Remarks on the global smooth solution of the 3D generalized magneto-micropolar equations

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Abstract: This paper is concerned with the global smooth solution of the 3D generalized magneto-micropolar equations. When the velocity dissipation is logarithmically hyperdissipative and the magnetic diffusion is fractional Laplacian, based on some new observations on the nonlinear structure for the magneto-micropolar equations, it is examined the system has a unique global smooth solution $(u,\, w,\, b) \in C([0,T]; H^s(\mathbb{R}^3))$.

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1. Introduction and main results

Consider the three-dimensional (3D) magneto-micropolar equations which is coupled with the incompressible Navier-Stokes equations, micro-rotational effects and magnetic effects.

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - (\nu+\kappa)\Delta u + \nabla p = (b\cdot\nabla)b + 2\kappa\nabla\times w, \\ \partial_t w + (u\cdot\nabla)w - \sigma\Delta w - \mu\nabla\nabla\cdot w + 4\kappa w = 2\kappa\nabla\times u, \\ \partial_t b + (u\cdot\nabla)b - \eta\Delta b = (b\cdot\nabla)u, \\ \nabla\cdot u = \nabla\cdot b = 0, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3), w = (w_1, w_2, w_3), b = (b_1, b_2, b_3)$ denote the velocity, the microrotation angular velocity and the magnetic field respectively. $p(x,t)$ the scalar pressure. ν, κ, σ, μ, η are viscous coefficients. When $w = 0$ or $b = 0$ or $w = b = 0$, the system (1.1) respectively reduces to the classic magnetohydrodynamics(MHD) equations, micropolar equations or the classic Navier-Stokes equations ([7, 11, 18, 21]).

Since the magneto-micropolar equations are coupled with the Navier-Stokes equations, the question of global regularity of the 3D system with large initial data is still a big open problem. Therefore more and more studies are focused on the global smooth solution of the 2D or 3D magneto-micropolar equations with partial dissipation. For the above three classic equations, there are many important studies on this direction.

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Here we only refer to parts of them, such as [3, 20] on the Navier-Stokes equations, [2, 12, 15, 24, 25] on the magnetohydrodynamics (MHD) equations and [4, 5, 6, 22] on the micropolar equations. As for the magneto-micropolar equations, Rojas-Medar [16] first investigated the existence of local smooth solution of the 3D system with full Laplacian dissipation. Yamazaki [26] studied the global regularity of 2D magneto-micropolar equations without angular viscosity. Regmi and Wu [14], Shang and Wu [17] studied the global regularity of 2D magneto-micropolar equations under the different isentropic dissipation cases and fractional dissipation cases. Jia, Xie and Dong [9] recently investigated the global smooth solution of 3D system with low amount of fractional dissipation. One may also refer to some recent global regularity results of 2D magneto-micropolar equations under different partial dissipation (see [8, 13, 19, 23, 27]).

The main purpose of this paper is to further consider the global smooth solution of the 3D isentropic dissipative magneto-micropolar equations. That is, we consider the following system coupled the logarithmically hyperdissipative velocity dissipation and the fractional magnetic diffusion

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \mathcal{M}^2 u + \nabla p = (b \cdot \nabla)b + \nabla \times w, \quad x \in \mathbb{R}^3, \ t > 0, \\ \partial_t w + (u \cdot \nabla)w + 2w = \nabla \times u, \\ \partial_t b + (u \cdot \nabla)b + (-\Delta)^\gamma b = (b \cdot \nabla)u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \ w(x,0) = w_0(x), \ b(x,0) = b_0(x), \end{cases} \tag{1.2}$$

where the Fourier multiplier $\mathcal{M}$ satisfies

$$\widehat{\mathcal{M}u}(\xi) = \frac{|\xi|^\alpha}{g(|\xi|)}\widehat{u}(\xi), \ \alpha \geq \frac{7}{4}, \ \alpha + \gamma \geq \frac{5}{2},$$

and the radially symmetric, non-decreasing function $g(\tau)$ satisfies

$$\int_e^\infty \frac{d\tau}{\tau\sqrt{\ln \tau} g^2(\tau)} = \infty, \quad \alpha = \frac{7}{4}, \tag{1.3}$$

$$\int_e^\infty \frac{d\tau}{\tau g^2(\tau)} = \infty, \quad \frac{7}{4} < \alpha < \frac{5}{2}, \tag{1.4}$$

We will show the following global existence results.

Theorem 1.1. *Let $(u_0, w_0, b_0) \in H^s(\mathbb{R}^3)$ with $s > \frac{5}{2}$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. then the system (1.2) admits a unique global smooth solution (u, w, b) such that for any given $T > 0$,*

$$(u, w, b) \in C([0,T]; H^s(\mathbb{R}^3)), \quad \mathcal{M}u \in L^2([0,T]; H^s(\mathbb{R}^3)), \quad b \in L^2([0,T]; H^{s+\gamma}(\mathbb{R}^3)).$$

It should be mentioned that the motivation of our study in logarithmical hyperdissipation case is partially borrowed the idea of Tao [20]. In this important work, Tao established the global regularity of a generalized Navier-Stokes equations

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \mathcal{D}^2 u + \nabla p = 0, \\ \nabla \cdot u = 0, \end{cases} \tag{1.5}$$

where the Fourier multiplier $\mathcal{D}$ satisfies

$$\widehat{\mathcal{D}u}(\xi) \geq \frac{|\xi|^{\frac{d+2}{4}}}{\mathfrak{g}(|\xi|)}\widehat{u}(\xi), \qquad \text{for large } |\xi|$$

and $\mathfrak{g} : R^+ \rightharpoonup R^+$ is a non-decreasing function satisfying

$$\int_1^\infty \frac{ds}{s\mathfrak{g}^4(s)} = +\infty.$$

Recently, Wu [24] also made an important progress on the global smooth solution of the generalized MHD equations with the logarithmical hyperdissipation of velocity fields and magnetic fields. Wang, Wu and Ye [22] recently examined the global regularity of the three-dimensional micropolar equations with the logarithmical hyperdissipation in velocity fields.

On comparison with the previous study, our results here are not completely parallel to those. The main new difficulty is how to improve the regularity of the angular velocity field w since we only look at the equations of w as transport equations. Fortunately, we have some new observations on the special structure of the generalized system (1.2). This observation allows us to derive a crucial higher order derivatives estimates of w step by step through dealing with some new a priori estimations of u and b ,

$$\begin{aligned}\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 + \|\Lambda^{\widetilde{\sigma}} w(t)\|_{L^2}^2 + &\int_0^t (\|\mathcal{M}\nabla u(\tau)\|_{L^2}^2 + \|\Lambda^\gamma \nabla b(\tau)\|_{L^2}^2)\, d\tau \\ &\leq C(t,\, u_0,\, w_0,\, b_0).\end{aligned}$$

with $\widetilde{\sigma} \in (0,\, \frac{3}{2})$.

Our study here also shows that the mechanism of the coupled logarithmical hyperdissipation and fractional dissipation to the regularity of solutions has independent interest.

2. *A priori* Estimates

It suffices to consider $\alpha + \gamma = \frac{5}{2}$ with $\frac{7}{4} \leq \alpha < \frac{5}{2}$. Let us start with the basic L^2-estimates and H^1 estimates.

Lemma 2.1. *Assume $(u_0,\, w_0,\, b_0)$ satisfies the assumptions stated in Theorem 1.1. If $\alpha + \gamma = \frac{5}{2}$ with $\frac{7}{4} \leq \alpha < \frac{5}{2}$, then the corresponding solution $(u,\, w,\, b)$ of (1.2) admits the following bound for any $t > 0$.*

$$\begin{aligned}\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + &\int_0^t (\|\Lambda^{\alpha-\sigma_1} u(\tau)\|_{L^2}^2 + \|\mathcal{M}u(\tau)\|_{L^2}^2 + \|\Lambda^\gamma b(\tau)\|_{L^2}^2)\, d\tau \\ &\leq C(t,\, u_0,\, w_0,\, b_0)\end{aligned} \tag{2.1}$$

for any $\sigma_1 \in (0,\, \alpha - 1)$.

Proof of Lemma 2.1. Taking the inner product of (1.2) by $(u,\, w,\, b)$ as well as adding the resulting equations together, we have

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + \|\mathcal{M}u\|_{L^2}^2 + \|\Lambda^\gamma b\|_{L^2}^2 + 2\|w\|_{L^2}^2$$

$$= \int_{\mathbb{R}^3} \Big((\nabla \times w) \cdot u + (\nabla \times u) \cdot w \Big) dx.$$

Thanks to the assumptions on g (more precisely, g grows logarithmically), one may conclude that for any given $\vartheta > 0$, there exists $N = N(\vartheta)$ satisfying

$$g(r) \le \widetilde{C} r^{\vartheta}, \quad \text{for any } r \ge N$$

with the constant $\widetilde{C} = \widetilde{C}(\vartheta)$. Therefore, straightforward computations give for any $\sigma_1 \in (0,\, \alpha - 1)$ that

$$\begin{aligned}
\|\mathcal{M}u\|_{L^2}^2 &= \int_{|\xi|<N(\sigma_1)} \frac{|\xi|^{2\alpha}}{g^2(|\xi|)} |\widehat{u}(\xi)|^2 \, d\xi + \int_{|\xi|\ge N(\sigma_1)} \frac{|\xi|^{2\alpha}}{g^2(|\xi|)} |\widehat{u}(\xi)|^2 \, d\xi \\
&\ge \int_{|\xi|\ge N(\sigma_1)} \frac{|\xi|^{2\alpha}}{\big[\widetilde{C}|\xi|^{\sigma_1}\big]^2} |\widehat{u}(\xi)|^2 \, d\xi \\
&= \int_{\mathbb{R}^3} \frac{|\xi|^{2\alpha}}{\big[\widetilde{C}|\xi|^{\sigma_1}\big]^2} |\widehat{u}(\xi)|^2 \, d\xi - \int_{|\xi|<N(\sigma_1)} \frac{|\xi|^{2\alpha}}{\big[\widetilde{C}|\xi|^{\sigma_1}\big]^2} |\widehat{u}(\xi)|^2 \, d\xi \\
&\ge \widetilde{C_1} \|\Lambda^{\alpha-\sigma_1} u\|_{L^2}^2 - \widetilde{C_2} \|u\|_{L^2}^2,
\end{aligned} \tag{2.2}$$

where $\widetilde{C_1}$ and $\widetilde{C_2}$ depend only on σ_1. Choosing $\sigma_1 \in (0,\, \frac{3}{2})$, we have that by combining all the above estimates

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + \frac{1}{2}\|\mathcal{M}u\|_{L^2}^2 + \frac{1}{2}\|\Lambda^{\gamma} b\|_{L^2}^2 + \widetilde{C_1}\|\Lambda^{\alpha-\sigma_1} u\|_{L^2}^2 \\
\le\ & \widetilde{C_2}\|u\|_{L^2}^2 + \int_{\mathbb{R}^3} \Big((\nabla \times w) \cdot u + (\nabla \times u) \cdot w \Big) dx \\
\le\ & \widetilde{C_2}\|u\|_{L^2}^2 + 2\|\nabla u\|_{L^2}\|w\|_{L^2} \\
\le\ & \widetilde{C_2}\|u\|_{L^2}^2 + 2\left(\|u\|_{L^2}^{\frac{\alpha-\sigma_1-1}{\alpha-\sigma_1}} \|\Lambda^{\alpha-\sigma_1} u\|_{L^2}^{\frac{1}{\alpha-\sigma_1}} \right) \|w\|_{L^2} \\
\le\ & \frac{\widetilde{C_1}}{2}\|\Lambda^{\alpha-\sigma_1} u\|_{L^2}^2 + C\left(\|u\|_{L^2}^2 + \|w\|_{L^2}^2\right).
\end{aligned}$$

Consequently, it implies

$$\begin{aligned}
\frac{d}{dt}\left(\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2\right) + \|\mathcal{M}u\|_{L^2}^2 + \|\Lambda^{\gamma} b\|_{L^2}^2 + \|\Lambda^{\alpha-\sigma_1} u\|_{L^2}^2 \\
\le C\left(\|u\|_{L^2}^2 + \|w\|_{L^2}^2 + \|b\|_{L^2}^2\right).
\end{aligned}$$

Making use of the Gronwall inequality, it directly yields (2.1). This completes the proof of Lemma 2.1. □

Lemma 2.2. *Assume $(u_0,\, w_0,\, b_0)$ satisfies the assumptions stated in Theorem 1.1. If $\alpha + \gamma = \frac{5}{2}$ with $\frac{7}{4} \le \alpha < \frac{5}{2}$, then the corresponding solution $(u,\, w,\, b)$ of (1.2) admits the following bound for any $t > 0$,*

$$\left\| \frac{\Lambda^{\alpha-1}}{g(\Lambda)} u(t) \right\|_{L^2}^2 + \int_0^t \left\| \frac{\Lambda^{2\alpha-1}}{g^2(\Lambda)} u(\tau) \right\|_{L^2}^2 d\tau \le C(t,\, u_0,\, w_0,\, b_0). \tag{2.3}$$

In particular, if $\alpha = \frac{7}{4}$, then it holds true

$$\int_0^t \left\| \frac{\Lambda^{\frac{5}{2}}}{g^2(\Lambda)} u(\tau) \right\|_{L^2}^2 d\tau \le C(t,\, u_0,\, w_0,\, b_0). \tag{2.4}$$

If $\frac{7}{4} < \alpha < \frac{5}{2}$, then it holds true

$$\int_0^t (\|\Lambda^{\frac{5}{2}} u(\tau)\|_{L^2}^2 + \|\nabla u(\tau)\|_{L^\infty}^2)\, d\tau \le C(t,\, u_0,\, w_0,\, b_0). \tag{2.5}$$

Proof of Lemma 2.2. Multiplying $(1.2)_1$ by $\frac{\Lambda^{2\alpha-2}}{g^2(\Lambda)} u$ and integrating it over the whole space, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \left\| \frac{\Lambda^{\alpha-1}}{g(\Lambda)} u(t) \right\|_{L^2}^2 + \left\| \frac{\Lambda^{2\alpha-1}}{g^2(\Lambda)} u \right\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3} \nabla \times w \cdot \frac{\Lambda^{2\alpha-2}}{g^2(\Lambda)} u\, dx - \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \frac{\Lambda^{2\alpha-2}}{g^2(\Lambda)} u\, dx + \int_{\mathbb{R}^3} (b \cdot \nabla b) \cdot \frac{\Lambda^{2\alpha-2}}{g^2(\Lambda)} u\, dx. \end{aligned}$$

It follows from the Young inequality that

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \nabla \times w \cdot \frac{\Lambda^{2\alpha-2}}{g^2(\Lambda)} u\, dx \right| &\le \|w\|_{L^2} \left\| \nabla \frac{\Lambda^{2\alpha-2}}{g^2(\Lambda)} u \right\|_{L^2} \\ &\le \frac{1}{8} \left\| \frac{\Lambda^{2\alpha-1}}{g^2(\Lambda)} u \right\|_{L^2}^2 + C\|w\|_{L^2}^2. \end{aligned}$$

Following the proof of (2.2), it is not difficult to check that for any $\sigma_2 \in (0,\, \alpha-1)$ and for any $\sigma_3 \in (0,\, 2\alpha-1)$, such that

$$\left\| \frac{\Lambda^{\alpha-1}}{g(\Lambda)} u \right\|_{L^2}^2 \ge C_1 \|\Lambda^{\alpha-1-\sigma_2} u\|_{L^2}^2 - C_2 \|u\|_{L^2}^2, \tag{2.6}$$

$$\left\| \frac{\Lambda^{2\alpha-1}}{g(\Lambda)} u \right\|_{L^2}^2 \ge C_3 \|\Lambda^{2\alpha-1-\sigma_3} u\|_{L^2}^2 - C_4 \|u\|_{L^2}^2. \tag{2.7}$$

Thanks to $\nabla \cdot u = 0$ and the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} \left| -\int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \frac{\Lambda^{2\alpha-2}}{g^2(\Lambda)} u\, dx \right| &\le C\|u \otimes u\|_{L^2} \left\| \nabla \frac{\Lambda^{2\alpha-2}}{g^2(\Lambda)} u \right\|_{L^2} \\ &\le C\|u\|_{L^4}^2 \left\| \frac{\Lambda^{2\alpha-1}}{g^2(\Lambda)} u \right\|_{L^2} \\ &\le C\|\Lambda u\|_{L^2} \|\Lambda^{\frac{1}{2}} u\|_{L^2} \left\| \frac{\Lambda^{2\alpha-1}}{g^2(\Lambda)} u \right\|_{L^2} \\ &\le \frac{1}{8} \left\| \frac{\Lambda^{2\alpha-1}}{g^2(\Lambda)} u \right\|_{L^2}^2 + C\|\Lambda u\|_{L^2}^2 \|\Lambda^{\frac{1}{2}} u\|_{L^2}^2 \\ &\le \frac{1}{8} \left\| \frac{\Lambda^{2\alpha-1}}{g^2(\Lambda)} u \right\|_{L^2}^2 + C\|u\|_{L^2}^2 \|\Lambda u\|_{L^2}^2 + C\|\Lambda u\|_{L^2}^2 \left\| \frac{\Lambda^{\alpha-1}}{g(\Lambda)} u \right\|_{L^2}^2, \end{aligned}$$

where in the last line we have applied (2.1) with $\sigma_1 = \alpha - 1$ and (2.6) with $\sigma_2 = \alpha - \frac{3}{2}$. Similarly, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3} (b\cdot\nabla b)\cdot \frac{\Lambda^{2\alpha-2}}{g^2(\Lambda)}u\,dx\right| &\le C\|b\otimes b\|_{L^2}\left\|\nabla\frac{\Lambda^{2\alpha-2}}{g^2(\Lambda)}u\right\|_{L^2}\\ &\le C\|b\|_{L^4}^2\left\|\frac{\Lambda^{2\alpha-1}}{g^2(\Lambda)}u\right\|_{L^2}\\ &\le \frac{1}{8}\left\|\frac{\Lambda^{2\alpha-1}}{g^2(\Lambda)}u\right\|_{L^2}^2 + C\|b\|_{L^4}^4.\end{aligned}$$

Combining the above estimates yields

$$\begin{aligned}&\frac{d}{dt}\left\|\frac{\Lambda^{\alpha-1}}{g(\Lambda)}u(t)\right\|_{L^2}^2 + \left\|\frac{\Lambda^{2\alpha-1}}{g^2(\Lambda)}u\right\|_{L^2}^2\\ &\le C\|\Lambda u\|_{L^2}^2\left\|\frac{\Lambda^{\alpha-1}}{g(\Lambda)}u\right\|_{L^2}^2 + C\|w\|_{L^2}^2 + C\|u\|_{L^2}^2\|\Lambda u\|_{L^2}^2 + C\|b\|_{L^4}^4.\end{aligned} \tag{2.8}$$

In order to close the above inequality, it sufficient to bound $\|b\|_{L^4}^4$. To this end, we multiply $(1.2)_3$ by $|b|^2 b$ to show

$$\frac{d}{dt}\|b(t)\|_{L^4}^4 + \int_{\mathbb{R}^3}\Lambda^{2\gamma}b(|b|^2 b)\,dx = \int_{\mathbb{R}^3} b\cdot\nabla u(|b|^2 b)\,dx.$$

We first notice that, for $0 < \gamma < \frac{3}{2}$

$$\int_{\mathbb{R}^3}\Lambda^{2\gamma}b(|b|^2 b)\,dx \ge C\|\Lambda^{\gamma}(|b|^2)\|_{L^2}^2 \ge C\|b\|_{L^{\frac{12}{3-2\gamma}}}^4.$$

The simple embedding inequality and the Hölder inequality ensure that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} b\cdot\nabla u(|b|^2 b)\,dx\right| &\le C\|\nabla u\|_{L^{\frac{6}{5-2\alpha}}}\|b\|_{L^{\frac{24}{1+2\alpha}}}^4\\ &\le C\|\Lambda^{\alpha}u\|_{L^2}\|b\|_{L^4}^2\|b\|_{L^{\frac{6}{\alpha-1}}}^2\\ &\le C\|\Lambda^{\alpha}u\|_{L^2}\|b\|_{L^4}^2\|b\|_{L^{\frac{12}{3-2\gamma}}}^2.\end{aligned}$$

One may check that for some $\sigma > 0$

$$\begin{aligned}\|\Lambda^{\alpha}u\|_{L^2} &\le \|S_N\Lambda^{\alpha}u\|_{L^2} + \sum_{l=N}^{\infty}\|\Delta_l\Lambda^{\alpha}u\|_{L^2}\\ &\le \|\chi(2^{-N}\xi)|\xi|^{\alpha}\widehat{u}(\xi)\|_{L^2} + C\sum_{l=N}^{\infty}2^{-l\sigma}\|\Delta_l\Lambda^{\alpha+\sigma}u\|_{L^2}\\ &\le C\left\|\chi(2^{-N}\xi)g(|\xi|)\frac{|\xi|^{\alpha}}{g(|\xi|)}\widehat{u}(\xi)\right\|_{L^2} + C\sum_{l=N}^{\infty}2^{-l\sigma}\|\Lambda^{\alpha+\sigma}u\|_{L^2}\\ &\le Cg(2^N)\left\|\frac{|\xi|^{\alpha}}{g(|\xi|)}\widehat{u}(\xi)\right\|_{L^2} + C2^{-N\sigma}\|\Lambda^{\alpha+\sigma}u\|_{L^2}\\ &\le Cg(2^N)\|\mathcal{M}u\|_{L^2} + C2^{-N\sigma}\|\Lambda^{\alpha-\sigma_1}u\|_{L^2}^{1-\varpi}\|\Lambda^{2\alpha-1-\sigma_3}u\|_{L^2}^{\varpi},\end{aligned}$$

where ϖ is given by

$$\varpi = \frac{\sigma + \sigma_1}{\alpha - 1 + \sigma_1 - \sigma_3} \in (0,\, 1).$$

This implies

$$\begin{aligned}\frac{d}{dt}\|b(t)\|_{L^4}^4 + \|b\|_{L^{\frac{12}{3-2\gamma}}}^4 &\le Cg(2^N)\,\|\mathcal{M}u\|_{L^2}\,\|b\|_{L^4}^2\|b\|_{L^{\frac{12}{3-2\gamma}}}^2 \\ &\quad + C2^{-N\sigma}\|\Lambda^{\alpha-\sigma_1}u\|_{L^2}^{1-\varpi}\|\Lambda^{2\alpha-1-\sigma_3}u\|_{L^2}^{\varpi}\|b\|_{L^4}^2\|b\|_{L^{\frac{12}{3-2\gamma}}}^2.\end{aligned} \tag{2.9}$$

For the sake of simplicity, we denote

$$X(t) := \left\|\frac{\Lambda^{\alpha-1}}{g(\Lambda)}u(t)\right\|_{L^2}^2 + \|b(t)\|_{L^4}^4,$$
$$Y(t) := \left\|\frac{\Lambda^{2\alpha-1}}{g^2(\Lambda)}u(t)\right\|_{L^2}^2 + \|b(t)\|_{L^{\frac{12}{3-2\gamma}}}^4,$$

then it follows from (2.8) and (2.9) that

$$\begin{aligned}\frac{d}{dt}X(t) + Y(t) &\le C(1 + \|\Lambda u\|_{L^2}^2)X(t) + C\|w\|_{L^2}^2 + C\|u\|_{L^2}^2\|\Lambda u\|_{L^2}^2 \\ &\quad + Cg(2^N)\,\|\mathcal{M}u\|_{L^2}\,X^{\frac12}(t)Y^{\frac12}(t) \\ &\quad + C2^{-N\sigma}\|\Lambda^{\alpha-\sigma_1}u\|_{L^2}^{1-\varpi}X^{\frac12}(t)Y^{\frac{1+\varpi}{2}}(t) \\ &\quad + C2^{-N\sigma}\|\Lambda^{\alpha-\sigma_1}u\|_{L^2}^{1-\varpi}X^{\frac12}(t)\|u\|_{L^2}^2.\end{aligned} \tag{2.10}$$

Now taking N as

$$2^{2N\sigma} \approx e + X(t),$$

we get from (2.10) that

$$\begin{aligned}\frac{d}{dt}X(t) + Y(t) &\le C(1 + \|\Lambda u\|_{L^2}^2)X(t) + C\|w\|_{L^2}^2 + C\|u\|_{L^2}^2\|\Lambda u\|_{L^2}^2 \\ &\quad + Cg\left([e + X(t)]^{\frac{1}{2\sigma}}\right)\|\mathcal{M}u\|_{L^2}\,X^{\frac12}(t)Y^{\frac12}(t) \\ &\quad + C\|\Lambda^{\alpha-\sigma_1}u\|_{L^2}^{1-\varpi}Y^{\frac{1+\varpi}{2}}(t) \\ &\le \frac12 Y(t) + C(1 + \|\Lambda u\|_{L^2}^2)X(t) + C\|w\|_{L^2}^2 + C\|u\|_{L^2}^2\|\Lambda u\|_{L^2}^2 \\ &\quad + C\,\|\mathcal{M}u\|_{L^2}^2\,g^2\left([e + X(t)]^{\frac{1}{2\sigma}}\right)(e + X(t)) \\ &\quad + C(\|\Lambda^{\alpha-\sigma_1}u\|_{L^2}^2 + 1)\|u\|_{L^2}^2 \\ &\le \frac12 Y(t) + \mathcal{Z}(t)g^2\left([e + X(t)]^{\frac{1}{2\sigma}}\right)(e + X(t)),\end{aligned} \tag{2.11}$$

where

$$\mathcal{Z}(t) := C\left[1 + \|\Lambda u\|_{L^2}^2 + \|w\|_{L^2}^2 + \|\mathcal{M}u\|_{L^2}^2 + (\|\Lambda^{\alpha-\sigma_1}u\|_{L^2}^2 + \|\Lambda u\|_{L^2}^2 + 1)\|u\|_{L^2}^2\right](t).$$

The estimate (2.1) implies that

$$\int_0^t \mathcal{Z}(\tau)\,d\tau \le C(t,\, u_0,\, w_0,\, b_0).$$

We thus have

$$\frac{d}{dt}X(t)+Y(t)\le \mathcal{Z}(t)g^2\left([e+X(t)]^{\frac{1}{2\sigma}}\right)(e+X(t)). \tag{2.12}$$

Now we deduce from (2.12) that

$$\int_{e+X(0)}^{e+X(t)}\frac{d\tau}{\tau g^2(\tau^{\frac{1}{2\sigma}})}\le\int_0^t\mathcal{Z}(\tau)\,d\tau\le C(t,\,u_0,\,w_0,\,b_0).$$

Noticing the following fact

$$\int_e^\infty\frac{d\tau}{\tau g^2(\tau^{\frac{1}{2\sigma}})}=2\sigma\int_{e^{\frac{1}{2\sigma}}}^\infty\frac{d\tau}{\tau g^2(\tau)}=\infty,$$

it follows that

$$X(t)\le C(t,\,u_0,\,w_0,\,b_0).$$

Coming back to (2.12), we also conclude

$$\int_0^t Y(\tau)\,d\tau\le C(t,\,u_0,\,w_0,\,b_0).$$

This leads to the bound (2.3). The desired estimate (2.5) is an easy consequence of (2.7) and the interpolation inequality. This completes the proof of the lemma 2.2. □

The following regular estimates are crucial for the proof of Theorem 1.1.

Lemma 2.3. *Assume* $(u_0,\,w_0,\,b_0)$ *satisfies the assumptions stated in Theorem 1.1. If* $\alpha+\gamma=\frac{5}{2}$ *with* $\frac{7}{4}\le\alpha<\frac{5}{2}$*, then the corresponding solution* $(u,\,w,\,b)$ *of* (1.2) *admits the following bound for any* $t>0$ *and for any* $\widetilde{\sigma}\in(0,\,\frac{3}{2})$*.*

$$\begin{aligned}\|\nabla u(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2+\|\Lambda^{\widetilde{\sigma}}w(t)\|_{L^2}^2+&\int_0^t(\|\mathcal{M}\nabla u(\tau)\|_{L^2}^2+\|\Lambda^\gamma\nabla b(\tau)\|_{L^2}^2)\,d\tau\\ \le& C(t,\,u_0,\,w_0,\,b_0).\end{aligned} \tag{2.13}$$

In particular, it implies

$$\int_0^t(\|\Lambda^{\frac{5}{2}}u(\tau)\|_{L^2}^2+\|\nabla u(\tau)\|_{L^\infty}^2)\,d\tau\le C(t,\,u_0,\,w_0,\,b_0). \tag{2.14}$$

Proof of Lemma 2.3. It follows from (2.5) that if $\alpha+\gamma=\frac{5}{2}$ with $\frac{7}{4}<\alpha<\frac{5}{2}$, then we already have (2.3), namely,

$$\int_0^t\left(\|\Lambda^{\frac{5}{2}}u(\tau)\|_{L^2}^2+\|\nabla u(\tau)\|_{L^\infty}^2\right)\,d\tau\le C(t,\,u_0,\,w_0,\,b_0). \tag{2.15}$$

As a matter of fact, the above estimate (2.15) immediately implies the higher regularity (see below for details). Consequently, it suffices to consider the endpoint case $\alpha+\gamma=\frac{5}{2}$ with $\alpha=\frac{7}{4}$. In this case, we only have (2.4), namely,

$$\int_0^t\left\|\frac{\Lambda^{\frac{5}{2}}}{g^2(\Lambda)}u(\tau)\right\|_{L^2}^2\,d\tau\le C(t,\,u_0,\,w_0,\,b_0). \tag{2.16}$$

Applying $\Lambda^{\widetilde{\sigma}}$ (for any $\widetilde{\sigma} \in (0, \frac{3}{2})$) to $(1.2)_2$ and taking the inner product with $\Lambda^{\widetilde{\sigma}} w$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^{\widetilde{\sigma}} w(t)\|_{L^2}^2 + 2\|\Lambda^{\widetilde{\sigma}} w\|_{L^2}^2 \\
= \;& \int_{\mathbb{R}^3} \Lambda^{\widetilde{\sigma}}\big(\nabla \times u\big)\Lambda^{\widetilde{\sigma}} w\, dx - \int_{\mathbb{R}^3} [\Lambda^{\widetilde{\sigma}}, u\cdot\nabla] w \Lambda^{\widetilde{\sigma}} w\, dx \\
:= \;& K_1 + K_2.
\end{aligned}$$

For K_1, Gagliardo-Nirenberg inequality and (2.2) implies

$$\begin{aligned}
K_1 \;&\leq\; C\|\Lambda^{\widetilde{\sigma}+1} u\|_{L^2}\|\Lambda^{\widetilde{\sigma}} w\|_{L^2} \\
&\leq\; \|\Lambda^{\widetilde{\sigma}} w\|_{L^2}^2 + C\left\|\frac{\Lambda^{\frac{5}{2}}}{g^2(\Lambda)} u\right\|_{L^2}^2 + C\|u\|_{L^2}^2.
\end{aligned}$$

For K_2, employing the classic Kato-Ponce commutator estimates ([10]) yields

$$\begin{aligned}
K_2 =\; & -\int_{\mathbb{R}^3} [\Lambda^{\widetilde{\sigma}}\partial_{x_i}, u_i] w \Lambda^{\widetilde{\sigma}} w\, dx \\
\leq\; & \big\|[\Lambda^{\widetilde{\sigma}}\partial_{x_i}, u_i] w\big\|_{L^2}\|\Lambda^{\widetilde{\sigma}} w\|_{L^2} \\
\leq\; & C\left(\|\nabla u\|_{L^\infty}\|\Lambda^{\widetilde{\sigma}} w\|_{L^2} + \|\Lambda^{\widetilde{\sigma}+1} u\|_{L^{\frac{3}{\widetilde{\sigma}}}}\|w\|_{L^{\frac{6}{3-2\widetilde{\sigma}}}}\right)\|\Lambda^{\widetilde{\sigma}} w\|_{L^2} \\
\leq\; & C\left(\|\nabla u\|_{L^\infty}\|\Lambda^{\widetilde{\sigma}} w\|_{L^2} + \|\Lambda^{\frac{5}{2}} u\|_{L^2}\|\Lambda^{\widetilde{\sigma}} w\|_{L^2}\right)\|\Lambda^{\widetilde{\sigma}} w\|_{L^2} \\
\leq\; & C\left(\|\nabla u\|_{L^\infty} + \|\Lambda^{\frac{5}{2}} u\|_{L^2}\right)\|\Lambda^{\widetilde{\sigma}} w\|_{L^2}^2.
\end{aligned}$$

Collecting the above estimates, we have for any $\widetilde{\sigma} \in (0, \frac{3}{2})$, such that

$$\frac{d}{dt}\|\Lambda^{\widetilde{\sigma}} w(t)\|_{L^2}^2 \leq C(1 + \|\nabla u\|_{L^\infty} + \|\Lambda^{\frac{5}{2}} u\|_{L^2})\|\Lambda^{\widetilde{\sigma}} w\|_{L^2}^2 + C\left\|\frac{\Lambda^{\frac{5}{2}}}{g^2(\Lambda)} u\right\|_{L^2}^2 + C\|u\|_{L^2}^2.$$

Applying the gradient operator ∇ to the equations of $(1.2)_1$ and $(1.2)_3$, multiplying them by ∇u and ∇b, respectively, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2\right) + \|\mathcal{M}\nabla u\|_{L^2}^2 + \|\Lambda^{\gamma}\nabla b\|_{L^2}^2 \\
&= \int_{\mathbb{R}^3} \nabla\nabla\times w\cdot\nabla u\, dx - \int_{\mathbb{R}^3}\nabla(u\cdot\nabla u)\cdot\nabla u\, dx - \int_{\mathbb{R}^3}\nabla(u\cdot\nabla b)\cdot\nabla b\, dx \\
&\quad + \int_{\mathbb{R}^3}\Big(\nabla(b\cdot\nabla u)\cdot\nabla b + \nabla(b\cdot\nabla b)\cdot\nabla u\Big)\, dx \\
&:= \sum_{k=1}^{4} A_k.
\end{aligned} \tag{2.17}$$

Following the argument used in proving (2.2), we have for any $\sigma_4 \in (0, \alpha]$, such that

$$\|\mathcal{M}\nabla u\|_{L^2}^2 \geq C_7\|\Lambda^{\alpha-\sigma_4}\nabla u\|_{L^2}^2 - C_8\|\nabla u\|_{L^2}^2. \tag{2.18}$$

In view of (2.18), we deduce that by taking $\sigma_4 \in (0, \widetilde{\sigma} + \alpha - 2]$,

$$A_1 \leq C\|\Lambda^{\widetilde{\sigma}} w\|_{L^2}\|\Lambda^{2-\widetilde{\sigma}}\nabla u\|_{L^2}$$

$$
\begin{aligned}
&\leq C\|\Lambda^{\widetilde{\sigma}}w\|_{L^2}(\|\nabla u\|_{L^2}+\|\Lambda^{\alpha-\sigma_4}\nabla u\|_{L^2})\\
&\leq C\|\Lambda^{\widetilde{\sigma}}w\|_{L^2}(\|\mathcal{M}\nabla u\|_{L^2}+\|\nabla u\|_{L^2})\\
&\leq \frac{1}{16}\|\mathcal{M}\nabla u\|_{L^2}^2+C(\|\Lambda^{\widetilde{\sigma}}w\|_{L^2}^2+\|\nabla u\|_{L^2}^2).
\end{aligned}
$$

Moreover, we also have

$$
\begin{aligned}
A_2 &\leq C\|\nabla u\|_{L^\infty}\|\nabla u\|_{L^2}^2,\\
A_3 &\leq C\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^2}^2,\\
A_4 &\leq C\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^2}^2.
\end{aligned}
$$

Collecting all the above estimates gives

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2+\|\Lambda^{\widetilde{\sigma}}w(t)\|_{L^2}^2)+\|\mathcal{M}\nabla u\|_{L^2}^2+\|\Lambda^{\alpha-\sigma_4}\nabla u\|_{L^2}^2\\
&\leq C(1+\|\nabla u\|_{L^\infty}+\|\Lambda^{\frac{5}{2}}u\|_{L^2})(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|\Lambda^{\widetilde{\sigma}}w\|_{L^2}^2)\\
&\quad+C\left\|\frac{\Lambda^{\frac{5}{2}}}{g^2(\Lambda)}u\right\|_{L^2}^2+C\|u\|_{L^2}^2.
\end{aligned}
$$

It should be noted that (2.5) and the Gronwall inequality imply the desired estimate (2.13). In order to handle the endpoint case $\alpha+\gamma=\frac{5}{2}$ with $\alpha=\frac{7}{4}$. Denoting

$$
\begin{aligned}
\widetilde{X}(t) &:= \|\nabla u(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2+\|\Lambda^{\widetilde{\sigma}}w(t)\|_{L^2}^2,\\
\widetilde{Y}(t) &:= \|\mathcal{M}\nabla u(t)\|_{L^2}^2+\|\Lambda^{\alpha-\sigma_4}\nabla u(t)\|_{L^2}^2+\|\Lambda^{\gamma}\nabla b\|_{L^2}^2,\\
\widetilde{H}(t) &:= C\left\|\frac{\Lambda^{\frac{5}{2}}}{g^2(\Lambda)}u(t)\right\|_{L^2}^2+C\|u(t)\|_{L^2}^2.
\end{aligned}
$$

We can deduce that

$$
\frac{d}{dt}\widetilde{X}(t)+\widetilde{Y}(t)\leq C\left(1+\|\nabla u\|_{L^\infty}+\|\Lambda^{\frac{5}{2}}u\|_{L^2}\right)\widetilde{X}(t)+\widetilde{H}(t). \tag{2.19}
$$

Employing the classic Littlewood-Paley decomposition (refer to [1] for details), one shows that

$$
\|\nabla u\|_{L^\infty}\leq \|\Delta_{-1}\nabla u\|_{L^\infty}+\sum_{l=0}^{N-1}\|\Delta_l\nabla u\|_{L^\infty}+\sum_{l=N}^{\infty}\|\Delta_l\nabla u\|_{L^\infty},
$$

where Δ_l ($l=-1,0,1,\cdots$) denote the nonhomogeneous dyadic blocks. Applying Bernstein inequality obeys

$$
\|\Delta_{-1}\nabla u\|_{L^\infty}\leq C\|u\|_{L^2},
$$

$$
\sum_{l=N}^{\infty}\|\Delta_l\nabla u\|_{L^\infty}\leq C\sum_{l=N}^{\infty}2^{l(\frac{3}{2}-\alpha+\sigma_4)}\|\Delta_l\Lambda^{\alpha-\sigma_4}\nabla u\|_{L^2}\leq C2^{N(\frac{3}{2}-\alpha+\sigma_4)}\|\Lambda^{\alpha-\sigma_4}\nabla u\|_{L^2},
$$

due to $\sigma_4\in(0,\,\alpha-\frac{3}{2})$.

Similarly,

$$\begin{aligned}\sum_{l=0}^{N-1}\|\Delta_l\nabla u\|_{L^\infty} &\le C\sum_{l=0}^{N-1}2^{\frac{5l}{2}}\|\Delta_l u\|_{L^2}\\ &\le C\sum_{l=0}^{N-1}\left\|\Delta_l\left(g^2(\Lambda)\frac{\Lambda^{\frac{5}{2}}}{g^2(\Lambda)}u\right)\right\|_{L^2}\\ &\le C\sum_{l=0}^{N-1}\left\|\varphi(2^{-l}\xi)g^2(|\xi|)\frac{|\xi|^{\frac{5}{2}}}{g^2(|\xi|)}\widehat{u}(\xi)\right\|_{L^2}\\ &\le C\sum_{l=0}^{N-1}g^2(2^l)\left\|\frac{|\xi|^{\frac{5}{2}}}{g^2(|\xi|)}\widehat{\Delta_l u}(\xi)\right\|_{L^2}\\ &\le C\left(\sum_{l=0}^{N-1}g^4(2^l)\right)^{\frac{1}{2}}\left(\sum_{l=0}^{N-1}\left\|\frac{|\xi|^{\frac{5}{2}}}{g^2(|\xi|)}\widehat{\Delta_l u}(\xi)\right\|_{L^2}^2\right)^{\frac{1}{2}}\\ &\le Cg^2(2^N)\left(\sum_{l=1}^{N-1}1\right)^{\frac{1}{2}}\left\|\frac{\Lambda^{\frac{5}{2}}}{g^2(\Lambda)}u\right\|_{L^2}\\ &\le Cg^2(2^N)\sqrt{N}\left\|\frac{\Lambda^{\frac{5}{2}}}{g^2(\Lambda)}u\right\|_{L^2},\end{aligned}$$

where we have the fact that g is a non-decreasing function.

Putting the above estimates altogether implies for any $\sigma_4\in(0,\,\alpha-\frac{3}{2})$, such that

$$\|\nabla u\|_{L^\infty}\le C\|u\|_{L^2}+Cg^2(2^N)\sqrt{N}\left\|\frac{\Lambda^{\frac{5}{2}}}{g^2(\Lambda)}u\right\|_{L^2}+C2^{N(\frac{3}{2}-\alpha+\sigma_4)}\|\Lambda^{\alpha-\sigma_4}\nabla u\|_{L^2},$$

$$\|\Lambda^{\frac{5}{2}}u\|_{L^2}\le C\|u\|_{L^2}+Cg^2(2^N)\sqrt{N}\left\|\frac{\Lambda^{\frac{5}{2}}}{g^2(\Lambda)}u\right\|_{L^2}+C2^{N(\frac{3}{2}-\alpha+\sigma_4)}\|\Lambda^{\alpha-\sigma_4}\nabla u\|_{L^2}.$$

By (2.19), we obtain

$$\frac{d}{dt}\widetilde{X}(t)+\widetilde{Y}(t)\le C\left(1+g^2(2^N)\sqrt{N}\left\|\frac{\Lambda^{\frac{5}{2}}}{g^2(\Lambda)}u\right\|_{L^2}+2^{-N\upsilon}\|\Lambda^{\alpha-\sigma_4}\nabla u\|_{L^2}\right)\widetilde{X}(t)+\widetilde{H}(t),$$

where

$$\upsilon:=\alpha-\sigma_4-\frac{3}{2}>0.$$

Taking N as

$$2^{N\upsilon}\approx e+\widetilde{X}(t),$$

we thus deduce

$$\frac{d}{dt}\widetilde{X}(t)+\widetilde{Y}(t)\le C\left(1+\left\|\frac{\Lambda^{\frac{5}{2}}}{g^2(\Lambda)}u\right\|_{L^2}\right)g^2\left([e+\widetilde{X}(t)]^{\frac{1}{\upsilon}}\right)\sqrt{\ln\left(e+\widetilde{X}(t)\right)}\left(e+\widetilde{X}(t)\right)$$

$$
\begin{aligned}
&+C\widetilde{Y}^{\frac{1}{2}}(t)+\widetilde{H}(t)\\
\le\ & C\left(1+\left\|\frac{\Lambda^{\frac{5}{2}}}{g^2(\Lambda)}u\right\|_{L^2}\right)g^2\left([e+\widetilde{X}(t)]^{\frac{1}{v}}\right)\sqrt{\ln\left(e+\widetilde{X}(t)\right)}\left(e+\widetilde{X}(t)\right)\\
&+\frac{1}{2}\widetilde{Y}(t)+C+\widetilde{H}(t).
\end{aligned}
$$

Consequently it gives

$$
\begin{aligned}
\frac{d}{dt}\widetilde{X}(t)+\widetilde{Y}(t)\le\ & C\left(1+\left\|\frac{\Lambda^{\frac{5}{2}}}{g^2(\Lambda)}u\right\|_{L^2}\right)g^2\left([e+\widetilde{X}(t)]^{\frac{1}{v}}\right)\sqrt{\ln\left(e+\widetilde{X}(t)\right)}\left(e+\widetilde{X}(t)\right)\\
&+C+\widetilde{H}(t).
\end{aligned}
$$

Due to

$$
g^2\left([e+\widetilde{X}(t)]^{\frac{1}{v}}\right)\sqrt{\ln\left(e+\widetilde{X}(t)\right)}\left(e+\widetilde{X}(t)\right)\ge 1,
$$

we have

$$
\begin{aligned}
&\frac{d}{dt}\widetilde{X}(t)+\widetilde{Y}(t)\\
\le\ & C\left(1+\|u\|_{L^2}^2+\left\|\frac{\Lambda^{\frac{5}{2}}}{g^2(\Lambda)}u\right\|_{L^2}^2\right)g^2\left([e+\widetilde{X}(t)]^{\frac{1}{v}}\right)\sqrt{\ln\left(e+\widetilde{X}(t)\right)}\left(e+\widetilde{X}(t)\right).
\end{aligned}
\tag{2.20}
$$

This along with (2.16) yields

$$
\int_{e+\widetilde{X}(0)}^{e+\widetilde{X}(t)}\frac{d\tau}{\tau\sqrt{\ln\tau}\,g^2(\tau^{\frac{1}{v}})}\le C\int_0^t\left(1+\|u\|_{L^2}^2+\left\|\frac{\Lambda^{\frac{5}{2}}}{g^2(\Lambda)}u\right\|_{L^2}^2\right)d\tau\le C(t,\,u_0,\,w_0,\,b_0).
$$

Notice that the condition

$$
\int_e^{\infty}\frac{d\tau}{\tau\sqrt{\ln\tau}\,g(\tau^{\frac{1}{v}})}=\sqrt{v}\int_{e^{\frac{1}{v}}}^{\infty}\frac{d\tau}{\tau\sqrt{\ln\tau}\,g(\tau)}=\infty.
$$

It is not hard to check that $\widetilde{X}(t)$ will keep boundedness for any given $t>0$, namely,

$$
\widetilde{X}(t)\le C(t,\,u_0,\,w_0,\,b_0).
$$

This together with (2.20) yields that

$$
\int_0^t\widetilde{Y}(\tau)\,d\tau\le C(t,\,u_0,\,w_0,\,b_0).
$$

Therefore, we have

$$
\begin{aligned}
\|\nabla u(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2+\|\Lambda^{\widetilde{\sigma}}w(t)\|_{L^2}^2+\int_0^t\left(\|\mathcal{M}\nabla u(\tau)\|_{L^2}^2+\|\Lambda^{\gamma}\nabla b(\tau)\|_{L^2}^2\right)d\tau&\\
\le\ C(t,\,u_0,\,w_0,\,b_0).&
\end{aligned}
$$

Thanks to (2.18) and the following fact

$$
\|\Lambda^{\frac{5}{2}}u\|_{L^2}+\|\nabla u\|_{L^\infty}\le C\|\Lambda^{\alpha-\sigma_4}\nabla u\|_{L^2}^2+C\|u\|_{L^2}^2,\quad \sigma_4\in\left(0,\,\alpha-\frac{3}{2}\right),
$$

the desired estimate (2.3) holds directly. This ends the proof of the lemma 2.3. □

3. The proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. To do so, applying Λ^s with $s > \frac{5}{2}$ to the system (1.2) and taking the L^2 inner product with $\Lambda^s u$, $\Lambda^s b$ and $\Lambda^s w$ respectively, adding them up, we can get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s w\|_{L^2}^2+\|\Lambda^s b\|_{L^2}^2\right)+\|\mathcal{M}\Lambda^s u\|_{L^2}^2+2\|\Lambda^s w\|_{L^2}^2+\|\Lambda^{\gamma+s} b\|_{L^2}^2\\
=&\int_{\mathbb{R}^3}\Big(\Lambda^s(\nabla\times u)\cdot\Lambda^s w+\Lambda^s(\nabla\times w)\cdot\ \Lambda^s u\Big)\,dx-\int_{\mathbb{R}^3}[\Lambda^s, u\cdot\nabla]u\ \cdot\Lambda^s u\,dx\\
&-\int_{\mathbb{R}^3}[\Lambda^s, u\cdot\nabla]w\ \cdot\Lambda^s w\,dx+\int_{\mathbb{R}^3}[\Lambda^s, b\cdot\nabla]b\ \cdot\Lambda^s u\,dx+\int_{\mathbb{R}^3}[\Lambda^s, b\cdot\nabla]u\ \cdot\Lambda^s b\,dx\\
&-\int_{\mathbb{R}^3}[\Lambda^s, u\cdot\nabla]b\ \cdot\Lambda^s b\,dx\\
:=&\sum_{k=1}^{6}J_k.
\end{aligned}\tag{3.1}$$

It is easy to check that for any $\sigma_5\in(0,\,\alpha)$,

$$\|\mathcal{M}\Lambda^s u\|_{L^2}^2\geq C_1\|\Lambda^{\alpha-\sigma_5}\Lambda^s u\|_{L^2}^2-C_2\|\Lambda^s u\|_{L^2}^2.\tag{3.2}$$

For $\sigma_5\in(0,\,\alpha-1)$, we have by (3.2)

$$\begin{aligned}
J_1&\leq C\|\Lambda^s w\|_{L^2}\|\Lambda^s\nabla u\|_{L^2}\\
&\leq C\|\Lambda^s w\|_{L^2}\left(\|\mathcal{M}\Lambda^s u\|_{L^2}+\|\Lambda^s u\|_{L^2}\right)\\
&\leq\frac{1}{16}\|\mathcal{M}\Lambda^s u\|_{L^2}^2+C\left(\|\Lambda^s w\|_{L^2}^2+\|\Lambda^s u\|_{L^2}\right).
\end{aligned}$$

The classic Kato-Ponce commutator estimates ([10]) yields

$$\begin{aligned}
J_2&\leq C\|[\Lambda^s, u\cdot\nabla]u\|_{L^2}\|\Lambda^s u\|_{L^2}\\
&\leq C\|\nabla u\|_{L^\infty}\|\Lambda^s u\|_{L^2}^2.
\end{aligned}$$

Using (3.2), it implies for $\widetilde{\sigma}-\sigma_5>\frac{5}{2}-\alpha$ that

$$\begin{aligned}
J_3=&\ -\int_{\mathbb{R}^3}[\Lambda^s\partial_{x_i}, u_i]w\Lambda^s w\,dx\\
\leq&\ \|[\Lambda^s\partial_{x_i}, u_i]w\|_{L^2}\|\Lambda^s w\|_{L^2}\\
\leq&\ C(\|\nabla u\|_{L^\infty}\|\Lambda^s w\|_{L^2}+\|\Lambda^{s+1}u\|_{L^{\frac{3}{\widetilde{\sigma}}}}\|w\|_{L^{\frac{6}{3-2\widetilde{\sigma}}}})\|\Lambda^s w\|_{L^2}\\
\leq&\ C(\|\nabla u\|_{L^\infty}\|\Lambda^s w\|_{L^2}+\|\Lambda^{\frac{5}{2}-\widetilde{\sigma}}\Lambda^s u\|_{L^2}\|w\|_{L^{\frac{6}{3-2\widetilde{\sigma}}}})\|\Lambda^s w\|_{L^2}\\
\leq&\ C(\|\nabla u\|_{L^\infty}\|\Lambda^s w\|_{L^2}+(\|\mathcal{M}\Lambda^s u\|_{L^2}+\|\Lambda^s u\|_{L^2})\|\Lambda^{\widetilde{\sigma}}w\|_{L^2})\|\Lambda^s w\|_{L^2}\\
\leq&\ \frac{1}{16}\|\mathcal{M}\Lambda^s u\|_{L^2}^2+C(1+\|\nabla u\|_{L^\infty}+\|\Lambda^{\widetilde{\sigma}}w\|_{L^2}^2)(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s w\|_{L^2}^2).
\end{aligned}$$

Now we take $\widetilde{p} > 2$ satisfying
$$\frac{1}{\widetilde{p}} \geq \frac{5\alpha - 5\sigma_5 - 3 - (2\alpha - 3 - 2\sigma_5)s}{6(\alpha - \sigma_5)}.$$
We remark that the above $\widetilde{p}$ would work by taking σ_5 suitable small. Consequently, it is not hard to check that
$$\begin{aligned}
J_4 \leq& C\|[\Lambda^s, b\cdot\nabla]b\|_{L^{\frac{2\widetilde{p}}{\widetilde{p}+2}}}\|\Lambda^s u\|_{L^{\frac{2\widetilde{p}}{\widetilde{p}-2}}}\\
\leq& C\|\nabla b\|_{L^{\widetilde{p}}}\|\Lambda^s b\|_{L^2}\|\Lambda^s u\|_{L^{\frac{2\widetilde{p}}{\widetilde{p}-2}}}\\
\leq& C\left(\|\nabla b\|_{L^2}^{1-\frac{3(\widetilde{p}-2)}{2(s-1)\widetilde{p}}}\|\Lambda^s b\|_{L^2}^{\frac{3(\widetilde{p}-2)}{2(s-1)\widetilde{p}}}\right)\|\Lambda^s b\|_{L^2}\left(\|\nabla u\|_{L^2}^{1-\frac{(s-1)\widetilde{p}+3}{(\alpha+s-1-\sigma_5)\widetilde{p}}}\|\Lambda^{\alpha-\sigma_5}\Lambda^s u\|_{L^2}^{\frac{(s-1)\widetilde{p}+3}{(\alpha+s-1-\sigma_5)\widetilde{p}}}\right)\\
\leq& C\left(\|\nabla b\|_{L^2}^{1-\frac{3(\widetilde{p}-2)}{2(s-1)\widetilde{p}}}\|\Lambda^s b\|_{L^2}^{\frac{3(\widetilde{p}-2)}{2(s-1)\widetilde{p}}}\right)\|\Lambda^s b\|_{L^2}\|\nabla u\|_{L^2}^{1-\frac{(s-1)\widetilde{p}+3}{(\alpha+s-1-\sigma_5)\widetilde{p}}}(\|\mathcal{M}\Lambda^s u\|_{L^2}+\|\Lambda^s u\|_{L^2})^{\frac{(s-1)\widetilde{p}+3}{(\alpha+s-1-\sigma_5)\widetilde{p}}}\\
\leq& \frac{1}{16}\|\mathcal{M}\Lambda^s u\|_{L^2}^2 + F(\|\nabla u\|_{L^2},\, \|\nabla b\|_{L^2})(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2),
\end{aligned}$$
where $F(x,\, y)$ is also a smooth increasing function with respect to variable x and y. Similarly, one has
$$\begin{aligned}
J_5,\, J_6 \leq& C\left(\|[\Lambda^s, u\cdot\nabla]b\|_{L^2} + \|[\Lambda^s, b\cdot\nabla]u\|_{L^2}\right)\|\Lambda^s b\|_{L^2}\\
\leq& C\left(\|\nabla u\|_{L^\infty}\|\Lambda^s b\|_{L^2} + \|\nabla b\|_{L^{\widetilde{p}}}\|\Lambda^s u\|_{L^{\frac{2\widetilde{p}}{\widetilde{p}-2}}}\right)\|\Lambda^s b\|_{L^2}\\
\leq& \frac{1}{16}\|\mathcal{M}\Lambda^s u\|_{L^2}^2 + F\left(\|\nabla u\|_{L^2},\, \|\nabla b\|_{L^2}\right)\left(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2\right)\\
&+ C\|\nabla u\|_{L^\infty}\|\Lambda^s b\|_{L^2}^2.
\end{aligned}$$
Collecting all the above estimates, we see that
$$\frac{d}{dt}\phi(t) + \|\mathcal{M}\Lambda^s u\|_{L^2}^2 + \|\Lambda^{s+\gamma} b\|_{L^2}^2 \leq C\psi(t)\left(e + \phi(t)\right),$$
where $\phi(t)$ and $\psi(t)$ are given by
$$\begin{aligned}
\phi(t) :=& \|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s w(t)\|_{L^2}^2 + \|\Lambda^s b(t)\|_{L^2}^2,\\
\psi(t) :=& \left(1 + \|\nabla u\|_{L^\infty} + \|\Lambda^{\widetilde{\sigma}} w\|_{L^2}^2 + F(\|\nabla u\|_{L^2},\, \|\nabla b\|_{L^2})\right)(t).
\end{aligned}$$
It follows from (2.13) and (2.3) that
$$\int_0^t \psi(\tau)\, d\tau \leq C(t,\, u_0,\, w_0,\, b_0).$$
The Gronwall inequality implies
$$\begin{aligned}
\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s w(t)\|_{L^2}^2 + \|\Lambda^s b(t)\|_{L^2}^2 +& \int_0^t (\|\mathcal{M}\Lambda^s u(\tau)\|_{L^2}^2 + \|\Lambda^{s+\gamma} b(\tau)\|_{L^2}^2)\, d\tau\\
&\leq C(t,\, u_0,\, w_0,\, b_0),
\end{aligned}$$
which concludes the proof of Theorem 1.1.

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