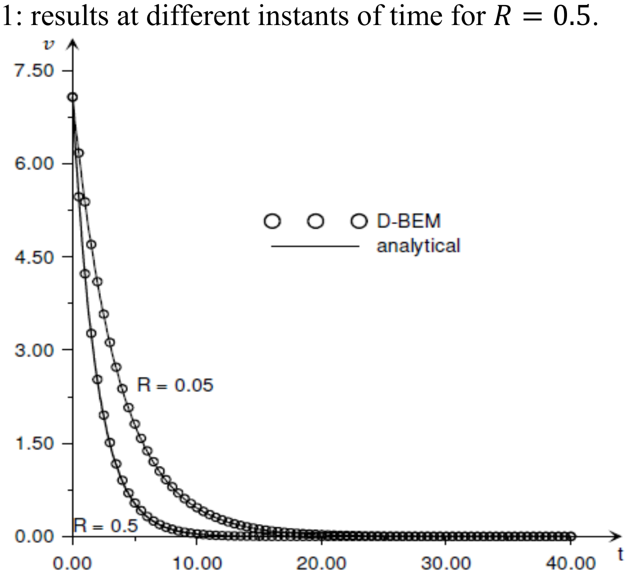

Figure 3. Example 1: results for $v(0.5, t)$.

Note that the ratio $\rho = \rho(r, \beta)$, defined by Equation (27), controls the rate of change of the variable of interest. When $\beta = 1$, the problem is reduced to the classical diffusion problem. Figure 3 depicts the results of the first and second analyses for $v(0.5, t)$. In all Figures, a good agreement is observed between the analytical solution and the DBEM results.

4.2 Domain under initial condition: hyperbolic cosine distribution – Example 2

In this example, a domain of unity length was considered again. The boundary conditions are:

$$\underline{v}'(0,t) = 0 \tag{28}$$

$$\underline{v}''(0,t) = 0 \tag{29}$$

$$\underline{v}''(1,t) = \frac{1}{a^2} v(1,t) \tag{30}$$

$$\underline{v}'''(1,t) = \frac{1}{a^2} \underline{v}'(1,t) \tag{31}$$

Note that the last two boundary conditions, at $x = 1$, are coupled. The initial condition field is given by:

$$v(x,t) = v_0 cosh\left(\frac{x}{a}\right) \tag{32}$$

The analytical solution reads

$$v(x,t) = v_0 exp\left(\frac{D\rho}{a^2} t\right)\ cosh\left(\frac{x}{a}\right) \tag{33}$$

with

$$\rho = \beta\left(1 - \frac{R}{a^2 D}(1-\beta)\right) \tag{34}$$

Two values were adopted for the parameter $q = 2$ and $q = 4$.

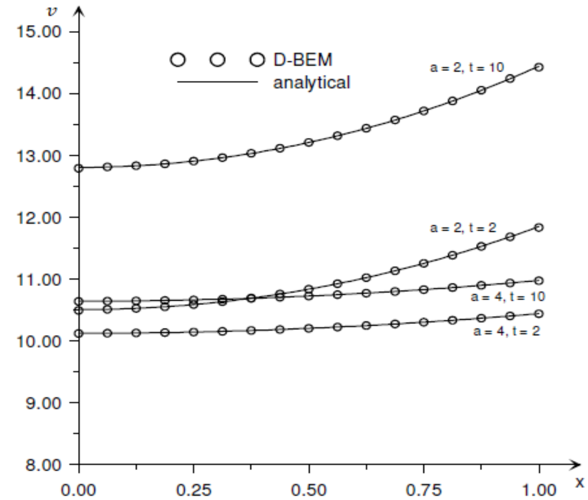

Figure 4. Example 2: results at different instants of time for $a = 2$ and $a = 4$, with $R = 0.05$.

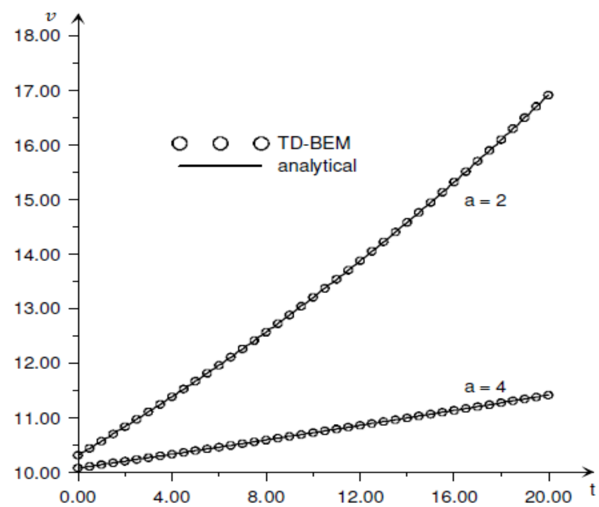

Figure 5. Example 2: results for $v(0.5, t)$ for $a = 2$ and $a = 4$ with R = 0.05.

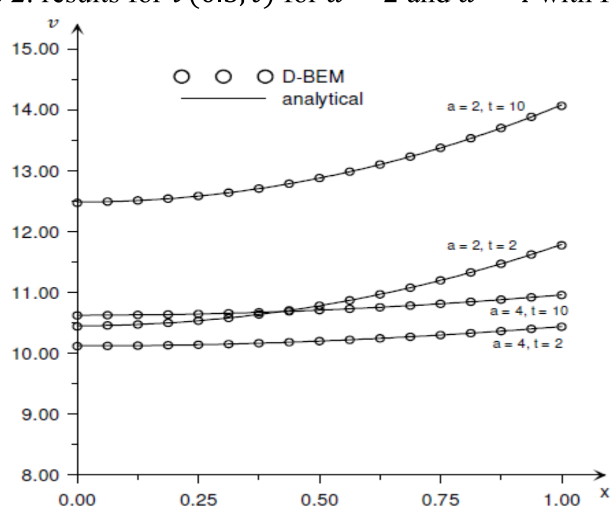

Figure 6. Example 2: results at different instants of time for $a = 2$ and $a = 4$, with R = 0.5.

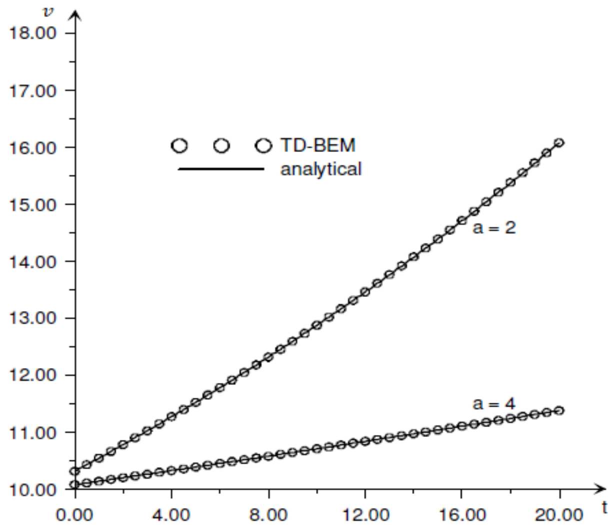

Figure 7. Example 2: results for $v(0.5, t)$ for $a = 2$ and $a = 4$, with $R = 0.5$.

Figures 4 and 5 depict results obtained with $R = 0.05$, whereas Figures 6 and 7 depict results obtained with $R = 0.5$. One can observe that for bigger values of the parameter q correspond smaller rates of increase of v, and that the parameter R does not have the influence it has had in the first example. Again, good agreement is observed between analytical and DBEM results.

Conclusions

This work is concerned with the solution of the anomalous diffusion equation, in one-dimension (1-D), by employing the DBEM. Although one-dimensional problems present a limited range of applications, their solutions always give the researcher experience for facing more complex problems in two and three dimensions. For the problem treated here, a fundamental solution, associated to the steady state problem, was found and a successful D-BEM formulation was developed. The results generated by the formulation are accurate and present good agreement with analytical solutions. Naturally, this is the first step towards the development of new DBEM formulations. The search for a time-dependent fundamental solution, at this time, seems to be very challenging, as well as the development of formulations for bi- and three-dimensional problems. Another problem that should deserve attention is the one to deal with anisotropic materials.

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In this paper, the datasets literary for this research are described in references: Jiang 2017; Simas, 2012; D'Angelo et al. 2003; Deleersnijder et al. 2006; Huang & Madey 1982; Bevilacqua et al. 2011; Bevilacqua et al. 2012; Bevilacqua et al. 2013; Brebbia et al. 1984; Carrer et al. 2009; Ferreira et al. 2010; Haggerty et al. 2004; Kindler et al. 2010; Jianhong & Xingfu 2001; LaBolle et al. 2006; Mainardi 1996; Maloszewski & Zuber 1990; Maloszewski & Zuber 1991; Saito 2018; Saito et al. 2019; Scherer et al. 2008; Scuciato et al. 2016;Smith 1985; Zhang et al. 2007

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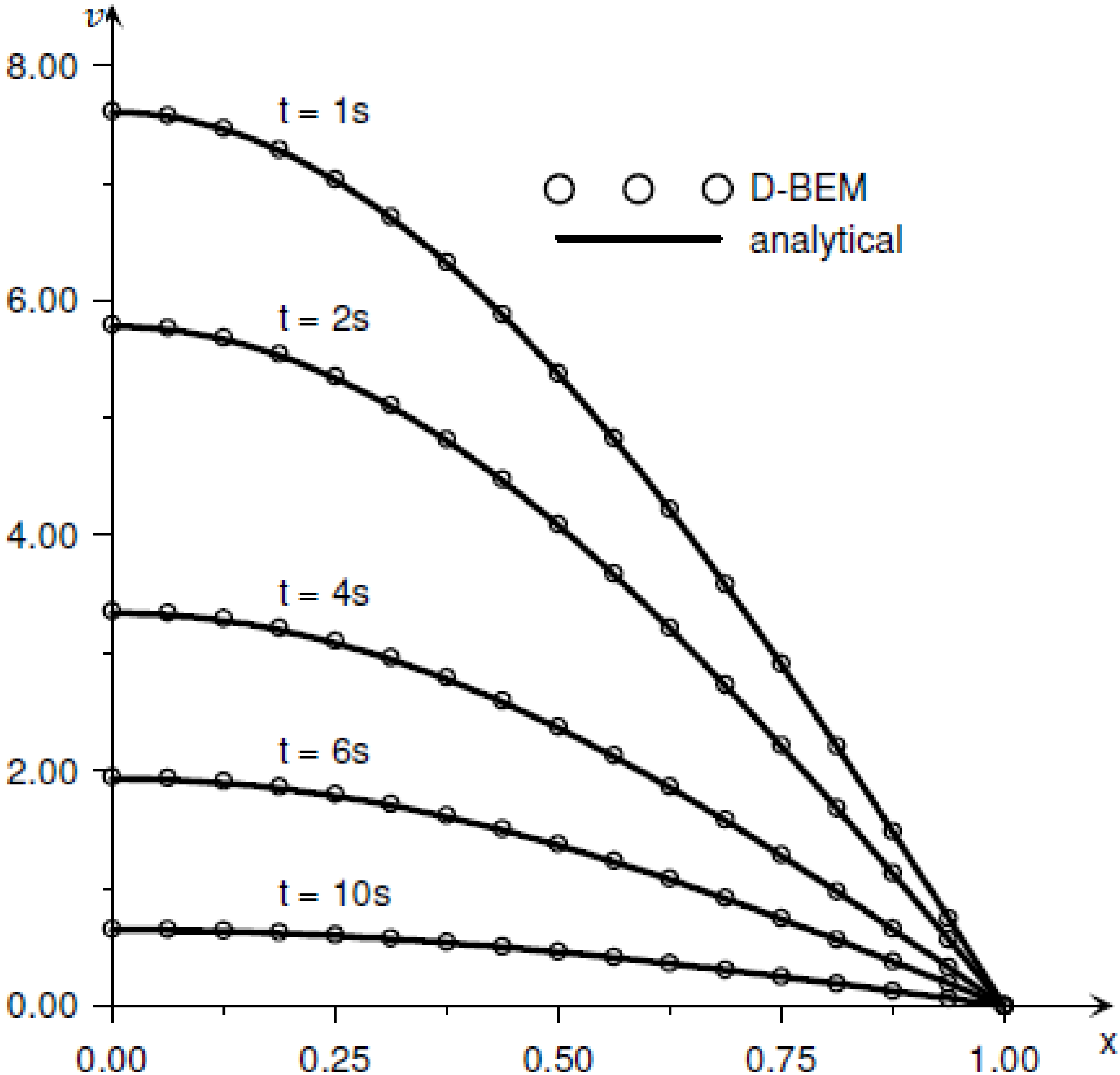
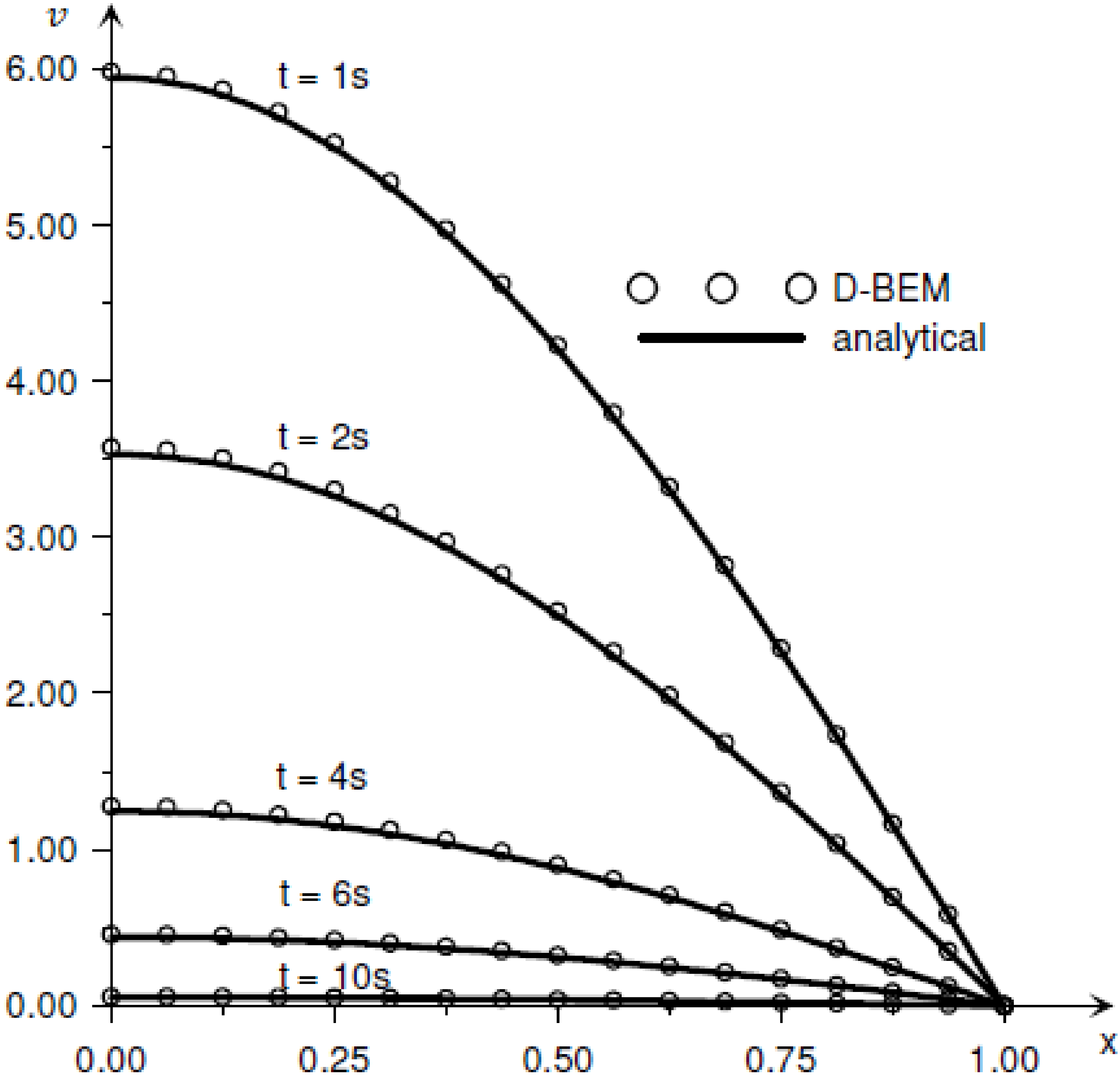
Figure 1. Example 1: results at different instants of time for R=0.05.



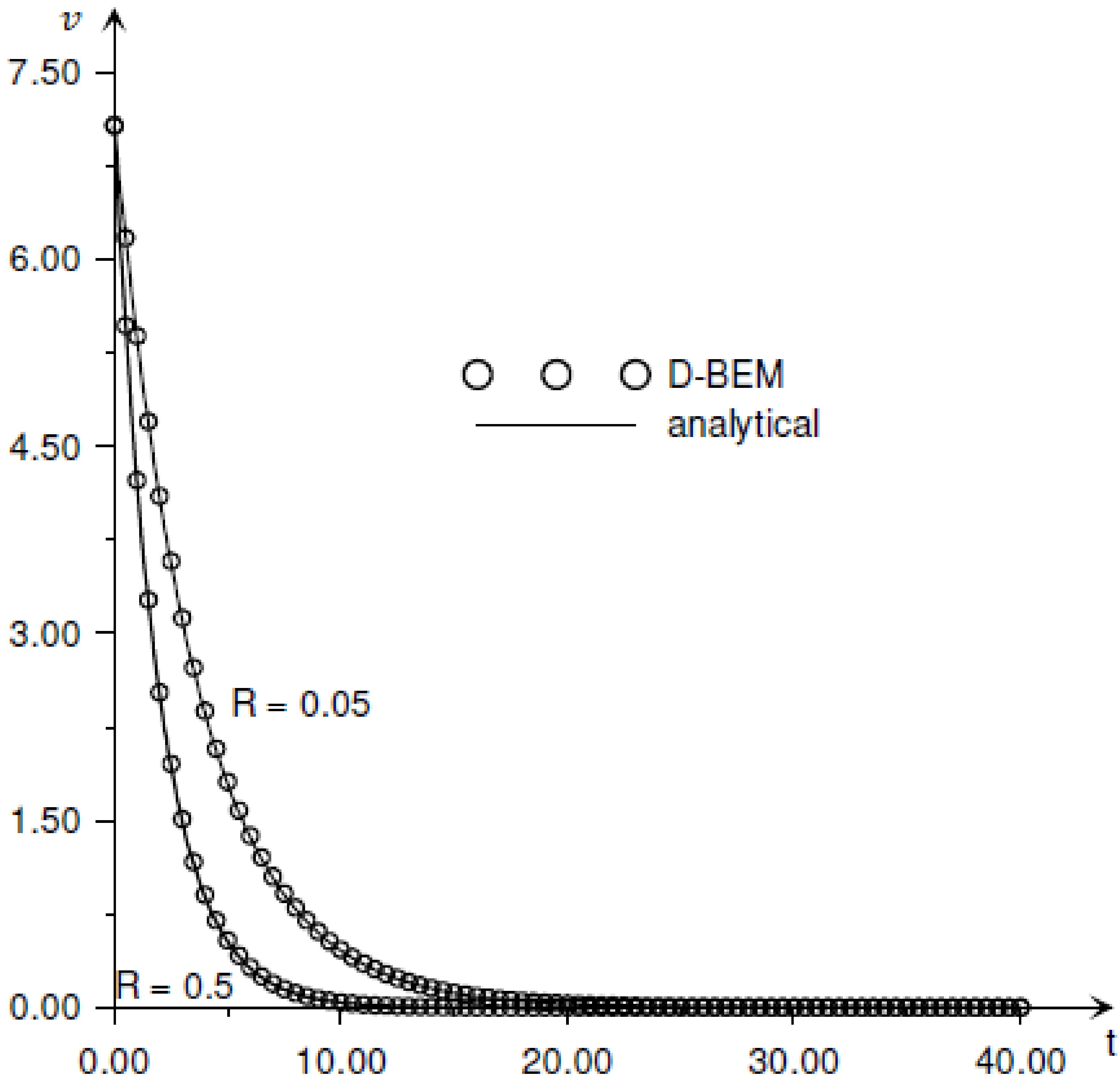
v
8.00
6.00
4.00
2.00
0.00
t = 1s
t = 2s
t = 4s
t = 6s
t = 10s
D-BEM
analytical
0.00
0.25
0.50
0.75
1.00
x

Figure 2. Example 1: results at different instants of time for R=0.5.



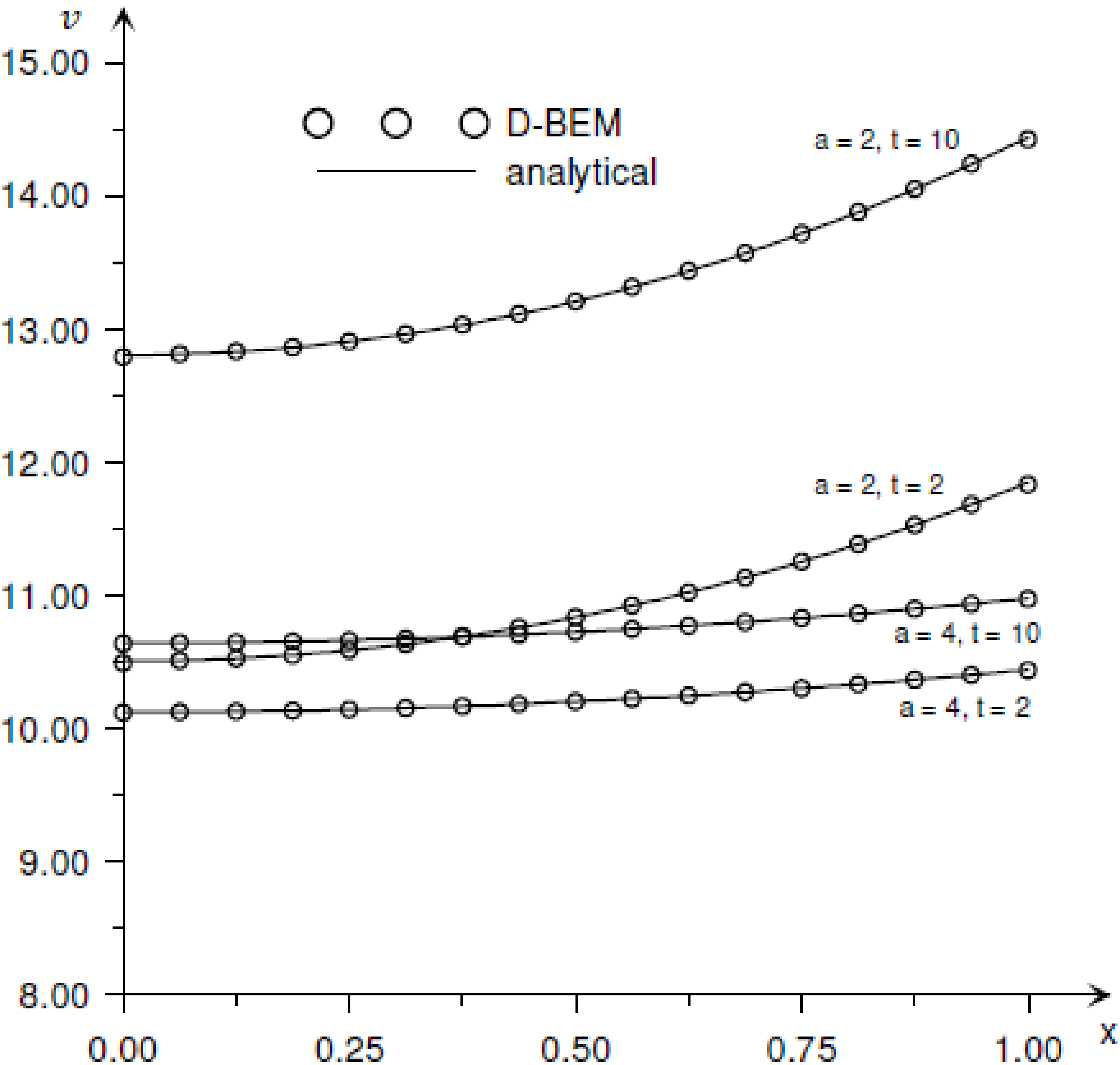
v
6.00
5.00
4.00
3.00
2.00
1.00
0.00
t = 1s
t = 2s
t = 4s
t = 6s
t = 10s
D-BEM
analytical
0.00
0.25
0.50
0.75
1.00
x

Figure 3. Example 1: results for v(0.5,t).



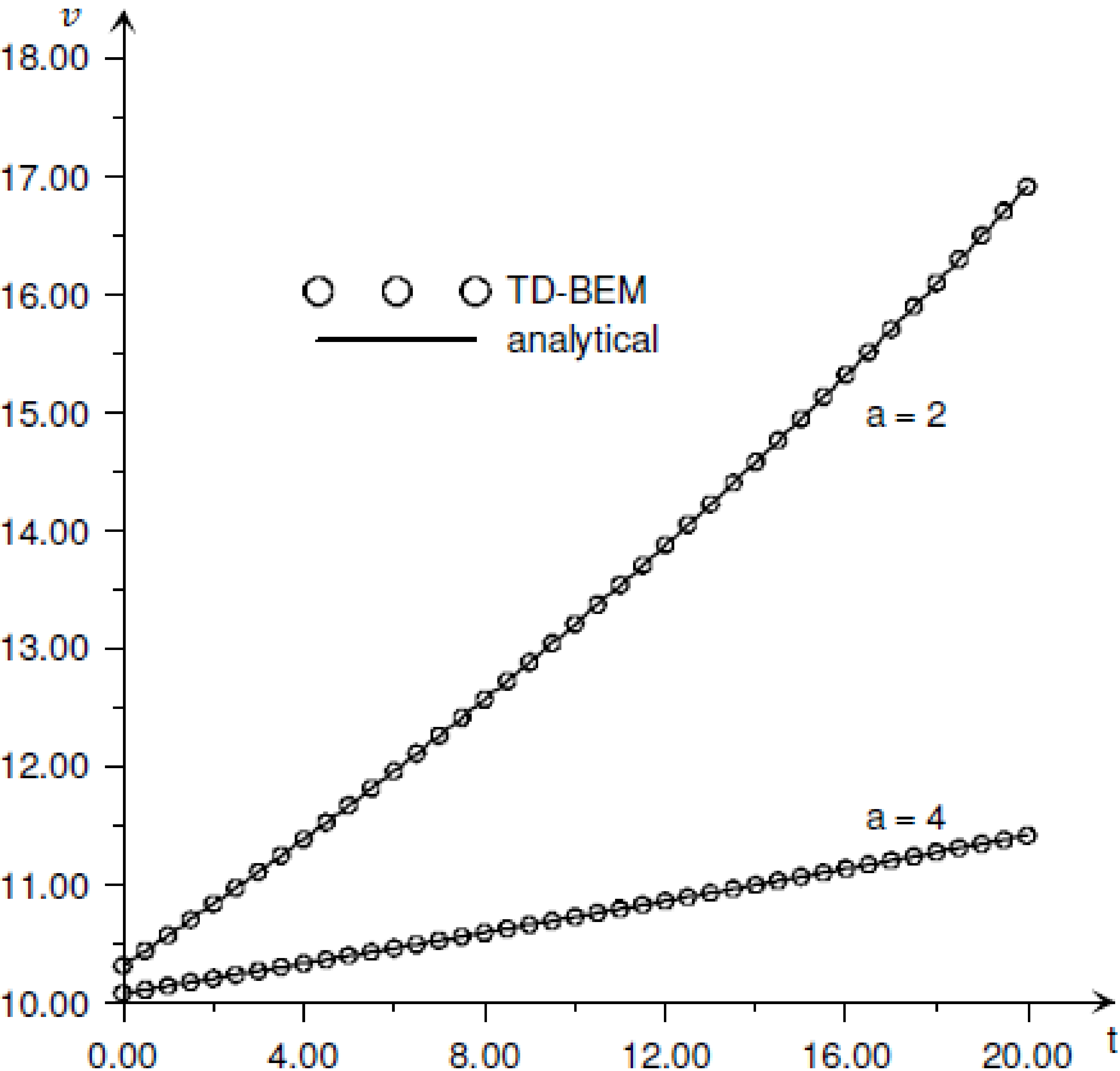
v
7.50
6.00
4.50
3.00
1.50
0.00
D-BEM
analytical
R = 0.05
R = 0.5
0.00
10.00
20.00
30.00
40.00
t

Figure 4. Example 2: results at different instants of time for a=2 and a=4, with R=0.05.



v
15.00
14.00
13.00
12.00
11.00
10.00
9.00
8.00
D-BEM
analytical
a = 2, t = 10
a = 2, t = 2
a = 4, t = 10
a = 4, t = 2
0.00
0.25
0.50
0.75
1.00
x

Figure 5. Example 2: results for v(0.5,t) for a=2 and a=4 with R = 0.05.



v
18.00
17.00
16.00
15.00
14.00
13.00
12.00
11.00
10.00
0.00
4.00
8.00
12.00
16.00
20.00
t
TD-BEM
analytical
a = 2
a = 4

Figure 6. Example 2: results at different instants of time for a=2 and a=4, with R = 0.5.

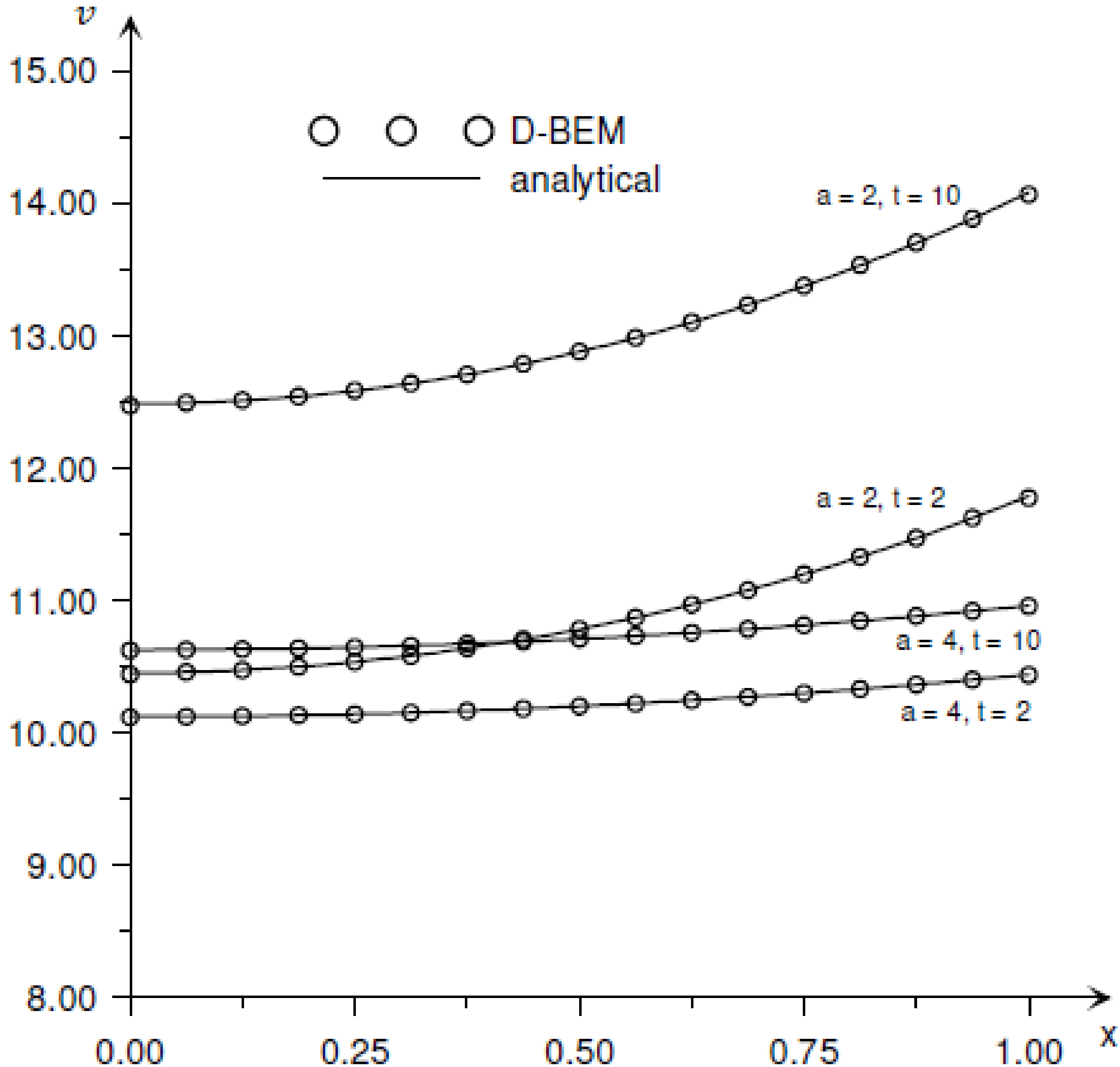
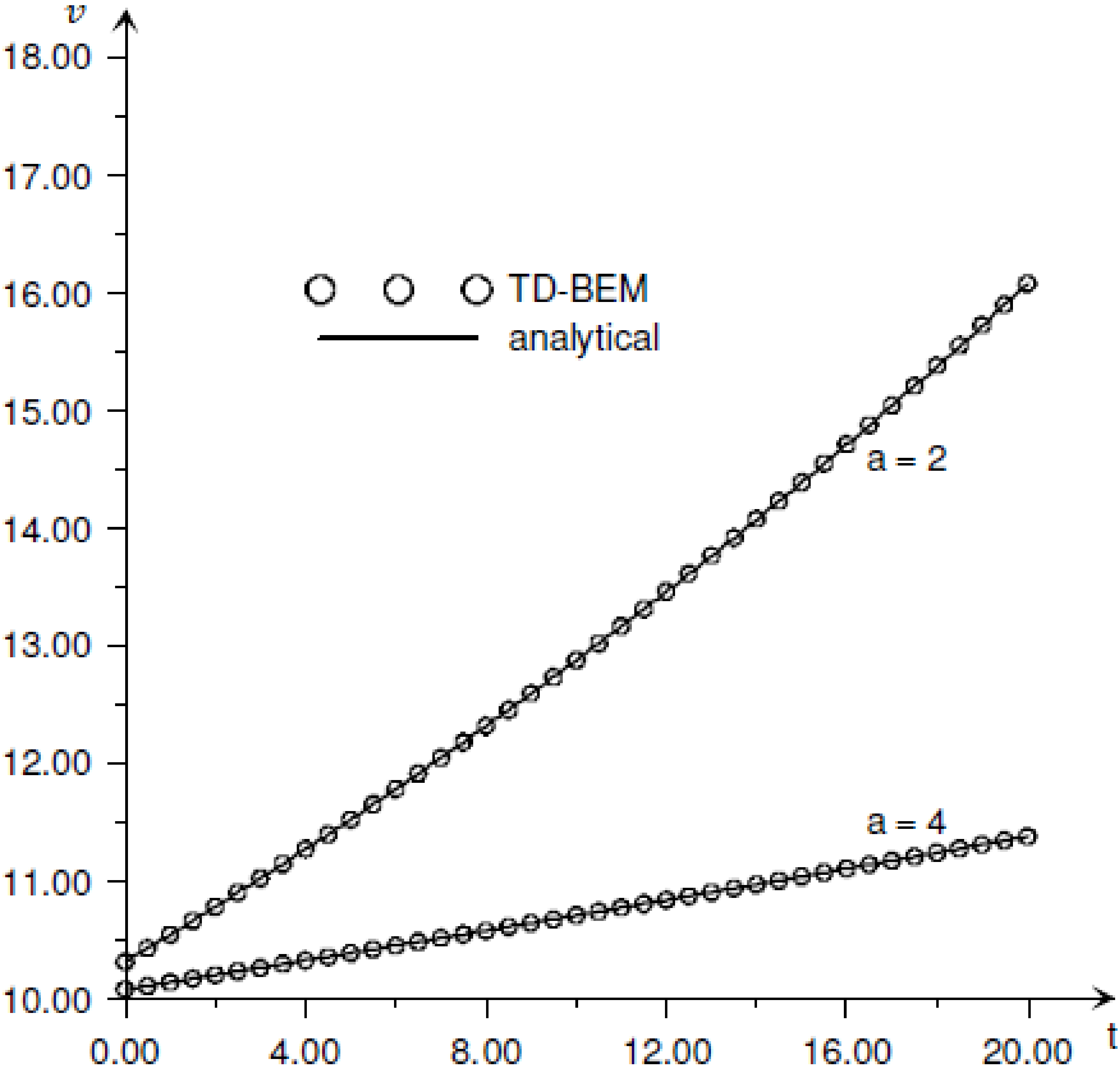

v
15.00
14.00
13.00
12.00
11.00
10.00
9.00
8.00
D-BEM
analytical
a = 2, t = 10
a = 2, t = 2
a = 4, t = 10
a = 4, t = 2
0.00
0.25
0.50
0.75
1.00
x

Figure 7. Example 2: results for v(0.5,t) for a=2 and a=4, with R=0.5.



v
18.00
17.00
16.00
15.00
14.00
13.00
12.00
11.00
10.00
0.00
4.00
8.00
12.00
16.00
20.00
t
TD-BEM
analytical
a = 2
a = 4