

# **An Analytical Model for the Clear-Sky Longwave Feedback**

Daniel D.B. Koll<sup>a</sup>, Nadir Jeevanjee<sup>b</sup>, Nicholas J. Lutsko<sup>c</sup>

<sup>a</sup> *Laboratory for Climate and Ocean-Atmosphere Studies, Dept. of Atmospheric and Oceanic Sciences; Peking University; Beijing; China*

<sup>b</sup> *Geophysical Fluid Dynamics Laboratory; Princeton, NJ; USA*

<sup>c</sup> *Scripps Institution of Oceanography; La Jolla, CA; USA*

8 ABSTRACT: Estimates of Earth’s clear-sky longwave feedback from climate models and obser-  
9 vations robustly give a value of approximately  $-2 \text{ W m}^{-2} \text{ K}^{-1}$ , suggesting that this feedback can be  
10 estimated from first principles. Here we derive an analytical model for Earth’s clear-sky longwave  
11 feedback based on a novel spectral decomposition that splits the feedback into components from  
12 surface emission,  $\text{CO}_2$ ,  $\text{H}_2\text{O}$ , and the  $\text{H}_2\text{O}$  continuum. Analytic expressions are given for each  
13 of these terms based on their underlying physics, and the model can also be framed in terms  
14 of Simpson’s Law and deviations therefrom. We validate the model by reproducing line-by-line  
15 radiative transfer calculations across a wide range of climates, as well as the spatial dependence of  
16 the clear-sky feedback from radiative kernels. The latter result motivates us to estimate the spatial  
17 pattern of Earth’s clear-sky longwave feedback from reanalysis data, which shows good agreement  
18 with climate model data. Together, these results show that Earth’s clear-sky longwave feedback,  
19 its spatial variations, and its state-dependence across past and future climates can be successfully  
20 understood from only a handful of physical principles.

21 SIGNIFICANCE STATEMENT: The climate feedback determines how much our planet warms  
22 due to changes in radiative forcing. For more than 50 years scientists have been predicting this  
23 feedback using complex numerical models. Except for cloud effects the numerical models widely  
24 agree, lending confidence to global warming predictions, but nobody has yet derived the feedback  
25 from simpler considerations. We show that Earth’s clearsky longwave feedback can be obtained  
26 using only basic spectroscopy and atmospheric physics. Our results confirm that numerical climate  
27 models get the right number for the right reasons, and allow us to explain regional and state  
28 variations of Earth’s climate feedback. These variations are difficult to understand solely from  
29 numerical models but are crucial for past and future climates.

## 30 1. Introduction

31 Although uncertainty in Earth’s climate sensitivity is largely driven by cloud feedbacks, its  
32 magnitude is dominated by the clear-sky longwave feedback,  $\lambda_{LW}$ . Observational estimates and  
33 climate models give a fairly narrow range between  $-1.8$  and  $-2.2 \text{ W m}^{-2} \text{ K}^{-1}$  for  $\lambda_{LW}$  (Andrews et al.  
34 2012; Chung et al. 2010; Kluft et al. 2019; Zhang et al. 2020a; Zelinka et al. 2020) and even early  
35 studies estimated  $\lambda_{LW}$  to be  $-2.2$ - $2.3 \text{ W m}^{-2} \text{ K}^{-1}$  (Manabe and Wetherald 1967; Budyko 1969),  
36 which is impressively close to the current best estimates. By contrast, the recent Sherwood et al.  
37 (2020) assessment estimated the total cloud feedback to be significantly smaller and less certain at  
38  $+0.45 \pm 0.33 \text{ W m}^{-2} \text{ K}^{-1}$ .

39 The robustness of the clear-sky longwave feedback suggests that it can be constrained from first  
40 principles. A simple model for  $\lambda_{LW}$ , grounded in the basic physics of radiative transfer, would  
41 provide definitive support for the value of  $-2 \text{ W m}^{-2} \text{ K}^{-1}$  derived from observations and climate  
42 models. It would also allow us to understand the state-dependence of  $\lambda_{LW}$ : at warm enough  
43 temperatures Earth’s atmosphere transitions to a runaway state, in which  $\lambda_{LW}$  becomes zero or  
44 even changes sign, but it is unclear how  $\lambda_{LW}$  varies between today’s value and the runaway limit.  
45 Similarly, there is a long-standing interest in using paleoclimate proxies to constrain climate sensi-  
46 tivity (Tierney et al. 2020), but this effort suffers from uncertainty regarding the state-dependence  
47 of climate feedbacks. At the local level, a simple model for  $\lambda_{LW}$  would explain the geographic  
48 variation of the local clear-sky longwave feedback and hence the contribution of the clear-sky feed-

back to the time-evolution of the global-mean climate feedback parameter in response to changing patterns of warming (Meraner et al. 2013; Armour et al. 2013; Andrews et al. 2015, 2018).

The earliest simple model for  $\lambda_{LW}$  was proposed by Simpson (1928), who suggested that an atmosphere that is optically thick due to water vapor absorption, but not in a runaway state, would still have a clear-sky longwave feedback that is approximately zero. Ingram (2010) resolved this “paradox” by separating out the parts of Earth’s outgoing radiation spectrum that are optically thick due to water vapor (and for which  $\lambda_{LW}$  is approximately zero) from the optically thin “window” region. Koll and Cronin (2018) subsequently quantified Ingram’s argument: using fixed relative humidity (RH), single-column calculations they showed that the net clear-sky longwave feedback is dominated by the surface component of the Planck feedback,  $\lambda_{LW} \approx \lambda_{\text{surf}}$ , which is simply the increase in surface emission that is able to escape to space through the spectral window. It follows that the other clear-sky longwave feedbacks – the atmospheric component of the Planck feedback, the lapse-rate feedback and the water vapor feedback – roughly cancel (Koll and Cronin 2018; Jeevanjee et al. 2021a). In a follow-up study, Zhang et al. (2020b) demonstrated that these fixed-RH results also apply to global climate model simulations, so long as the global histogram of column RH remains invariant under warming.

The match between  $\lambda_{LW}$  and the surface component of the Planck feedback is not exact, however. Raghuraman et al. (2019) found in radiative calculations of present-day Earth’s tropics that the surface Planck feedback  $\lambda_{\text{surf}}$  only accounts for about 60% of  $\lambda_{LW}$ , implying a gap in the argument of Koll and Cronin (2018). Similarly, Seeley and Jeevanjee (2021) showed that in hot, high- $\text{CO}_2$  climates  $\lambda_{\text{surf}}$  becomes negligible yet  $\lambda_{LW}$  does not go to zero. As the surface warms the atmosphere is still able to increase its emission to space in spectral regions that are dominated by  $\text{CO}_2$ . This emission mostly comes from the upper atmosphere, and gives rise to a spectral  $\text{CO}_2$  “radiator fin” feedback. The existence of the  $\text{CO}_2$  radiator fin feedback means  $\lambda_{LW}$  must depend on  $\text{CO}_2$  concentration, and thus must have  $\text{CO}_2$  state-dependence. Moreover, because  $\text{CO}_2$  predominantly radiates energy to space from the upper parts of the atmosphere, the magnitude of the  $\text{CO}_2$  feedback should be sensitive to additional parameters such as the atmospheric lapse rate. So while  $\lambda_{\text{surf}}$  gives a reasonable first-order approximation to  $\lambda_{LW}$ , more terms are needed to describe  $\lambda_{LW}$  quantitatively.

78 In this study, we aim to provide a simple model of Earth’s feedback that can quantitatively  
 79 capture the magnitude of  $\lambda_{LW}$  as well as its state-dependence and regional variations. The model is  
 80 derived from line-by-line radiative transfer equations, and decomposes  $\lambda_{LW}$  into a surface Planck  
 81 feedback ( $\lambda_{\text{surf}}$ ), a CO<sub>2</sub> “radiator fin” feedback ( $\lambda_{\text{CO}_2}$ ), a non-Simpsonian water vapor band feedback  
 82 ( $\lambda_{\text{H}_2\text{O}}$ ), and a destabilizing water vapor continuum feedback ( $\lambda_{\text{cnt}}$ ). Although these feedbacks are  
 83 less familiar, they represent the spectroscopic factors governing the longwave feedback, and are  
 84 amenable to analysis. As shown below, expressions can be derived for each of the spectral  
 85 feedbacks starting from the basic equations of radiative transfer and idealized models of H<sub>2</sub>O and  
 86 CO<sub>2</sub> spectroscopy. These expressions can be interpreted as a global-mean model for  $\lambda_{LW}$  or in  
 87 terms of local feedbacks (Feldl and Roe 2013; Armour et al. 2013; Bloch-Johnson et al. 2020).  
 88 That is, we treat each atmospheric column as an isolated 1D system whose longwave feedback only  
 89 depends on its local surface temperature. We validate the model (and the utility of the spectral  
 90 decomposition) by comparing it against two sets of calculations, one with a line-by-line radiation  
 91 code and one with a radiative kernel.

92 Our model of  $\lambda_{LW}$  is based on spectroscopic thinking and hence represents a different perspective  
 93 than the traditional decomposition which breaks the clear-sky longwave feedback into Planck,  
 94 Lapse-rate and Water Vapor feedbacks (e.g., Soden et al. 2008; Sherwood et al. 2020; Zelinka  
 95 et al. 2020). The traditional decomposition has been an invaluable tool for understanding  $\lambda_{LW}$  and  
 96 for diagnosing the physics governing outgoing longwave radiation in climate models. However,  
 97 the traditional feedback decomposition is also deceptive, since it obscures large cancellations  
 98 between the atmospheric part of the Planck feedback, the Lapse-rate feedback and the Water Vapor  
 99 feedback (Held and Shell 2012; Koll and Cronin 2018; Jeevanjee et al. 2021a). By obscuring the  
 100 cancellations, the traditional decomposition hides the basic physics governing  $\lambda_{LW}$  and can give a  
 101 false impression of the uncertainty of climate models. The same cancellations also make it difficult  
 102 to understand the state-dependence of  $\lambda_{LW}$  – Planck, Lapse-rate and Water Vapor feedbacks all  
 103 increase in a warmer climate, but it is far from obvious how these changes add up to affect  $\lambda_{LW}$ .  
 104 Building on previous discussions of spectral feedbacks (e.g., Huang et al. 2010, 2014; Pan and  
 105 Huang 2018; Seeley and Jeevanjee 2021; Kluft et al. 2021), our goal in this paper is to show that  
 106 these issues with the traditional decomposition can be resolved by viewing  $\lambda_{LW}$  in terms of its  
 107 spectral components instead.

108 The layout of the rest of this paper is as follows. Section 2 discusses several preliminaries  
 109 which are necessary for the main derivations, namely: an idealized Clausius-Clapeyron relation,  
 110 an analytical approximation for moist lapse rates and idealized band models for H<sub>2</sub>O and CO<sub>2</sub>  
 111 spectroscopy. Section 3 lays out our spectral framework and introduces the emission-level ap-  
 112 proximation, our spectral decomposition of  $\lambda_{LW}$ , and a description of our numerical line-by-line  
 113 calculations. Section 4 derives analytical expressions for Earth’s emission temperature in different  
 114 parts of the spectrum, which are then used in Section 5 to derive analytical feedbacks. Our ex-  
 115 pressions compare favorably against the state-dependence of  $\lambda_{LW}$  from line-by-line calculations.  
 116 Next, Section 6 uses these results to understand the spatial pattern of Earth’s clear-sky longwave  
 117 feedback. We first generate global maps of Earth’s clear-sky longwave feedback using a radiative  
 118 kernel plus climate model data from a preindustrial and a warmed climate. We show that our  
 119 analytical expressions recover the same feedback patterns when using the same climate model  
 120 data. Moreover, similar patterns can also be obtained solely from reanalysis data, which means the  
 121 spatial pattern of  $\lambda_{LW}$  is largely predictable only using knowledge of Earth’s current climate. The  
 122 manuscript closes in Section 7 with a conclusion and broader discussion of the results.

## 123 2. Preliminaries

124 Our goal is to derive the longwave feedback of a cloud-free vertical column of atmosphere.  
 125 The column’s state can be specified using five parameters:  $T_s$ ,  $\gamma_{lr}$ , RH,  $q_{CO_2}$  and  $T_{strat}$ .  $T_s$  is the  
 126 column’s surface temperature,  $\gamma_{lr} \equiv d \ln T / d \ln p$  is the column’s temperature lapse rate, RH is the  
 127 column relative humidity,  $q_{CO_2}$  is the column’s CO<sub>2</sub> mass mixing ratio and  $T_{strat}$  is the stratospheric  
 128 temperature. We idealize the state of the vertical column by treating  $\gamma_{lr}$ , RH, and  $q_{CO_2}$  as bulk  
 129 parameters which are uniform in the vertical dimension; all are defined more precisely below.  
 130 Similarly, we approximate the stratosphere as isothermal.

### 131 a. Clausius-Clapeyron

136 The Clausius-Clapeyron relation governs the temperature-dependence of the saturation vapor  
 137 pressure  $e^*(T)$  and is an essential element of our analytical model. The Clausius-Clapeyron  
 138 relation is often solved by ignoring the temperature-dependence of the latent heat of vaporization,

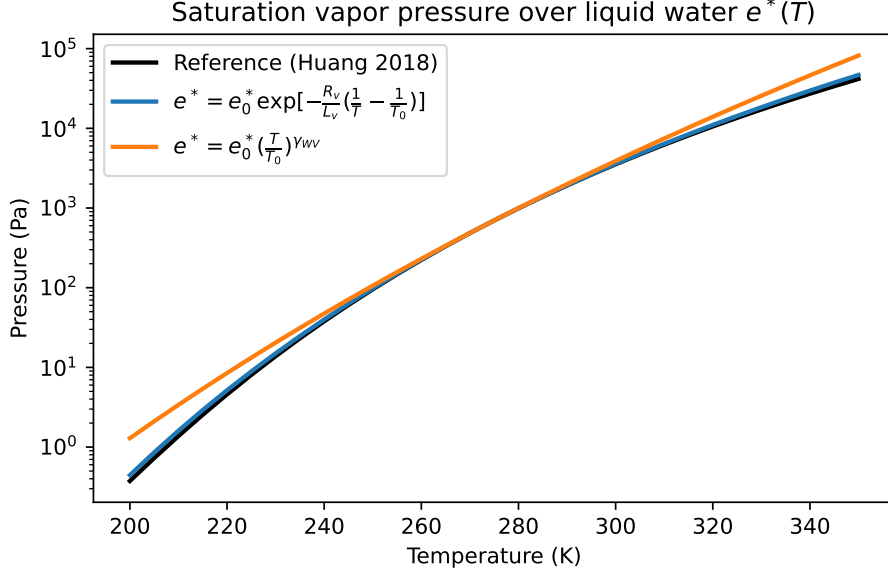


FIG. 1. Different approximations to the Clausius-Clapeyron relation. Black: fit based on experimental data (Huang 2018). Blue: the commonly-used quasi-exponential approximation. Orange: the power law approximation used in this work. The saturation vapor pressure is with respect to liquid water. In this plot  $(T_0, e_0^*)$  are set equal to the triple point values of  $\text{H}_2\text{O}$ , so  $\gamma_{\text{wv}} = 19.8$ .

$d \ln e^* / d \ln T = L_v(T) / (R_v T) \approx L_v(T_0) / (R_v T)$ , which leads to the quasi-exponential approximation

$$e^* \approx e_0^*(T_0) \exp \left[ -\frac{L_v(T_0)}{R_v} \left( \frac{1}{T} - \frac{1}{T_0} \right) \right]. \quad (1)$$

This quasi-exponential form does not lead to closed-form analytical expressions in the equations of radiative transfer, however, so we require a simpler form of the Clausius-Clapeyron relation. We obtain this by approximating the Clausius-Clapeyron relation further as  $d \ln e^* / d \ln T = L_v(T) / (R_v T) \approx \text{const}$ , which leads to a simple power law between temperature and saturation vapor pressure (Koll and Cronin 2019),

$$e^* \approx e_0^*(T_0) \left( \frac{T}{T_0} \right)^{\gamma_{\text{wv}}}, \quad (2)$$

where

$$\gamma_{\text{wv}} \equiv \frac{L_v(T_0)}{R_v T_0}. \quad (3)$$

Here  $T_0$  is an arbitrary reference temperature around which we are approximating the saturation vapor pressure as a power law. We emphasize that  $T_0$  is effectively a thermodynamic constant and does not change with surface warming. The non-dimensional power law exponent is generally large and reflects the steep rise of  $e^*$  with temperature; at Earth-like temperatures,  $\gamma_{\text{wv}} \approx 20$ . The fractional increase in saturation vapor pressure per unit warming is  $d \ln e^* / dT = \gamma_{\text{wv}} / T \sim 7\%/\text{K}$ , in line with other Clausius-Clapeyron approximations.

Figure 1 compares the approximations in Equations 1 and 2 against a fit that is based on experimental data (Huang 2018). Considering that a typical tropical atmospheric column spans the vertical temperature range 200 – 300 K, the quasi-exponential approximation is quite accurate, whereas our power law approximation only matches more detailed fits to within a factor of two or so. Nevertheless, as shown below, this accuracy is good enough to match numerical calculations.

### b. Bulk moist lapse rate

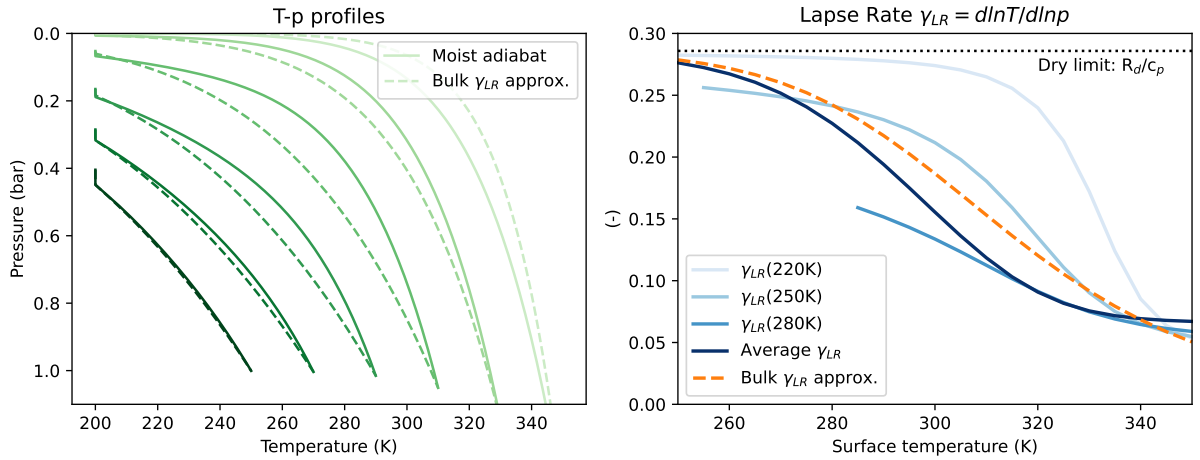


FIG. 2. Moist adiabatic lapse rates versus our analytical approximation. Left: Pressure-Temperature profiles following a moist adiabat (solid) and following the bulk lapse rate approximation (dashed). Right: numerical lapse rates computed at different temperature levels inside the troposphere (light blue), compared with the bulk approximation in Equation (8) (orange). The average of the numerical lapse rates  $\gamma_{\text{lr}}$  (dark blue) is a mass-weighted mean taken over the troposphere.



163 The vertical temperature-pressure profile of an atmospheric column can be specified via the  
 164 lapse-rate exponent

$$\gamma_{lr} = d \ln T / d \ln p, \quad (4)$$

165 where  $p$  is pressure. In the limit of a completely dry column the lapse rate exponent is vertically  
 166 uniform,  $\gamma_{lr} = R_d/c_p \approx 2/7$ . For a moist atmosphere  $\gamma_{lr}$  varies both as a function of temperature  
 167 and pressure, but due to the latent heat release in a convecting parcel it is generally smaller than  
 168 the dry lapse rate:  $\gamma_{lr} \leq R_d/c_p$ .

169 In order to obtain analytically tractable expressions we would like to treat  $\gamma_{lr}$  as constant in the  
 170 vertical even for a moist column, so we diagnose a bulk  $\gamma_{lr}$  using the surface and tropopause values  
 171 of  $(T, p)$ :

$$\gamma_{lr} \approx \frac{\ln(T_{tp}/T_s)}{\ln(p_{tp}/p_s)}. \quad (5)$$

172 Assuming that the tropopause temperature stays constant in response to surface temperature  
 173 changes, in accord with the FAT/FiTT hypothesis (Hartmann and Larson 2002; Seeley et al.  
 174 2019), then all that is needed is an expression for how  $p_{tp}$  depends on  $T_s$ . We can derive such an  
 175 expression by first obtaining an expression for the tropopause height  $z_{tp}$ , following Romps (2016).  
 176 From MSE conservation along an undilute moist adiabat between the surface and tropopause,

$$z_{tp} \approx \frac{1}{g} (c_p(T_s - T_{tp}) + L_v q_s^*), \quad (6)$$

177 where  $q_s^*$  is the mass mixing ratio of water at saturation,  $q^*$ , evaluated at the surface and we neglect  
 178  $q^*$  at the tropopause.  $p_{tp}$  can then be obtained as

$$p_{tp} = p_s e^{-z_{tp}/H}, \quad (7)$$

179 where  $H$  is the scale height of pressure ( $= \frac{R_d T_{av}}{g}$ ) and  $T_{av} \equiv (T_s + T_{tp})/2$ . Plugging this into (5) yields

$$\gamma_{lr} \approx \frac{R_d T_{av} \ln(T_s/T_{tp})}{c_p(T_s - T_{tp}) + L_v q_s^*}. \quad (8)$$

180 One can show that Equation 8 correctly reduces to the dry lapse rate  $\gamma_{lr} = R_d/c_p$  by setting  $q_s^* = 0$   
 181 and series expanding the logarithm, assuming  $T_s - T_{tp} \ll T_{tp}$ . In practice the latter assumption is

not strictly true but the resulting deviation from the dry adiabat is small even for a 100 K difference between surface and tropopause.

According to the bulk approximation,  $\gamma_{\text{lr}}$  is constant in the vertical and only varies in response to climatic changes (e.g., changes in surface or tropopause temperature). We can therefore write the column’s vertical temperature-pressure profile as a power law,

$$T(p) = T_s \left( \frac{p}{p_s} \right)^{\gamma_{\text{lr}}} . \quad (9)$$

Figure 2 (left) compares temperature-pressure profiles based on the bulk lapse rate approximation to moist adiabatic profiles. The moist adiabats are obtained by numerically integrating a generalized form of the moist adiabat which does not approximate water vapor as a dilute substance and thus remains valid at high temperatures (Ding and Pierrehumbert 2016). The temperature profiles given by Equation 9 produce a reasonable fit to the moist adiabatic profiles, though they tend to be colder throughout most of the troposphere than the moist adiabats at surface temperatures below 340 K. The tropopause temperatures and pressures are accurately reproduced by the power law profiles.

Figure 2 (right) shows the resulting  $T_s$ -dependence of  $\gamma_{\text{lr}}$ . The analytical approximation captures the  $T_s$ -dependence of a numerical mass-weighted column-average  $\gamma_{\text{lr}}$  relatively well, though the general agreement can obscure significant differences at individual atmospheric levels. For example, our analytical approximation of  $\gamma_{\text{lr}}$  deviates by more than a factor of two from the moist-adiabatic  $\gamma_{\text{lr}}$  at the atmospheric level that corresponds to  $T = 220$  K. We will show below that these details of atmospheric lapse rates do not have a major impact on Earth’s longwave feedback at low surface temperatures, but they become increasingly important above  $\sim 300$  K.

### c. $H_2O$ and $CO_2$ spectroscopy

The third ingredient for our derivations is a model of  $H_2O$  and  $CO_2$  spectroscopy. We follow previous studies and model the absorption cross-sections of  $H_2O$  and  $CO_2$  as log-linear band shapes. Despite the simplicity of these models, they are able to explain numerous features of Earth’s climate, including the logarithmic nature of  $CO_2$  forcing, the temperature dependence of Earth’s surface feedback and the vertical structure of radiative cooling (Crisp et al. 1986; Pierrehumbert 2010; Wilson and Gea-Banacloche 2012; Koll and Cronin 2018; Jeevanjee and Fueglistaler 2020; Romps et al. 2022). Because we explore feedbacks over a wide range of temperatures, we additionally

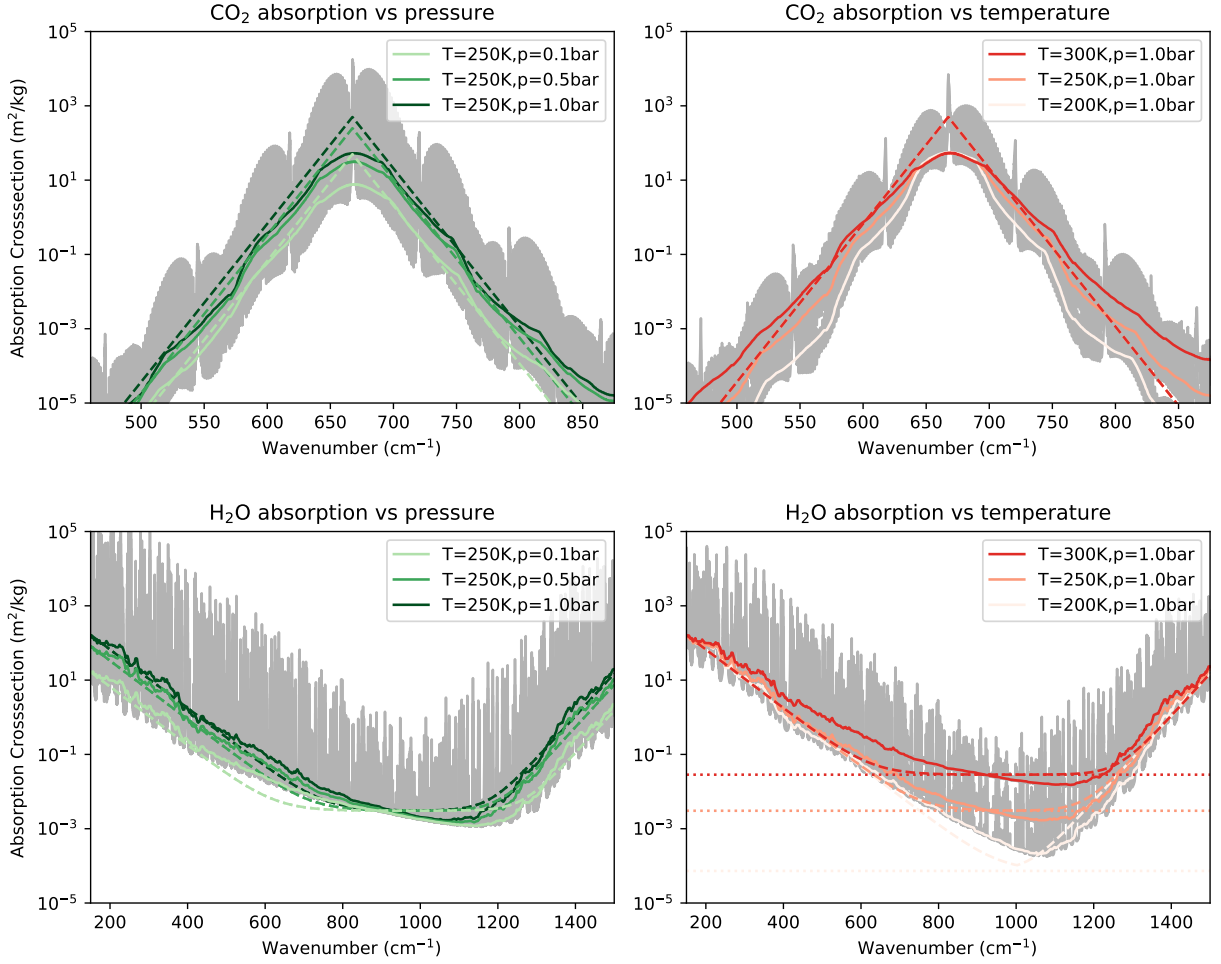


FIG. 3. Idealized band models compared against the absorption cross-sections of CO<sub>2</sub> (top row) and H<sub>2</sub>O (bottom). Grey envelopes show cross-sections computed at line-by-line spectral resolution, solid lines are the cross-sections smoothed by a median filter with width 25 cm<sup>-1</sup>. Dashed lines are our band models for CO<sub>2</sub> and H<sub>2</sub>O bands (the sum of line and continuum absorption), while dotted lines show the grey H<sub>2</sub>O continuum model only.

need to account for the H<sub>2</sub>O continuum. We do so by approximating the continuum as a grey absorber.

For CO<sub>2</sub> line absorption the absorption cross-section is

$$\kappa_{\text{CO}_2} = \kappa_0 \left( \frac{p}{p_0} \right) \exp \left( -\frac{|\nu - \nu_0|}{l_\nu} \right), \quad (10)$$

where  $\kappa_0$  is the absorption cross-section in the center of the band,  $p_0$  is a reference pressure,  $\nu$  is wavenumber,  $\nu_0$  the wavenumber of the center of the band and  $l_\nu$  the decay rate of the absorption cross-section in wavenumber space. Previous work fitted these parameters to the CO<sub>2</sub> absorption spectrum at a reference pressure of  $p_0 = 0.1$  bar (Jeevanjee et al. 2021b), here we rescale the fits to a reference pressure of  $p_0 = 1$  bar. The resulting values are  $\kappa_0 = 500 \text{ m}^2/\text{kg}$ ,  $\nu_0 = 667.5 \text{ cm}^{-1}$  and  $l_\nu = 10.2 \text{ cm}^{-1}$ .

H<sub>2</sub>O line absorption can similarly be modeled using a log-linear shape, though one has to account for the fact that H<sub>2</sub>O has two bands which are relevant for Earth's longwave feedback. The rotation band determines H<sub>2</sub>O absorption at wavenumbers less than  $1000 \text{ cm}^{-1}$ , the vibration-rotation band at wavenumbers larger than  $1000 \text{ cm}^{-1}$ . We model these two bands as

$$\kappa_{\text{H}_2\text{O},line} = \left(\frac{p}{p_0}\right) \max \left[ \kappa_{\text{rot}} \exp\left(-\frac{|\nu - \nu_{\text{rot}}|}{l_{\text{rot}}}\right), \kappa_{\text{v-r}} \exp\left(-\frac{|\nu - \nu_{\text{v-r}}|}{l_{\text{v-r}}}\right) \right]. \quad (11)$$

The factor  $p/p_0$  in front of both H<sub>2</sub>O and CO<sub>2</sub> cross-sections reflects pressure broadening: under present-Earth conditions CO<sub>2</sub> and H<sub>2</sub>O absorption lines become wider due to collisions of those molecules with the background air (N<sub>2</sub> or O<sub>2</sub>). This has the overall effect that both gases become more efficient absorbers at higher pressure.

In contrast to the CO<sub>2</sub> and H<sub>2</sub>O bands, the H<sub>2</sub>O continuum is dominated by self broadening so the continuum cross-section is independent of pressure and instead scales as  $\propto e = \text{RHe}^*$ . Although continuum absorption in reality is not uniform with respect to wavenumber, its spectral dependence is significantly weaker than the H<sub>2</sub>O or CO<sub>2</sub> bands. We therefore approximate the continuum as a grey absorber and write

$$\kappa_{\text{H}_2\text{O},cnt} = \kappa_{\text{cnt}} \text{RH} \frac{e^*(T)}{e_0^*} \left(\frac{T}{T_0}\right)^{-a}, \quad (12)$$

where the dimensionless exponent  $a$  captures the direct temperature dependence which acts to weaken the continuum (Pierrehumbert 2010). The total H<sub>2</sub>O cross-section is the sum of line and continuum absorption,  $\kappa_{\text{H}_2\text{O}} = \kappa_{\text{H}_2\text{O},line} + \kappa_{\text{H}_2\text{O},cnt}$ . Because the line opacity decreases exponentially away from H<sub>2</sub>O band centers, the total opacity becomes largely dominated by the continuum in the window region around  $\sim 1000 \text{ cm}^{-1}$ .

Our model of H<sub>2</sub>O spectroscopy has eight parameters:  $\kappa_{\text{rot}}$ ,  $l_{\text{rot}}$ ,  $\nu_{\text{rot}}$ ,  $\kappa_{\text{v-r}}$ ,  $l_{\text{v-r}}$ ,  $\nu_{\text{v-r}}$ ,  $\kappa_{\text{cnt}}$ ,  $a$ . We set  $\nu_{\text{rot}} = 150 \text{ cm}^{-1}$  and  $\nu_{\text{v-r}} = 1500 \text{ cm}^{-1}$ , and fit the remaining parameters using the median-smoothed H<sub>2</sub>O cross-sections shown in Figure 3 across the wavenumber range  $150 \text{ cm}^{-1} \leq \nu \leq 1500 \text{ cm}^{-1}$ . The results are sensitive to the smoothing procedure, that is whether one uses a geometric mean or a median. Because the average transmission across a spectral band tends to be dominated by the most optically thin frequencies (Pierrehumbert 2010), we use a median filter. To perform the fits we use the non-linear least-squares algorithm `scipy.optimize.curve_fit`, with a reference temperature of  $T_0=300 \text{ K}$ . We first fit the parameters  $\kappa_{\text{rot}}$ ,  $l_{\text{rot}}$ ,  $\kappa_{\text{v-r}}$ ,  $l_{\text{v-r}}$  to H<sub>2</sub>O line opacities only, and then use these parameters to fit  $\kappa_{\text{cnt}}$  and  $a$  to H<sub>2</sub>O cross-sections that include both line and continuum opacity. The resulting values are  $\kappa_0 = 165 \text{ m}^2/\text{kg}$ ,  $l_0 = 55 \text{ cm}^{-1}$ ,  $\kappa_1 = 15 \text{ m}^2/\text{kg}$ ,  $l_1 = 38 \text{ cm}^{-1}$ ,  $\kappa_{\text{cnt}} = 3 \times 10^{-3} \text{ m}^2/\text{kg}$  and  $a = 7$ , which broadly match the H<sub>2</sub>O fits previously reported in Jeevanjee and Fueglistaler (2020). Table 1 summarizes the thermodynamic and spectral parameters used in this paper.

Figure 3 compares the idealized band models with line-by-line absorption cross-sections. Overall, the shape of the cross-sections is captured fairly well. The median CO<sub>2</sub> and H<sub>2</sub>O cross-sections scale linearly with total pressure, as expected for pressure-broadening. The increasing H<sub>2</sub>O absorption in response to warming around  $1000 \text{ cm}^{-1}$  is also qualitatively captured by our grey continuum model, even though the H<sub>2</sub>O continuum itself is actually not grey.

Figure 3 (right plots) also shows that the slopes of the CO<sub>2</sub> and H<sub>2</sub>O bands flatten as temperature increases, with roughly constant opacity in the band centers but increasing opacity in the band wings. This behavior is not captured by our simple models. Physically, absorption band slopes can depend on temperature due to the shifting population of different molecular excitation states. For example, the wings of the  $667 \text{ cm}^{-1}$  CO<sub>2</sub> band consist of multiple smaller bands that correspond to transitions between excited states of CO<sub>2</sub> (so-called hot bands). In contrast, the center of the CO<sub>2</sub> band is dominated by transitions to/from the ground state of CO<sub>2</sub>. As temperature rises more CO<sub>2</sub> molecules leave the ground state and access excited states, which increases the probability of transitions involving excited states, which in turn preferentially increases the opacity in the wings of the CO<sub>2</sub> band. To keep our parameterizations simple, however, we do not attempt to model the temperature dependence of the band slopes.

TABLE 1. List of parameters and, where applicable, assumed values.

Parameter name	Explanation	Assumed value
Thermodynamic parameters		
$T_0$	Reference temperature for saturation vapor pressure power-law	300 K
$\gamma_{wv}$	Exponent in saturation vapor pressure power-law	18
$\gamma_{lr}$	Exponent in bulk lapse rate temperature-pressure power-law	Fit to data, or computed using Eqn. 8
Spectral parameters		
$p_0$	Reference pressure for absorption cross-sections	1bar
$\kappa_0$	Absorption cross-section in center of CO <sub>2</sub> band	500 m <sup>2</sup> /kg
$\nu_0$	Wavenumber of the center of the CO <sub>2</sub> band	667.5 cm <sup>-1</sup>
$l_\nu$	Decay rate of the CO <sub>2</sub> absorption cross-section in wavenumber space	10.2 cm <sup>-1</sup>
$\kappa_{rot}$	Absorption cross-section in center of H <sub>2</sub> O rotation band	165 m <sup>2</sup> /kg
$\nu_{rot}$	Wavenumber of the center of the H <sub>2</sub> O rotation band	150 cm <sup>-1</sup>
$l_{rot}$	Decay rate of the H <sub>2</sub> O absorption cross-section in wavenumber space in the rotation band	55 cm <sup>-1</sup>
$\kappa_{v-r}$	Absorption cross-section in center of H <sub>2</sub> O vibration-rotation band	15 m <sup>2</sup> /kg
$\nu_{v-r}$	Wavenumber of the center of the H <sub>2</sub> O vibration-rotation band	1500 cm <sup>-1</sup>
$l_{v-r}$	Decay rate of the H <sub>2</sub> O absorption cross-section in wavenumber space in the vibration-rotation band	38 cm <sup>-1</sup>
$\kappa_{rot}$	Grey absorption cross-section of H <sub>2</sub> O continuum	$3 \times 10^{-3}$ m <sup>2</sup> /kg
$a$	Exponent of H <sub>2</sub> O continuum temperature dependence	7
Analytic model parameters		
$T_{strat}$	Stratospheric temperature	200 K
$c_{surf}$	Scaling constant for surface feedback	0.8 (bulk lapse rate)/0.8 (moist adiabat)
$c_{H_2O}$	Scaling constant for H <sub>2</sub> O feedback	0.6 (bulk lapse rate)/1.0 (moist adiabat)
$c_{cnt}$	Scaling constant for continuum feedback	0.4 (bulk lapse rate)/0.4 (moist adiabat)
$c_{CO_2}$	Scaling constant for CO <sub>2</sub> feedback	0.7 (bulk lapse rate)/0.9 (moist adiabat)

### 3. Spectral Framework

#### *a. The emission-level approximation*

To decompose the net longwave feedback into multiple spectroscopic feedbacks we first need to consider the spectrally-integrated outgoing longwave flux (OLR) of a vertical column. Using the monochromatic optical thickness  $\tau$  as the vertical coordinate, which varies from  $\tau = 0$  at the TOA to  $\tau = \tau_{LW}$  at the surface, and the wavenumber  $\nu$  as the spectral coordinate, OLR is equal to (e.g. Pierrehumbert 2010)

$$OLR = \int_0^\infty \pi B_\nu(T_s) e^{-\tau_{LW}} d\nu + \int_0^\infty \int_0^{\tau_{LW}} \pi B_\nu(T(\tau)) e^{-\tau} d\tau d\nu. \quad (13)$$

The optical thicknesses  $\tau$  and  $\tau_{LW}$  are functions of  $\nu$ , so the order of integration cannot be switched. Physically, the first term corresponds to the surface's emission to space, while the second term corresponds to an integral of the emission coming from each vertical level in the atmosphere.

280 The emission-level or radiating-level approximation states that the atmosphere's emission to  
 281 space (the second integral in Equation 13) originates from the vertical level at which optical  
 282 thickness  $\tau$  is order unity. The intuition behind the emission-level approximation is that levels of  
 283 the atmosphere for which  $\tau \ll 1$  are so optically thin that they do not contribute much to the TOA  
 284 flux, while most emission from levels with  $\tau \gg 1$  is absorbed by the overlying atmosphere and  
 285 so its contribution to the TOA flux is also small. The emission level has been defined at slightly  
 286 different values of  $\tau$ , but all definitions agree on a value that is of order unity (Pierrehumbert 2010;  
 287 Jeevanjee et al. 2021b). For simplicity, we define the emission level here as the level at which  
 288  $\tau = 1$ . The temperature at this level is then the emission level temperature,  $T_{\text{rad}} \equiv T(\tau = 1)$ , so

$$\text{OLR} \approx \int_0^\infty \pi B_\nu(T_s) e^{-\tau_{LW}} d\nu + \int_0^\infty \pi B_\nu(T_{\text{rad}}(\nu)) d\nu. \quad (14)$$

289 Given the emission-level approximation, the clear-sky longwave feedback is determined by how  
 290 the surface emission and the atmospheric emission change in response to warming,

$$\begin{aligned} -\lambda_{LW} &= \frac{d\text{OLR}}{dT_s} \\ &\approx \int_0^\infty \pi \frac{dB_\nu}{dT} \Big|_{T_s} e^{-\tau_{LW}} d\nu + \int_0^\infty \pi \frac{dB_\nu}{dT} \Big|_{T_{\text{rad}}} \frac{dT_{\text{rad}}}{dT_s} d\nu. \end{aligned} \quad (15)$$

291 The minus sign ensures consistency with the sign convention used in most climate studies: OLR  
 292 typically increases in response to surface warming, so  $\lambda_{LW} < 0$ . Note that Equation 15 does not  
 293 contain any terms  $\propto d\tau_{LW}/dT_s$  because the resulting contribution to change in the surface emission  
 294 decreases with warming at exactly the same rate as the atmospheric emission increases (this can  
 295 be seen by differentiating Eqn. 13 first before applying the emission-level approximation).

### 296 *b. Spectral feedback decomposition*

297 The net feedback in Equation 15 can be decomposed into multiple spectral regions or bands.  
 298 The surface term dominates in the window region where  $\tau_{LW} < 1$  and the feedback is primarily a  
 299 function of surface temperature  $T_s$ . The atmospheric emission dominates where  $\tau_{LW} > 1$ , and its  
 300 magnitude primarily depends on the derivative  $dT_{\text{rad}}/dT_s$ . As we show below,  $dT_{\text{rad}}/dT_s$  differs  
 301 depending on the opacity source at a given wavenumber, so we split the spectral integral into four

302 terms:

$$\begin{aligned}
-\lambda_{LW} &= \int_{\text{surf}} \pi \frac{dB_\nu}{dT} \Big|_{T_s} e^{-\tau_{LW}} d\nu + \int_{\text{CO}_2} \pi \frac{dB_\nu}{dT} \Big|_{T_{\text{CO}_2}} \frac{dT_{\text{CO}_2}}{dT_s} d\nu + \\
&\quad \int_{\text{H}_2\text{O}} \pi \frac{dB_\nu}{dT} \Big|_{T_{\text{H}_2\text{O}}} \frac{dT_{\text{H}_2\text{O}}}{dT_s} d\nu + \int_{\text{cnt}} \pi \frac{dB_\nu}{dT} \Big|_{T_{\text{cnt}}} \frac{dT_{\text{cnt}}}{dT_s} d\nu \\
&= -(\lambda_{\text{surf}} + \lambda_{\text{CO}_2} + \lambda_{\text{H}_2\text{O}} + \lambda_{\text{cnt}}),
\end{aligned} \tag{16}$$

303 where  $T_{\text{CO}_2}$ ,  $T_{\text{H}_2\text{O}}$  and  $T_{\text{cnt}}$  are the emission temperatures in the  $\text{CO}_2$  band, the  $\text{H}_2\text{O}$  band, and the  
304  $\text{H}_2\text{O}$  continuum respectively. We refer to the four feedback terms as the surface Planck feedback  
305 ( $\lambda_{\text{surf}}$ ), the  $\text{CO}_2$  radiator fin feedback ( $\lambda_{\text{CO}_2}$ ), the non-Simpsonian  $\text{H}_2\text{O}$  feedback ( $\lambda_{\text{H}_2\text{O}}$ ), and the  
306  $\text{H}_2\text{O}$  continuum feedback ( $\lambda_{\text{cnt}}$ ).

307 Our spectral decomposition complements the traditional feedback decomposition which splits  
308  $\lambda_{LW}$  into Planck, Lapse-Rate, and Water Vapor (or Relative Humidity) feedbacks. The surface  
309 feedback  $\lambda_{\text{surf}}$  measures the OLR increase due to surface warming while keeping the atmosphere  
310 fixed. This term is identical to the surface contribution to the Planck feedback or “surface kernel”  
311 in the traditional decomposition (Soden et al. 2008). As for the atmospheric feedback, Equation  
312 15 shows that it depends on the *total* derivative of  $T_{\text{rad}}$ , that is, on  $dT_{\text{rad}}/dT_s$ . The traditional  
313 decomposition can be interpreted as splitting the total derivative  $dT_{\text{rad}}/dT_s$  up into various partial  
314 derivatives (uniform warming versus lapse-rate versus water vapor changes), while using a single,  
315 spectrally-averaged  $T_{\text{rad}}$ . In contrast, our decomposition splits the atmosphere’s feedback into  
316 three different bands, but still retains the total derivative  $dT_{\text{rad}}/dT_s$  in each band. In principle  
317 our decomposition could be split further to recover the traditional decomposition. That is, one  
318 could further decompose  $dT_{\text{rad}}/dT_s$  in each band into partial derivatives of  $T_{\text{rad}}$  that correspond to  
319 vertically-uniform warming, lapse-rate warming, and water-vapor changes – see Jeevanjee et al.  
320 (2021a) for more details. Here, however, we do not pursue this approach because our analytical  
321 expressions are general enough to predict  $T_{\text{rad}}$  and the total derivative  $dT_{\text{rad}}/dT_s$  from first principles.

322 We use relative humidity as the state variable throughout this paper, so the analytical results  
323 are compatible with papers that argue for the use of relative humidity in feedback decompositions  
324 instead of specific humidity (Held and Shell 2012; Jeevanjee et al. 2021a). In the fixed-RH  
325 framework the traditional Water Vapor feedback is replaced by a Relative Humidity feedback,  
326 which measures the clear-sky feedback due to RH changes. It is worth noting that the RH feedback



is small in individual climate models, and its multi-model mean is close to zero (Zelinka et al. 2020). In the derivations below we therefore treat RH as a parameter that varies according to the base state of an atmospheric column, and can thus affect  $\lambda_{LW}$ , but whose value is assumed constant under surface warming.

### c. Line-by-line calculations

To calculate spectral feedbacks numerically we use a 1D line-by-line model, PyRADs (Koll and Cronin 2018). The model’s radiative transfer includes HITRAN2016 CO<sub>2</sub> and H<sub>2</sub>O absorption data as well as the H<sub>2</sub>O component of the MTCKD continuum (Mlawer et al. 2012; Gordon et al. 2017). Calculations cover the spectral range 0.1-2500 cm<sup>-1</sup> with a resolution of  $\Delta\nu = 0.01$  cm<sup>-1</sup>, while the vertical resolution is 50 points in log-pressure. The two-stream equations require specifying an average zenith angle for the radiative fluxes, and PyRADs assumes  $\cos(\bar{\theta}) = 3/5$ . To quantitatively compare our theoretical scalings against line-by-line calculations the optical thickness in all scalings therefore needs to be divided by  $\cos(\bar{\theta})$  (this is equivalent to multiplying the surface gravity  $g$  by  $\cos(\bar{\theta})$ ).

In the 1D calculations we assume the atmospheric temperature profile follows either a moist adiabat or a power law temperature-pressure profile that is consistent with our bulk lapse rate approximation. In both cases the troposphere is capped by a tropopause at 200 K, while the overlying stratosphere is isothermal at the same temperature. Relative humidity in the troposphere is vertically uniform while the H<sub>2</sub>O mass fraction in the stratosphere is set equal to its value at the tropopause. CO<sub>2</sub> is treated as uniformly mixed in the vertical and fixed with respect to surface temperature. Because we are considering a wide range of surface temperatures, across which the tropopause pressure varies substantially, we vary the vertical grid-spacing in PyRADs. The model top pressure is set to a slightly lower value than the estimated tropopause pressure based on our bulk lapse rate formulation, which ensures the model’s top is always in the stratosphere and the tropopause is well resolved.

The spectrally-resolved feedback is the difference in the spectrally-resolved outgoing longwave flux,  $\text{OLR}_\nu$ , between a base state and a perturbed state with warmed surface and atmosphere,

$$-\lambda_\nu = \frac{\text{OLR}_\nu(T_s + \Delta T_s, \vec{T} + \Delta \vec{T}) - \text{OLR}_\nu(T_s, \vec{T})}{\Delta T_s}. \quad (17)$$

354 We use  $\Delta T_s = 1$  K, while  $\Delta \vec{T}$  denotes the atmospheric temperature perturbation caused by the  
 355 surface warming  $\Delta T_s$ . Because relative humidity is kept fixed, the atmospheric warming  $\vec{T} + \Delta \vec{T}$   
 356 also implies an increase in specific humidity.

357 Previous work used various approaches to assign physical significance to line-by-line output.  
 358 Seeley and Jeevanjee (2021) defined CO<sub>2</sub> versus H<sub>2</sub>O bands based on the column-integrated,  
 359 spectrally-smoothed optical thickness of CO<sub>2</sub> and H<sub>2</sub>O. However, as we show below, the behavior  
 360 of H<sub>2</sub>O differs strongly between the H<sub>2</sub>O bands and the H<sub>2</sub>O continuum, and it is difficult to  
 361 distinguish these terms based on column-integrated optical thicknesses. For example, the H<sub>2</sub>O  
 362 continuum might have a larger integrated optical thickness at some wavenumber than the H<sub>2</sub>O  
 363 bands, but because continuum absorption decays more rapidly with altitude than line absorption  
 364 ( $\kappa_{\text{cnt}} \propto e^*(T)$  versus  $\kappa_{\text{H}_2\text{O}} \propto p$ ) the emission at the level where  $\tau \sim 1$  could still be determined by  
 365 the H<sub>2</sub>O bands.

366 Instead we first split the net feedback into its contributions from the surface versus atmosphere.  
 367 The spectrally-resolved surface feedback is the feedback in response to surface-only warming while  
 368 keeping the atmosphere fixed,

$$-\lambda_{\text{surf}}^{\nu} = \frac{\text{OLR}_{\nu}(T_s + \Delta T_s, \vec{T}) - \text{OLR}_{\nu}(T_s, \vec{T})}{\Delta T_s}. \quad (18)$$

369 If we integrate  $\lambda_{\text{surf}}^{\nu}$  over all wavenumbers we get the surface feedback  $\lambda_{\text{surf}}$ , equivalent to the  
 370 surface kernel of Soden et al. (2008). The atmospheric feedback is equal to the difference between  
 371  $\lambda_{\nu}$  and  $\lambda_{\text{surf}}^{\nu}$ ,

$$-\lambda_{\text{atm}}^{\nu} = \frac{\text{OLR}_{\nu}(T_s, \vec{T} + \Delta \vec{T}) - \text{OLR}_{\nu}(T_s, \vec{T})}{\Delta T_s}. \quad (19)$$

372 We split  $\lambda_{\text{atm}}^{\nu}$  into different bands based on the spectrally-resolved emission pressures of CO<sub>2</sub>,  
 373 H<sub>2</sub>O, and the H<sub>2</sub>O continuum. For each absorber PyRADS computes the optical thickness as  
 374 a function of pressure and wavenumber,  $\tau(p, \nu)$ . We define the CO<sub>2</sub> emission pressure as the  
 375 pressure at which the optical thickness of CO<sub>2</sub> is equal to unity,

$$\tau_{\text{CO}_2}(p_{\text{rad}}, \nu) = 1, \quad (20)$$

which can be solved in each wavenumber bin to find  $p_{\text{rad}}(\nu)$  (in practice we interpolate to find the pressure at which  $\log[\tau] = 0$ ). The emission pressures of H<sub>2</sub>O and the H<sub>2</sub>O continuum are determined for each wavenumber bin in the same manner. The CO<sub>2</sub> feedback  $\lambda_{\text{co}_2}$  is then the integral of  $\lambda_{\text{atm}}^\nu$  over all wavenumbers at which CO<sub>2</sub> has the smallest emission pressure, the H<sub>2</sub>O feedback  $\lambda_{\text{H}_2\text{O}}$  is the integral of  $\lambda_{\text{atm}}^\nu$  over all wavenumbers at which H<sub>2</sub>O has the smallest emission pressure, and so on. We note that this approach is justified if one emitter clearly dominates the atmosphere's emission at a given wavenumber, such that its emission pressure  $p_{\text{rad}}$  is much lower than that of any other emitters, but could be misleading if two emitters have very similar emission pressures. In practice, however, H<sub>2</sub>O and CO<sub>2</sub> absorption cross-sections decrease quasi-exponentially away from their band centers (see Section 2), which means the wavenumber range over which two absorbers can have a similar emission pressure is limited.

#### 4. Emission temperatures

The feedbacks are set by the temperatures at the  $\tau = 1$  levels, so we seek analytical expressions for the emission temperatures  $T_{\text{co}_2}$ ,  $T_{\text{H}_2\text{O}}$  and  $T_{\text{cnt}}$ . The optical thickness of a generic absorber is

$$\tau = \int \kappa q \frac{dp}{g}, \quad (21)$$

where  $\kappa$  is the absorption cross-section and  $q$  is the absorber's mass-specific concentration. We use this equation to derive expressions for the emission temperatures by first writing the optical thickness in each band as a function of atmospheric temperature, then inverting these relations to find the emission temperature at the  $\tau = 1$  level.

##### a. CO<sub>2</sub>

CO<sub>2</sub> is well-mixed in the atmosphere so its mass-specific concentration  $q_{\text{co}_2}$  is vertically uniform. As discussed in Section 2, its absorption cross-section depends linearly on pressure due to pressure broadening and can be written as  $\kappa_{\text{co}_2}(\nu, p) = \kappa_{\text{co}_2}^*(\nu)(p/p_0)$ .  $\kappa_{\text{co}_2}^*$  captures the wavenumber-dependence of the CO<sub>2</sub> absorption cross-section,  $\kappa_{\text{co}_2}^* \propto \exp(-|\nu - \nu_0|/l_\nu)$ , while  $p_0$  is an arbitrary reference pressure. For simplicity we set  $p_0$  equal to the surface pressure  $p_s$ , so  $\kappa_{\text{co}_2}^*(\nu) = \kappa_{\text{co}_2}(\nu, p_s)$ .

400 The optical thickness at a vertical level with temperature and pressure  $(T, p)$  is then

$$\begin{aligned}
 \tau_{\text{CO}_2} &= \int_0^p \kappa_{\text{CO}_2}^* \left( \frac{p'}{p_s} \right) q_{\text{CO}_2} \frac{dp'}{g}, \\
 &= \frac{\kappa_{\text{CO}_2}^*}{2g p_s} q_{\text{CO}_2} p^2, \\
 &= \frac{\kappa_{\text{CO}_2}^* p_s}{2g} q_{\text{CO}_2} \left( \frac{p}{p_s} \right)^2 \\
 &= \frac{\kappa_{\text{CO}_2}^* p_s}{2g} q_{\text{CO}_2} \left( \frac{T}{T_s} \right)^{2/\gamma_{\text{lr}}} \\
 &\equiv \tau_{\text{CO}_2}^*(\nu) q_{\text{CO}_2} \times \left( \frac{T}{T_s} \right)^{2/\gamma_{\text{lr}}}, \tag{22}
 \end{aligned}$$

401 where we have used the bulk lapse rate in the fourth step. Note that all spectroscopic parameters  
 402 as well  $p_s$  and  $g$  are combined into a reference optical thickness,  $\tau_{\text{CO}_2}^*(\nu)$ , which encapsulates how  
 403  $\text{CO}_2$  absorption varies with respect to wavenumber  $\nu$ , surface pressure  $p_s$ , and gravity  $g$ , but which  
 404 can be treated as constant in response to warming.

#### 405 *b. Non-Simpsonian $\text{H}_2\text{O}$*

406 As for  $\text{CO}_2$ , the absorption cross-section of  $\text{H}_2\text{O}$  scales linearly with pressure and can be written  
 407 as  $\kappa_{\text{H}_2\text{O}}(\nu, p) = \kappa_{\text{H}_2\text{O}}^*(\nu)(p/p_s)$ . We use the Clausius-Clapeyron power law approximation to write  
 408 the saturation specific humidity as  $q^* \approx R_d/R_v \times e_0^*/p \times (T/T_0)^{\gamma_{\text{wv}}}$  and the specific humidity as

409  $q = \text{RH} \times q^*$ . The optical thickness of  $\text{H}_2\text{O}$  at a level  $(T, p)$  is then

$$\begin{aligned}
\tau_{\text{H}_2\text{O}} &= \int_0^p \kappa_{\text{H}_2\text{O}}^* \left( \frac{p'}{p_s} \right) q \frac{dp'}{g}, \\
&\approx \text{RH} \frac{\kappa_{\text{H}_2\text{O}}^* e_0^*}{g} \frac{R_d}{R_v} \times \int_0^p \left( \frac{p'}{p_s} \right) \left( \frac{T'}{T_0} \right)^{\gamma_{\text{wv}}} \frac{dp'}{p'} \\
&= \text{RH} \frac{\kappa_{\text{H}_2\text{O}}^* e_0^*}{g} \frac{R_d}{R_v} \times \int_0^T \left( \frac{T'}{T_s} \right)^{1/\gamma_{\text{lr}}} \left( \frac{T'}{T_0} \right)^{\gamma_{\text{wv}}} \frac{1}{\gamma_{\text{lr}}} \frac{dT'}{T'} \\
&= \text{RH} \frac{\kappa_{\text{H}_2\text{O}}^* e_0^*}{g} \frac{R_d}{R_v} \frac{1}{\gamma_{\text{lr}}} \left( \frac{T_0}{T_s} \right)^{1/\gamma_{\text{lr}}} \times \int_0^T \left( \frac{T'}{T_0} \right)^{\gamma_{\text{wv}} + \frac{1}{\gamma_{\text{lr}}}} \frac{dT'}{T'} \\
&= \text{RH} \frac{\kappa_{\text{H}_2\text{O}}^* e_0^*}{g} \frac{R_d}{R_v} \frac{1}{1 + \gamma_{\text{wv}} \gamma_{\text{lr}}} \times \left( \frac{T}{T_0} \right)^{\frac{1 + \gamma_{\text{wv}} \gamma_{\text{lr}}}{\gamma_{\text{lr}}}} \left( \frac{T_0}{T_s} \right)^{1/\gamma_{\text{lr}}} \\
&\equiv \text{RH} \tau_{\text{H}_2\text{O}}^*(\nu) \frac{1}{1 + \gamma_{\text{wv}} \gamma_{\text{lr}}} \times \left( \frac{T}{T_0} \right)^{\frac{1 + \gamma_{\text{wv}} \gamma_{\text{lr}}}{\gamma_{\text{lr}}}} \left( \frac{T_0}{T_s} \right)^{1/\gamma_{\text{lr}}}. \tag{23}
\end{aligned}$$

410 where in the second step we have used the Clausius-Clapeyron power law and also replaced the  
411 water vapor concentration in the stratosphere with the water vapor concentration of a moist adiabat  
412 that extends all the way to the top-of-atmosphere. We again define a reference optical thickness,  
413  $\tau_{\text{H}_2\text{O}}^*(\nu)$ , which encapsulates how  $\text{H}_2\text{O}$  line absorption varies with respect to wavenumber  $\nu$ , and  
414 gravity  $g$ , but which is independent of temperature.

### 415 *c. $\text{H}_2\text{O}$ Continuum*

416 Absorption by the  $\text{H}_2\text{O}$  continuum strengthens in response to increasing water vapor concen-  
417 trations and weakens in response to warming,  $\kappa_{\text{H}_2\text{O},\text{cnt}} = \kappa_{\text{cnt}} \times \text{RH} \, e^*(T)/e^*(T_0) \times (T/T_0)^{-a}$ . The

418 optical thickness of the continuum is then

$$\begin{aligned}
\tau_{\text{cnt}} &= \text{RH} \int_0^p \kappa_{\text{cnt}} \frac{e^*(T')}{e^*(T_0)} \left( \frac{T'}{T_0} \right)^{-a} q \frac{dp'}{g}, \\
&\approx \text{RH}^2 \frac{\kappa_{\text{cnt}} e_0^*}{g} \frac{R_d}{R_v} \times \int_0^T \left( \frac{T'}{T_0} \right)^{2\gamma_{\text{wv}}-a} \frac{dp'}{p}, \\
&= \text{RH}^2 \frac{\kappa_{\text{cnt}} e_0^*}{g} \frac{R_d}{R_v} \times \int_0^T \left( \frac{T'}{T_0} \right)^{2\gamma_{\text{wv}}-a} \frac{1}{\gamma_{\text{lr}}} \frac{dT'}{T'}, \\
&= \text{RH}^2 \frac{\kappa_{\text{cnt}} e_0^*}{g} \frac{R_d}{R_v} \frac{1}{(2\gamma_{\text{wv}}-a)\gamma_{\text{lr}}} \times \left( \frac{T}{T_0} \right)^{2\gamma_{\text{wv}}-a}, \\
&\equiv \text{RH}^2 \tau_{\text{cnt}}^* \frac{1}{(2\gamma_{\text{wv}}-a)\gamma_{\text{lr}}} \times \left( \frac{T}{T_0} \right)^{2\gamma_{\text{wv}}-a}, \tag{24}
\end{aligned}$$

419 where the second and third step make the same assumptions as our derivation for the H<sub>2</sub>O band.  
420 Here the reference optical thickness,  $\tau_{\text{cnt}}^*$  encapsulates how the H<sub>2</sub>O self-continuum varies with  
421 respect to gravity  $g$  but has no dependence on wavenumber or temperature.

#### 422 *d. Emission temperatures*

423 By setting  $\tau = 1$  and inverting the above relations, we arrive at the emission temperatures in the  
424 CO<sub>2</sub> band, the H<sub>2</sub>O band and the H<sub>2</sub>O self-continuum:

$$T_{\text{co}_2} = T_s \left( \frac{1}{\tau_{\text{co}_2}^*(\nu) q_{\text{co}_2}} \right)^{\gamma_{\text{lr}}/2} \tag{25a}$$

$$T_{\text{H}_2\text{O}} = T_0 \left( \frac{1 + \gamma_{\text{wv}} \gamma_{\text{lr}}}{\tau_{\text{H}_2\text{O}}^*(\nu) \text{RH}} \right)^{\frac{\gamma_{\text{lr}}}{1 + \gamma_{\text{wv}} \gamma_{\text{lr}}}} \left( \frac{T_s}{T_0} \right)^{\frac{1}{1 + \gamma_{\text{wv}} \gamma_{\text{lr}}}} \tag{25b}$$

$$T_{\text{cnt}} = T_0 \left( \frac{(2\gamma_{\text{wv}} - a) \gamma_{\text{lr}}}{\tau_{\text{cnt}}^* \text{RH}^2} \right)^{\frac{1}{2\gamma_{\text{wv}} - a}}. \tag{25c}$$

425 To interpret these emission temperatures, consider whether a given emitter stabilizes or destabi-  
426 lizes Earth's climate. For CO<sub>2</sub> it is easy to see that the feedback is always stabilizing. Ignoring  
427 lapse rate changes we have  $T_{\text{co}_2} \propto T_s$ , so  $dT_{\text{co}_2}/dT_s > 0$ . More intuitively, the optical thickness of  
428 CO<sub>2</sub> can be written as

$$\tau_{\text{co}_2} \propto \left( \frac{T}{T_s} \right)^{2/\gamma_{\text{lr}}} = \left( \frac{p}{p_s} \right)^2. \tag{26}$$

429 The emission level of CO<sub>2</sub> is therefore a fixed function of pressure. Given that the atmosphere's  
 430 temperature at a fixed pressure level always increases in response to surface warming,  $T_{\text{CO}_2}$  also  
 431 has to increase under warming. This effect can be thought of as a spectral "radiator fin", and is  
 432 valid also if the lapse rate  $\gamma_{\text{lr}}$  varies in response to surface warming. It implies that even if the  
 433 atmosphere stops emitting more at all other wavenumbers, so  $dT_{\text{rad}}/dT_s = 0$  outside the CO<sub>2</sub> band,  
 434 the presence of CO<sub>2</sub> still allows the atmosphere to shed more energy to space in response to surface  
 435 warming (Seeley and Jeevanjee 2021).

436 Next, our expressions suggest that the feedback from H<sub>2</sub>O is small and, to first order, might  
 437 be negligible. Equation 25b shows  $T_{\text{H}_2\text{O}} \propto T_s^{1/(1+\gamma_{\text{wv}}\gamma_{\text{lr}})}$ , where representative values for Earth's  
 438 tropics are  $\gamma_{\text{wv}} \sim 20$  and  $\gamma_{\text{lr}} \sim 1/7$ , so the H<sub>2</sub>O emission temperature only depends weakly on  
 439 surface temperature,  $T_{\text{H}_2\text{O}} \propto T_s^{1/4}$ . This small exponent is closely related to Simpson's "paradox"  
 440 (Ingram 2010) or Simpson's "law" (Jeevanjee et al. 2021a), which state that  $T_{\text{H}_2\text{O}}$  is approximately  
 441 independent of surface temperature. In the limit  $\gamma_{\text{wv}}\gamma_{\text{lr}} = d \ln e^* / d \ln p \gg 1$ , that is, if water vapor  
 442 increases much faster in the vertical than the total atmospheric mass, then

$$T_{\text{H}_2\text{O}} \approx T_0 \left( \frac{\gamma_{\text{wv}}\gamma_{\text{lr}}}{\tau_{\text{H}_2\text{O}}^*(\nu)RH} \right)^{\frac{1}{\gamma_{\text{wv}}}}, \quad (27)$$

443 and  $T_{\text{H}_2\text{O}}$  ceases to depend on  $T_s$ . If the lapse rate is also independent of  $T_s$  we recover Simpson's  
 444 law:

$$\frac{dT_{\text{H}_2\text{O}}}{dT_s} \approx 0. \quad (28)$$

445 In reality, however, water vapor dominates most of the atmospheric emission and even minor  
 446 deviations from Simpson's law can have a non-negligible impact on the longwave feedback.  
 447 Deviations arise because the H<sub>2</sub>O optical thickness is sensitive to pressure broadening and because  
 448 changes in  $\gamma_{\text{lr}}$  modify the total water vapor path inside the atmospheric column. For present-day  
 449 Earth the net impact of these changes is to increase the H<sub>2</sub>O emission temperature under surface  
 450 warming,  $T_{\text{H}_2\text{O}} \propto T_s^{1/4}$ , such that  $dT_{\text{H}_2\text{O}}/dT_s > 0$  which means the H<sub>2</sub>O bands tend to stabilize  
 451 Earth's climate.

Finally,  $T_{\text{cnt}}$  has no direct dependence on surface temperature, but is sensitive to lapse rate changes. If we take the continuum's emission temperature (Eqn. 25c), and assume that the direct temperature-dependence of the continuum  $a \sim 7$  is much smaller than its temperature-dependence due to the Clausius-Clapeyron relation,  $2\gamma_{\text{wv}} \sim 40$ , we have

$$T_{\text{cnt}} \propto T_0 \times [\gamma_{\text{lr}}]^{1/(2\gamma_{\text{wv}})}. \quad (29)$$

Because the lapse rate  $\gamma_{\text{lr}}$  decreases under surface warming we have  $dT_{\text{cnt}}/dT_s = dT_{\text{cnt}}/d\gamma_{\text{lr}} \times d\gamma_{\text{lr}}/dT_s < 0$ . Physically, this effect can be understood by considering the impact of  $\gamma_{\text{lr}}$  on the atmosphere's total water vapor path. If one decreases the lapse rate  $\gamma_{\text{lr}}$  while keeping  $T_s$  fixed, the atmospheric column can store more water vapor. To still maintain an optical thickness of unity then requires that the continuum's emission level moves to colder temperatures. Our expressions thus predict that the  $\text{H}_2\text{O}$  continuum gives rise to a destabilizing feedback.

#### *e. Comparison against LBL calculations*

Equations 25a - 25c predict how Earth's emission temperature varies in response to changes in  $T_s$ ,  $q_{\text{CO}_2}$ ,  $\gamma_{\text{lr}}$  and RH. To test these equations we perform four sets of numerical experiments with PyRADS in which we variously change  $T_s$ ,  $q_{\text{CO}_2}$ ,  $\gamma_{\text{lr}}$ , and RH while holding the other parameters fixed. The default values are  $T_s = 290$  K, 400 ppm of  $\text{CO}_2$ ,  $\gamma_{\text{lr}} = 2/7$ , and  $\text{RH} = 0.8$ . To match our underlying assumptions we assume a bulk tropospheric lapse rate, so  $T = T_s(p/p_s)^{\gamma_{\text{lr}}}$ , which means the temperature profile differs from an adiabat if  $\gamma_{\text{lr}} < 2/7$ . The troposphere is capped by an isothermal stratosphere which is kept fixed at  $T_{\text{strat}} = 200$  K. Note that in Equations 25a - 25c the dependence on wavenumber only enters through the reference optical thicknesses  $\tau_{\text{CO}_2}^*$ ,  $\tau_{\text{H}_2\text{O}}^*$ , and  $\tau_{\text{cnt}}^*$ , which are evaluated using the cross-sections from Section 2. Because the cross-sections were fitted independently, the analytical  $T_{\text{rad}}$  expressions do not contain any free tuning parameters.

To compare the analytical results against line-by-line calculations we first numerically compute the top-of-atmosphere spectral flux  $\text{OLR}_\nu$  for a given set of  $(T_s, q_{\text{CO}_2}, \gamma_{\text{lr}}, \text{RH})$ . We then smooth  $\text{OLR}_\nu$  with a median filter of width  $50 \text{ cm}^{-1}$ , before inverting it using the Planck function to find the atmosphere's emission temperature (also known as brightness temperature) at a given wavenumber.



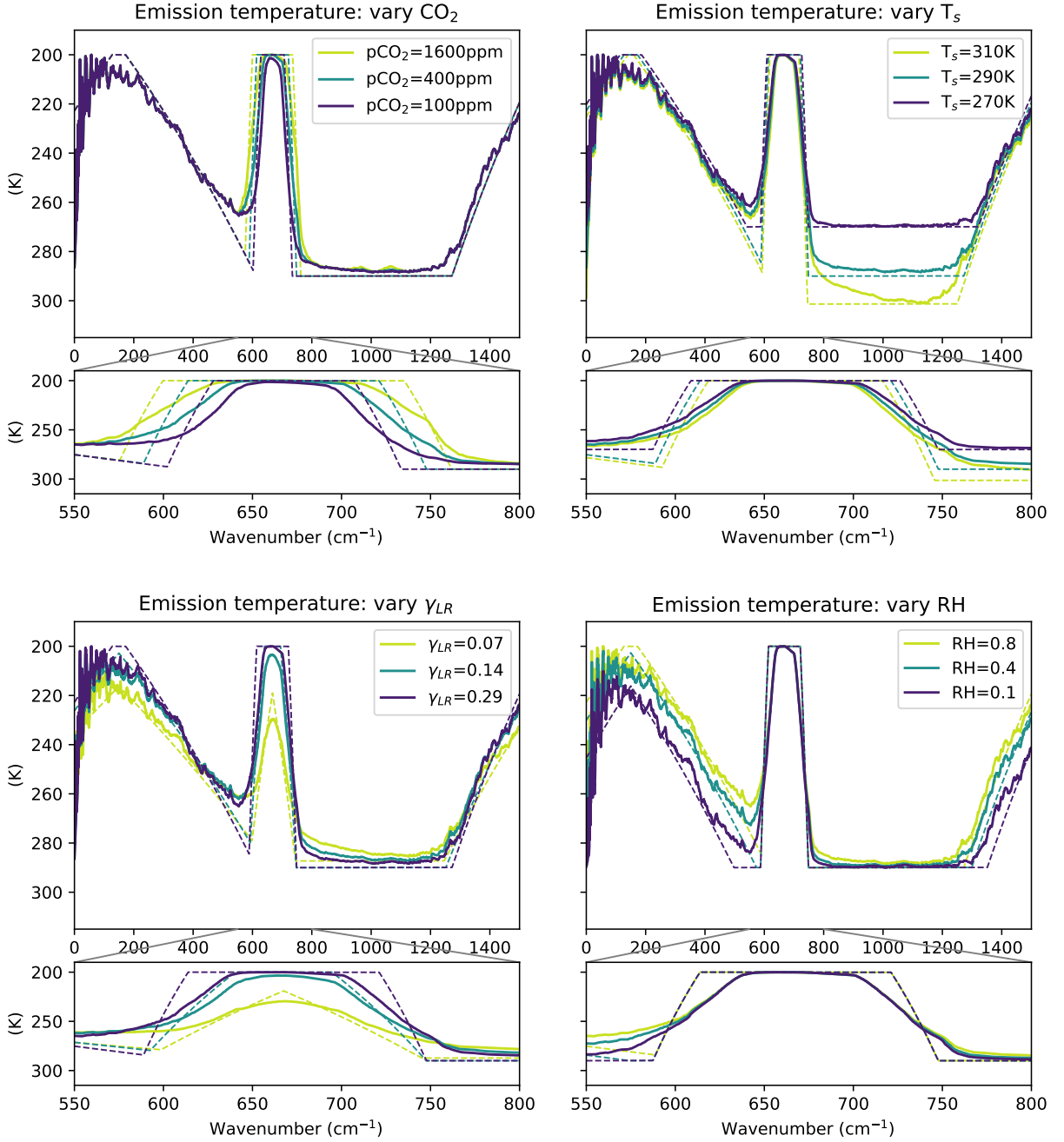
477 Finally, we combine our analytical expressions into a single emission temperature via

$$T_{\text{rad}} = \max \left[ T_{\text{strat}}, \min \left[ T_s, T_{\text{CO}_2}, T_{\text{H}_2\text{O}}, T_{\text{cnt}} \right] \right], \quad (30)$$

478 to compare directly with temperatures from line-by-line calculations.

483 Figure 4 shows that the analytical results compare favorably against numerical calculations. Even  
484 though the analytical  $T_{\text{rad}}$  shapes are idealized compared to the numerical calculations, the overall  
485 response of  $T_{\text{rad}}$  to perturbations is captured well. First, increasing  $\text{CO}_2$  concentration lowers  $T_{\text{rad}}$   
486 around  $667 \text{ cm}^{-1}$ , which corresponds to the wings of the  $\text{CO}_2$  band. This is simply a spectrally  
487 resolved view of how increasing  $\text{CO}_2$  acts as a radiative forcing (Jeevanjee et al. 2021b). Second,  
488 warming the surface while keeping all other parameters fixed has multiple effects. The main impact  
489 is to increase the emission temperature in the window region between  $\sim 800$  and  $1200 \text{ cm}^{-1}$ . In  
490 addition there are secondary impacts: surface warming also shrinks the width of the  $\text{CO}_2$  band  
491 and slightly increases the emission temperature in the  $\text{H}_2\text{O}$  bands below  $600 \text{ cm}^{-1}$  and above  $1300$   
492  $\text{cm}^{-1}$  (this latter effect is hard to see in Figure 4). The increased emission in the  $\text{H}_2\text{O}$  bands  
493 shows that Simpson’s law in Equation 28 is not exact, an effect that is captured by our analytical  
494 expressions. Third, reducing the lapse rate  $\gamma_{\text{lr}}$  preserves the width of the  $\text{CO}_2$  band, but it flattens  
495 the steepness of its slopes and increases the emission temperature in the center of the band. In the  
496  $\text{H}_2\text{O}$  bands, a smaller  $\gamma_{\text{lr}}$  while keeping  $T_s$  fixed also leads to a non-Simpsonian increase of the  
497 emission temperature in the  $\text{H}_2\text{O}$  bands. In contrast to the  $\text{H}_2\text{O}$  bands, the emission temperature  
498 of the  $\text{H}_2\text{O}$  continuum around  $1000 \text{ cm}^{-1}$  decreases as  $\gamma_{\text{lr}}$  is reduced. As discussed above, this is  
499 because the atmospheric water path increases with a smaller  $\gamma_{\text{lr}}$ , which reduces  $T_{\text{cnt}}$ . The feedback  
500 of the  $\text{H}_2\text{O}$  continuum therefore has the opposite sign as the  $\text{H}_2\text{O}$  bands, in line with the analytical  
501 results. Finally, reducing the relative humidity increases  $T_{\text{rad}}$  in all regions dominated by water  
502 vapor, both in the  $\text{H}_2\text{O}$  bands below  $600 \text{ cm}^{-1}$ /above  $1300 \text{ cm}^{-1}$  and in the  $\text{H}_2\text{O}$  continuum around  
503  $1000 \text{ cm}^{-1}$ , while the  $\text{CO}_2$  band is unaffected.

504 Overall, Figure 4 underlines that comparatively simple physics is sufficient to explain the  
505 spectrally-resolved response of  $T_{\text{rad}}$  to different climate perturbations. To connect Figure 4 back to  
506 the total clear-sky longwave feedback we only need to consider how emission temperature changes  
507 play out once we average them into spectral bands, and how multiple bands add up to determine  
508 the net longwave feedback.



479 FIG. 4. Analytical emission temperatures (dashed), compared against numerical line-by-line results smoothed  
 480 with a median filter of width  $50 \text{ cm}^{-1}$ . Large rows show the entire infrared spectrum, small rows are zoomed  
 481 in on the  $\text{CO}_2$  band. The y-axes are flipped so that emission temperature decreases going up, the same way  
 482 temperature decreases with altitude in Earth's atmosphere.

## 5. Analytical Feedbacks

Having derived expressions for the emission temperature in different parts of the LW spectrum, and verified these expressions against line-by-line calculations, we can now derive analytic expressions for the four spectral feedbacks:  $\lambda_{\text{surf}}$ ,  $\lambda_{\text{CO}_2}$ ,  $\lambda_{\text{H}_2\text{O}}$  and  $\lambda_{\text{cnt}}$ . Because each spectral feedback is defined as an integral over a relevant wavenumber range (Eqn. 16) we first need to estimate the width of different spectral bands.

### a. Band widths

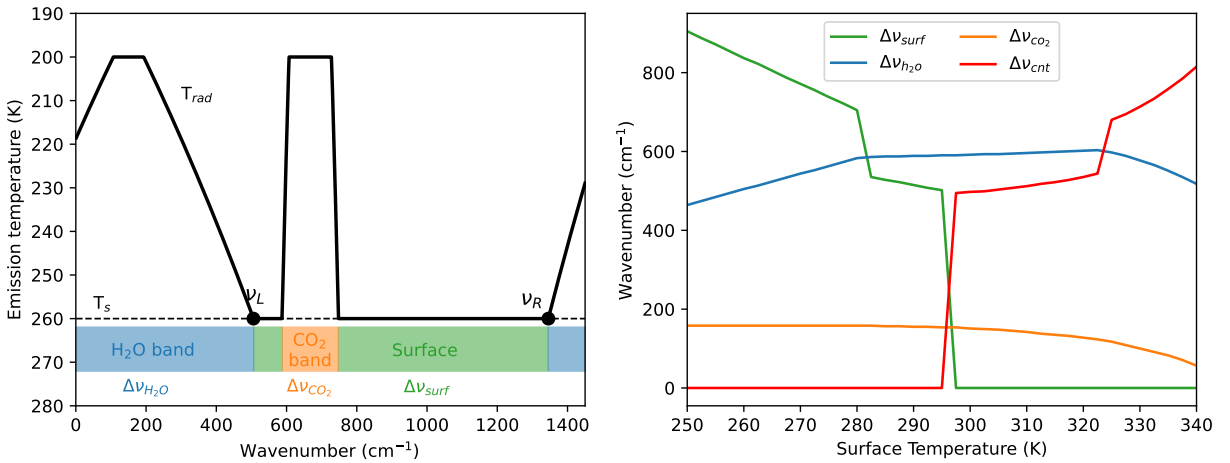


FIG. 5. Illustration of spectral band widths. The emission temperature is equal to the emission temperature of whichever emitter is coldest,  $T_{\text{rad}} = \min[T_{\text{CO}_2}, T_{\text{H}_2\text{O}}, T_{\text{cnt}}, T_s]$ , or the stratospheric temperature. Left: Lines show the analytical  $T_{\text{rad}}$  (solid) and surface temperature  $T_s$  (dashed), while colored regions illustrate which emitters dominate in which band. The calculation shown uses  $T_s = 260$  K, RH = 0.8, and 400ppm of CO<sub>2</sub>. Right: Band widths as a function of surface temperature, numerically calculated based on our emission temperature expressions. Here  $\Delta\nu_{\text{H}_2\text{O}}$  refers only to the rotational band at wavenumbers lower than 1000 cm<sup>-1</sup>.

In line with the emission-level approximation and our results in Section 4, we define an absorption band as the spectral range in which a given absorber has the coldest emission temperature and therefore dominates the emission to space. For example, the CO<sub>2</sub> band covers all wavenumbers in which  $T_{\text{CO}_2} < \min[T_{\text{H}_2\text{O}}, T_{\text{cnt}}, T_s]$ , as illustrated in Figure 5. The width of the CO<sub>2</sub> band is then determined by the two wavenumbers  $\nu^{\text{edge}}$  at which the emission temperature of CO<sub>2</sub> intersects the emission temperatures of its neighboring absorbers:  $T_{\text{CO}_2}(\nu^{\text{edge}}) = \min[T_{\text{H}_2\text{O}}(\nu^{\text{edge}}), T_{\text{cnt}}, T_s]$ .

## 1) CO<sub>2</sub> BAND WIDTH

To estimate the width of the CO<sub>2</sub> band we consider three situations: (1) the CO<sub>2</sub> concentration  $q_{\text{CO}_2}$  is so low that even in the center of the CO<sub>2</sub> band the optical thickness is less than one; (2) a cold (or dry) atmosphere in which there is no overlap between the CO<sub>2</sub> and H<sub>2</sub>O bands; and (3) a warm (or moist) atmosphere in which there is some overlap between CO<sub>2</sub> and H<sub>2</sub>O.

First, at low CO<sub>2</sub> concentrations the band width of CO<sub>2</sub> is simply equal to zero. When does this situation occur? From the optical thickness of CO<sub>2</sub> (Eqn. 22), the column-integrated optical thickness in the middle of the CO<sub>2</sub> band is equal to  $\tau_{\text{CO}_2}(\nu_0, T_s) = q_{\text{CO}_2} \tau_{\text{CO}_2}^*(\nu_0)$  so

$$\Delta\nu_{\text{CO}_2} = 0, \quad \text{if } q_{\text{CO}_2} \tau_{\text{CO}_2}^*(\nu_0) < 1. \quad (31)$$

Second, at non-negligible CO<sub>2</sub> concentration and cold surface temperatures, water vapor is a weak absorber and CO<sub>2</sub>-H<sub>2</sub>O overlap is negligible. In this case the edge of the CO<sub>2</sub> band can be defined as the wavenumber  $\nu^{\text{cold}}$  at which  $T_{\text{CO}_2}$  intersects with the surface temperature  $T_s$ ,  $T_{\text{CO}_2}(\nu^{\text{cold}}) = T_s$ . The emission temperature of CO<sub>2</sub> is equal to  $T_{\text{CO}_2} = T_s \times (\tau_{\text{CO}_2}^* q_{\text{CO}_2})^{-\gamma_{\text{r}}/2}$  (Eqn. 25a), while our model of CO<sub>2</sub> spectroscopy states  $\tau_{\text{CO}_2}(\nu)^* \propto \exp(-|\nu - \nu_0|/l_\nu)$  (Eqn. 10). Combining the two equations yields

$$\nu^{\text{cold}} = \nu_0 \pm l_\nu \log(q_{\text{CO}_2} \tau_{\text{CO}_2}^*(\nu_0)). \quad (32)$$

Here  $\tau_{\text{CO}_2}^*(\nu_0) = \kappa_0(\nu_0)p_s/(2g)$  is the reference optical thickness in the center of the CO<sub>2</sub> band. The overall width of the CO<sub>2</sub> band at cold temperatures is therefore

$$\Delta\nu_{\text{CO}_2}^{\text{cold}} = 2l_\nu \log(q_{\text{CO}_2} \tau_{\text{CO}_2}^*(\nu_0)). \quad (33)$$

To estimate the order of magnitude of  $\Delta\nu_{\text{CO}_2}^{\text{cold}}$  we use  $\kappa_0 = 500 \text{ m}^2 \text{ kg}^{-1}$  (from Section 2c) and a  $q_{\text{CO}_2}$  that corresponds to 400ppm of CO<sub>2</sub>. The optical thickness in the center of the CO<sub>2</sub> band is  $\tau_{\text{CO}_2}^*(\nu_0) \sim 1600$ . This large optical thickness decreases exponentially with wavenumber away from  $\nu_0$ , so that  $T_{\text{CO}_2}$  equals  $T_s$  only  $\sim 80 \text{ cm}^{-1}$  away from  $\nu_0$ . Because CO<sub>2</sub>'s band shape is symmetric about  $\nu_0$ , the present-day CO<sub>2</sub> band width is thus less than  $200 \text{ cm}^{-1}$ .

Third, at high temperatures the emission from the surface is replaced by H<sub>2</sub>O emission, so  $T_{\text{CO}_2}(\nu^{\text{hot}}) = T_{\text{H}_2\text{O}}(\nu^{\text{hot}})$ . Because the CO<sub>2</sub> band slope is much steeper than the H<sub>2</sub>O band slope (see Fig. 3) we further approximate  $T_{\text{H}_2\text{O}}$  as constant across the CO<sub>2</sub> band and equal to its value in the CO<sub>2</sub> band center  $T_{\text{H}_2\text{O}}(\nu) \approx T_{\text{H}_2\text{O}}(\nu_0)$ . Combining the emission temperature of CO<sub>2</sub> (Eqn. 25a) with our model of CO<sub>2</sub> spectroscopy (Eqn. 10),

$$\nu^{\text{hot}} = \nu_0 \pm l_\nu \log \left[ q_{\text{CO}_2} \tau_{\text{CO}_2}^*(\nu_0) \left( \frac{T_{\text{H}_2\text{O}}(\nu_0)}{T_s} \right)^{\frac{2}{\gamma_{\text{lr}}}} \right], \quad (34)$$

where the emission temperature of H<sub>2</sub>O can be evaluated using Eqn. 25b. Physically speaking, the H<sub>2</sub>O emission temperature is colder than the surface,  $T_{\text{H}_2\text{O}}(\nu_0)/T_s < 1$ , so our model correctly captures the fact that H<sub>2</sub>O-CO<sub>2</sub> overlap decreases the width of the CO<sub>2</sub> band. Taking into account all three regimes, the overall width of the CO<sub>2</sub> band is therefore

$$\Delta \nu_{\text{CO}_2} = \begin{cases} 0, & \text{if } q_{\text{CO}_2} \tau_{\text{CO}_2}^*(\nu_0) < 1 \\ 2 \times \min(\nu^{\text{hot}} - \nu_0, \nu^{\text{cold}} - \nu_0), & \text{if } q_{\text{CO}_2} \tau_{\text{CO}_2}^*(\nu_0) \geq 1. \end{cases} \quad (35)$$

## 2) H<sub>2</sub>O BAND WIDTH

To determine H<sub>2</sub>O's band width the potential overlap with CO<sub>2</sub> matters less because (at present-day CO<sub>2</sub> concentrations) the CO<sub>2</sub> band is too narrow to block a significant portion of the emission by H<sub>2</sub>O. However, at high temperatures competition between the H<sub>2</sub>O bands and the H<sub>2</sub>O continuum becomes important, so we again consider two temperature regimes. At cold temperatures continuum absorption is negligible and we solve  $T_{\text{H}_2\text{O}}(\nu^{\text{cold}}) = T_s$ . Combining the emission temperature of H<sub>2</sub>O (Eqn. 25b) with our H<sub>2</sub>O band model (Eqn. 11), this leads to

$$\nu_L^{\text{cold}} = \nu_{\text{rot}} + l_{\text{rot}} \log \left( \frac{\text{RH} \tau_{\text{rot}}^*(\nu_{\text{rot}})}{1 + \gamma_{\text{wv}} \gamma_{\text{lr}}} \left( \frac{T_s}{T_0} \right)^{\gamma_{\text{wv}}} \right), \quad (36a)$$

$$\nu_R^{\text{cold}} = \nu_{\text{v-r}} - l_{\text{v-r}} \log \left( \frac{\text{RH} \tau_{\text{v-r}}^*(\nu_{\text{v-r}})}{1 + \gamma_{\text{wv}} \gamma_{\text{lr}}} \left( \frac{T_s}{T_0} \right)^{\gamma_{\text{wv}}} \right), \quad (36b)$$

where  $\nu_L$  is the left edge of the window below  $\sim 1000 \text{ cm}^{-1}$ , and  $\nu_R$  is the right edge of the window above  $\sim 1000 \text{ cm}^{-1}$  (see Figure 5). The two H<sub>2</sub>O bands have different spectral slopes, and subscript "rot" denotes quantities that are related to the rotational H<sub>2</sub>O band at wavenumbers

below  $1000 \text{ cm}^{-1}$  while subscript “v-r” denotes quantities related to the vibrational-rotational  $\text{H}_2\text{O}$  band at wavenumbers above  $1000 \text{ cm}^{-1}$  (see Section 2). At high temperatures the continuum cuts off emission from the surface so the  $\text{H}_2\text{O}$  band edge  $\nu^{hot}$  is determined by  $T_{\text{H}_2\text{O}}(\nu^{hot}) = T_{\text{cnt}}$ . Using the emission temperature of  $\text{H}_2\text{O}$  (Eqn. 25b) and our  $\text{H}_2\text{O}$  band model, we find

$$\nu_L^{hot} = \nu_{\text{rot}} + l_{\text{rot}} \log \left[ \frac{\text{RH} \tau_{\text{rot}}^*(\nu_{\text{rot}})}{1 + \gamma_{\text{wv}} \gamma_{\text{lr}}} \left( \frac{T_0}{T_s} \right)^{1/\gamma_{\text{lr}}} \left( \frac{T_{\text{cnt}}}{T_0} \right)^{\frac{1+\gamma_{\text{wv}} \gamma_{\text{lr}}}{\gamma_{\text{lr}}}} \right], \quad (37a)$$

$$\nu_R^{hot} = \nu_{\text{v-r}} - l_{\text{v-r}} \log \left[ \frac{\text{RH} \tau_{\text{v-r}}^*(\nu_{\text{v-r}})}{1 + \gamma_{\text{wv}} \gamma_{\text{lr}}} \left( \frac{T_0}{T_s} \right)^{1/\gamma_{\text{lr}}} \left( \frac{T_{\text{cnt}}}{T_0} \right)^{\frac{1+\gamma_{\text{wv}} \gamma_{\text{lr}}}{\gamma_{\text{lr}}}} \right], \quad (37b)$$

where the continuum emission temperature is given by Eqn. 25c. Combining both temperature regimes, the window width due to  $\text{H}_2\text{O}$  absorption is therefore

$$\begin{aligned} \Delta \nu_{\text{surf}}(T_s, \text{RH}, \gamma_{\text{lr}}) &= \nu_R - \nu_L \\ &= \max(\nu_R^{cold}, \nu_R^{hot}) - \min(\nu_L^{cold}, \nu_L^{hot}). \end{aligned} \quad (38)$$

Similar to the  $\text{CO}_2$  band width, Equations 36 and 37 become invalid at very low RH or  $T_s$  because in those situations  $\text{H}_2\text{O}$  ceases to be optically thick at all wavenumbers (mathematically, this happens when RH or  $T_s$  become small enough that the logarithms in Eqns. 36 and 37 change sign). We do not consider the limit  $\text{RH} \rightarrow 0$  in this paper, but care should be taken when applying our results to extremely dry or cold atmospheres.

Finally, our feedback expression for the  $\text{H}_2\text{O}$  feedback requires us to separately specify the width of the rotational  $\text{H}_2\text{O}$  band below  $1000 \text{ cm}^{-1}$ . This width can be estimated by assuming that the rotational band always extends from  $0 \text{ cm}^{-1}$  to the left edge of the window region  $\nu_L$  (see Figure 5). Doing so presumes that  $\text{H}_2\text{O}$  is always optically thick at low wavenumbers around  $\nu = 0 \text{ cm}^{-1}$ . While this assumption again breaks down in very cold or dry climates (the maximum absorption in the rotational band occurs around  $\nu \sim 150 \text{ cm}^{-1}$ , not  $0 \text{ cm}^{-1}$ , so low wavenumbers could become optically thin even if the band center is still optically thick), in those climates the  $\text{H}_2\text{O}$  feedback becomes negligible anyway. The width of the rotational  $\text{H}_2\text{O}$  band is then

$$\Delta \nu_{\text{H}_2\text{O}}(T_s, \text{RH}, \gamma_{\text{lr}}) \approx \nu_L - 0 = \min(\nu_L^{cold}, \nu_L^{hot}), \quad (39)$$

587 where the wavenumber  $\nu_L$  denotes the left edge of the surface window (see above), as well as the  
 588 right edge of the rotational H<sub>2</sub>O band.

### 589 *b. Surface Feedback*

590 The surface feedback is given by

$$-\lambda_{\text{surf}} = \int_{\text{surf}} \pi \frac{dB_\nu}{dT} \Big|_{T_s} e^{-\tau_{LW}} d\nu. \quad (40)$$

591 The column-integrated optical thickness at a single frequency is the sum over all absorbers at  
 592 that frequency,  $\tau_{LW}(\nu) = \tau_{\text{H}_2\text{O}}(\nu) + \tau_{\text{CO}_2}(\nu) + \tau_{\text{cnt}}$ . However, the optical thickness of H<sub>2</sub>O and CO<sub>2</sub>  
 593 drops off exponentially as a function of wavenumber away from their band centers. Thus most  
 594 frequencies are either so optically thick with respect to H<sub>2</sub>O and CO<sub>2</sub> that all surface radiation is  
 595 absorbed by the atmosphere (and hence does not contribute to the surface feedback), or so optically  
 596 thin that we can ignore H<sub>2</sub>O and CO<sub>2</sub>. Inside the window we therefore only consider absorption by  
 597 the grey continuum,  $\tau_{LW} \approx \tau_{\text{cnt}}$ , while the H<sub>2</sub>O and CO<sub>2</sub> bands primarily affect the surface feedback  
 598 by setting the width of the window.

599 To determine the width of the window we first consider an atmosphere without CO<sub>2</sub>. As discussed  
 600 above, in this case the window region is set the H<sub>2</sub>O bands, with  $\nu_L$  denoting the left window edge  
 601 around  $\sim 700 \text{ cm}^{-1}$  and  $\nu_R$  the right window edge around  $\sim 1200 \text{ cm}^{-1}$ . The H<sub>2</sub>O continuum is  
 602 grey and so can be taken out of the spectral integral,

$$-\lambda_{\text{surf}} \approx e^{-\tau_{\text{cnt}}(T_s)} \int_{\nu_L}^{\nu_R} \pi \frac{dB_\nu}{dT} \Big|_{T_s} d\nu.$$

603 We approximate the integral by treating the Planck function derivative as constant with respect  
 604 to wavenumber, evaluated at the central wavenumber  $\tilde{\nu}$  of the window region, so  $\int dB_\nu/dT d\nu \propto$   
 605  $dB_{\tilde{\nu}}/dT \times \Delta\nu$ . In reality the Planck derivative is not constant with wavenumber, so our approxima-  
 606 tion should only be treated as a scaling which we account for by including a scaling constant  $c_{\text{surf}}$ .  
 607 The magnitude of  $c_{\text{surf}}$  is further discussed below. The result is

$$-\lambda_{\text{surf}} \approx c_{\text{surf}} \times \pi \frac{dB_{\tilde{\nu}}}{dT} \Big|_{T_s} e^{-\tau_{\text{cnt}}(T_s)} \Delta\nu_{\text{surf}},$$

where  $\Delta\nu_{\text{surf}} = \nu_R - \nu_L$  is the window region width due to H<sub>2</sub>O line absorption (see Eqn. 38), and we determine the central wavenumber of the window as  $\tilde{\nu} = (\nu_R + \nu_L)/2$ .

Next, we add the effect of CO<sub>2</sub>-surface spectral blocking. After all, even if the atmosphere contained no water vapor whatsoever, part of the surface's emission would still be absorbed by CO<sub>2</sub> and thus have no effect on the TOA feedback. We account for the potential overlap between the surface and CO<sub>2</sub> by simply subtracting the CO<sub>2</sub> band width from the H<sub>2</sub>O-only window width,

$$\Delta\tilde{\nu}_{\text{surf}} = \max \left[ 0, \Delta\nu_{\text{surf}}(T_s, \text{RH}, \gamma_{\text{lr}}) - \Delta\nu_{\text{CO}_2}(q_{\text{CO}_2}) \right], \quad (41)$$

where  $\Delta\nu_{\text{CO}_2}$  is defined above (Eqn. 35) and the tilde distinguishes the window width here from the H<sub>2</sub>O-only window width. Our final expression for the surface feedback is thus

$$-\lambda_{\text{surf}} \approx c_{\text{surf}} \times \pi \frac{dB_{\tilde{\nu}}}{dT} \Big|_{T_s} e^{-\tau_{\text{cnt}}(T_s)} \Delta\tilde{\nu}_{\text{surf}}. \quad (42)$$

### c. Non-Simpsonian H<sub>2</sub>O feedback

The H<sub>2</sub>O feedback is given by

$$-\lambda_{\text{H}_2\text{O}} = \int_{\text{H}_2\text{O}} \pi \frac{dB_{\nu}}{dT} \Big|_{T_{\text{H}_2\text{O}}} \frac{dT_{\text{H}_2\text{O}}}{dT_s} d\nu. \quad (43)$$

As sketched out in Figure 5, we consider the rotational H<sub>2</sub>O band as ranging from  $\nu \approx 0$  to the left edge of the window,  $\nu_L$ . We do not consider the potential feedback from the vibration-rotation band at wavenumbers higher than  $\sim 1250 \text{ cm}^{-1}$  and, for purposes of the H<sub>2</sub>O feedback, also ignore CO<sub>2</sub>-H<sub>2</sub>O overlap effects.

The derivative of  $T_{\text{H}_2\text{O}}$  can be solved analytically. If water vapor behaved strictly according to Simpson's law then  $dT_{\text{H}_2\text{O}}/dT_s = 0$  and the H<sub>2</sub>O feedback would be zero. Simpson's law is only



an approximation, however, so

$$\begin{aligned}
\frac{dT_{\text{H}_2\text{O}}}{dT_s} &= \frac{\partial T_{\text{H}_2\text{O}}}{\partial T_s} + \frac{\partial T_{\text{H}_2\text{O}}}{\partial \gamma_{\text{lr}}} \frac{d\gamma_{\text{lr}}}{dT_s} \\
&= \frac{1}{1 + \gamma_{\text{wv}}\gamma_{\text{lr}}} \frac{T_{\text{H}_2\text{O}}}{T_s} + \\
&\quad \frac{\gamma_{\text{wv}}\gamma_{\text{lr}} - \gamma_{\text{wv}} \log\left(\frac{T_s}{T_0}\right) + \log\left(\frac{1 + \gamma_{\text{wv}}\gamma_{\text{lr}}}{\text{RH}\tau_0^*}\right)}{(1 + \gamma_{\text{wv}}\gamma_{\text{lr}})^2} T_{\text{H}_2\text{O}} \times \frac{d\gamma_{\text{lr}}}{dT_s}.
\end{aligned} \tag{44}$$

One could also explicitly write out the lapse rate derivative  $d\gamma_{\text{lr}}/dT_s$ , but the resulting expressions are long and do not lead to additional physical insight, so in practice we evaluate  $d\gamma_{\text{lr}}/dT_s$  numerically. To estimate a typical value for  $dT_{\text{H}_2\text{O}}/dT_s$  we ignore lapse rate changes, that is, the second term in Equation 44. Assuming values representative of Earth's tropics,  $1 + \gamma_{\text{wv}}\gamma_{\text{lr}} = 1 + 1/7 \times 20 \sim 4$ , and representative temperatures  $T_{\text{H}_2\text{O}} \sim 240$  K (see Figure 4) and  $T_s \sim 300$  K, a characteristic value for  $dT_{\text{H}_2\text{O}}/dT_s$  is thus

$$\frac{dT_{\text{H}_2\text{O}}}{dT_s} \sim \frac{1}{4} \times \frac{240}{300} = \frac{1}{5}, \tag{45}$$

in line with the numerical results of Jeevanjee et al. (2021a).

Next, we treat the  $\text{H}_2\text{O}$  feedback similar to the surface feedback. We assume the integrand of the spectral feedback integral is approximately constant with respect to wavenumber, and equal to its value at a central frequency  $\tilde{\nu}$ . The feedback is then

$$\begin{aligned}
-\lambda_{\text{H}_2\text{O}} &= \int_0^{\nu_L} \pi \frac{dB_\nu}{dT} \Big|_{T_{\text{H}_2\text{O}}} \frac{dT_{\text{H}_2\text{O}}}{dT_s} d\nu \\
&\approx c_{\text{H}_2\text{O}} \times \pi \frac{dB_{\tilde{\nu}}}{dT} \Big|_{T_{\text{H}_2\text{O}}(\tilde{\nu})} \times \frac{dT_{\text{H}_2\text{O}}}{dT_s} \Big|_{\tilde{\nu}} \times \Delta\nu_{\text{H}_2\text{O}},
\end{aligned} \tag{46}$$

where  $\Delta\nu_{\text{H}_2\text{O}} = \nu_L$  is the width of the  $\text{H}_2\text{O}$  band,  $\tilde{\nu} = \nu_L/2$  is the central wavenumber of the  $\text{H}_2\text{O}$  band, and  $c_{\text{H}_2\text{O}}$  is again a scaling constant to account for the fact that we are replacing a spectral integral with simple multiplication.

638 *d. H<sub>2</sub>O continuum feedback*

639 The H<sub>2</sub>O continuum feedback is

$$-\lambda_{\text{cnt}} = \int_{\text{cnt}} \pi \frac{dB_\nu}{dT} \Big|_{T_{\text{cnt}}} \frac{dT_{\text{cnt}}}{dT_s} d\nu. \quad (47)$$

640 We apply the same logic as for the surface and H<sub>2</sub>O feedbacks. The derivative  $dT_{\text{cnt}}/dT_s$  can be  
 641 solved for analytically: because  $T_{\text{cnt}}$  has no direct dependence on  $T_s$ , we have

$$\begin{aligned} \frac{dT_{\text{cnt}}}{dT_s} &= \frac{\partial T_{\text{cnt}}}{\partial \gamma_{\text{lr}}} \frac{d\gamma_{\text{lr}}}{dT_s} \\ &= \frac{T_{\text{cnt}}}{\gamma_{\text{lr}}(2\gamma_{\text{wv}} - a)} \frac{d\gamma_{\text{lr}}}{dT_s}. \end{aligned} \quad (48)$$

642 One important difference between the continuum and the other feedbacks is that the continuum  
 643 is transparent across all wavenumbers at low surface temperatures, and only becomes optically  
 644 thick at high surface temperatures. We approximate the continuum's emissivity as  $1 - e^{-\tau_{\text{cnt}}}$ , which  
 645 correctly captures the limiting behavior of an emitter at small and large optical thickness ( $\tau_{\text{cnt}} \ll 1$   
 646 versus  $\tau_{\text{cnt}} \gg 1$ ). The continuum can only dominate the atmosphere's emission at wavenumbers at  
 647 which CO<sub>2</sub> and H<sub>2</sub>O absorption is weak, so we set the effective width of the continuum equal to  
 648 the width of the window region  $\Delta\tilde{\nu}_{\text{surf}}$ , defined above. The continuum feedback is then

$$\begin{aligned} -\lambda_{\text{cnt}} &= \int_{\text{cnt}} \pi \frac{dB_\nu}{dT} \Big|_{T_{\text{cnt}}} \frac{dT_{\text{cnt}}}{dT_s} d\nu \\ &\approx c_{\text{cnt}} \times \pi \frac{dB_{\tilde{\nu}}}{dT} \Big|_{T_{\text{cnt}}} \times \frac{dT_{\text{cnt}}}{dT_s} \times \Delta\tilde{\nu}_{\text{surf}} (1 - e^{-\tau_{\text{cnt}}}) \end{aligned} \quad (49)$$

649 where  $c_{\text{cnt}}$  is again a scaling constant. The sign of  $\lambda_{\text{cnt}}$  is positive because the bulk lapse rate  
 650 decreases with warming,  $d\gamma_{\text{lr}}/dT_s < 0$ . As discussed above, this means the H<sub>2</sub>O continuum acts  
 651 as a positive/destabilizing feedback and has the opposite sign of the negative/stabilizing H<sub>2</sub>O  
 652 feedback.

653 *e. CO<sub>2</sub> radiator fin feedback*

654 For the CO<sub>2</sub> feedback we introduce an idealized “ditch” model, as shown in Figure 6. Our  
 655 approach is closely related to the CO<sub>2</sub> forcing models of Wilson and Gea-Banacloche (2012) and

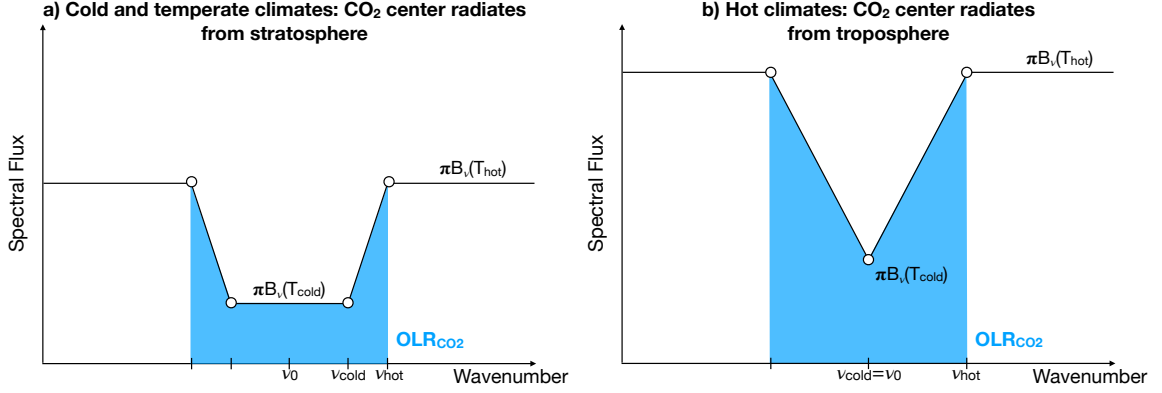


FIG. 6. A CO<sub>2</sub> “ditch” model: the CO<sub>2</sub> band emits  $\pi B_\nu(T_{\text{cold}})$  in its center, its flanks emit  $\pi B_\nu(T_{\text{hot}})$ , and the slopes in-between are approximated as linear and symmetric. Shaded blue area is the OLR contribution from the CO<sub>2</sub> band. Left: in cold climates or at high CO<sub>2</sub> abundances the CO<sub>2</sub> band center radiates from the stratosphere. Right: in hot climates or at low CO<sub>2</sub> abundances the CO<sub>2</sub> band center radiates from the troposphere.

Jeevanjee et al. (2021b) – in Appendix A we show that this ditch model also be used to rederive the results of those previous studies, underlining the close relationship between forcing and feedbacks.

We approximate the CO<sub>2</sub> band as symmetric around the central frequency  $\nu_0 = 667 \text{ cm}^{-1}$ . The center of the band emits  $\pi B_\nu(T_{\text{cold}})$  while outside the band the emission is  $\pi B_\nu(T_{\text{hot}})$ . Here  $T_{\text{cold}}$  and  $T_{\text{hot}}$  are cold and hot emission temperatures, while  $\nu_{\text{hot}}$  and  $\nu_{\text{cold}}$  denote the edges of the CO<sub>2</sub> ditch. At low and moderate surface temperatures the CO<sub>2</sub> band center around  $667 \text{ cm}^{-1}$  radiates from the stratosphere, so  $T_{\text{cold}}$  is equal to the stratospheric temperature. However, this situation is no longer true at high surface temperatures. Physically, the tropopause rises as the surface warms. If we warm the surface while holding CO<sub>2</sub> concentration fixed (this is implicit in the feedback definition, which is the change in OLR per change in surface temperature while keeping CO<sub>2</sub> fixed), more parts of the CO<sub>2</sub> band that were previously in the stratosphere have to start radiating from the troposphere. Eventually, even the CO<sub>2</sub> band center radiates from the troposphere so the rectangular CO<sub>2</sub> ditch turns into a triangle (see Fig. 6b). Here we leave our expressions general to allow for either situation.

The CO<sub>2</sub> band is relatively narrow, so we can neglect the wavenumber dependence of the Planck function and evaluate it at the center of the CO<sub>2</sub> band,  $\pi B_\nu(T) \approx \pi B_{\nu_0}(T)$ . Treating the slopes of the CO<sub>2</sub> ditch as piecewise-linear, the OLR from the CO<sub>2</sub> band is then simply the area of the ditch

shape in Figure 6a,

$$\begin{aligned}\text{OLR}_{\text{CO}_2} &= 2 \int_{\nu_0}^{\nu_{\text{hot}}} \pi B_{\nu_0}(T_{\text{CO}_2}) d\nu \\ &= [\pi B_{\nu_0}(T_{\text{hot}}) + \pi B_{\nu_0}(T_{\text{cold}})] (\nu_{\text{hot}} - \nu_{\text{cold}}) + 2\pi B_{\nu_0}(T_{\text{cold}})(\nu_{\text{cold}} - \nu_0).\end{aligned}\quad (50)$$

It follows that the OLR change in response to some climate perturbation is

$$\begin{aligned}\Delta \text{OLR}_{\text{CO}_2} &= \text{OLR}'_{\text{CO}_2} - \text{OLR}_{\text{CO}_2} \\ &= [\pi B_{\nu_0}(T'_{\text{hot}}) + \pi B_{\nu_0}(T'_{\text{cold}})] (\nu'_{\text{hot}} - \nu'_{\text{cold}}) - [\pi B_{\nu_0}(T_{\text{hot}}) + \pi B_{\nu_0}(T_{\text{cold}})] (\nu_{\text{hot}} - \nu_{\text{cold}}) + \\ &\quad 2\pi B_{\nu_0}(T'_{\text{cold}})(\nu'_{\text{cold}} - \nu_0) - 2\pi B_{\nu_0}(T_{\text{cold}})(\nu_{\text{cold}} - \nu_0),\end{aligned}\quad (51)$$

where primes indicate perturbed variables. For the CO<sub>2</sub> feedback the relevant perturbation is a change in surface temperature  $\Delta T_s$ , while for the forcing the relevant perturbation is a change in  $q_{\text{CO}_2}$  (see Appendix A). If  $\Delta T_s$  is small enough we can series expand and drop higher-order terms. For example, the perturbation of the emission at the CO<sub>2</sub> band edge is

$$\pi B_{\nu_0}(T'_{\text{hot}}) = \pi B_{\nu_0}(T_{\text{hot}}) + \pi \left. \frac{dB_{\nu_0}}{dT} \right|_{T_{\text{hot}}} \frac{dT_{\text{hot}}}{dT_s} \Delta T_s,$$

with similar expressions for  $T'_{\text{cold}}$ ,  $\nu'_{\text{cold}}$ , and  $\nu'_{\text{cold}}$ . Plugging back into Equation 51, the feedback of the CO<sub>2</sub> ditch is

$$\begin{aligned}-\lambda_{\text{CO}_2} &= \lim_{\Delta T_s \rightarrow 0} \frac{\Delta \text{OLR}_{\text{CO}_2}}{\Delta T_s} \\ &= \left[ \pi \left. \frac{dB_{\nu_0}}{dT} \right|_{T_{\text{hot}}} \frac{dT_{\text{hot}}}{dT_s} + \pi \left. \frac{dB_{\nu_0}}{dT} \right|_{T_{\text{cold}}} \frac{dT_{\text{cold}}}{dT_s} \right] (\nu_{\text{hot}} - \nu_{\text{cold}}) \\ &\quad + [\pi B_{\nu_0}(T_{\text{hot}}) + \pi B_{\nu_0}(T_{\text{cold}})] \left( \frac{d\nu_{\text{hot}}}{dT_s} - \frac{d\nu_{\text{cold}}}{dT_s} \right) \\ &\quad + 2\pi \left. \frac{dB_{\nu_0}}{dT} \right|_{T_{\text{cold}}} \frac{dT_{\text{cold}}}{dT_s} (\nu_{\text{cold}} - \nu_0) + 2B_{\nu_0}(T_{\text{cold}}) \frac{d\nu_{\text{cold}}}{dT_s}.\end{aligned}\quad (52)$$

Equation 52 gives the most general expression for the feedback of the CO<sub>2</sub> ditch. To evaluate this expression we need to specify how the emission temperatures  $T_{\text{hot}}$ ,  $T_{\text{cold}}$  and the CO<sub>2</sub> band edges  $\nu_{\text{hot}}$ ,  $\nu_{\text{cold}}$  vary as a function of surface temperature.

At cold surface temperatures we again ignore H<sub>2</sub>O absorption around the CO<sub>2</sub> band so  $T_{\text{hot}} = T_s$ . Similarly, the tropopause is low and the CO<sub>2</sub> band center radiates from the stratosphere, so  $T_{\text{cold}} = T_{\text{strat}}$  and  $dT_{\text{cold}}/dT_s = 0$ . As in Section 5a, we find the band edges  $\nu_{\text{hot}}$  and  $\nu_{\text{cold}}$  by solving  $T_{\text{CO}_2}(\nu_{\text{hot}}) = T_s$  and  $T_{\text{CO}_2}(\nu_{\text{cold}}) = T_{\text{strat}}$ . The results are  $\nu_{\text{hot}} = \nu_0 + l_\nu \log[\tau_{\text{CO}_2}^*(\nu_0)q_{\text{CO}_2}]$ , and  $\nu_{\text{cold}} = \nu_0 + l_\nu \log[\tau_{\text{CO}_2}^*(\nu_0)q_{\text{CO}_2}(T_{\text{strat}}/T_s)^{2/\gamma_{\text{lr}}}]$ . We can see that the hot CO<sub>2</sub> band edge does not change under surface warming,  $d\nu_{\text{hot}}/dT_s = 0$ , while the sensitivity of the cold or stratospheric band edge to surface warming is

$$\begin{aligned} \frac{d\nu_{\text{cold}}}{dT_s} &= \left. \frac{\partial \nu_{\text{cold}}}{\partial T_s} \right|_{\gamma_{\text{lr}}} + \left. \frac{\partial \nu_{\text{cold}}}{\partial \gamma_{\text{lr}}} \right|_{T_s} \frac{d\gamma_{\text{lr}}}{dT_s} \\ &= -\frac{2l_\nu}{\gamma_{\text{lr}}T_s} + \frac{2l_\nu}{\gamma_{\text{lr}}^2} \log\left(\frac{T_s}{T_{\text{strat}}}\right) \frac{d\gamma_{\text{lr}}}{dT_s}. \end{aligned} \quad (53)$$

The lapse rate change  $d\gamma_{\text{lr}}/dT_s$  is always negative, so if we warm the surface while holding CO<sub>2</sub> concentration fixed the portion of the CO<sub>2</sub> band inside the stratosphere shrinks,  $d\nu_{\text{cold}}/dT_s < 0$ . This is again a simple consequence of a rising tropopause. As the surface warms, the tropopause moves to lower pressures, thus moving more of CO<sub>2</sub>'s emission from the stratosphere into the tropopause. Plugging back into Equation 52, the CO<sub>2</sub> feedback at cold surface temperatures is

$$-\lambda_{\text{CO}_2}^{\text{cool}} = \pi \frac{dB_{\nu_0}}{dT} \bigg|_{T_s} \frac{2}{\gamma_{\text{lr}}} \log\left(\frac{T_s}{T_{\text{strat}}}\right) + [\pi B_{\nu_0}(T_s) - \pi B_{\nu_0}(T_{\text{strat}})] \times \left( \frac{2l_\nu}{\gamma_{\text{lr}}T_s} - \frac{2l_\nu}{\gamma_{\text{lr}}^2} \log\left(\frac{T_s}{T_{\text{strat}}}\right) \frac{d\gamma_{\text{lr}}}{dT_s} \right) \quad (54)$$

At high surface temperatures the CO<sub>2</sub> band center moves into the tropopause and the rectangular ditch turns into a triangle (see lower left in Fig. 4, and sketch in Fig. 6b). We set  $\nu_{\text{cold}} = \nu_0$ , where the central wavenumber  $\nu_0$  is set by the spectroscopic properties of CO<sub>2</sub> and so is fixed under surface warming ( $d\nu_{\text{cold}}/dT_s = 0$ ). The emission temperature in the center of the CO<sub>2</sub> band is now  $T_{\text{cold}} = T_{\text{CO}_2}(\nu_0)$ , where  $T_{\text{CO}_2}$  is the emission temperature of CO<sub>2</sub> (Eqn. 25a). The crucial difference between high and low surface temperatures is that once the CO<sub>2</sub> band center moves into the tropopause  $T_{\text{cold}}$  is no longer constant,

$$\begin{aligned} \frac{dT_{\text{CO}_2}(\nu_0)}{dT_s} &= \left. \frac{\partial T_{\text{CO}_2}(\nu_0)}{\partial T_s} \right|_{\gamma_{\text{lr}}} + \left. \frac{\partial T_{\text{CO}_2}(\nu_0)}{\partial \gamma_{\text{lr}}} \right|_{T_s} \frac{d\gamma_{\text{lr}}}{dT_s} \\ &= \frac{T_{\text{CO}_2}(\nu_0)}{T_s} - \frac{T_{\text{CO}_2}(\nu_0)}{2} \log[q_{\text{CO}_2} \tau_{\text{CO}_2}^*(\nu_0)] \frac{d\gamma_{\text{lr}}}{dT_s}. \end{aligned} \quad (55)$$

707 The outer edges of the CO<sub>2</sub> band at high temperatures are set by water vapor absorption,  
 708  $T_{\text{hot}} = \min[T_{\text{H}_2\text{O}}(\nu_0), T_{\text{cnt}}]$ . We treat H<sub>2</sub>O as Simpsonian, so  $dT_{\text{hot}}/dT_s \approx 0$ , and also ignore non-  
 709 Simpsonian shifts in the outer CO<sub>2</sub> band edge,  $d\nu_{\text{hot}}/dT_s \approx 0$ . Plugging back into Equation 52, the  
 710 feedback at high surface temperatures is then

$$\begin{aligned}
 -\lambda_{\text{CO}_2}^{\text{hot}} &= \pi \left. \frac{dB_{\nu_0}}{dT} \right|_{T_{\text{cold}}} \frac{dT_{\text{cold}}}{dT_s} (\nu_{\text{hot}} - \nu_{\text{cold}}) \\
 &= \pi \left. \frac{dB_{\nu_0}}{dT} \right|_{T_{\text{cold}}} \frac{dT_{\text{cold}}}{dT_s} l_{\nu} \log \left[ \tau_{\text{CO}_2}^*(\nu_0) q_{\text{CO}_2} \left( \frac{T_{\text{hot}}}{T_s} \right)^{\frac{2}{\gamma_{\text{tr}}}} \right].
 \end{aligned} \tag{56}$$

711 Finally, when does the CO<sub>2</sub> band center change from a stratospheric radiator at low  $T_s$  to a  
 712 tropospheric radiator at high  $T_s$ , which also determines the transition between  $\lambda_{\text{CO}_2}^{\text{cool}}$  and  $\lambda_{\text{CO}_2}^{\text{hot}}$ ?  
 713 Based on line-by-line calculations with 400 ppm of CO<sub>2</sub>, Appendix B shows that the smoothed  
 714 emission temperature in the CO<sub>2</sub> band center moves out of the stratosphere at surface temperatures  
 715 above 310 K. We therefore identify 310 K as the transition point between the low-temperature  
 716 and high-temperature CO<sub>2</sub> feedback regimes. Note, however, that this value also depends on CO<sub>2</sub>  
 717 concentration.

718 Multiplying the low-temperature regime with a scaling constant  $c_{\text{CO}_2}$ , similar to our other spectral  
 719 feedbacks, the overall CO<sub>2</sub> radiator fin feedback is thus

$$\lambda_{\text{CO}_2} = \begin{cases} c_{\text{CO}_2} \times \lambda_{\text{CO}_2}^{\text{cool}} & \text{if } T_s \leq 310 \text{ K} \\ \lambda_{\text{CO}_2}^{\text{hot}} + b & \text{if } T_s > 310 \text{ K.} \end{cases} \tag{57}$$

720 where we choose the constant  $b$  to ensure that  $\lambda_{\text{CO}_2}$  remains continuous at 310 K (in practice  $b$  is  
 721 always of order unity,  $b \sim 0.5$ ).

## 722 *f. Validation against LBL calculations*

723 To test our analytical feedback expressions we again use 1D calculations with PyRADS. One  
 724 potential issue is that our derivations use the bulk lapse rate approximation, and so might differ  
 725 from realistic feedbacks. Appendix C compares feedbacks calculated with a moist adiabat to  
 726 feedbacks with bulk lapse rate profiles. Overall, the bulk lapse rate approximation only introduces  
 727 minor errors in  $\lambda_{\text{LW}}$  over the temperature range 250 – 320 K, while errors in individual spectral

feedbacks as well as in  $\lambda_{LW}$  become notable at  $\sim 320$  K. We therefore consider the bulk lapse rate approximation sufficiently accurate below 320 K, but care should be taken when applying our analytical expressions to extremely hot climates. To better match the derivations, the PyRADS calculations here also use vertical profiles with a bulk lapse rate, so  $T = T_s(p/p_s)^{\gamma_{lr}}$ . We explore the surface temperature-dependence of spectral feedbacks at high and low relative humidity (RH=0.8 and RH=0.1), without CO<sub>2</sub> and with 400 ppm of CO<sub>2</sub>, for four sets of calculations total.

To compare our analytical expressions against the 1D calculations we need to specify the scaling constants  $c_{\text{surf}}$ ,  $c_{\text{H}_2\text{O}}$ ,  $c_{\text{cnt}}$ , and  $c_{\text{CO}_2}$ . We pick these constants to match the 1D calculations at RH=0.8 and 400 ppm of CO<sub>2</sub>. The temperature-dependence varies significantly between different feedbacks, so we choose  $c_{\text{surf}}$  to match  $\lambda_{\text{surf}}$  at low temperatures ( $T_s = 250$  K),  $c_{\text{cnt}}$  to match  $\lambda_{\text{cnt}}$  at high temperatures ( $T_s = 330$  K), and  $c_{\text{H}_2\text{O}}$  and  $c_{\text{CO}_2}$  to match  $\lambda_{\text{H}_2\text{O}}$  and  $\lambda_{\text{CO}_2}$  around Earth's present-day mean temperature ( $T_s = 290$  K). Table 1 shows the resulting values. In this Section we choose the scaling constants to match the 1D calculations with bulk lapse rates. We note that Section 6 considers a feedback calculation specifically for present-day Earth, so in that Section we use slightly-different scaling constants which match 1D calculations with moist adiabatic lapse rates (see Appendix C). Regardless of the exact values, however, the scaling constants are always of order unity.

Figure 7 shows that our analytical expressions successfully capture the basic state-dependence of  $\lambda_{LW}$  as well as of its spectral constituents. The longwave feedback  $\lambda_{LW}$  is sensitive to changes in surface temperature, but it also varies in response to humidity and CO<sub>2</sub> changes. Comparing between different panels in Figure 7,  $\lambda_{LW}$  becomes larger with decreasing relative humidity (also see McKim et al. 2021). In contrast, CO<sub>2</sub> evens out the temperature-dependence of  $\lambda_{LW}$ , by decreasing  $\lambda_{LW}$  at cold temperatures and increasing  $\lambda_{LW}$  at high temperatures. Importantly, the analytical expressions capture most of the variation in  $\lambda_{LW}$  and recover the state-dependence of the net longwave feedback.

To understand the overall behavior of  $\lambda_{LW}$  we turn to the individual spectral feedbacks. The surface feedback  $\lambda_{\text{surf}}$  is generally the dominant term in the spectral decomposition. Without CO<sub>2</sub>,  $\lambda_{\text{surf}}$  makes up at least 90% of  $\lambda_{LW}$  below 300 K. The presence of CO<sub>2</sub> decreases  $\lambda_{\text{surf}}$  but even in this case  $\lambda_{\text{surf}}$  makes up at least 60% of  $\lambda_{LW}$  below 300 K. Our analytical expressions thus agree with previous studies which argued that Earth's longwave feedback is dominated by the

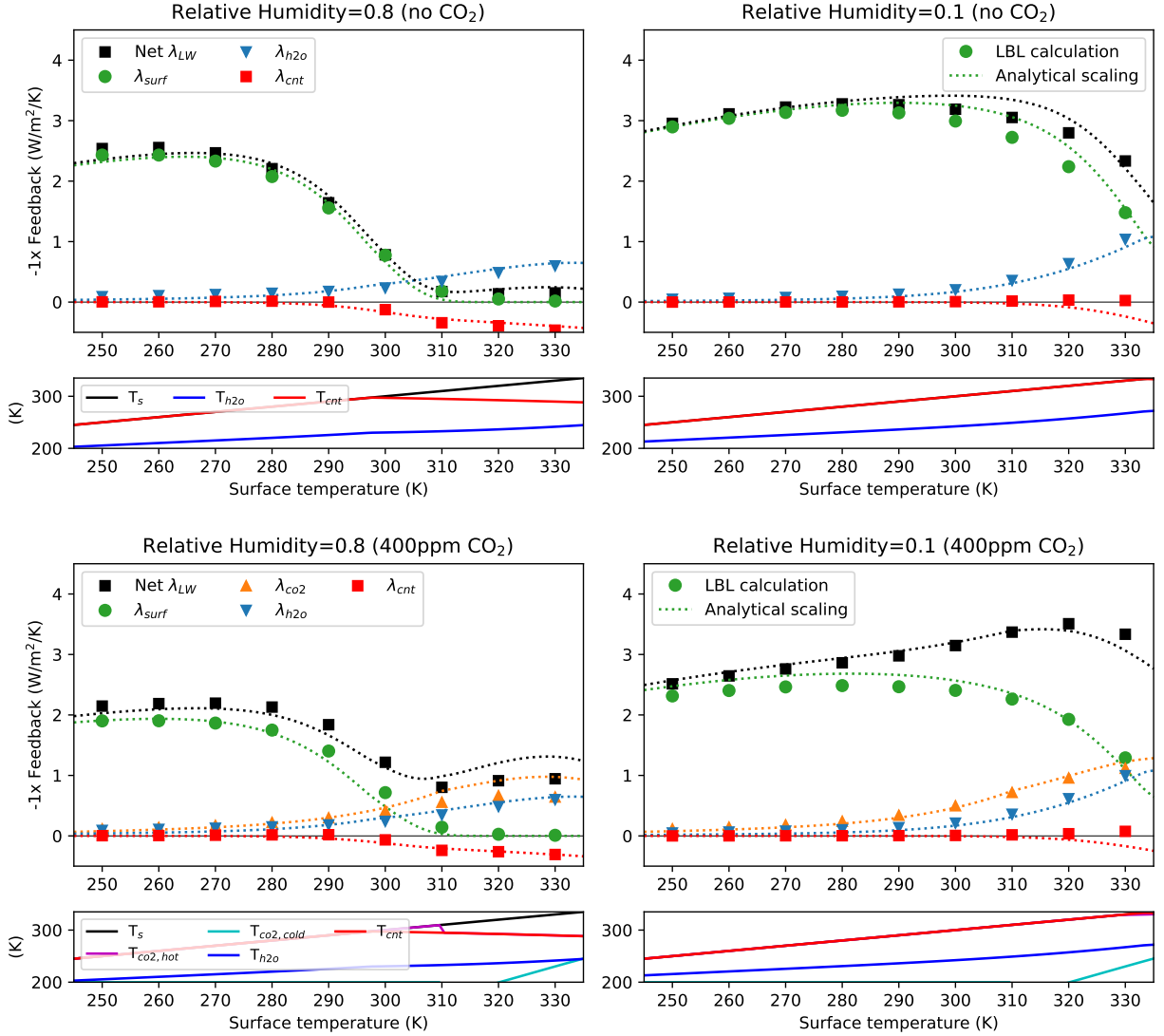


FIG. 7. Spectral feedbacks calculated using a bulk lapse rate (symbols), compared against the analytical scalings (lines). Top row shows calculations without CO<sub>2</sub>, bottom row 400ppm of CO<sub>2</sub>. The large panels show feedbacks while small panels show the corresponding analytical emission temperatures.

surface feedback (Koll and Cronin 2018; Raghuraman et al. 2019). This situation changes at high temperatures, however, once the surface window closes, at which point  $\lambda_{LW}$  becomes dominated by atmospheric feedbacks.

Next, in line with Section 4, the CO<sub>2</sub> radiator fin feedback acts to stabilize Earth's climate, and its importance increases with surface temperature. Below 300 K,  $\lambda_{co2}$  contributes less than 20% of the total feedback. However,  $\lambda_{co2}$  rapidly grows with surface temperature such that at 330 K and



TABLE 2. Summary of main theoretical results.

Emission temperatures	
$T_{\text{co}_2}(\nu) =$	$T_s \left( \frac{1}{\tau_{\text{co}_2}^*(\nu) q_{\text{co}_2}} \right)^{\gamma_{\text{lr}}/2}$
$T_{\text{H}_2\text{O}}(\nu) =$	$T_0 \left( \frac{1+\gamma_{\text{wv}} \gamma_{\text{lr}}}{\tau_{\text{H}_2\text{O}}^*(\nu) \text{RH}} \right)^{\frac{\gamma_{\text{lr}}}{1+\gamma_{\text{wv}} \gamma_{\text{lr}}}} \left( \frac{T_s}{T_0} \right)^{\frac{1}{1+\gamma_{\text{wv}} \gamma_{\text{lr}}}}$
$T_{\text{cnt}} =$	$T_0 \left( \frac{(2\gamma_{\text{wv}}-a) \gamma_{\text{lr}}}{\tau_{\text{cnt}}^* \text{RH}^2} \right)^{\frac{1}{2\gamma_{\text{wv}}-a}}$
Feedbacks	
$-\lambda_{\text{surf}} =$	$c_{\text{surf}} \times \pi \frac{dB_{\tilde{\nu}}}{dT} \Big _{T_s} \Delta \tilde{\nu}_{\text{surf}} e^{-\tau_{\text{cnt}}}$
$-\lambda_{\text{H}_2\text{O}} =$	$c_{\text{H}_2\text{O}} \times \pi \frac{dB_{\tilde{\nu}}}{dT} \Big _{T_{\text{H}_2\text{O}}(\tilde{\nu})} \times \frac{dT_{\text{H}_2\text{O}}(\tilde{\nu})}{dT_s} \times \Delta \nu_{\text{H}_2\text{O}}$
$-\lambda_{\text{cnt}} =$	$c_{\text{cnt}} \times \pi \frac{dB_{\tilde{\nu}}}{dT} \Big _{\tilde{\nu}, T_{\text{cnt}}} \times \frac{dT_{\text{cnt}}}{dT_s} \times \Delta \tilde{\nu}_{\text{surf}} (1 - e^{-\tau_{\text{cnt}}})$
$-\lambda_{\text{co}_2} =$	$\begin{cases} c_{\text{co}_2} \times \frac{2\pi}{\gamma_{\text{lr}}} \frac{dB_{\nu_0}}{dT} \Big _{T_s} \log \left( \frac{T_s}{T_{\text{strat}}} \right) - [\pi B_{\nu_0}(T_s) - \pi B_{\nu_0}(T_{\text{strat}})] \times \frac{d\nu_{\text{cold}}}{dT_s} & \text{at low } T_s \\ \pi \frac{dB_{\nu_0}}{dT} \Big _{T_{\text{cold}}} \frac{dT_{\text{cold}}}{dT_s} (\nu_{\text{hot}} - \nu_{\text{cold}}) + b & \text{at high } T_s \end{cases}$

767 high relative humidity  $\lambda_{\text{co}_2}$  makes up almost 70% of  $\lambda_{LW}$ . Interestingly, if RH is high,  $\lambda_{\text{co}_2}$  becomes  
 768 equal to  $\lambda_{\text{surf}}$  at surface temperatures around  $\sim 305$  K. Extrapolating from these 1D calculations to  
 769 Earth's spatial feedback pattern, we can expect that Earth's feedback is dominated by the surface  
 770 in most regions, but that atmospheric feedbacks become important in the inner tropics – an issue  
 771 we explore in detail in Section 6.

772 Finally, again in line with our analytical results, the two water vapor feedbacks  $\lambda_{\text{H}_2\text{O}}$  and  $\lambda_{\text{cnt}}$   
 773 have opposing signs. At high relative humidity  $\lambda_{\text{H}_2\text{O}}$  and  $\lambda_{\text{cnt}}$  partially cancel. In contrast, at low  
 774 relative humidity  $\lambda_{\text{cnt}}$  becomes negligible while  $\lambda_{\text{H}_2\text{O}}$  only changes moderately – a non-Simpsonian  
 775 effect. The different sensitivity to RH arises because the continuum's optical thickness scales as  
 776  $\tau_{\text{cnt}} \propto \text{RH}^2$ , whereas the optical thickness in the water vapor bands only scales as  $\tau_{\text{H}_2\text{O}} \propto \text{RH}$ .  
 777 Decreases in relative humidity therefore increase  $\lambda_{LW}$  both by increasing the surface feedback  
 778  $\lambda_{\text{surf}}$  and by reducing  $\lambda_{\text{cnt}}$ , so that  $\text{H}_2\text{O}$  acts as a net stabilizing feedback. Comparing  $\lambda_{\text{H}_2\text{O}}$  and  
 779  $\lambda_{\text{co}_2}$  at present-day  $\text{CO}_2$  levels, we see that the two feedbacks are roughly equal in magnitude.  
 780 Non-Simpsonian  $\text{H}_2\text{O}$  effects are thus about as important as the  $\text{CO}_2$  radiator fin for Earth's current  
 781 longwave feedback.

## 6. The spatial pattern of $\lambda_{LW}$

In the previous two sections we found that our analytical model, summarized in Table 2, successfully captures the behavior of Earth’s emission temperature  $T_{\text{rad}}$  as well as the state-dependence of  $\lambda_{LW}$ . In this section we perform another validation by reproducing the spatial map of  $\lambda_{LW}$  for Earth’s present-day climate. First, we generate a reference map of  $\lambda_{LW}$  using the radiative kernel technique (Soden et al. 2008), which diagnoses  $\lambda_{LW}$  by combining a radiative kernel with the forced response from climate model output. Here these quantities are evaluated using preindustrial and 4xCO<sub>2</sub> simulations from the HadGEM2 climate model (Collins et al. 2011), described below. Next, we generate maps of  $\lambda_{LW}$  using our analytical expressions and variable amounts of input data from HadGEM2. We find that the spatial pattern of  $\lambda_{LW}$  can be largely constrained using only knowledge about the preindustrial climate, that is, without any knowledge of the climate’s forced response. As a best estimate of Earth’s true clear-sky longwave feedback we therefore also generate a map of  $\lambda_{LW}$  using only our analytical expressions and the observationally-constrained ERA5 reanalysis dataset (Hersbach et al. 2020).

### *a. Description of Kernel and Input Data*

We follow established methods for the kernel calculation. We use the HadGEM2 radiative kernel and, for consistency with the analytical model (which assumes the stratosphere is isothermal and at a fixed temperature), we set the kernel to zero inside the stratosphere. The tropopause is defined as in Soden et al. (2008): the tropopause pressure  $p_{tp}$  linearly increases with latitude, from 100hPa at the equator to 300hPa at the poles. The analytical model also assumes relative humidity stays fixed under surface warming, so we do not include relative humidity changes in the kernel calculation. Doing so is justified because RH feedbacks only make a minor contribution to  $\lambda_{LW}$  in individual climate models, and the RH feedback moreover tends to cancel in the multi-model mean (Zelinka et al. 2020). To compute the forced response we use HadGEM2 climatologies from the CMIP5 archive for a preindustrial control simulation and an abrupt-4XCO<sub>2</sub> simulation. By multiplying the kernel with the forced response one obtains a map of the change in top-of-atmosphere (TOA) radiation (Soden et al. 2008). To compute a feedback one additionally needs to normalize the change in TOA radiation by a change in temperature. Consistent with our derivations we compute local-local feedback maps, that is, we divide the local change in OLR deduced from the kernel by

811 the local change in surface temperature (Feldl and Roe 2013; Armour et al. 2013; Bloch-Johnson  
812 et al. 2020).

817 We compare the spatial map of  $\lambda_{LW}$  from the kernel against maps of  $\lambda_{LW}$  from our analytical  
818 expressions. The analytical expressions require six input parameters: CO<sub>2</sub> abundance, surface  
819 temperature  $T_s$ , stratosphere temperature  $T_{strat}$ , relative humidity RH, temperature lapse rate  $\gamma_{lr}$ , and  
820 the change in lapse rate under surface warming  $d\gamma_{lr}/dT_s$ . Except for the lapse rate change  $d\gamma_{lr}/dT_s$ ,  
821 all these inputs can be obtained from a single climate state (e.g., the HadGEM2 preindustrial state)  
822 and do not require any knowledge of the climate’s forced response. CO<sub>2</sub> is set to be spatially  
823 uniform at 400 ppm (results are highly similar if using a preindustrial 285 ppm instead); the surface  
824 temperature  $T_s$  is taken as the air temperature at 2m; and the stratospheric temperature  $T_{strat}$  is  
825 set equal to the temperature at the tropopause pressure level,  $T_{strat} = T(p_{tp})$ , using the above  
826 tropopause definition of Soden et al. (2008). The relative humidity RH is set equal to the column  
827 relative humidity, defined as the ratio between the atmospheric column’s water vapor path and its  
828 water vapor path at saturation (e.g., Bretherton et al. 2005),

$$\text{RH} = \frac{\text{WVP}}{\text{WVP}^*}, \quad (58)$$

$$= \frac{\int_{p_{tp}}^{p_s} q \, dp/g}{\int_{p_{tp}}^{p_s} q^* \, dp/g}. \quad (59)$$

829 Here the vertical integral is taken from the tropopause  $p_{tp}$  down to the surface to exclude the  
830 strongly sub-saturated stratosphere. One could in principle also approximate RH using other  
831 measures of atmospheric humidity; however, the column relative humidity is a natural choice  
832 because it correctly captures the atmosphere’s total water vapor path, which in turn determines the  
833 width of the window region and  $\lambda_{\text{surf}}$ .

834 Next, the lapse rate  $\gamma_{lr} = d \ln T / d \ln p$  varies strongly in the vertical. We compute the mean lapse  
835 rate using a mass-weighted vertical average,

$$\gamma_{lr} = \frac{1}{p_1 - p_{tp}} \int_{p_{tp}}^{p_1} \frac{p}{T} \frac{dT}{dp} dp, \quad (60)$$

836 where the average is taken from the tropopause level  $p_{tp}$  down to  $p_1 = 850$  hPa. We exclude the  
837 stratosphere and near-surface layers to avoid inversions (our derivations break down if  $\gamma_{lr} < 0$ ).

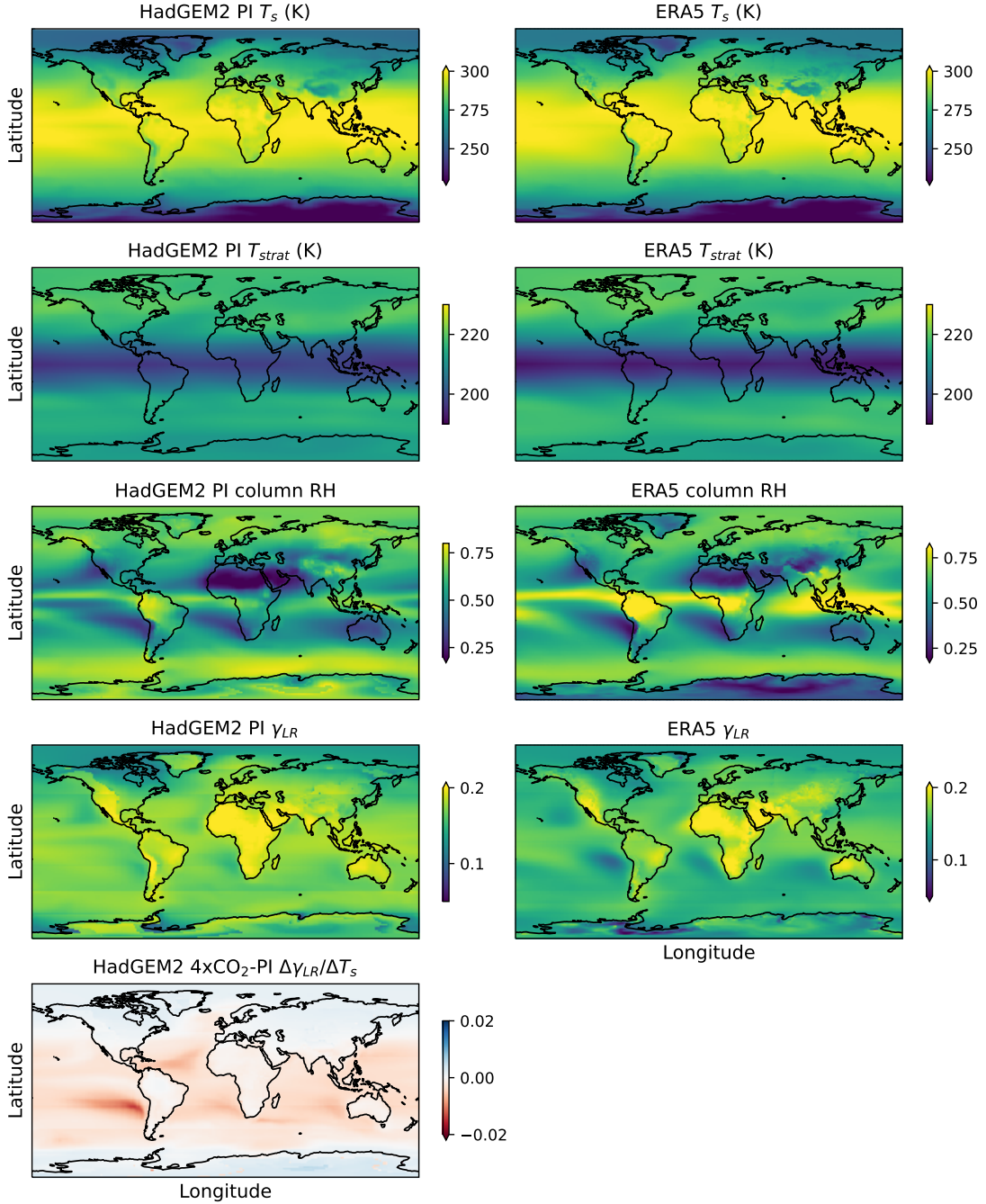


FIG. 8. Input data used to evaluate the analytical feedback model. Left column: input from the HadGEM2 GCM. The top four panels are from a preindustrial (PI) simulation, the bottom panel shows the lapse rate difference  $\Delta\gamma_{lr}/\Delta T_s$  between 4xCO<sub>2</sub> and PI simulations. Right column: input from ERA5 reanalysis averaged over the years 1980-1999.

One could also evaluate  $\gamma_{lr}$  using Equation 5 and an appropriate tropopause definition; however, this approach makes the inferred lapse rates highly sensitive to the specific tropopause definition, which we side-step by using Equation 60 instead.

Finally, the only input in our analytical expressions that requires information about the climate’s forced response is the change in lapse rate  $d\gamma_{lr}/dT_s$ . We use two different approaches. First, we compute  $d\gamma_{lr}/dT_s$  as the change in  $\gamma_{lr}$  between abrupt-4x and preindustrial HadGEM2 simulations. As in feedback calculations that are based on radiative kernels (e.g., Soden et al. 2008), this approach is purely diagnostic, since one can only deduce  $\lambda_{LW}$  when the climate’s forced response is already known. We also demonstrate a second approach in which we do not use any information about the climate’s forced response. To do so we assume  $d\gamma_{lr}/dT_s$  is locally moist-adiabatic, i.e., we compute  $d\gamma_{lr}/dT_s$  using a bulk lapse rate evaluated at the local surface temperature. Doing so is only an approximation, but it has the advantage that it allows one to predict  $\lambda_{LW}$  solely using information about a climate’s current state (preindustrial climate model data or reanalysis data of present-day Earth).

Figure 8 shows the input data maps used to evaluate the analytical expressions. The left column is based on the HadGEM2 simulations, the right column is based on the ERA5 reanalysis data. Spatial variations are most notable in the maps of surface temperature  $T_s$ , column relative humidity RH, and mean lapse rate  $\gamma_{lr}$ , which show qualitatively (if not necessarily quantitatively) similar patterns between HadGEM2 and ERA5. In contrast, apart from minor midlatitude stationary wave patterns, the stratospheric temperature  $T_{strat}$  is fairly symmetric zonally and varies by only about 20 K between the equator and poles. The lapse rate change under surface warming,  $\Delta\gamma_{lr}/\Delta T_s$ , requires us to know a climate’s forced response and so is only shown for HadGEM2. In agreement with previous studies,  $\Delta\gamma_{lr}/\Delta T_s$  exhibits an equator-pole contrast – at low latitudes  $\gamma_{lr}$  decreases, as expected for a moist adiabat, while at high latitudes  $\gamma_{lr}$  increases under global warming (e.g., Payne et al. 2015; Cronin and Jansen 2016; Stuecker et al. 2018).

#### *b. Model validation: Kernel versus analytical feedback maps*

The maps of  $\lambda_{LW}$  resulting from the kernel and analytical calculations are shown in Figure 9. The kernel-derived map of  $\lambda_{LW}$  shows strong spatial contrasts between high latitudes, subtropics, and inner tropics (Fig. 9, top left). The longwave feedback is smallest in the inner tropics, especially

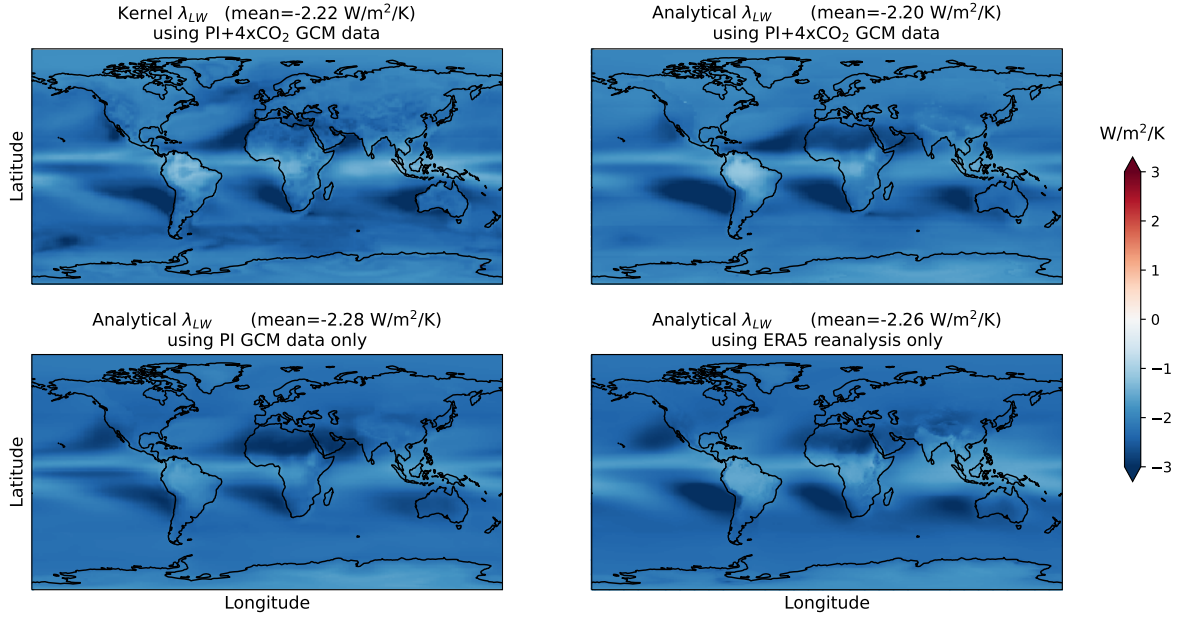
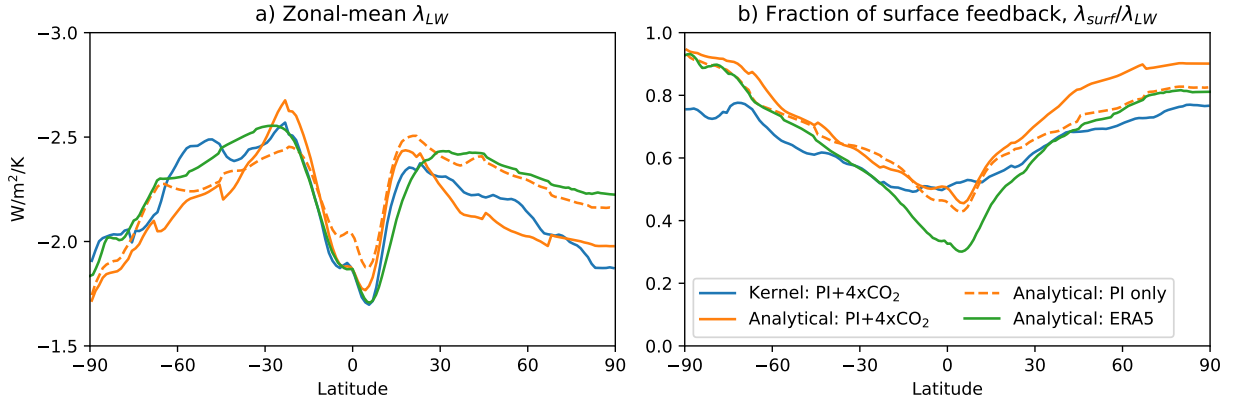


FIG. 9. Spatial pattern of  $\lambda_{LW}$  deduced from varying amounts of input data. Top left: Reference. Computed using a radiative kernel, plus preindustrial (PI) and abrupt-4x CO<sub>2</sub> GCM simulations. Top right: from our analytical expressions, plus PI and 4xCO<sub>2</sub> GCM data. Bottom left: from our analytical expressions and PI GCM data only. Bottom right: from our analytical expressions and ERA5 reanalysis data only. Means shown above each subpanel are area-weighted global means, and do not take into account the pattern of warming.

in the intertropical convergence zone (ITCZ), while it is largest in the subtropics, especially in subtropical deserts like the Sahara and over eastern ocean basins. At mid and high latitudes the magnitude of  $\lambda_{LW}$  is close to the global mean. Zonal contrasts are largest in the subtropics and more muted at higher latitudes.

Importantly, all analytical calculations recover the same spatial patterns as the radiative kernel:  $\lambda_{LW}$  approaches zero in the ITCZ, is largest in the subtropics, and intermediate at high latitudes. Comparing the global area-weighted means of  $\lambda_{LW}$ , we find that the analytical expressions also get the mean feedback magnitude right. The mean of  $\lambda_{LW}$  is  $-2.2 \text{ W m}^{-2} \text{ K}^{-1}$  in the kernel-derived map, while it varies between  $-2.2$  and  $-2.3 \text{ W m}^{-2} \text{ K}^{-1}$  in the analytical maps. Note that to compute the true global-mean  $\lambda_{LW}$ , the local feedback maps in Figure 9 should be weighted by the pattern of the forced surface temperature response (Feldl and Roe 2013; Armour et al. 2013). Doing so is ambiguous when the forced response is not known, however, so we compare area-weighted means instead.

885 The agreement between kernel and analytical feedbacks in Figure 9 is less good at regional  
 886 scales. For example, even the analytical calculation that uses both preindustrial (PI) and 4xCO<sub>2</sub>  
 887 HadGEM2 data, and thus should agree most closely with the kernel, shows noticeable deviations  
 888 from the kernel map over the Sahara and over the Eastern Pacific off the coast of Peru. Similarly,  
 889 the kernel-derived map of  $\lambda_{LW}$  shows clear small-scale variation over extratropical oceans, in  
 890 particular over the Southern Ocean and North Atlantic, while all analytical maps show little-to-no  
 891 variation in these regions.



892 FIG. 10. a) Zonal mean of the  $\lambda_{LW}$  maps in Figure 9. b) Zonal mean fraction of the surface feedback to the  
 893 net feedback,  $\lambda_{surf}/\lambda_{LW}$ .

894 To further analyze the match between kernel and analytical feedbacks, Figure 10 shows the zonal  
 895 means of  $\lambda_{LW}$  and the fraction of  $\lambda_{LW}$  that is due to the surface feedback  $\lambda_{surf}$ . Kernel and analytical  
 896 calculations agree well in the ITCZ but less so at high latitudes, with differences in the zonal-mean  
 897  $\lambda_{LW}$  reaching up to 10% over the Arctic. Similarly, the analytical calculations qualitatively recover  
 898 the partition between surface and atmospheric feedbacks (panel b) but deviate at regional scales.  
 899 In the HadGEM2 kernel  $\lambda_{surf}/\lambda_{LW}$  is about 60% in the tropics and about 80% at the poles, which is  
 900 comparable to the value of 60% reported by Raghuraman et al. (2019). In contrast, our analytical  
 901 model tends to modestly over-predict the surface feedback, with  $\lambda_{surf}/\lambda_{LW}$  varying from about 60%  
 902 in the tropics to over 90% at the poles. One plausible reason for the deviations in  $\lambda_{surf}/\lambda_{LW}$  is that  
 903 our expressions do not consider ozone. Ozone absorbs effectively inside the window region, which  
 904 should reduce the window width  $\Delta\nu_{surf}$  and thus also the surface feedback  $\lambda_{surf}$ . Nevertheless,  
 905 both kernel and analytical calculations agree that  $\lambda_{LW}$  is dominated by  $\lambda_{surf}$  across most of the

globe, while atmospheric feedbacks only start to rival  $\lambda_{\text{surf}}$  in the inner tropics and particularly inside the ITCZ.

Despite some noticeable inaccuracies at regional scales, we conclude that the analytical expressions are sufficiently accurate to be useful for understanding the spatial pattern of  $\lambda_{LW}$ . To do so we break  $\lambda_{LW}$  up into its spectral constituents next.

### c. Spectral decomposition

Figure 11 shows the spectral decomposition of our analytical feedback maps. In agreement with Section 5, where we found that  $\lambda_{LW}$  is dominated by  $\lambda_{\text{surf}}$  across a wide surface temperature range, the spatial variation in  $\lambda_{LW}$  is also largely dominated by the spatial variation in  $\lambda_{\text{surf}}$ . The surface feedback is large at high latitudes, peaks in the subtropics, and decreases to less than a fourth of its peak value in the ITCZ. This contrasts with what one might naively expect for a blackbody: if the atmosphere was completely transparent the surface feedback would simply be  $4\sigma T_s^3$  so one might expect that  $\lambda_{\text{surf}}$  is smallest at high latitudes and largest at the equator (also see Henry and Merlis 2019). In reality  $\lambda_{\text{surf}}$  does increase with surface temperature,  $\lambda_{\text{surf}} \propto \pi B_\nu(T_s)$ , but it also rapidly decreases with increasing relative humidity due to the closing of the window and the onset of the continuum,  $\lambda_{\text{surf}} \propto \Delta\nu_{\text{surf}} \exp(-\tau_{\text{cnt}})$ .

In contrast to the surface, atmospheric feedbacks are small at the poles and large in the tropics. This is partly because atmospheric feedbacks are sensitive to both  $\gamma_{\text{lr}}$  and  $d\gamma_{\text{lr}}/dT_s$ , largely via the change in emission temperatures  $dT_{\text{H}_2\text{O}}/dT_s$  and  $dT_{\text{CO}_2}/dT_s$ , whereas the surface feedback only depends on  $\gamma_{\text{lr}}$ , via the influence of the lapse rate on the column water vapor path which in turn sets the window width  $\Delta\nu_{\text{surf}}$  (see Table 2). Since  $d\gamma_{\text{lr}}/dT_s$  has a distinct equator-pole gradient (Fig. 8), the spatial pattern of lapse-rate changes thus helps create a pole-equator contrast in atmospheric feedbacks. In agreement with Section 5, where we found that  $\lambda_{\text{CO}_2}$  and  $\lambda_{\text{H}_2\text{O}}$  are quite similar in magnitude, the maps of  $\lambda_{\text{CO}_2}$  and  $\lambda_{\text{H}_2\text{O}}$  are also highly similar and the global mean values are within 10% in all cases. This is a coincidence caused by Earth’s present-day  $\text{CO}_2$  concentration, and may not be the case in other climates (e.g., for paleoclimates).

We further analyze the spectral feedback maps in Figure 11 by computing their spatial correlation with the inputs in Figure 8. For simplicity we only discuss correlations here based on ERA5, but similar correlations also hold in HadGEM data. In the tropics (equatorward of 30 degrees)  $\lambda_{\text{surf}}$



939 is strongly correlated with RH, with a correlation coefficient  $r = 0.95$ . Note the sign of the  
 940 correlation – larger values of RH coincide with a larger (i.e., less negative)  $\lambda_{\text{surf}}$ . In contrast, the  
 941 net atmospheric feedback  $\lambda_{\text{CO}_2} + \lambda_{\text{H}_2\text{O}} + \lambda_{\text{cnt}}$  is strongly correlated with the lapse rate  $\gamma_{\text{lr}}$ ,  $r = 0.83$ .  
 942 Here the sign of the correlation is again positive, such that a smaller lapse rate coincides with  
 943 a more negative atmospheric feedback. These correlations are much more significant than any  
 944 intrinsic correlations in the input data. For example, in the tropics the column relative humidity  
 945 is essentially independent of  $\gamma_{\text{lr}}$  ( $r = 0.14$ ) and only weakly correlated with surface temperature  
 946 ( $r = 0.43$ ). The surface feedback is therefore most sensitive to the closing of the window region,  
 947 which in the tropics is dominated by the spatial pattern of RH, while the atmospheric feedback  
 948 primarily responds to the atmospheric lapse rate  $\gamma_{\text{lr}}(T_s)$  and, if known, its change under warming  
 949  $d\gamma_{\text{lr}}/dT_s$ . The varying sensitivity to RH versus  $\gamma_{\text{lr}}$  is reflected in the individual feedback maps.  
 950 For example,  $\lambda_{\text{surf}}$  is large both over the Sahara and over the Eastern Pacific close to the coast of  
 951 Peru because both regions are extremely dry (see Fig. 8). In contrast,  $\lambda_{\text{CO}_2}$  and  $\lambda_{\text{H}_2\text{O}}$  are relatively  
 952 small over the Sahara but large over the Eastern Pacific, due to the fact that  $\gamma_{\text{lr}}$  is large over the  
 953 Sahara but small over the Eastern Pacific.

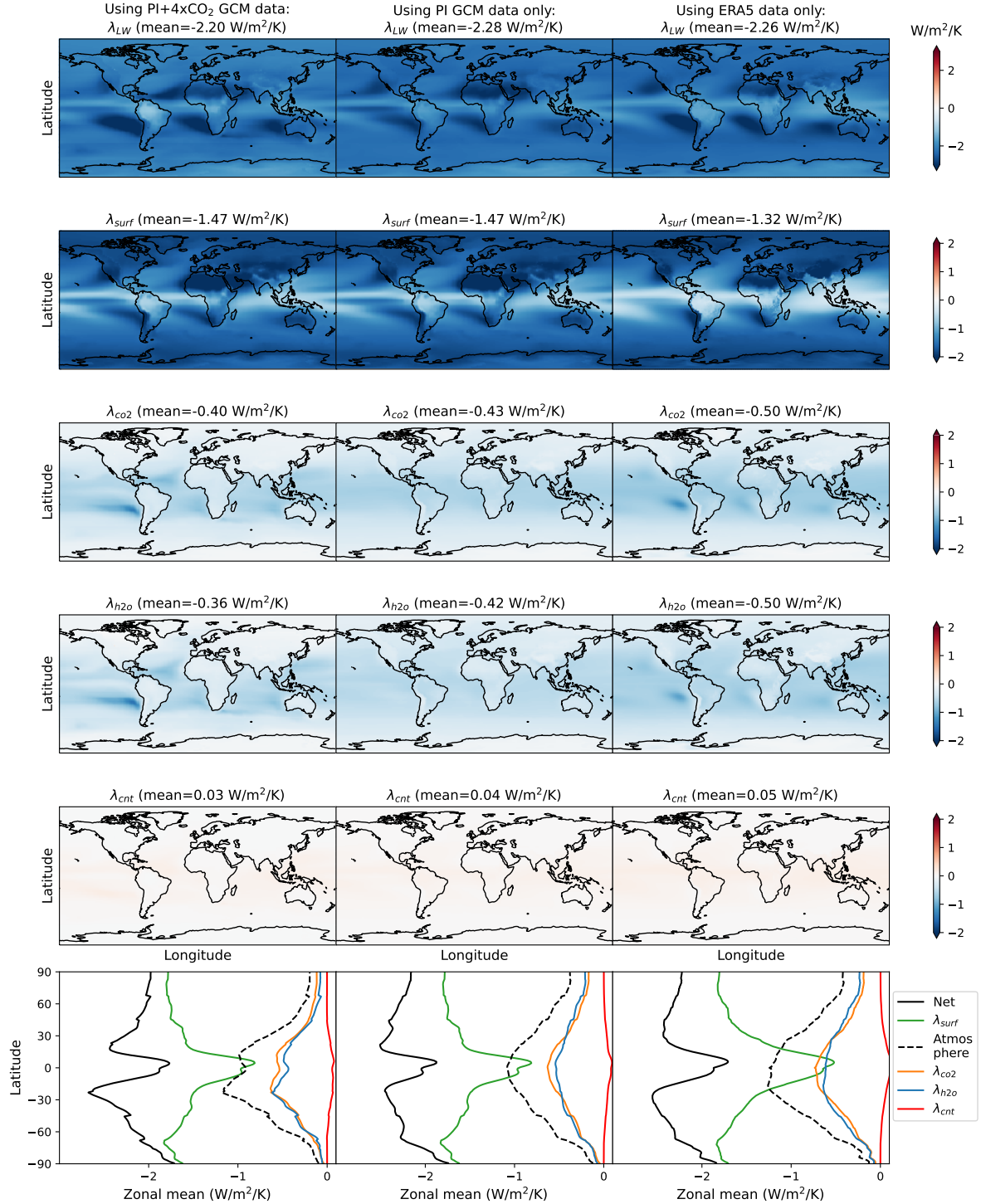


FIG. 11. Net feedback  $\lambda_{LW}$  and its spectral components. Left column: calculation based on input data from preindustrial (PI) and 4xCO<sub>2</sub> GCM simulations. Middle column: calculation only using preindustrial (PI) GCM input data. Right column: calculation only using ERA5 reanalysis input data. Note the different color axes between different rows. Bottom row shows zonal means of the overlying rows.

## 7. Discussion & Conclusions

In this paper, we have presented a novel decomposition of Earth’s clear-sky longwave feedback  $\lambda_{LW}$  into four spectral components: a surface Planck feedback ( $\lambda_{\text{surf}}$ ), a CO<sub>2</sub> “radiator fin” feedback ( $\lambda_{\text{CO}_2}$ ), a non-Simpsonian water vapor band feedback ( $\lambda_{\text{H}_2\text{O}}$ ), and a destabilizing water vapor continuum feedback ( $\lambda_{\text{cnt}}$ ). We have derived simple analytic expressions for each of these spectral feedbacks, which accurately reproduce the results of line-by-line calculations and can also be used to diagnose the local feedback map of climate models. In fact, we have been able to predict Earth’s spatial feedback map solely using reanalysis data. This requires assuming lapse rate changes are moist adiabatic, but our analysis suggests that even with this approximation one can obtain a reasonable estimate. In order to fully predict Earth’s true global-mean  $\lambda_{LW}$  the spatial feedback maps derived from our expressions must be weighted by the equilibrium spatial pattern of warming, which is not known for Earth’s present climate. Nevertheless, we have shown that from a radiative perspective the factors determining  $\lambda_{LW}$  can all be understood from first principles, adding further support to the close agreement for  $\lambda_{LW}$  between observations and climate models.

The picture of Earth’s clear-sky longwave feedback that emerges from this perspective is relatively simple, consisting of a surface feedback plus atmospheric feedbacks from CO<sub>2</sub> and H<sub>2</sub>O. At present the surface feedback  $\lambda_{\text{surf}}$  is the most important contributor in the global-mean and at most latitudes, with its relative importance determined by the distribution of atmospheric water vapor.  $\lambda_{\text{surf}}$  is largest in the dry subtropics, consistent with the view that these are the locus of Earth’s stabilizing longwave feedback (Pierrehumbert 1995; McKim et al. 2021), and smallest in the deep tropics, where the surface’s emission is blocked by the H<sub>2</sub>O continuum. The CO<sub>2</sub> and H<sub>2</sub>O feedbacks  $\lambda_{\text{CO}_2}$  and  $\lambda_{\text{H}_2\text{O}}$  play a supporting role to  $\lambda_{\text{surf}}$  at mid and high latitudes, but they dominate Earth’s feedback in the deep tropics. The H<sub>2</sub>O continuum provides a negligible feedback at present, but the continuum itself is still important through its influence on  $\lambda_{\text{surf}}$ .

This picture is arguably a more intuitive starting point for reasoning about different climates than the traditional decomposition of  $\lambda_{LW}$  into Planck, Lapse Rate and Water Vapor feedbacks. As discussed by Cronin and Dutta (in revision at QJRM), it is already non-trivial to accurately estimate the supposedly-simple Planck feedback from first principles. Similarly, one can qualitatively reason that Lapse Rate and Water Vapor feedbacks both increase in magnitude under global warming, but these are large and of opposite sign, so it is difficult to predict their net change and, by extension,

984 the  $T_s$ -dependence of  $\lambda_{LW}$ , in the traditional decomposition without resorting to numerical models.  
985 The strong cancellations between Planck, Lapse Rate and Water Vapor feedbacks can be alleviated  
986 by considering traditional feedbacks in a fixed relative humidity framework (Ingram 2010; Held  
987 and Shell 2012), but this comes at the cost that the state-dependence of the Planck feedback is no  
988 longer trivial to understand at fixed RH.

989 In contrast, the state-dependence of  $\lambda_{LW}$  is fairly straightforward to understand from a spectral  
990 perspective, at least in broad brushstrokes. For present-day Earth the  $T_s$ -dependence of  $\lambda_{LW}$  is  
991 dominated by the surface in most regions. If relative humidity is fixed,  $\lambda_{\text{surf}}$  increases at very  
992 cold temperatures, peaks around 260 – 290 K depending on RH, and then decreases again (see  
993 Section 5). The decrease is rapid at high RH due to the H<sub>2</sub>O continuum, but much slower at  
994 low RH. Atmospheric feedbacks also have state-dependence. All of them increase in magnitude  
995 as the atmosphere warms, and are further amplified by a weakening lapse rate. In the tropics  
996 the state-dependence of  $\lambda_{LW}$  is thus set by the interplay between a decreasing surface feedback  
997 and increasing atmospheric feedbacks. This can lead to surprising dynamics – at high RH,  $\lambda_{\text{surf}}$   
998 decreases more rapidly with warming than  $\lambda_{\text{CO}_2}$  and  $\lambda_{\text{H}_2\text{O}}$  increase. As a result  $\lambda_{LW}$  becomes  
999 non-monotonic with warming and develops a local minimum around  $\sim 310$  K, which leads to a  
1000 local maximum in climate sensitivity (Seeley and Jeevanjee 2021).

1001 The state-dependence of  $\lambda_{LW}$  at temperatures far above  $\sim 310$  K is beyond the scope of this  
1002 paper, but a spectral perspective points to the importance of stabilizing H<sub>2</sub>O and CO<sub>2</sub> bands versus  
1003 the destabilizing H<sub>2</sub>O continuum as Earth approaches the runaway greenhouse. The main caveat  
1004 here is that Earth’s net feedback does not necessarily stay dominated by  $\lambda_{LW}$  at very high surface  
1005 temperatures, while atmospheric feedbacks also become more complicated at high temperatures  
1006 due to effects such as non-dilute thermodynamics and surface pressure changes (Goldblatt et al.  
1007 2013; Ramirez et al. 2014).

1008 There are several remaining shortcomings in our analysis of  $\lambda_{LW}$  that are beyond the scope of  
1009 this paper. Perhaps the largest is our assumption of an atmosphere that is approximately moist  
1010 adiabatic, such that temperature has to monotonically decrease with altitude. In the real world  
1011 inversions are common, particularly in polar regions and over subtropical oceans. Comparable to  
1012 the long-standing discussion about how to interpret the Lapse Rate feedback at high latitudes in the  
1013 traditional decomposition (e.g., Cai and Lu 2009; Payne et al. 2015; Stuecker et al. 2018; Boeke

et al. 2021), we therefore expect that our approach here only provides a first stab at understanding the processes which shape  $\lambda_{LW}$  in inversion regions.

Another assumption is that we ignore stratospheric changes, even though stratospheric cooling induced by rising CO<sub>2</sub> levels is a major and robust signal of anthropogenic warming (e.g., Vallis et al. 2014). It is notable that the radiative changes due to stratospheric cooling are also hard to intuitively explain using traditional feedbacks. Climate model analyses typically treat the stratosphere’s fast radiative adjustment to CO<sub>2</sub> changes as a process distinct from Planck, Lapse Rate, and Water Vapor feedbacks. Our derivations here sidestep this issue and treat  $T_{strat}$  as a fixed parameter.

Similarly, our derivations ignore the potential feedback from relative humidity changes. In reality there is no guarantee that relative humidity changes will remain negligible under global warming, let alone that they can be neglected when trying to understand paleoclimates. In principle our analysis starting from the emission level approximation can be extended to estimate the feedbacks associated with changes in either RH or  $T_{strat}$ ; RH changes would lead to a feedback term proportional to  $\partial T_{rad}/\partial RH$ , while stratospheric changes would lead to a feedback term proportional to  $\partial T_{rad}/\partial T_{strat}$ . We hope to explore the consequences of such changes in future work.

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*Data availability statement.* HadGEM2 GCM data is publicly available in CMIP data archives. The HadGEM2 radiative kernel is available at <https://archive.researchdata.leeds.ac.uk/382>. ERA5 reanalysis data are available from the Copernicus Climate Change Service. Scripts to compute our analytical feedbacks will be posted online once the manuscript is accepted for publication.

## APPENDIX A

### CO<sub>2</sub> Forcing

The CO<sub>2</sub> ditch model can be used to explain the CO<sub>2</sub> forcing in addition to the CO<sub>2</sub> radiator fin feedback. This section rederives the CO<sub>2</sub> forcing expressions from Wilson and Gea-Banacloche

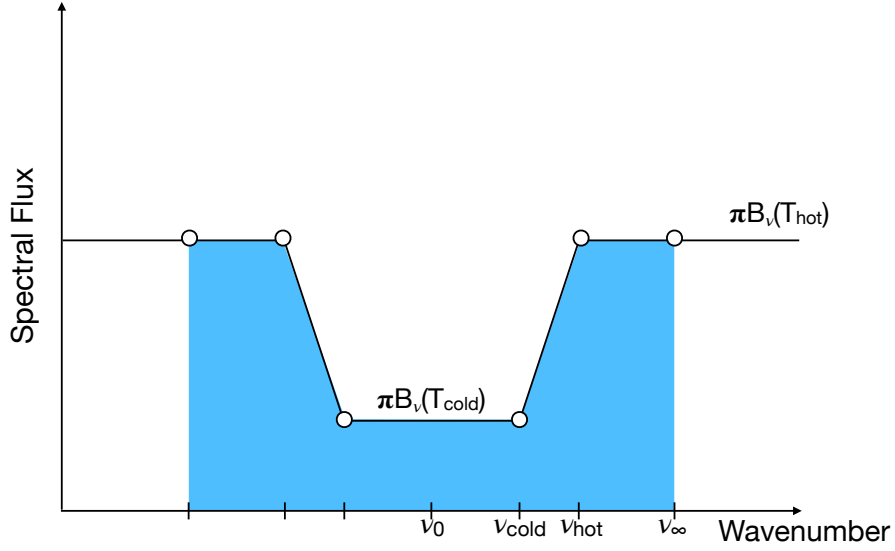


FIG. A1. CO<sub>2</sub> ditch model for the CO<sub>2</sub> forcing. The shaded blue area is the OLR contribution from the CO<sub>2</sub> band as well as neighboring spectral regions. The band edges  $\nu_{\text{hot}}$  and  $\nu_{\text{cold}}$  vary in response to CO<sub>2</sub> concentration  $q_{\text{CO}_2}$ , while  $\nu_{\infty}$  is sufficiently far away from the CO<sub>2</sub> band to be constant with respect to  $q_{\text{CO}_2}$ .

(2012) and Jeevanjee et al. (2021b), which are valid as long as the CO<sub>2</sub> band center radiates from the stratosphere. Note that our CO<sub>2</sub> feedback model only considers OLR changes inside the CO<sub>2</sub> band (see Figure 6). This is because the effect of CO<sub>2</sub> on  $\lambda_{\text{H}_2\text{O}}$  or  $\lambda_{\text{surf}}$  is separately considered in the derivation of those feedbacks. Forcing is defined as the OLR change integrated across all wavenumbers, however, so here we need to consider the expanded shaded region shown in Figure A1. The OLR integrated across this expanded region,  $\text{OLR}_+$ , is

$$\begin{aligned}
 \text{OLR}_+ &= 2 \int_{\nu_0}^{\nu_{\infty}} \pi B_{\nu_0}(T_{\text{rad}}) d\nu \\
 &= [\pi B_{\nu_0}(T_{\text{hot}}) + \pi B_{\nu_0}(T_{\text{cold}})] (\nu_{\text{hot}} - \nu_{\text{cold}}) + 2\pi B_{\nu_0}(T_{\text{cold}}) (\nu_{\text{cold}} - \nu_0) + \\
 &\quad 2\pi B_{\nu_0}(T_{\text{hot}}) (\nu_{\infty} - \nu_{\text{hot}}).
 \end{aligned} \tag{A1}$$

1051 The forcing from a doubling of CO<sub>2</sub> is then

$$\begin{aligned}
F_{\text{CO}_2}^{2x} &= -\frac{d\text{OLR}_+}{d\log_2(q_{\text{CO}_2})} \\
&= -\ln(2)\frac{d\text{OLR}_+}{d\ln q_{\text{CO}_2}} \\
&= -\ln(2)\left(\left[\pi B_{\nu_0}(T_{\text{hot}}) + \pi B_{\nu_0}(T_{\text{cold}})\right]\left(\frac{d\nu_{\text{hot}}}{d\ln q_{\text{CO}_2}} - \frac{d\nu_{\text{cold}}}{d\ln q_{\text{CO}_2}}\right) + 2\pi B_{\nu_0}(T_{\text{cold}})\frac{d\nu_{\text{cold}}}{d\ln q_{\text{CO}_2}} - \right. \\
&\quad \left. 2\pi B_{\nu_0}(T_{\text{hot}})\frac{d\nu_{\text{hot}}}{d\ln q_{\text{CO}_2}}\right) \tag{A2}
\end{aligned}$$

1052 The minus sign in the first line ensures that forcing is positive when OLR decreases, while the  
1053 base-2 logarithm is necessary because forcing is defined with respect to a CO<sub>2</sub> doubling. In the  
1054 second step we then change the logarithm's base to the natural logarithm, while in the third step  
1055 we treat the emission temperatures  $T_{\text{hot}}$  and  $T_{\text{cold}}$  as constant. This is valid because the derivative  
1056 of OLR with respect to  $q_{\text{CO}_2}$  is taken at fixed  $T_s$  (i.e., at fixed surface temperature, the temperature  
1057 outside the CO<sub>2</sub> band and in the stratosphere are both independent of CO<sub>2</sub> concentration).

1058 The CO<sub>2</sub> band edges are defined by  $T_{\text{CO}_2}(\nu_{\text{hot}}) = T_{\text{hot}}$  and  $T_{\text{CO}_2}(\nu_{\text{cold}}) = T_{\text{strat}}$ . Solving for  $\nu_{\text{hot}}$  and  
1059  $\nu_{\text{cold}}$  we find

$$\nu_{\text{hot}} = \nu_0 + l_\nu \log \left[ q_{\text{CO}_2} \tau_{\text{CO}_2}^*(\nu_0) \left( \frac{T_{\text{hot}}}{T_s} \right)^{2/\gamma_{\text{lr}}} \right] \tag{A3}$$

$$\nu_{\text{cold}} = \nu_0 + l_\nu \log \left[ q_{\text{CO}_2} \tau_{\text{CO}_2}^*(\nu_0) \left( \frac{T_{\text{strat}}}{T_s} \right)^{2/\gamma_{\text{lr}}} \right]. \tag{A4}$$

1060 We can see that the CO<sub>2</sub> band edges shift equally in response to a CO<sub>2</sub> increase:

$$\frac{d\nu_{\text{hot}}}{d\ln q_{\text{CO}_2}} = \frac{d\nu_{\text{cold}}}{d\ln q_{\text{CO}_2}} = l_\nu. \tag{A5}$$

1061 It follows that the first term proportional to  $d\nu_{\text{hot}}/d\ln q_{\text{CO}_2} - d\nu_{\text{cold}}/d\ln q_{\text{CO}_2}$  in Equation A2 is zero.

1062 The CO<sub>2</sub> forcing is thus

$$F_{\text{CO}_2}^{2x} = 2\ln(2)l_\nu (\pi B_{\nu_0}(T_{\text{hot}}) - \pi B_{\nu_0}(T_{\text{cold}})), \tag{A6}$$

which is identical to the analytical CO<sub>2</sub> forcing model in Jeevanjee et al. (2021b) (their Equations 7 and 14).

## APPENDIX B

### Transition from stratospheric to tropospheric CO<sub>2</sub> radiator fin

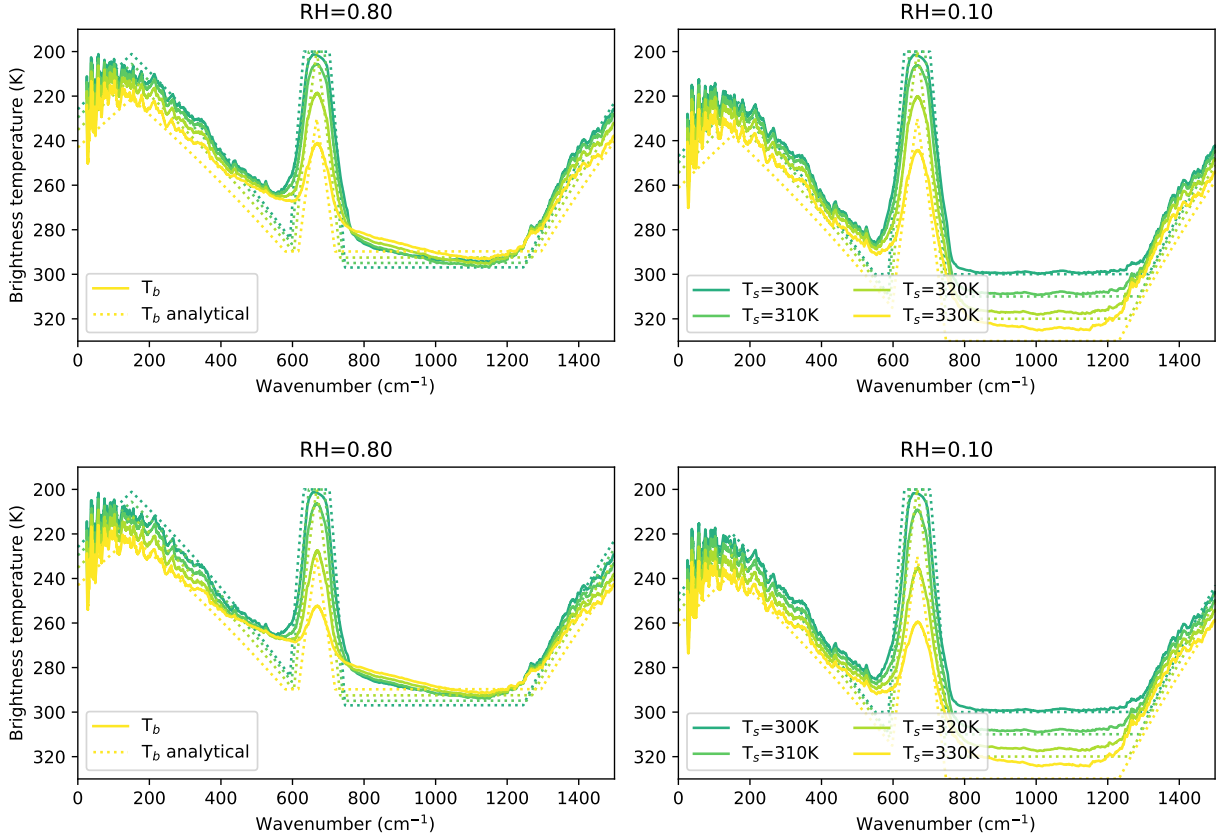


FIG. B1. Brightness temperatures computed from line-by-line calculations and smoothed with a 50 cm<sup>-1</sup> median filter (solid), versus analytical emission temperatures (dashed). Top row: calculations use a bulk lapse-rate profile,  $T(p) = T_s(p/p_s)^{\gamma_r}$ . Bottom row: calculations use a moist adiabat.

At high surface temperatures the CO<sub>2</sub> band center transitions from mainly radiating from the stratosphere to mainly radiating from the troposphere. Figure B1 shows smoothed brightness temperatures  $T_b$  computed from the 1D line-by-line calculations described in Section 5, with a CO<sub>2</sub> volume-mixing ratio of 400 ppm. In the middle of the CO<sub>2</sub> band, at about 667 cm<sup>-1</sup>, CO<sub>2</sub> radiates from the troposphere at surface temperatures above ~ 310 K. In rough agreement with



the line-by-line results, our analytical CO<sub>2</sub> brightness temperatures predict this transition happens at a surface temperature of  $\sim 320$  K (dashed lines in Fig. B1). In practice we therefore use a transition temperature of  $T_{s,0} = 310$  K for 400 ppm of CO<sub>2</sub> to determine when CO<sub>2</sub> changes from a stratospheric to a tropospheric radiator.

## APPENDIX C

### Impact of realistic lapse rates on $\lambda_{LW}$

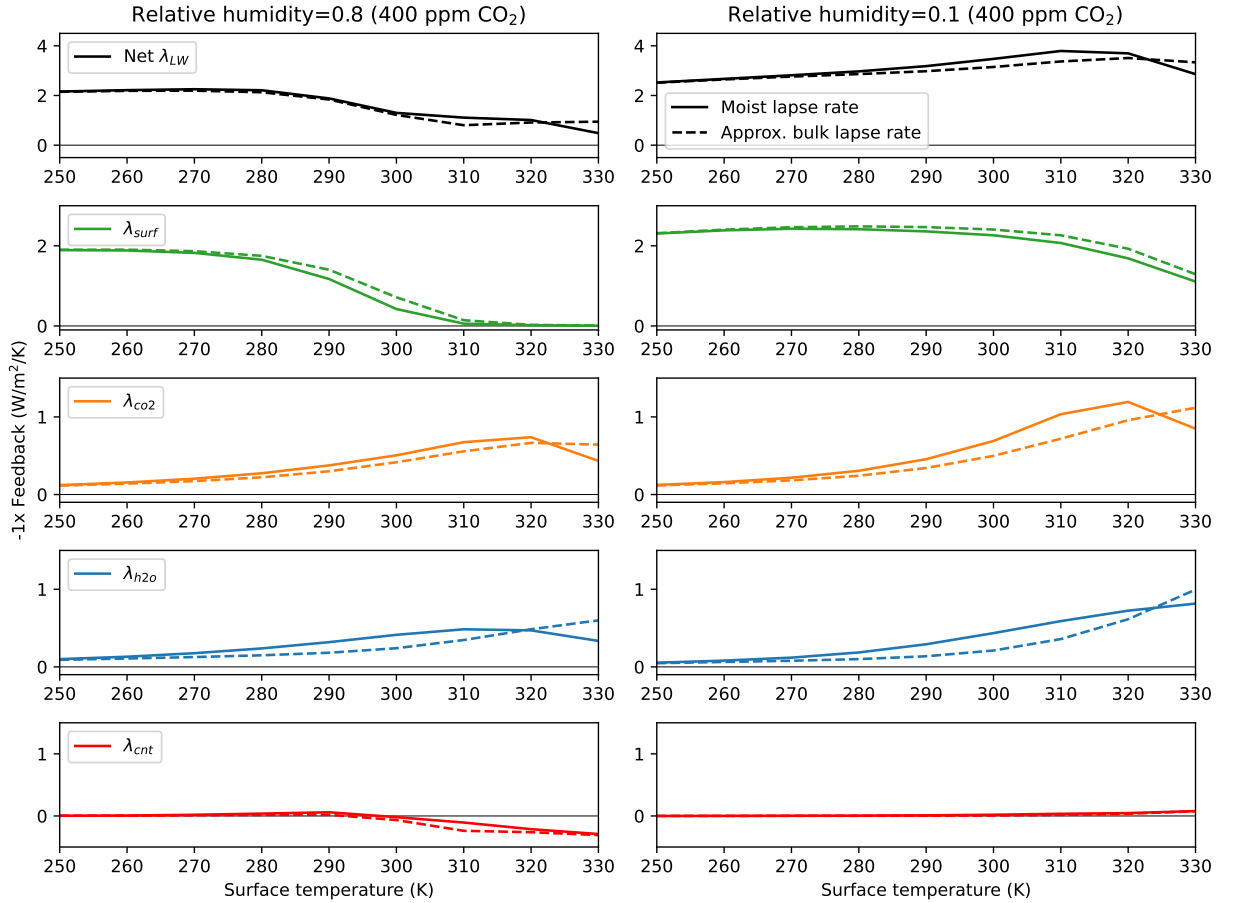


FIG. C1. The impact of the bulk lapse rate approximation on longwave feedbacks is modest below  $\sim 320$  K, but becomes significant at high temperatures. Solid lines are numerical feedbacks calculated assuming the atmosphere follows a moist adiabatic profile, dashed lines are numerical feedbacks calculated assuming the atmosphere follows our bulk lapse rate approximation. Note the change in y-axes between different rows.

Figure C1 compares a set of 1D calculations with a moist adiabat to a set of numerical calculations that fixes the atmospheric temperature-pressure profile according to our bulk lapse rate approximation. The lapse rate approximation only has a small impact on  $\lambda_{LW}$  at surface temperatures below 300 K. The error in individual feedbacks is more significant below 300 K, which means errors in surface and atmospheric feedbacks due to differences in the lapse rate  $\gamma_{lr}$  largely cancel. The cancellation of errors can be partly explained by the effect of  $\gamma_{lr}$  on the atmospheric water vapor path. For example, an erroneously large value for  $\gamma_{lr}$  means the upper atmosphere is too cold, and thus contains less water vapor, than a realistic moist adiabat. This shrinks the width of the window region while increasing the width of the H<sub>2</sub>O bands, which renders  $\lambda_{surf}$  too large and  $\lambda_{H_2O}$  too small, in line with the results shown in Figure C1.

The impact of the bulk lapse rate approximation becomes more significant above 300 K, with clear errors in the temperature-dependence of individual feedbacks above 320 K. Overall, we therefore consider the bulk lapse rate approximation suitable for approximating  $\lambda_{LW}$  below  $\sim 320$  K.

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