Decorrelation is not dissociation: there is no rational solution to the Brutsaert-Nieber parameter association problem

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Abstract. The coefficient (k) of the Brutsaert-Niber equation ($-dQ/dt = kQ^{\alpha}$, Q being discharge at time t) cannot provide information on streamflow dynamics independently because its unit depends on the exponent α. One way to address this challenge is to compute k after fixing α at α median, which may involve large fitting errors. A recent study has therefore adopted a method to decorrelate k from α by resealing Q and demonstrated that the decorrelated coefficients hold useful information on streamflow dynamics. Here, I argue that decorrelation is not dissociation and propose a framework to evaluate the parameter estimation methods quantitatively. Analysis of real as well as synthetic recession curves suggests that under no circumstance the decorrelation method is superior to the fixed exponent method. To conclude, there seems to be no rational solution to the Brutsaert-Niber parameter association problem.

1. Introduction

A large number of natural phenomena can be expressed by means of a power-law equation [*Rodríguez-Iturbe and Rinaldo*, 2001; *Pinto et al.*, 2012]. In hydrological science, recession flows in rivers are commonly described as [*Brutsaert and Nieber*, 1977]

$$-\frac{dQ}{dt} = kQ^{\alpha} \tag{1}$$

where Q is the discharge at the river cross section at time t. Accurate estimation of the parameters α and k of the Brutsaert-Nieber (BN) equation (Eq. (1)) is essential for many hydrological applications as they provide crucial information on basin storage, which otherwise cannot be observed [e.g., *Brutsaert and Nieber*, 1977; *Wang and Cai*, 2009; *Biswal and Marani*, 2010; *Shaw and Riha*, 2012; *Doulatyari et al.*, 2015; *Dralle et al.*, 2015; *Li and Nieber*, 2017; *Santos et al.*, 2018; *Reddyvaraprasad et al.*, 2020]. Because of observational errors and subjectivities involved with recession analysis, a large number of methods exist to estimate α and k [e.g., *Stoelzle et al.*, 2012; *Chen and Krajewski*, 2016; *Roques et al.*, 2017; *Dralle et al.*, 2017; *Jachens et al.*, 2020]. Traditional methods assume that the relationship between $-dQ/dt$ and Q is static or one-to-one for a basin, which also means that $(-dQ/dt, Q)$ data points can be plotted altogether in log-log plane to estimate α and k. However, many studies have recently supported the notion that $-dQ/dt$-Q relationship changes significantly across recession events, which implies we should analyze recession curves individually [*Biswal and Marani*, 2010; *Shaw and Riha*, 2012; *Mutzner et al.*, 2013; *Biswal and Marani*, 2014; *Dralle et al.*, 2017]

Individual recession curve analysis, however, has its own limitations, particular with respect to estimation of k. While α is dimensionless, the unit (scale) of k is dependent on α. Thus, the values of k obtained for a basin are quite meaningless as they cannot be compared with each other. One solution in this regard is to fix the value of α at α median (α_m) for each recession curve of the basin and estimate the BN coefficient (k_m) [*Biswal and Marani*, 2014; *Biswal and Nagesh Kumar*, 2014; *Bart and Hope*, 2014; *Reddyvaraprasad et al.*, 2020], which will be henceforth called as the 'fixed exponent method.' Of course, this method will add curve-fitting error when α is different from α_m. *Dralle et al.* [2015] therefore followed a method to decorrelate α and k, which, according to them, generates decorrelated BN coefficients (k_{dc}s) that can be compared with each other for obtaining meaningful hydrological information. The decorrelation method was originally proposed by *Bergner and Zouhar* [2000] who studied fatigue behavior of materials. Using observed discharge data from 16 US catchments *Dralle et al.* [2015] showed that the decorrelated BN coefficients can explain seasonal variability of streamflow. Nevertheless, they did not quantitatively analyze how much benefit the decorrelation method provides, particularly with respect to the fixed exponent method.

In this study, it is argued that there is no rational solution to the BN parameter parameter estimation problem. I first analytically show that the decorrelation method does not actually dissociates k from α. Thereafter, I compare the decorrelation method with the fixed exponent method using observed as well as synthetic recession flow data to understand their usefulness.

2. Data and methodological details

2.1. Observed and synthetic recession flow curves

Now that we know both the decorrelation method and the fixed exponent method intend to produce recession coefficients that independently carry meaningful information, it will be only logical to compare them using observed data. For this purpose, available daily discharge times series data were collected from 31 USGS basins (Table S1). To further strengthen the analysis I also generated synthetic recession curves with certain known characteristics (Table S1). A recession curve is defined here as a monotonically decreasing discharge time series spanning at least 5 days. Q and $-dQ/dt$ for i-th day are computed as: $Q_i = (Q_i + Q_{i+1})/2$ and $-dQ/dt = Q_i - Q_{i+1}$ [*Brutsaert and Nieber*, 1977].

2.2. The BN parameter decorrelation method

To estimate α and k following the least square linear regression method, we need to take the logarithm of both sides of Eq. (1): $\ln -dQ/dt = \ln k + \alpha \ln Q$. If we change the unit of Q such that $Q' = Q/Z$, the equation will be $\ln -dQ'/dt = \ln k' + \alpha \ln Q'$, where $\ln k' = \ln k + (\alpha - 1)\ln Z$ or $k' = kZ^{\alpha-1}$. Because of the presence of both α and Z in the equation, we can expect the correlation between k' and α (R^2) to change if Z changes. Therefore, for a set of (α, k) values it is possible to find a Z (Z_{dc}) such that the correlation (R^2) vanishes, i.e. $\sum(\ln k' - \overline{\ln k'})(\alpha - \overline{\alpha})/\sqrt{\sum(\ln k' - \overline{\ln k'})^2 \sum(\alpha - \overline{\alpha})^2} = 0$. For this condition to be satisfied both $\sum(\ln k' - \overline{\ln k'})^2$ and $\sum(\alpha - \overline{\alpha})^2$ should be nonzero but $\sum(\ln k' - \overline{\ln k'})(\alpha - \overline{\alpha})$ zero. The overline sign stands for mean value. Now replacing $\ln k'$ with $\ln k + (\alpha - 1)\ln Z_{dc}$, one can obtain the condition for decorrelation [*Bergner and Zouhar*, 2000]

$$Z_{dc} = \exp\left(\frac{\sum(\ln k - \overline{\ln k})(\alpha - \overline{\alpha})}{\sum(\alpha - \overline{\alpha})^2}\right) \quad (2)$$

Eq. (2) suggests that decorrelation of k from α is possible provided that the underlying assumptions are satisfied (Figure 1). The real question here, however, is what BN-parameter decorrelation actually means.

2.3. Decorrelation is not dissociation!

The BN parameters of a re-scaled recession curve $((-dQ'/dt_1,\, Q'_1),\, (-dQ'/dt_2,\, Q'_2)...$ $(-dQ'/dt_N,\, Q'_N))$ can be obtained minimizing the sum of squared errors of the logarithmic quantities $E = \sum(\ln -dQ'/dt - \alpha \ln Q' - \ln k')^2$. For E to be minimum, its partial derivative with respect to $\ln k'$ has to be zero, which yields the equation:

$$\ln k' = \frac{\sum \ln -dQ'/dt}{N} - \alpha \frac{\sum \ln Q'}{N} \tag{3}$$

According to Eq. (3), k' will be dissociated from α only when $\sum \ln Q' = 0$, which gives the condition $Z = \widehat{Q}$, where $\widehat{}$ symbolizes geometric mean. Since all the recession curves of a basin are not expected have the same $\widehat{Q}$, no single value of Z can dissociate the BN coefficient from the expoent for every recession curve (Figure 1c). In other words, although Eq. (2) decorrelates the BN parameters, it does not disassociates them or make them independent of each other. On the other hand, instead of focusing on parameter dissociation, the fixed exponent method allows only k_m to vary.

2.4. The purpose of BN coefficient estimation

To understand how recession coefficient provides information on discharge, let's integrate Eq. (1): $-\int_{Q_0}^{Q} Q^{-\alpha} = k \int_0^t dt$, which gives us

$$Q = Q_0 \left[1 + (\alpha - 1)ktQ_0^{\alpha-1}\right]^{1/(1-\alpha)} \tag{4}$$

where Q_0 is discharge at the beginning of the recession event. Eq. (4) shows k and α combinedly cannot provide full information on streamflow variability due to the presence of Q_0. However, the effect of Q_0 on Q decreases with t, and when t is sufficiently large such that $(\alpha - 1)ktQ_0^{\alpha-1} >> 1$, $Q = [(\alpha - 1)kt]^{1/(1-\alpha)}$. When t is held constant, say at T, for all the recession events of a basin [*Biswal and Marani*, 2014],

$$Q_T \propto [(\alpha - 1)k]^{1/(1-\alpha)} \tag{5}$$

where Q_T is Q at time T. The above equation provides a quantitative description of how k and α compete with each other to control Q_T. Considering Eq. (5) as the theoretical basis, this study proposes that any BN coefficient estimation method can be evaluated in terms of the correlation (R^2) between Q_T and the estimated BN coefficient. The performances of the decorrelation method and the fixed exponent method are respectively denoted as R^2_{dc} and R^2_m.

The dependency of Q_T on α will also not vanish even if we decorrelate k from α because, as we showed in the previous subsection, k_{dc} cannot be free from α for all recession events. The only scenario in which Q_T variation across recession events will be fully explained by k alone is when α does not vary (Eq. (5)). Although the fixed exponent method does not allow α to vary, it too does not fully eliminate the dependency of Q_T on α. From Eq. (3) we can obtain the relationship between k_m and k as $k_m = k\widehat{Q}^{\alpha_m - \alpha}$, which shows the dependency of k_m on α when $\alpha \neq \alpha_m$. If, for example, k is constant for a set of recession curves, k_m will vary because of α.

3. Results and discussion

Since the definition of recession curve here allows inclusion of events having minimum length of 5 days, the highest value of T can be 5. R^2 between the original k (k_o, computed in in mm$^{1-\alpha}$/day$^{2-\alpha}$ unit) and Q_5 is viewed as the baseline performance (R^2_o). The scatter plot between k_o and Q_5 is shown for a sample basin in FIgure 02a. Large amount of scatter ($R^2_o = 0.16$) in the plot suggests k_o obtained from discharge values in mm/day cannot provide reliable information on Q_5 for the basin. The decorrelation method provides more useful hydrologic information, which is highlighted by the presence of relatively low amount of scatter in the Q_5 vs. k_{dc} plot for the basin ($R^2_{dc} = 0.6$). However, R^2_m for the basin (0.94) is greater than R^2_{dc}. In fact, for no real or synthetic basin studied here $R^2_{dc} > R^2_m$ (see Tables S1 and S2 of the supplementary material). Interestingly, R^2_o is greater than R^2_{dc} for 6 out of the 31 study basins (Table S1). Although a purely mathematical explanation for this is beyond the scope of this study, one thing is clear that Z_{dc} is not the optimum value of Z that obtains k' with most useful hydrolological information. Overall, the results reported above suggest the fixed exponent method is more reliable than the decorrelation method in providing information on streamflow dynamics.

If the criterion $(\alpha - 1)ktQ_0^{\alpha-1} >> 1$ is satisfied, Q_5 will be a function of k (and hence k_m) according to Eq. (5) provided that the exponent α does not vary. On the contrary, the decorrelation method is not applicable when α does not vary. Thus, to create an appropriate scenario to compare the two methods, a set of synthetic recession curves was chosen with α varying within a narrow range between 1.94 and 1.97; in comparison, k_o was allowed to vary between 0.56 and 62.73. As predicted by Eq. (5), Q_5 has a near perfect relationship with k_o (Figure 3a). Not surprisingly the Q_5 vs. k_m plot displays almost no scatter (Figure 3c), highlighting the fact that the fixed α method is appropriate when α does not vary much. However, the Q_5 vs. k_{dc} plot (Figure 3b) has a lot more scatter than the Q_5 vs. k_o plot (Figure 3a), which further strengthens the notion that the deccorelation method sometimes add noise rather than information to the analysis.

It should be recalled that the decorrelation method is also not applicable when k does not vary. The question that may invoke curiosity is what if k varies within a narrow range.

Moreover, the fixed exponent method is not expected to perform in such a scenario. To address these concerns, a set of recession curves was chosen with k_o ranging between 13.12 and 18.19 and α between 1.59 and 7.28. Although R_o^2 is 0.98 for the synthetic basin (Figure 3d), the plot does not seem to tell the actual relationship between k_o and Q_5 since according to Eq. (5) Q_5 should exhibit an inverse relationship with k_o, contrary to the direct relationship shown in the plot. Q_5 vs. k_{dc} plot, on the other hand, shows a combination of direct and inverse relationships between the two variables (Figure 3e), suggesting that the decorrelation method may change the fundamental nature of the relationship between the BN coefficient and streamflow. Figure 3f shows the plot between k_m and Q_5, which correctly describes the inverse relationship between the two variables, perhaps because of the way the fixed exponent method uses α to obtain k_m (see Subsection 2.4).

It should be acknowledged that the analyses conducted here might have been associated with several uncertainties such as those related to definition of recession curves and numerical errors associated with estimation of $-dQ/dt$ [*Stoelzle et al.*, 2012; *Roques et al.*, 2017; *Dralle et al.*, 2017; *Jachens et al.*, 2020]. However, there seems to be no reason to believe that the conclusions will radically alter if another study is conducted as the synthetic recession curves too led to the same conclusions. One may also wonder if the comparative analysis performed in this study itself is biased. To my knowledge no previous study has proposed a framework to objectively evaluate the decorrelation method. *Dralle et al.* [2015], who first time used the decorrelation method to estimate BN coefficient, provided visual (i.e. not quantitative) evidence that k_{dc} is better than k_o at explaining seasonal streamflow dynamics. This study, on the other hand, provides a framework to perform the same comparative analysis quantitatively.

4. Concluding remarks

Precise value of streamflow at any given time can be obtained if k, α and Q_0 are known. However, we need to focus on the scale (the unit of Q) if the objective is to extract hydrologic information only from k because rescaling of Q by a factor Z^{-1} will result in a new coefficient $k' = kZ^{\alpha-1}$. Depending on the value of Z, the correlation between k' and α may strengthen or weaken. *Dralle et al.* [2015] argued that the rescaling exercise will be truly effective if we choose a Z such that k' is completely decorrelated from α. On the contrary, this study analytically demonstrated that decorrelation is not dissociation, i.e. zero correlation between k' and α does not necessarily mean that k' is independent of α. To be precise, k' will be dissociated from α for a recession curve only if $Z = \widehat{Q}$. Since it is very unlikely that all the recession events of a basin will have the same $\widehat{Q}$, dissociation of k' from α, as the decorrelation method indents to do, by a single value of Z is not feasible. On the other hand, instead of attempting to decorrelate k from α, the fixed exponent method effectively utilizes α to obtain a coefficient (k_m) that provides better hydrological information.

To evaluate BN coefficient estimation methods, this study proposed a quantitative framework that appreciates the fact that the role of Q_0 diminishes with time t when $\alpha > 1$. The effectiveness of a BN coefficient estimation method can thus be characterized by the R^2 between BN coefficient and discharge at a large time. Analysis of observed as well as synthetic recession curves could not come across a single case for which the decorrelation method is more useful than the fixed exponent method, supporting the notion that decorrelation is not dissociation. Moreover, R^2_{dc} is not always greater than R^2_o, which raises additional doubts regarding the ability of the decorrelation method to explain streamflow dynamics. Using synthetic recession curves with special properties this study also threw some light on the workings of the two methods. Overall, this study asserts that it is not possible to obtain BN coefficients for a basin that are independent of the exponents. If the objective is to obtain coefficients that independently carry meaningful hydrological information, the fixed exponent method can do a reasonable job.

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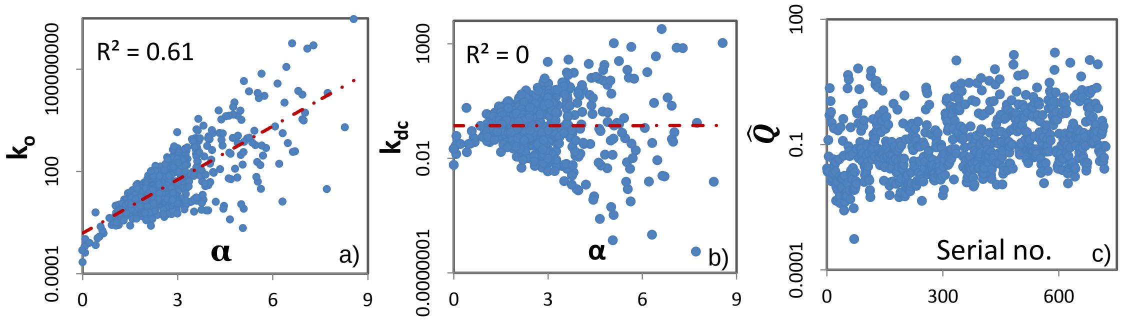

Figure 1. The correlation between k and α depends on the scale (unit) of measurement. a) The scatter plot between α and k_o (obtained from discharge values in mm/day) shows a robust relationship ($R^2 = 0.61$) exists between the two. With $Z_{dc} = 11.25$ the decorrelation method generated BN coefficients (k_{dc}s) having no correlation with αs (b). However, k_{dc} will be independent of α only when $Z_{dc} = \widehat{Q}$. Since $\widehat{Q}$ is expected to not remain constant for a basin, the decorrelation method cannot dissociate k and α (see Subsection 2.3). c) $\widehat{Q}$ vs. recession curve serial number plot for the basin shows wide variation of $\widehat{Q}$ across events. The figure was prepared using the results from a sample USGS basin (ID: 07160500).

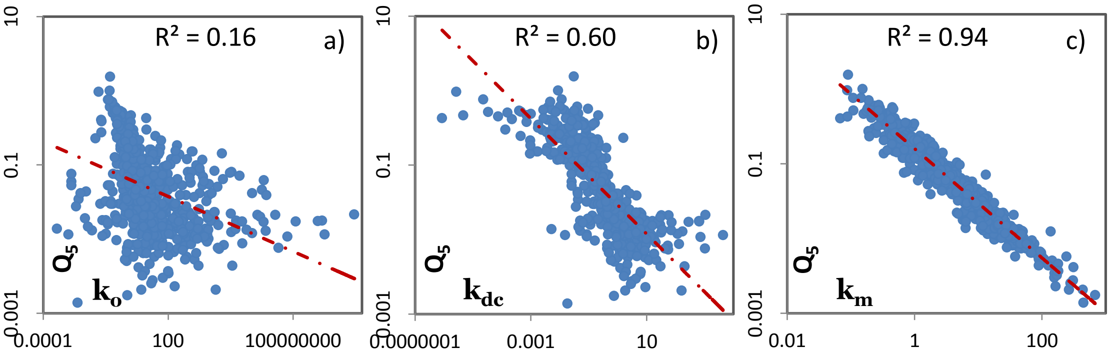

Figure 2. Comparision of the BN coefficient estimation methods for the sample basin. a) Q_5 (mm/day) vs. k_o (in mm$^{1-\alpha}$/day$^{2-\alpha}$) scatter plot exhibits very weak correlation, indicating that k_o alone cannot provide information on streamflow dynamics very effectively. b) Q_5-k_o relationship is substantially stronger, indicating the effectiveness of the decorrelation method for the basin. However, Q_5 vs. k_m plot displays the least amount of scatter, suggesting that the fixed exponent method is more reliable than the decorrelation method for this case.

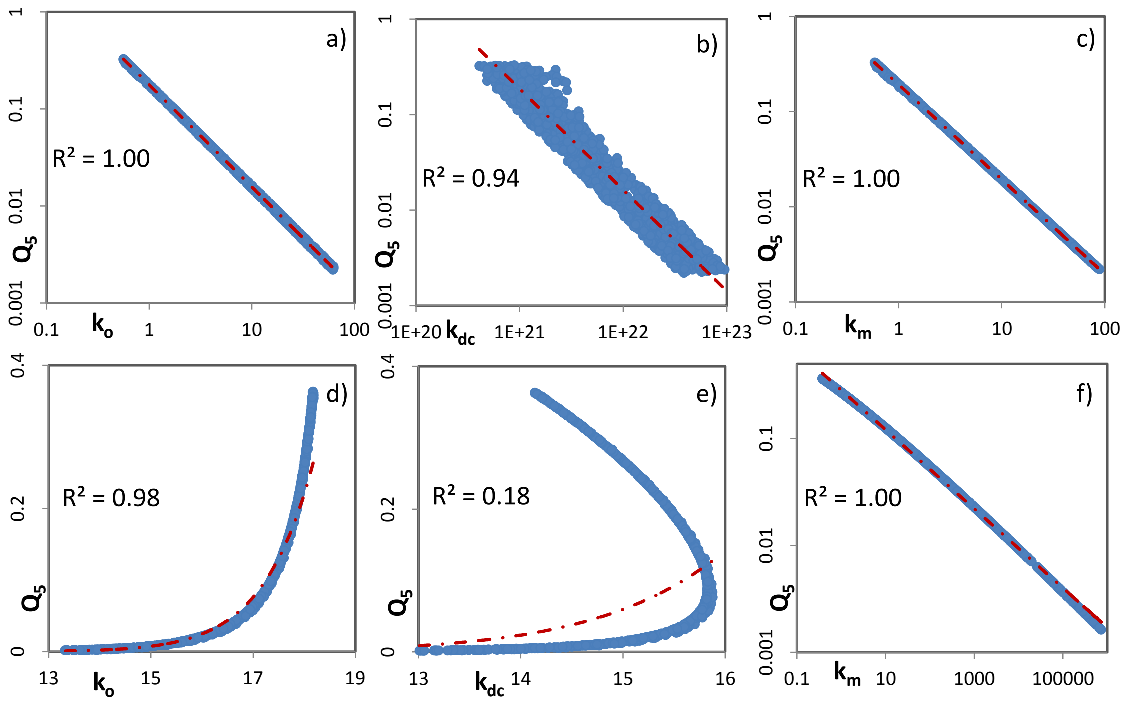

Figure 3. Results from the numerical experiments on two synthetic basins with special characteristics: one exhibits little α variation but wide k_o variation (a-c) and the other has the exactly opposite characteristics (d-f). a) Q_5 (mm/day) vs. k_o (in mm$^{1-\alpha}$/day$^{2-\alpha}$) scatter plot extends support to Eq. (5)). The fixed exponent method is expected to perform when α varies little (c). However, the decorrelation method ($Z_{dc} = e^{49}$) adds noise rather than information (b). d) Q_5 vs. k_o plot for the other synthetic basin shows the two variables exhibiting a direct relationship, in contrast to what Eq. (5) says. The decorrelation method with a weak rescaling factor ($Z_{dc} = 1.04$) corrected the relationship only partly (e). On the other hand, the fixed exponent method depicted the relationship quite well, suggesting that it can perform even when α exhibits wide variation.