The time-harmonic electromagnetic wave scattering by a bi-periodic elastic body

Tielei Zhu*, Changkun Wei†, Jiaqing Yang‡

Abstract

This paper concentrates on an interaction scattering problem between the time-harmonic electromagnetic waves and an unbounded periodic elastic medium. The uniqueness results of the interaction problem are established for small frequencies or all frequencies except a discrete set in both the absorbing and non-absorbing medium, and then the existence of solutions is derived by the classical Fredholm alternative. The perfectly matched layer (PML) method is proposed to truncate the unbounded scattering domain to a bounded computational domain. We prove the well-posedness of the solution for the truncated PML problem, where a homogeneous boundary condition is imposed on the outer boundary of the PML. The exponential convergence of the PML method is established in terms of the thickness and parameters of the PML. The proof is based on the PML extension and the exponential decay properties of the modified fundamental solution.

Keywords: Electromagnetic field, elastic waves, periodic structure, well-posedness, PML, exponential convergence.

1 Introduction

Scattering theory in periodic structures, also called diffraction grating problem has been studied extensively in the last decades and has many important applications in micro-optics [4, 5, 24, 29]. There are usually two fundamental methods to study the well-posedness of diffraction grating problems, including the integral equation technique and the variational methods. For example, the existence of the unique solution for the diffraction gratings of electromagnetic and elastic waves could be found in [1, 16, 35, 41], by using the integral equation technique. Similar results were established in [2, 3, 5, 24, 26, 27, 44] by using the variational methods involved with the Dirichlet to Neumann (DtN) mappings. Most of the existing works about the diffraction grating problems were confined to the pure electromagnetic, elastic diffraction, and fluid-solid interaction in periodic structures.

The interaction scattering problems between multi-physical fields, e.g. fluid-solid interaction, electromagnetic and elastic waves interaction, have existed widely in practical applications [15,

*School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi, 710049, China. (zhutielei@stu.xjtu.edu.cn)

†The corresponding author, School of Mathematics and Statistics, Beijing Jiaotong University, Beijing, 100044, China.(ckwei@bjtu.edu.cn)

‡School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi, 710049, China.(jiaq.yang@xjtu.edu.cn)

25, 32, 38, 40, 43, 45] so that the related mathematical models and analysis need to be developed crucially. However, the mathematical analysis is quite rare for the interface interaction problem of electromagnetic field and elastic waves, due to the quite challenge of the interaction between two kinds of complicated vector fields. The first mathematical model was established in [15] for the case of bounded elastic body, where Cakoni and Hsiao proved the uniqueness of the direct problem under special transmission coefficients in the interface conditions. Based on the framework in [15], Gatica *et al.* [28] and Bernardo *et al.* [11] considered the reduced model and analyzed the corresponding finite element schemes, respectively. Recently, Zhu *et al.* [48], under more general transmission conditions, gave a rigorous proof for the well-posedness by the variational method and for the first time proved the uniqueness of the inverse problem. Here, we refer to a related work [46] on the analysis for the electromagnetic and elastic waves interaction problem in the time domain.

The perfectly matched layer (PML) method is a fast and effective method for solving unbounded scattering problems which was originally proposed by Bérenger in 1994 for Maxwell's equations [12]. Since then, various constructions of the PML absorbing layers have been proposed and investigated and the PML method attracts great interests for mathematicians to study the convergence analysis for the wave scattering problems; see, e.g. [13, 14, 19–21, 47] for the acoustic, elastic and electromagnetic scattering problems. For the diffraction grating problems in the periodic structures, Chen and Wu [18] firstly proved the exponential convergence of the PML method for the acoustic diffraction in two-dimensional gratings. The PML technique was further developed by Bao and Wu [6] for the electromagnetic diffraction by a bi-periodic structure. Recently, Jiang *et al.* [33] studied the PML analysis for the elastic scattering by a bi-periodic rigid surface. We remark that their proofs depend on the explicit formulas of the exact DtN mapping and the DtN mapping related to the PML problem.

In this paper, we focus on the theoretical analysis for the interaction problem between the time-harmonic electromagnetic waves and an unbounded periodic elastic medium. We emphasize several difficulties to study this type of interaction problems. Firstly, the electromagnetic and elastic domains are unbounded, which naturally brings some problems in mathematical analysis and numerical computation. Secondly, the compact embedding of $H(\mathrm{curl},\cdot)$ into $\boldsymbol{L}^2(\cdot)$ in a bounded domain does not hold such that the Fredholm alternative theorem can not be applied directly. Thirdly, the uniqueness in diffraction grating problems is not available in general for all frequencies, which is obviously different from the case of scattering by a bounded elastic body. We obtain the following results:

- *Existence and uniqueness of solution.* If special transmission coefficients are considered in the model, one of which is corresponding to the frequency-domain case adopted in [46] for the time-domain analysis, we prove the uniqueness of the interaction problem with lossless medium and non-absorbing medium ($\operatorname{Im}\varepsilon_r = 0$ and $\operatorname{Im}\rho = 0$) for small frequencies or all frequencies except a discrete set. If general transmission coefficients are studied, an aforementioned uniqueness result can be obtained in the case of the lossy electromagnetic medium ($\operatorname{Im}\varepsilon_r > 0$) and the uniqueness result for all frequencies can also be derived in the case of lossy elastic medium ($\operatorname{Im}\rho > 0$). In combination with the above uniqueness results and the Fredholm alternative theorem, we obtain the existence of solutions for the interaction problem.
- *Convergence of PML.* The PML method is proposed to solve the interaction problem numerically. The complex coordinate stretching to derive the PML problem is [22] (for

details see (4.2) below)

$$
\begin{aligned}
\widetilde{x}_1 &:= x_1, \\
\widetilde{x}_2 &:= x_2, \\
\widetilde{x}_3 &:= \begin{cases} \displaystyle\int_0^{x_3} [1 + (\zeta_1 + \mathrm{i})\sigma_1(t)]\,\mathrm{d}t & x_3 > b, \\ x_3 & -b \le x_3 \le b, \\ \displaystyle\int_0^{x_3} [1 + (\zeta_2 + \mathrm{i})\sigma_2(t)]\,\mathrm{d}t & x_3 < -b, \end{cases}
\end{aligned}
$$

where $\zeta_j \ge 0$ are constants to be specified and $\sigma_j(t)$ are the PML medium properties. Different from deducing the explicit formulas of the exact DtN mapping and the DtN mapping related to the PML problem, we adopted the PML extension, which is essentially an integral representation in the complex stretching coordinates, to prove the exponential convergence of the PML method. In our proof, the exponential decay of the modified fundamental solution plays an important role. The main obstacle in using the explicit formulas of the two DtN mappings lies in the complexity of the Rayleigh expansion in elastic scattering which contains p-part and s-part waves, such that a quite complex symbol matrix of eight order needs to be solved. Besides, the DtN mapping adopted by Jiang *et al.* [33] is a special form with respect to the traction $\widetilde{T}\boldsymbol{u} = \mu^*\partial\boldsymbol{u}/\partial\nu + (\lambda+\mu^*)\nu\mathrm{div}\boldsymbol{u}$ which is obviously different with the traction $T\boldsymbol{u} = 2\mu^*\partial\boldsymbol{u}/\partial\nu + \lambda\nu\mathrm{div}\boldsymbol{u} + \mu^*\nu \times \mathrm{curl}\boldsymbol{u}$ appearing in our interaction model problem. Thus we cannot apply their conclusions directly for the PML analysis in elastic domain. Under the assumptions for fictitious absorbing coefficients σ_j, the parameters ζ_j, and the uniqueness of the PML equations in PML layers, we prove the exponential convergence of the solution for the truncated PML problem, where the homogeneous Dirichlet boundary conditions are imposed on the outer boundary of the PML layers.

The paper is organized as follows. In section 2, we introduce the mathematical model and some function spaces. In section 3, the uniqueness and existence of solution for the interaction problem is proved. In section 4, we propose the PML for the interaction problem and prove the exponential convergence of the PML. Some conclusions are given in section 5.

2 Problem formulation

2.1 Mathematical model

In this subsection, we introduce the mathematical formulation of the model problem. Throughout this paper, let $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ and $x' := (x_1, x_2)^T \in \mathbb{R}^2$ be the first two variables of x. As seen in Figure 1, let $D^- := \{x \in \mathbb{R}^3 : x_3 < f(x_1, x_2)\}$ be the elastic medium with the boundary Γ, which is characterized by Lamé constants λ and μ^* and density $\rho(x) \in L^\infty(D^-)$ with $\mathrm{Re}\,\rho > 0$ and $\mathrm{Im}\,\rho \ge 0$, where $f(x_1, x_2) \in C^2(\mathbb{R}^2)$ is a 2π-periodic function in the x_1 and x_2 directions and the density $\rho(x)$ is 2π-periodic in the x_1 and x_2 directions, i.e.,

$$\rho(x_1 + 2n_1\pi, x_2 + 2n_2\pi, x_3) = \rho(x_1, x_2, x_3)$$

for $x \in D^-$ and $n_1, n_2 \in \mathbb{Z}$. The region above D^- is denoted by D^+, which is characterized by electric permittivity $\varepsilon(x) \in L^\infty(D^+)$ with $\mathrm{Re}\,\varepsilon(x) > 0$ and $\mathrm{Im}\,\varepsilon(x) \ge 0$, magnetic permeability

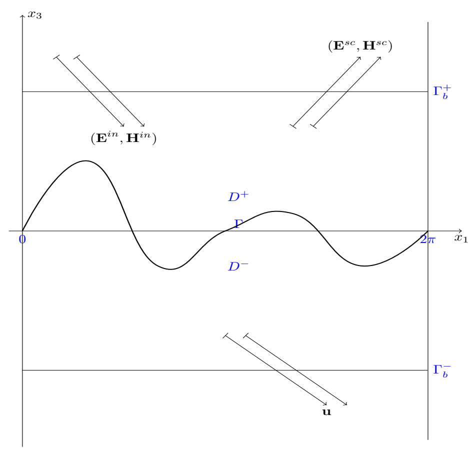

Figure 1: Interaction between electromagnetic wave and a bounded elastic body with periodic structure

$\mu(x) \in L^\infty(D^+)$ with $\mu(x) > 0$ and electric conductivity $\sigma(x) \in L^\infty(D^+)$, which satisfy

$$\begin{aligned}\varepsilon(x_1 + 2n_1\pi, x_2 + 2n_2\pi, x_3) &= \varepsilon(x_1, x_2, x_3),\\ \mu(x_1 + 2n_1\pi, x_2 + 2n_2\pi, x_3) &= \mu(x_1, x_2, x_3),\\ \sigma(x_1 + 2n_1\pi, x_2 + 2n_2\pi, x_3) &= \sigma(x_1, x_2, x_3)\end{aligned}$$

for $x \in D^+$ and $n_1,\ n_2 \in \mathbb{Z}$. Further, suppose that there exists $b > 0$ such that $\varepsilon(x) = \varepsilon_0$, $\mu(x) = \mu_0$ and $\sigma(x) = 0$ hold for $x_3 > b$ with $\varepsilon_0 > 0$ the electric permittivity and $\mu_0 > 0$ the magnetic permeability in the background electromagnetic medium and $\rho(x) = \rho_0$ holds for $x_3 < -b$ with $\rho_0 > 0$ the density in the background elastic medium.

Given $\alpha = (\alpha_1, \alpha_2)^T \in \mathbb{R}^2$, define

$$\eta = \begin{cases}(\kappa^2 - |\alpha|^2)^{\frac{1}{2}} & |\alpha| \le \kappa,\\ \mathrm{i}(|\alpha|^2 - \kappa^2)^{\frac{1}{2}} & |\alpha| > \kappa.\end{cases}$$

The incident time-harmonic electromagnetic plane wave with frequency ω is given by

$$\boldsymbol{E}^{in}(x, d, p) = (p \times d)e^{\mathrm{i}\kappa x\cdot d}, \quad \boldsymbol{H}^{in}(x, d, p) = pe^{\mathrm{i}\kappa x\cdot d}, \tag{2.1}$$

where $\kappa = \omega\sqrt{\varepsilon_0\mu_0}$ is the wave-number, $d = 1/\kappa(\alpha^T, -\eta)^T$ and the polarization vector $p \in \mathbb{R}^3$ satisfies $p \cdot d = 0$. The total electromagnetic fields $(\boldsymbol{E}, \boldsymbol{H})$ are the superpositions of the incident field $(\boldsymbol{E}^{in}, \boldsymbol{H}^{in})$ and scattered field $(\boldsymbol{E}^{sc}, \boldsymbol{H}^{sc})$, i.e., $\boldsymbol{E} = \boldsymbol{E}^{in} + \boldsymbol{E}^{sc}$ and $\boldsymbol{H} = \boldsymbol{H}^{in} + \boldsymbol{H}^{sc}$, which satisfy the Maxwell's equations in D^+

$$\mathrm{curl}\, \boldsymbol{E} - \mathrm{i}\kappa\mu_r\boldsymbol{H} = \boldsymbol{0}, \quad \mathrm{curl}\, \boldsymbol{H} + \mathrm{i}\kappa\varepsilon_r\boldsymbol{E} = \boldsymbol{0} \quad \text{in } D^+, \tag{2.2}$$

where

$$\varepsilon_r(x) := \frac{\varepsilon(x)}{\varepsilon_0} + \mathrm{i}\frac{\sigma(x)}{\varepsilon_0\omega}, \quad \mu_r(x) := \frac{\mu(x)}{\mu_0}.$$

The scattered field $(\boldsymbol{E}^{sc}, \boldsymbol{H}^{sc})$ needs to be α-quasi-periodic with respect to x_1 and x_2 directions in the sense that $e^{-\mathrm{i}\kappa\alpha\cdot x'}\boldsymbol{H}^{sc}(x)$ and $e^{-\mathrm{i}\kappa\alpha\cdot x'}\boldsymbol{E}^{sc}(x)$ are periodic in x_1 and x_2 directions and $\boldsymbol{H}^{sc}$ satisfies the Rayleigh expansion

$$\boldsymbol{H}^{sc}(x) = \sum_{n\in\mathbb{Z}^2} \boldsymbol{H}_n e^{\mathrm{i}(\alpha_n\cdot x' + \eta_n x_3)} \qquad \text{for} \quad x_3 > b, \tag{2.3}$$

where $\alpha_n = (\alpha_n^{(1)}, \alpha_n^{(2)})^T := (\alpha_1 + n_1, \alpha_2 + n_2)^T \in \mathbb{R}^2$,

$$\eta_n := \begin{cases} (\kappa^2 - |\alpha_n|^2)^{\frac{1}{2}} & |\alpha_n| \le \kappa, \\ \mathrm{i}(|\alpha_n|^2 - \kappa^2)^{\frac{1}{2}} & |\alpha_n| > \kappa, \end{cases}$$

and $\boldsymbol{H}_n := (\boldsymbol{H}_n^{(1)}, \boldsymbol{H}_n^{(2)}, \boldsymbol{H}_n^{(3)})^T \in \mathbb{C}^3$ are Rayleigh coefficients with $(\alpha_n^T, \eta_n)^T \cdot \boldsymbol{H}_n = 0$.

Since the incident electromagnetic wave is upon the elastic medium and the elastic deformation occurs in view of physical properties of the elastic body on the surface Γ, the elastic field $\boldsymbol{u}$ satisfies Navier's equations in D^-

$$\triangle^* \boldsymbol{u} + \rho\omega^2 \boldsymbol{u} = \boldsymbol{0} \quad \text{in } D^-, \tag{2.4}$$

where $\triangle^* = \mu^*\triangle + (\lambda+\mu^*)\mathrm{grad}\,\mathrm{div}$. Throughout this paper, we assume $\mu^* > 0$ and $3\lambda + 2\mu^* > 0$. Similarly, the elastic field $\boldsymbol{u}$ needs to be α-quasi-periodic with respect to x_1 and x_2 directions and satisfies the Rayleigh expansion

$$\boldsymbol{u}(x) = \sum_{n\in\mathbb{Z}^2} A_{p,n} \begin{pmatrix} \alpha_n \\ -\beta_n \end{pmatrix} e^{\mathrm{i}(\alpha_n\cdot x' - \beta_n x_3)} + \boldsymbol{A}_{s,n} e^{\mathrm{i}(\alpha_n\cdot x' - \gamma_n x_3)} \qquad \text{for} \quad x_3 < -b, \tag{2.5}$$

where β_n and γ_n are defined analogous to η_n with κ replaced by the compressional and shear wave-numbers $\kappa_p := \omega\sqrt{\rho_0/(\lambda + 2\mu^*)}$ and $\kappa_s := \omega\sqrt{\rho_0/\mu^*}$, respectively

$$\beta_n := \begin{cases} (\kappa_p^2 - |\alpha_n|^2)^{\frac{1}{2}} & |\alpha_n| \le \kappa_p, \\ \mathrm{i}(|\alpha_n|^2 - \kappa_p^2)^{\frac{1}{2}} & |\alpha_n| > \kappa_p, \end{cases} \qquad \gamma_n := \begin{cases} (\kappa_s^2 - |\alpha_n|^2)^{\frac{1}{2}} & |\alpha_n| \le \kappa_s, \\ \mathrm{i}(|\alpha_n|^2 - \kappa_s^2)^{\frac{1}{2}} & |\alpha_n| > \kappa_s, \end{cases}$$

and $A_{p,n} \in \mathbb{C}$, $\boldsymbol{A}_{s,n} \in \mathbb{C}^3$ are the corresponding Rayleigh coefficients with $(\alpha_n^T, -\gamma_n)^T \cdot \boldsymbol{A}_{s,n} = 0$.

Based on the Voigt's model [11, 15, 28, 38], the transmission conditions between the electromagnetic field and elastic filed on the interface are

$$\begin{aligned} T\boldsymbol{u} - b_1\nu \times \boldsymbol{H}^{sc} &= b_1\nu \times \boldsymbol{H}^{in} \quad \text{on } \Gamma, \\ \nu \times \boldsymbol{u} - b_2\nu \times \boldsymbol{E}^{sc} &= b_2\nu \times \boldsymbol{E}^{in} \quad \text{on } \Gamma, \end{aligned} \tag{2.6}$$

where b_1, $b_2 \in \mathbb{C}$, $b_1 b_2 \ne 0$, ν denotes the outward unit normal vector of D^- and $T\boldsymbol{u} = 2\mu^*\partial\boldsymbol{u}/\partial\nu + \lambda\nu\mathrm{div}\boldsymbol{u} + \mu^*\nu \times \mathrm{curl}\boldsymbol{u}$ denotes the traction operator.

2.2 Preliminaries

In this subsection, we introduce some α-quasi-periodic function spaces and Dirichlet-to-Neumann (DtN) mappings about the electromagnetic fields and elastic field. For more details on classical function spaces and their traces, we refer to [7–10]

Because of the periodicity of the problem, we only need to consider a single periodic cell and identify the $D^\pm$ and Γ with their single periodic cells. Then we define

$$D_b^\pm := \{x \in D^\pm : 0 < x_1, x_2 < 2\pi, |x_3| < b\}, \quad \Gamma_b^\pm := \{x \in \mathbb{R}^3 : 0 < x_1, x_2 < 2\pi, x_3 = \pm b\}.$$

Let $\Omega = D_b^\pm$, $\Sigma = \Gamma$ (or $\Sigma = \Gamma_b^\pm$) and ν is the normal vector of Σ. For $s \in \mathbb{R}$, let $H^s(\Omega)$ (resp. $H^s(\Sigma)$) denotes the standard scalar Sobolev spaces defined on Ω (resp. Σ) and $\boldsymbol{H}^s(\Omega)$ (resp. $\boldsymbol{H}^s(\Sigma)$) denotes the associated vector function spaces in the three-dimensional case. Moreover, we introduce the following function spaces

$$\begin{aligned} \boldsymbol{H}_t^s(\Sigma) &:= \{\phi \in \boldsymbol{H}^s(\Sigma) : \phi \cdot \nu = 0\}, \quad s \in \mathbb{R}, \\ H(\mathrm{curl}, \Omega) &:= \{\boldsymbol{B} \in \boldsymbol{L}^2(\Omega) : \mathrm{curl}\, \boldsymbol{B} \in \boldsymbol{L}^2(\Omega)\}, \end{aligned}$$

and α-quasi-periodic function spaces

$$\begin{aligned} H_\alpha^1(\Omega) &:= \{\varphi \in H^1(\Omega) : \varphi \text{ is } \alpha\text{-quasi-periodic w.r.t. } x_1 \text{ and } x_2 \text{ directions}\} \\ \boldsymbol{H}_\alpha^1(\Omega) &:= \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v} \text{ is } \alpha\text{-quasi-periodic w.r.t. } x_1 \text{ and } x_2 \text{ directions}\} \\ \boldsymbol{H}_{t,\alpha}^s(\Sigma) &:= \{\boldsymbol{v} \in \boldsymbol{H}_t^s(\Sigma) : \boldsymbol{v} \text{ is } \alpha\text{-quasi-periodic w.r.t. } x_1 \text{ and } x_2 \text{ directions}\}, \quad s \in \mathbb{R}, \\ H_\alpha(\mathrm{curl}, \Omega) &:= \{\boldsymbol{B} \in H(\mathrm{curl}, \Omega) : \boldsymbol{B} \text{ is } \alpha\text{-quasi-periodic w.r.t. } x_1 \text{ and } x_2 \text{ directions}\}. \end{aligned}$$

The trace spaces of $H_\alpha(\mathrm{curl}, D_b^+)$ are given by

$$\begin{aligned} H_\alpha^{-\frac{1}{2}}(\mathrm{Div}, \Sigma) &:= \{\mu \in \boldsymbol{H}_{t,\alpha}^{-\frac{1}{2}}(\Sigma) : \mathrm{Div}_\Sigma\, \mu \in H_\alpha^{-\frac{1}{2}}(\Sigma)\}, \\ H_\alpha^{-\frac{1}{2}}(\mathrm{Curl}, \Sigma) &:= \{\mu \in \boldsymbol{H}_{t,\alpha}^{-\frac{1}{2}}(\Sigma) : \mathrm{Curl}_\Sigma\, \mu \in H_\alpha^{-\frac{1}{2}}(\Sigma)\}, \end{aligned}$$

where Div_Σ and Curl_Σ denote the surface divergence and the surface curl with respect to the surface Σ, respectively (cf .[10] for the case of Lipschitz domain). For convenience, we also write $\nabla_\Sigma \cdot$ for Div_Σ, where ∇_Σ is the surface gradient defined by $\nabla_\Sigma := (\nu \times \nabla) \times \nu$.

Let γ_t and γ_T be trace operators from $H_\alpha(\mathrm{curl}, D_b^+)$ into $H_\alpha^{-1/2}(\mathrm{Div}, \Gamma)$ and $H_\alpha^{-1/2}(\mathrm{Curl}, \Gamma)$, respectively and make analogous definitions for γ_t^+ and γ_T^+ from $H_\alpha(\mathrm{curl}, D_b^+)$ into $H_\alpha^{-1/2}(\mathrm{Div}, \Gamma_b^+)$ and $H_\alpha^{-1/2}(\mathrm{Curl}, \Gamma_b^+)$, respectively. The duality pair between $H_\alpha^{-1/2}(\mathrm{Div}, \Gamma)$ and $H_\alpha^{-1/2}(\mathrm{Curl}, \Gamma)$ is defined by

$$\langle \varphi, \overline{\psi} \rangle_\Gamma := \int_{D_b^+} (\mathrm{curl}\, \boldsymbol{B} \cdot \overline{\boldsymbol{w}} - \mathrm{curl}\, \overline{\boldsymbol{w}} \cdot \boldsymbol{B})\, \mathrm{d}x$$

where $\boldsymbol{B}$, $\boldsymbol{w} \in H_\alpha(\mathrm{curl}, D_b^+)$ satisfy $\gamma_t \boldsymbol{B} = \varphi$, $\gamma_t^+ \boldsymbol{B} = 0$, $\gamma_T \boldsymbol{w} = \psi$ and $\gamma_T^+ \boldsymbol{w} = 0$. Notice that $H_\alpha^{-1/2}(\mathrm{Div}, \Gamma)$ is a subspace of $\boldsymbol{H}_{t,\alpha}^{-1/2}(\Gamma)$. In fact, let $\varphi \in H_\alpha^{-1/2}(\mathrm{Div}, \Gamma)$ and $\psi \in \boldsymbol{H}_{t,\alpha}^{1/2}(\Gamma)$,

$$\langle \varphi, \overline{\psi} \rangle_{\boldsymbol{H}^{-\frac{1}{2}} \times \boldsymbol{H}^{\frac{1}{2}}} := \int_{D_b^+} (\mathrm{curl}\, \boldsymbol{w} \cdot \overline{\boldsymbol{v}} - \mathrm{curl}\, \overline{\boldsymbol{v}} \cdot \boldsymbol{w})\, \mathrm{d}x$$

where $\boldsymbol{w} \in H_\alpha(\mathrm{curl}, D_b^+)$ and $\boldsymbol{v} \in \boldsymbol{H}_\alpha^1(D_b^+)$ satisfy $\gamma_t \boldsymbol{w} = \varphi$, $\gamma_t^+ \boldsymbol{w} = 0$, $\gamma_T \boldsymbol{v} = \psi$ and $\gamma_T^+ \boldsymbol{v} = 0$.

In order to truncate D^+, we introduce DtN mapping $\mathcal{R} : H_\alpha^{-1/2}(\mathrm{Div}, \Gamma_b^+) \to H_\alpha^{-1/2}(\mathrm{Curl}, \Gamma_b^+)$ corresponding to the electromagnetic fields by

$$(\mathcal{R}\widetilde{\boldsymbol{H}})(x') := e_3 \times (\mathrm{curl}\, \boldsymbol{H} \times e_3) \quad \text{on } \Gamma_b^+,$$

where $\widetilde{\boldsymbol{H}} \in H_\alpha^{-1/2}(\mathrm{Div}, \Gamma_b^+)$, $e_3 = (0,0,1)^T$ denotes the unit outward normal vector of Γ_b^+ and $\boldsymbol{H}(x)$ is a quasi-periodic solution of the following exterior problem

$$\begin{cases} \mathrm{curl}^2\, \boldsymbol{H} - \kappa^2 \boldsymbol{H} = \mathbf{0} & x_3 > b, \\ e_3 \times \boldsymbol{H} = \widetilde{\boldsymbol{H}}(x') & \text{on } \Gamma_b^+, \\ \boldsymbol{H}(x) = \sum_{n\in\mathbb{Z}^2} \boldsymbol{H}_n e^{\mathrm{i}(\alpha_n\cdot x' + \eta_n x_3)} & x_3 > b, \end{cases} \tag{2.7}$$

where $\boldsymbol{H}_n := (\boldsymbol{H}_n^{(1)}, \boldsymbol{H}_n^{(2)}, \boldsymbol{H}_n^{(3)})^T \in \mathbb{C}^3$ are the Rayleigh coefficients with $(\alpha_n^T, \eta_n)^T \cdot \boldsymbol{H}_n = 0$.

Lemma 2.1. (cf. [29, equations (13)-(16)]). *The definition of $\mathcal{R}$ shows that the DtN mapping $\mathcal{R}$ has the following properties:*

1. *$\mathcal{R} : H_\alpha^{-1/2}(\mathrm{Div}, \Gamma_b^+) \to H_\alpha^{-1/2}(\mathrm{Curl}, \Gamma_b^+)$ is a linear and bounded operator and can be expressed as*

$$(\mathcal{R}\widetilde{\boldsymbol{E}})(x') = -\sum_{n\in\mathbb{Z}^2} \frac{1}{\mathrm{i}\eta_n}\{\kappa^2 \widetilde{\boldsymbol{E}}_n - (\alpha_n \cdot \widetilde{\boldsymbol{E}}_n)\alpha_n\} e^{\mathrm{i}\alpha_n\cdot x'}$$

   where $\widetilde{\boldsymbol{E}}(x') = \sum_{n\in\mathbb{Z}^2} \widetilde{\boldsymbol{E}}_n e^{\mathrm{i}\alpha_n\cdot x'}$.

2. *The following formulas hold.*

$$\begin{aligned} \mathrm{Re}\langle \mathcal{R}\widetilde{\boldsymbol{E}}, \overline{\widetilde{\boldsymbol{E}}}\rangle_{\Gamma_b^+} &= 4\pi^2 \sum_{|\alpha_n|>\kappa} \frac{1}{|\eta_n|}(\kappa^2|\widetilde{\boldsymbol{E}}_n|^2 - |\alpha_n \cdot \widetilde{\boldsymbol{E}}_n|^2), \\ -\mathrm{Re}\langle \mathcal{R}\widetilde{\boldsymbol{E}}, \overline{\widetilde{\boldsymbol{E}}}\rangle_{\Gamma_b^+} &\geq C_1 \|\mathrm{div}\widetilde{\boldsymbol{E}}\|^2_{H_\alpha^{-\frac{1}{2}}(,\Gamma_b^+)} - C_2\|\widetilde{\boldsymbol{E}}\|^2_{H_\alpha^{-\frac{1}{2}}(\Gamma_b^+)}, \\ \mathrm{Im}\langle \mathcal{R}\widetilde{\boldsymbol{E}}, \overline{\widetilde{\boldsymbol{E}}}\rangle_{\Gamma_b^+} &= 4\pi^2 \sum_{|\alpha_n|<\kappa} \frac{1}{|\eta_n|}(\kappa^2|\widetilde{\boldsymbol{E}}_n|^2 - |\alpha_n \cdot \widetilde{\boldsymbol{E}}_n|^2) \geq 0, \end{aligned}$$

   where C_1 and C_2 are positive constants and $\widetilde{\boldsymbol{E}}(x') = \sum_{n\in\mathbb{Z}^2} \widetilde{\boldsymbol{E}}_n e^{\mathrm{i}\alpha_n\cdot x'}$.

In order to truncate D^-, we introduce another DtN mapping $\mathcal{T} : \boldsymbol{H}_\alpha^{1/2}(\Gamma_b^-) \to \boldsymbol{H}_\alpha^{-1/2}(\Gamma_b^-)$ corresponding to the elastic field

$$\mathcal{T}\widetilde{\boldsymbol{v}}(x') := T\boldsymbol{v} \quad \text{on } \Gamma_b^-$$

where $\widetilde{\boldsymbol{v}} \in \boldsymbol{H}_\alpha^{1/2}(\Gamma_b^-)$ and $\boldsymbol{v}$ is a quasi-periodic solution of the following problem

$$\begin{cases} \triangle^* \boldsymbol{v} + \rho\omega^2 \boldsymbol{v} = \mathbf{0} & x_3 < -b, \\ \boldsymbol{v} = \widetilde{\boldsymbol{v}} & \text{on } \Gamma_b^-, \\ \boldsymbol{v}(x) = \sum_{n\in\mathbb{Z}^2} V_{p,n} \begin{pmatrix} \alpha_n \\ -\beta_n \end{pmatrix} e^{\mathrm{i}(\alpha_n\cdot x' - \beta_n x_3)} + \boldsymbol{V}_{s,n} e^{\mathrm{i}(\alpha_n\cdot x' - \gamma_n x_3)} & x_3 < -b, \end{cases} \tag{2.8}$$

with $V_{p,n} \in \mathbb{C}$ and $\boldsymbol{V}_{s,n} \in \mathbb{C}^3$ the corresponding Rayleigh coefficients and $(\alpha_n^T, -\gamma_n)^T \cdot \boldsymbol{V}_{s,n} = 0$.

Let $\boldsymbol{v}(x') = \sum_{n\in\mathbb{Z}^2} \boldsymbol{v}_n e^{\mathrm{i}\alpha_n\cdot x'} \in \boldsymbol{H}^{1/2}(\Gamma_b^-)$ with $\boldsymbol{v}_n \in \mathbb{C}^3$ and the direct computation shows $\mathcal{T}$ can be expressed as

$$\mathcal{T}\boldsymbol{v}(x') = \sum_{n\in\mathbb{Z}^2} \mathrm{i}W_n \boldsymbol{v}_n e^{\mathrm{i}\alpha_n\cdot x'},$$

where the coefficient matrix $W_n \in \mathbb{C}^{3\times 3}$ is given by

$$W_n := \frac{1}{|\alpha_n|^2 + \beta_n\gamma_n}\begin{pmatrix} a_n & b_n & -c_n \\ b_n & d_n & -e_n \\ c_n & e_n & f_n \end{pmatrix},$$
$$a_n = \mu[(\gamma_n - \beta_n)(\alpha_n^{(2)})^2 + \kappa_s^2\beta_n], \qquad b_n = -\mu\alpha_n^{(1)}\alpha_n^{(2)}(\gamma_n - \beta_n),$$
$$c_n = (2\mu\alpha_n^2 - \rho\omega^2 + 2\mu\gamma_n\beta_n)\alpha_n^{(1)}, \quad e_n = (2\mu\alpha_n^2 - \rho\omega^2 + 2\mu\gamma_n\beta_n)\alpha_n^{(2)},$$
$$d_n = \mu[(\gamma_n - \beta_n)(\alpha_n^{(1)})^2 + \kappa_s^2\beta_n], \qquad f_n = \gamma_n\omega^2\rho.$$

Lemma 2.2. (cf. [27, 31]). *The DtN mapping $\mathcal{T}$ has the following properties:*

1. *$-\mathcal{T}$ can be decomposed into the sum of a positive definite operator $\mathcal{T}_1$ and a finite rank operator $\mathcal{T}_2$ from $\boldsymbol{H}_\alpha^{1/2}(\Gamma_b^-)$ to $\boldsymbol{H}_\alpha^{-1/2}(\Gamma_b^-)$, i.e.,*

$$\mathrm{Re}\int_{\Gamma_b^-} \mathcal{T}_1\boldsymbol{u}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}s \ge 0.$$

2. *The following equality holds.*

$$\mathrm{Im}\int_{\Gamma_b^-} \mathcal{T}\boldsymbol{u}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}s = 4\pi^2\left(\sum_{|\alpha_n|<\kappa_p}\beta_n|A_{p,n}|^2\rho\omega^2 + \sum_{|\alpha_n|<\kappa_s}\gamma_n|\boldsymbol{A}_{s,n}|^2\mu^*\right).$$

3. *There exist a sufficiently small frequency ω_0 and a constant $C > 0$ such that for any $\omega \in (0, \omega_0]$,*

$$-\mathrm{Re}\int_{\Gamma_b^-} \mathcal{T}\boldsymbol{u}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}s \ge C\|\boldsymbol{u}\|^2_{\boldsymbol{H}_\alpha^{1/2}(\Gamma_b^-)},$$

if $\boldsymbol{u}_n = 0$ holds for $n = (0,0)^T$, where $\boldsymbol{u}_n$ is the Fourier coefficient of $e^{-\mathrm{i}\alpha\cdot x'}\boldsymbol{u}|_{\Gamma_b^-}$.

Throughout this paper we assume $|\alpha_n|^2 + \beta_n\gamma_n \neq 0$ and $\eta_n \neq 0$ for all $n \in \mathbb{Z}^2$. Based on the above DtN mappings, we can get the following boundary-value problem equivalent with the original scattering problem (2.1)-(2.6).

$$\begin{cases} \mathrm{curl}\,(\varepsilon_r^{-1}\mathrm{curl}\,\boldsymbol{H}) - \kappa^2\mu_r\boldsymbol{H} = \boldsymbol{0} & \text{in } D_b^+, \\ \triangle^*\boldsymbol{u} + \rho\omega^2\boldsymbol{u} = \boldsymbol{0} & \text{in } D_b^-, \\ T\boldsymbol{u} - b_1\nu\times\boldsymbol{H} = \boldsymbol{0} & \text{on } \Gamma, \\ \nu\times(\varepsilon_r^{-1}\mathrm{curl}\,\boldsymbol{H}) + \dfrac{\mathrm{i}\kappa}{b_2}\nu\times\boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma, \\ \gamma_T^+(\mathrm{curl}\,\boldsymbol{H}) - \mathcal{R}(e_3\times\boldsymbol{H}) = \gamma_T^+(\mathrm{curl}\,\boldsymbol{H}^{in}) - \mathcal{R}(e_3\times\boldsymbol{H}^{in}) & \text{on } \Gamma_b^+, \\ T\boldsymbol{u} - \mathcal{T}\boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma_b^-. \end{cases} \tag{2.9}$$

Remark 2.3. *Obviously, the solution to the original problem* (2.1)-(2.6) *satisfy the problem* (2.9)*. Conversely, the solution to the probelm* (2.9) *can be extended to the domains $U_{b^\pm} := \{x \in \mathbb{R}^3 : \pm x_3 > b\}$ by the solutions in the definitions of* (2.7) *and* (2.8)*.*

3 The well-posedness

In this section, we aim to prove the well-posedness of the model problem (2.9) by the variational method under two transmission conditions (2.6), where the uniqueness of solutions can be derived except a discrete set of frequencies for the absorbing ($\operatorname{Im}\varepsilon_r > 0$ or $\operatorname{Im}\rho > 0$) and non-absorbing medium ($\operatorname{Im}\varepsilon_r = 0$ and $\operatorname{Im}\rho = 0$) under some assumptions about the coefficients in the transmission conditions and the existence of solutions can be obtained by the application of Fredholm alternative.

3.1 Variational formula

The problem (2.9) is a special case of the following boundary-value problem. Given $\boldsymbol{f}_1 \in \boldsymbol{H}_\alpha^{-1/2}(\Gamma)$, $\boldsymbol{f}_2 \in H_\alpha^{-1/2}(\mathrm{Div}, \Gamma)$ and $\boldsymbol{f}_3 \in H_\alpha^{-1/2}(\mathrm{Curl}, \Gamma_b^+)$, find $\boldsymbol{u} \in \boldsymbol{H}_\alpha^1(D_b^-)$ and $\boldsymbol{H} \in H_\alpha(\mathrm{curl}, D_b^+)$ satisfying

$$
\begin{cases}
\mathrm{curl}\,(\varepsilon_r^{-1}\mathrm{curl}\,\boldsymbol{H}) - \kappa^2\mu_r\boldsymbol{H} = \boldsymbol{0} & \text{in } D_b^+, \\
\triangle^*\boldsymbol{u} + \rho\omega^2\boldsymbol{u} = \boldsymbol{0} & \text{in } D_b^-, \\
T\boldsymbol{u} - b_1\nu \times \boldsymbol{H} = \boldsymbol{f}_1 & \text{on } \Gamma, \\
\nu \times (\varepsilon_r^{-1}\mathrm{curl}\,\boldsymbol{H}) + \dfrac{\mathrm{i}\kappa}{b_2}\nu \times \boldsymbol{u} = \boldsymbol{f}_2 & \text{on } \Gamma, \\
\gamma_T^+(\mathrm{curl}\,\boldsymbol{H}) - \mathcal{R}(e_3 \times \boldsymbol{H}) = \boldsymbol{f}_3 & \text{on } \Gamma_b^+, \\
T\boldsymbol{u} - \mathcal{T}\boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma_b^-.
\end{cases}
\tag{3.1}
$$

From Green's and Betti's formulas [10, 23, 36], it follows that

$$
\begin{aligned}
0 &= \int_{D_b^+} \Big(\mathrm{curl}\,(\varepsilon_r^{-1}\mathrm{curl}\,\boldsymbol{H}) - \kappa^2\mu_r\boldsymbol{H}\Big) \cdot \overline{\boldsymbol{w}}\,\mathrm{d}x \\
&= \int_{D_b^+} (\varepsilon_r^{-1}\mathrm{curl}\,\boldsymbol{H} \cdot \mathrm{curl}\,\overline{\boldsymbol{w}} - \kappa^2\mu_r\boldsymbol{H} \cdot \overline{\boldsymbol{w}})\,\mathrm{d}x + \Big(\int_{\Gamma_b^+} - \int_\Gamma\Big)\nu \times (\varepsilon_r^{-1}\mathrm{curl}\,\boldsymbol{H}) \cdot \overline{\boldsymbol{w}}\,\mathrm{d}s, \\
0 &= -\int_{D_b^-} (\triangle^*\boldsymbol{u} + \rho\omega^2\boldsymbol{u}) \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x = \int_{D_b^-} (\mathcal{E}(\boldsymbol{u}, \overline{\boldsymbol{v}}) - \rho\omega^2\boldsymbol{u} \cdot \overline{\boldsymbol{v}})\,\mathrm{d}x - \Big(\int_{\Gamma_b^-} + \int_\Gamma\Big)T\boldsymbol{u} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}s,
\end{aligned}
$$

for $\boldsymbol{w} \in H_\alpha(\mathrm{curl}, D_b^+)$ and $\boldsymbol{v} \in \boldsymbol{H}_\alpha^1(D_b^-)$ with

$$
\mathcal{E}(\boldsymbol{u}, \boldsymbol{v}) := 2\mu^* \sum_{j,k=1}^{3} \partial_j u_k \partial_k v_j + \lambda \mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{v} - \mu^*\mathrm{curl}\,\boldsymbol{u} \cdot \mathrm{curl}\,\boldsymbol{v}.
$$

The transmission conditions and DtN mappings combining with the above formulas imply

$$
\begin{aligned}
&\int_{D_b^+} (\varepsilon_r^{-1}\mathrm{curl}\,\boldsymbol{H} \cdot \mathrm{curl}\,\overline{\boldsymbol{w}} - \kappa^2\mu_r\boldsymbol{H} \cdot \overline{\boldsymbol{w}})\,\mathrm{d}x - \int_{\Gamma_b^+} \mathcal{R}(e_3 \times \boldsymbol{H}) \cdot e_3 \times \overline{\boldsymbol{w}}\,\mathrm{d}s + \frac{\mathrm{i}\kappa}{b_2}\int_\Gamma \nu \times \boldsymbol{u} \cdot \overline{\boldsymbol{w}}\,\mathrm{d}s \\
&= \int_\Gamma \boldsymbol{f}_2 \cdot \overline{\boldsymbol{w}}\,\mathrm{d}s + \int_{\Gamma_b^+} \boldsymbol{f}_3 \cdot (e_3 \times \overline{\boldsymbol{w}})\,\mathrm{d}x, \\
&\frac{-\mathrm{i}\kappa}{b_1\overline{b}_2}\int_{D_b^-} (\mathcal{E}(\boldsymbol{u}, \overline{\boldsymbol{v}}) - \rho\omega^2\boldsymbol{u} \cdot \overline{\boldsymbol{v}})\,\mathrm{d}x - \frac{-\mathrm{i}\kappa}{b_1\overline{b}_2}\int_{\Gamma_b^-} \mathcal{T}\boldsymbol{u} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}s - \frac{-\mathrm{i}\kappa}{\overline{b}_2}\int_\Gamma \gamma_t\boldsymbol{H} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}s = \frac{-\mathrm{i}\kappa}{b_1\overline{b}_2}\int_\Gamma \boldsymbol{f}_1 \cdot \overline{\boldsymbol{v}}\,\mathrm{d}s.
\end{aligned}
$$

for $\boldsymbol{w} \in H_\alpha(\mathrm{curl}, D_b^+)$ and $\boldsymbol{v} \in \boldsymbol{H}_\alpha^1(D_b^-)$.

For simplicity, define $Q := \boldsymbol{H}_\alpha^1(D_b^-)$ and $X := H_\alpha(\mathrm{curl}, D_b^+)$. Then we get the variational formula of (3.1): Given $\boldsymbol{f}_1 \in \boldsymbol{H}_\alpha^{-1/2}(\Gamma)$, $\boldsymbol{f}_2 \in H_\alpha^{-1/2}(\mathrm{Div}, \Gamma)$ and $\boldsymbol{f}_3 \in H_\alpha^{-1/2}(\mathrm{Curl}, \Gamma_b^+)$, find $\boldsymbol{u} \in Q$ and $\boldsymbol{H} \in X$ satisfying

$$\mathcal{A}((\boldsymbol{u}, \boldsymbol{H}), (\boldsymbol{v}, \boldsymbol{w})) = \mathcal{F}((\boldsymbol{v}, \boldsymbol{w})) \quad \forall (\boldsymbol{v}, \boldsymbol{w}) \in Q \times X, \tag{3.2}$$

where

$$\begin{aligned}
\mathcal{A}((\boldsymbol{u}, \boldsymbol{H}), (\boldsymbol{v}, \boldsymbol{w})) :=& \frac{-\mathrm{i}\kappa}{b_1 \bar{b}_2} \int_{D_b^-} (\mathcal{E}(\boldsymbol{u}, \overline{\boldsymbol{v}}) - \rho\omega^2 \boldsymbol{u} \cdot \overline{\boldsymbol{v}})\, \mathrm{d}x - \frac{-\mathrm{i}\kappa}{b_1 \bar{b}_2} \int_{\Gamma_b^-} \mathcal{T}\boldsymbol{u} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}s \\
&- \Big(\int_{D_b^+} (\varepsilon_r^{-1} \mathrm{curl}\, \boldsymbol{H} \cdot \mathrm{curl}\, \overline{\boldsymbol{w}} - \kappa^2 \mu_r \boldsymbol{H} \cdot \overline{\boldsymbol{w}})\, \mathrm{d}x - \int_{\Gamma_b^+} \mathcal{R}(\gamma_t^+ \boldsymbol{H}) \cdot \gamma_t^+ \overline{\boldsymbol{w}}\, \mathrm{d}s \Big) \\
&+ \frac{\mathrm{i}\kappa}{b_2} \langle \gamma_t \overline{\boldsymbol{w}}, \gamma_T \boldsymbol{u} \rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}} \times H_{\mathrm{Curl}}^{-\frac{1}{2}}} + \frac{\mathrm{i}\kappa}{\bar{b}_2} \langle \gamma_t \boldsymbol{H}, \gamma_T \overline{\boldsymbol{v}} \rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}} \times H_{\mathrm{Curl}}^{-\frac{1}{2}}}, \\
\mathcal{F}((\boldsymbol{v}, \boldsymbol{w})) :=& \frac{-\mathrm{i}\kappa}{b_1 \bar{b}_2} \langle \boldsymbol{f}_1, \gamma_T \overline{\boldsymbol{v}} \rangle_{\boldsymbol{H}^{-\frac{1}{2}} \times \boldsymbol{H}^{\frac{1}{2}}} - \langle \boldsymbol{f}_2, \gamma_T \overline{\boldsymbol{w}} \rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}} \times H_{\mathrm{Curl}}^{-\frac{1}{2}}} - \int_{\Gamma_b^+} \boldsymbol{f}_3 \cdot (e_3 \times \overline{\boldsymbol{w}})\, \mathrm{d}s.
\end{aligned}$$

3.2 Uniqueness

Analogous to the Jones frequencies in the fluid-solid interaction [40], we introduce the following system

$$\begin{cases} \triangle^* \boldsymbol{u} + \rho\omega^2 \boldsymbol{u} = \boldsymbol{0} & \text{in } D^-, \\ \nu \times \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma, \\ T\boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma, \\ \boldsymbol{u} \text{ satisfies the Rayleigh expansion} & \end{cases} \tag{3.3}$$

and define pathological frequencies set $\mathcal{P}(\omega)$:=$\{\omega$: frequency ω for which (3.3) has a nontrivial solution$\}$.

Remark 3.1. *In view of the uniqueness of elastic diffraction in a periodic structure about boundary condition of the second kind [27, Theorem 3], we know that $\mathcal{P}(\omega)$ is at most a discrete set and there exists small enough $\omega_1 > 0$ such that $(0, \omega_1] \not\subset \mathcal{P}(\omega)$.*

Now, we are in the position to arrive at the following uniqueness result except a discrete set of frequencies in the cases of lossy and lossless medium under some assumptions.

Theorem 3.2.

1. *Assume $\mathrm{Re}(b_1\bar{b}_2) = 0$ and $\mathrm{Im}(b_1\bar{b}_2) < 0$.*
   (1) Suppose $\varepsilon_r \in C(D_b^+)$ and $\mu_r \in C^1(D_b^+)$. If $\omega \notin \mathcal{P}(\omega)$ and there exists some $x_0 \in D^+$ such that $\mathrm{Im}\, \varepsilon_r(x_0) > 0$, then the problem (3.1) has at most one solution.
   (2) Suppose $\rho \in C(D_b^-)$. If there exists $x_0 \in D^-$ such that $\mathrm{Im}\, \rho(x_0) > 0$, then the problem (3.1) has at most one solution for all $\omega > 0$.
2. *If $\varepsilon_r = 1$, $\mu_r = 1$, $\rho = \rho_0$, $b_1 = c_1$, $b_2 = \mathrm{i}c_2/\kappa$ and $|\alpha| < \kappa \le \kappa_p$, there exists small enough $\omega_2 > 0$ such that the problem (3.1) has at most one solution for $\omega \in (0, \omega_2]$, where $c_1 > 0$ and $c_2 > 0$ are constants independent of ω.*

Proof. Let $\boldsymbol{f}_1 = \boldsymbol{0}$, $\boldsymbol{f}_2 = \boldsymbol{0}$ and $\boldsymbol{f}_3 = \boldsymbol{0}$.

1. Assume $\mathrm{Re}(b_1\overline{b}_2) = 0$ and $\mathrm{Im}(b_1\overline{b}_2) < 0$ hold. Choosing $\boldsymbol{v} = \boldsymbol{u}$ and $\boldsymbol{w} = -\boldsymbol{H}$ in (3.2) and taking the imaginary part yield

$$\begin{aligned} 0 = &-\frac{-\mathrm{i}\kappa}{b_1\overline{b}_2}\int_{D_b^-}(\mathrm{Im}\,\rho)\omega^2|\boldsymbol{u}|^2\,\mathrm{d}x - \frac{-\mathrm{i}\kappa}{b_1\overline{b}_2}\mathrm{Im}\int_{\Gamma_b^-}\mathcal{T}\boldsymbol{u}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}s \\ &+\int_{D_b^+}(\mathrm{Im}\,\varepsilon_r^{-1})|\mathrm{curl}\,\boldsymbol{H}|^2\,\mathrm{d}x - \mathrm{Im}\int_{\Gamma_b^+}\mathcal{R}(e_3\times\boldsymbol{H})\cdot(e_3\times\overline{\boldsymbol{H}})\,\mathrm{d}s. \end{aligned} \tag{3.4}$$

(1) If there exists some $x_0 \in D^+$ such that $\mathrm{Im}\,\varepsilon_r(x_0) > 0$, then $\mathrm{curl}\,\boldsymbol{H} = \boldsymbol{0}$ in some neighbourhood of x_0. It follows from the Maxwell's equations that $\boldsymbol{E} = \boldsymbol{0}$ in some neighbourhood of x_0, which, according to the unique continuation principle (cf. [23, 42]) and $\mu_r \in C^1(D_b^+)$, shows $\boldsymbol{E} = \boldsymbol{0}$ in D^+ . Then it means that $\boldsymbol{u}$ satisfies (3.3) and hence the assumption $\omega \notin \mathcal{P}(\omega)$ leads to $\boldsymbol{u} = \boldsymbol{0}$ in D^-.

(2) If there exists some $x_0 \in D^-$ such that $\mathrm{Im}\,\rho(x_0) > 0$, then $\boldsymbol{u} = \boldsymbol{0}$ in some neighborhood of x_0. Obviously, the unique continuation principle (cf. [34]) and $\rho \in C(D_b^-)$ show $\boldsymbol{u} = \boldsymbol{0}$ in D^-. Similarly, from the unique continuation principle ([23, 42]), we have $\boldsymbol{E} = \boldsymbol{0}$ and $\boldsymbol{H} = \boldsymbol{0}$ in D^+.

2. We consider the case that there hold $\varepsilon_r = 1$, $\mu_r = 1$, $b_1 = c_1$, $b_2 = \mathrm{i}c_2/\kappa$ and $|\alpha| < \kappa \le \kappa_p$. Then $(\boldsymbol{u}, \boldsymbol{H})$ satisfies the following variational formulation:

$$\begin{aligned} &\frac{\kappa^2}{c_1c_2}\int_{D_b^-}(\mathcal{E}(\boldsymbol{u},\overline{\boldsymbol{v}}) - \rho_0\omega^2\boldsymbol{u}\cdot\overline{\boldsymbol{v}})\,\mathrm{d}x - \frac{\kappa^2}{c_1c_2}\int_{\Gamma_b^-}\mathcal{T}\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}s - \frac{\kappa^2}{c_2}\langle\gamma_t\boldsymbol{H},\gamma_T\overline{\boldsymbol{v}}\rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}}\times H_{\mathrm{Curl}}^{-\frac{1}{2}}} \\ &-\Big(\int_{D_b^+}(\mathrm{curl}\,\boldsymbol{H}\cdot\mathrm{curl}\,\overline{\boldsymbol{w}} - \kappa^2\boldsymbol{H}\cdot\overline{\boldsymbol{w}})\,\mathrm{d}x - \int_{\Gamma_b^+}\mathcal{R}(e_3\times\boldsymbol{H})\cdot(e_3\times\overline{\boldsymbol{w}})\,\mathrm{d}s \\ &-\frac{\kappa^2}{c_2}\langle\gamma_t\overline{\boldsymbol{w}},\gamma_T\boldsymbol{u}\rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}}\times H_{\mathrm{Curl}}^{-\frac{1}{2}}}\Big) = 0 \end{aligned} \tag{3.5}$$

for any $(\boldsymbol{v}, \boldsymbol{w}) \in Q \times X$. Note if $b_1 = c_1$ and $b_2 = \mathrm{i}c_2/\kappa$ hold then the coefficients b_1 and b_2 satisfy $\mathrm{Re}(b_1\overline{b}_2) = 0$ and $\mathrm{Im}(b_1\overline{b}_2) < 0$, and (3.4) is thus available for $(\boldsymbol{u}, \boldsymbol{H})$. So from (3.4), Lemma 2.1 and 2.2, there holds $\boldsymbol{H}_n = \boldsymbol{0}$ for $|\alpha_n| < \kappa$, $A_{p,n} = 0$ for $|\alpha_n| < \kappa_p$ and $\boldsymbol{A}_{s,n} = \boldsymbol{0}$ for $|\alpha_n| < \kappa_s$. This, combined with the fact that $|\alpha| < \kappa \le \kappa_p$, implies that $\boldsymbol{H}_n = \boldsymbol{0}$, $A_{p,n} = 0$ and $\boldsymbol{A}_{s,n} = \boldsymbol{0}$ for $n = (0,0)^T$.

In what follows, to analyze (3.5) and obtain the uniqueness result we shall introduce the Helmholtz decomposition (see [30, Lemma 4.1] for more details about this decomposition). For this, we define two spaces

$$S := \Big\{\varphi \in H_\alpha^1(D_b^+) : \int_{\Gamma_b^+}\varphi\,\mathrm{d}s = 0\Big\} \text{ and } X_0 := \Big\{\boldsymbol{H}_0 \in X : \int_{D_b^+}\boldsymbol{H}_0\cdot\nabla\overline{\psi}\,\mathrm{d}x = 0,\ \forall\psi \in S\Big\}. \tag{3.6}$$

Then we have the Helmholtz decomposition $X = X_0 \bigoplus \nabla S$. It is easy to see that (cf. [39, Cor. 4.8]) if $\boldsymbol{H}_0 \in X_0$, then there exists a constant $C > 0$ such that

$$\|\mathrm{curl}\,\boldsymbol{H}_0\|_{\boldsymbol{L}_\alpha^2(D_b^+)} \ge C\|\boldsymbol{H}_0\|_{\boldsymbol{L}_\alpha^2(D_b^+)}. \tag{3.7}$$

Using the Helmholtz decomposition, we obtain that for any $\boldsymbol{H}, \boldsymbol{w} \in X$ there exist $\boldsymbol{H}_0$, $\boldsymbol{w}_0 \in X_0$ and φ, $\psi \in S$ such that $\boldsymbol{H} = \boldsymbol{H}_0 + \nabla\varphi$ and $\boldsymbol{w} = \boldsymbol{w}_0 + \nabla\psi$. Then we can rewrite the variational formula (3.5) as

$$
\begin{aligned}
&\frac{\kappa^2}{c_1c_2}\Big(\int_{D_b^-}(\mathcal{E}(\boldsymbol{u},\overline{\boldsymbol{v}})-\rho\omega^2\boldsymbol{u}\cdot\overline{\boldsymbol{v}})\,\mathrm{d}x-\int_{\Gamma_b^-}\mathcal{T}\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}s\Big)-\frac{\kappa^2}{c_2}\langle\gamma_t\boldsymbol{H}_0,\gamma_T\overline{\boldsymbol{v}}\rangle_{H_{\mathrm{Div}}^{-\frac12}\times H_{\mathrm{Curl}}^{-\frac12}}\\
&+\kappa^2\int_{D_b^+}\nabla\varphi\cdot\nabla\overline{\psi}\,\mathrm{d}x+\int_{\Gamma_b^+}\mathcal{R}(e_3\times\nabla\varphi)\cdot(e_3\times\nabla\overline{\psi})\,\mathrm{d}s+\int_{\Gamma_b^+}\mathcal{R}(e_3\times\boldsymbol{H}_0)\cdot(e_3\times\nabla\overline{\psi})\,\mathrm{d}s\\
&-\frac{\kappa^2}{c_2}\langle\gamma_t\nabla\varphi,\gamma_T\overline{\boldsymbol{v}}\rangle_{H_{\mathrm{Div}}^{-\frac12}\times H_{\mathrm{Curl}}^{-\frac12}}+\frac{\kappa^2}{c_2}\langle\gamma_t\nabla\overline{\psi},\gamma_T\overline{\boldsymbol{u}}\rangle_{H_{\mathrm{Div}}^{-\frac12}\times H_{\mathrm{Curl}}^{-\frac12}}\\
&-\Big(\int_{D_b^+}(\operatorname{curl}\boldsymbol{H}_0\cdot\operatorname{curl}\overline{\boldsymbol{w}}_0-\kappa^2\boldsymbol{H}_0\cdot\overline{\boldsymbol{w}}_0)\,\mathrm{d}x-\int_{\Gamma_b^+}\mathcal{R}(e_3\times\boldsymbol{H}_0)\cdot(e_3\times\overline{\boldsymbol{w}}_0)\,\mathrm{d}s\\
&-\int_{\Gamma_b^+}\mathcal{R}(e_3\times\nabla\varphi)\cdot(e_3\times\overline{\boldsymbol{w}}_0)\,\mathrm{d}s-\frac{\kappa^2}{c_2}\langle\gamma_t\overline{\boldsymbol{w}}_0,\gamma_T\boldsymbol{u}\rangle_{H_{\mathrm{Div}}^{-\frac12}\times H_{\mathrm{Curl}}^{-\frac12}}\Big)=0.
\end{aligned}
\tag{3.8}
$$

for any $\boldsymbol{v}\in Q$, $\boldsymbol{w}_0\in X_0$ and $\psi\in S$. Choosing $\boldsymbol{v}=\boldsymbol{0}$, $\psi=0$ and $\boldsymbol{w}_0=\boldsymbol{H}_0$ in (3.8) gives

$$
\begin{aligned}
&\int_{D_b^+}(|\operatorname{curl}\boldsymbol{H}_0|^2-\kappa^2|\boldsymbol{H}_0|^2)\,\mathrm{d}x-\int_{\Gamma_b^+}\mathcal{R}(e_3\times\boldsymbol{H}_0)\cdot(e_3\times\overline{\boldsymbol{H}}_0)\,\mathrm{d}s\\
&-\int_{\Gamma_b^+}\mathcal{R}(e_3\times\nabla\varphi)\cdot(e_3\times\overline{\boldsymbol{H}}_0)\,\mathrm{d}s-\frac{\kappa^2}{c_2}\langle\gamma_t\overline{\boldsymbol{H}}_0,\gamma_T\boldsymbol{u}\rangle_{H_{\mathrm{Div}}^{-\frac12}\times H_{\mathrm{Curl}}^{-\frac12}}=0.
\end{aligned}
\tag{3.9}
$$

Taking the boundedness of $\mathcal{R}$ into consideration, we have

$$
-\mathrm{Re}\int_{\Gamma_b^+}\mathcal{R}(e_3\times\boldsymbol{H}_0)\cdot(e_3\times\overline{\boldsymbol{H}}_0)\,\mathrm{d}s\geq-C\kappa^2\|e_3\times\boldsymbol{H}_0\|^2_{\boldsymbol{H}_\alpha^{-1/2}(\Gamma_b^+)}. \tag{3.10}
$$

where $C>0$ is a constant independent with $\boldsymbol{H}_0$ and κ. Suppose

$$
\begin{pmatrix}\widetilde{F}(x')\\0\end{pmatrix}:=e_3\times\boldsymbol{H}_0|_{\Gamma_b^+}\quad\text{and}\quad\begin{pmatrix}\widetilde{Q}(x')\\0\end{pmatrix}:=e_3\times\nabla\varphi|_{\Gamma_b^+}
$$

and then we have

$$
\widetilde{F}(x')=\sum_{n\in\mathbb{Z}^2}\widetilde{F}_ne^{\mathrm{i}\alpha_n\cdot x'}\quad\text{and}\quad\widetilde{Q}(x')=\sum_{n\in\mathbb{Z}^2}\begin{pmatrix}-\alpha_n^{(2)}\\\alpha_n^{(1)}\end{pmatrix}\widetilde{Q}_ne^{\mathrm{i}\alpha_n\cdot x'}
$$

with $\widetilde{F}_n\in\mathbb{C}^2$ and $\widetilde{Q}_n\in\mathbb{C}$ and

$$
\begin{aligned}
\Big|\int_{\Gamma_b^+}\mathcal{R}(e_3\times\nabla\varphi)\cdot(e_3\times\overline{\boldsymbol{H}}_0)\,\mathrm{d}s\Big|&=4\pi^2\Big|-\sum_{n\in\mathbb{Z}^2}\frac{\widetilde{Q}_n}{\mathrm{i}\eta_n}\kappa^2\begin{pmatrix}-\alpha_n^{(2)}\\\alpha_n^{(1)}\end{pmatrix}\cdot\overline{\widetilde{F}}_n\Big|\\
\leq&C\kappa^2\Big(\sum_{n\in\mathbb{Z}^2}(1+|\alpha_n|^2)^{\frac12}|\widetilde{Q}_n|^2\Big)^{\frac12}\Big(\sum_{n\in\mathbb{Z}^2}(1+|\alpha_n|^2)^{-\frac12}|\widetilde{F}_n|^2\Big)^{\frac12}\\
\leq&C\kappa^2\|e_3\times\nabla\varphi\|_{\boldsymbol{H}_\alpha^{-\frac12}(\Gamma_b^+)}\|e_3\times\boldsymbol{H}_0\|_{\boldsymbol{H}_\alpha^{-\frac12}(\Gamma_b^+)},
\end{aligned}
\tag{3.11}
$$

where $C>0$ is a constant independent with κ. Similarly, in a way similar with (3.11), it is simple to show

$$
\Big|\int_{\Gamma_b^+}\mathcal{R}(e_3\times\boldsymbol{H}_0)\cdot(e_3\times\nabla\overline{\varphi})\,\mathrm{d}s\Big|\leq C\kappa^2\|e_3\times\nabla\varphi\|_{\boldsymbol{H}_\alpha^{-\frac12}(\Gamma_b^+)}\|e_3\times\boldsymbol{H}_0\|_{\boldsymbol{H}_\alpha^{-\frac12}(\Gamma_b^+)}. \tag{3.12}
$$

If κ is sufficiently small, taking the real part in (3.9) and combining (3.7), (3.10)-(3.11) and the trace theorems, we can show

$$0 \geq \|\boldsymbol{H}_0\|_X^2 - C\kappa^2 \|e_3 \times \nabla\varphi\|^2_{\boldsymbol{H}_\alpha^{-\frac{1}{2}}(\Gamma_b^+)} - C\kappa^2 \|\gamma_T \boldsymbol{u}\|^2_{\boldsymbol{H}_\alpha^{\frac{1}{2}}(\Gamma)}. \tag{3.13}$$

Letting $\boldsymbol{v} = \boldsymbol{u}$, $\psi = \varphi$ and $\boldsymbol{w}_0 = \boldsymbol{0}$ in (3.8), we observe

$$\begin{aligned} 0 =& \frac{\kappa^2}{c_1 c_2} \Big(\int_{D_b^-} (\mathcal{E}(\boldsymbol{u}, \overline{\boldsymbol{u}}) - \rho\omega^2 |\boldsymbol{u}|^2) \,\mathrm{d}x - \int_{\Gamma_b^-} \mathcal{T}\boldsymbol{u} \cdot \overline{\boldsymbol{u}} \,\mathrm{d}s \Big) \\ &+ \kappa^2 \int_{D_b^+} \nabla\varphi \cdot \nabla\overline{\varphi} \,\mathrm{d}x + \int_{\Gamma_b^+} \mathcal{R}(e_3 \times \nabla\varphi) \cdot (e_3 \times \nabla\overline{\varphi}) \,\mathrm{d}s \\ &+ \int_{\Gamma_b^+} \mathcal{R}(e_3 \times \boldsymbol{H}_0) \cdot (e_3 \times \nabla\overline{\varphi}) \,\mathrm{d}s - \frac{\kappa^2}{c_2} \langle \gamma_t \boldsymbol{H}_0, \gamma_T \overline{\boldsymbol{u}} \rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}} \times H_{\mathrm{Curl}}^{-\frac{1}{2}}} \\ &- \frac{\kappa^2}{c_2} \langle \gamma_t \nabla\varphi, \gamma_T \overline{\boldsymbol{u}} \rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}} \times H_{\mathrm{Curl}}^{-\frac{1}{2}}} + \frac{\kappa^2}{c_2} \langle \gamma_t \nabla\overline{\varphi}, \gamma_T \boldsymbol{u} \rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}} \times H_{\mathrm{Curl}}^{-\frac{1}{2}}}. \end{aligned} \tag{3.14}$$

Observe that

$$-\frac{\kappa^2}{c_2} \langle \gamma_t \nabla\varphi, \gamma_T \overline{\boldsymbol{u}} \rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}} \times H_{\mathrm{Curl}}^{-\frac{1}{2}}} + \frac{\kappa^2}{c_2} \langle \gamma_t \nabla\overline{\varphi}, \gamma_T \boldsymbol{u} \rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}} \times H_{\mathrm{Curl}}^{-\frac{1}{2}}} \tag{3.15}$$

is pure imaginary number and (see Lemma 2.1)

$$\mathrm{Re} \int_{\Gamma_b^+} \mathcal{R}(e_3 \times \nabla\varphi) \cdot (e_3 \times \nabla\overline{\varphi}) \,\mathrm{d}s \geq 0. \tag{3.16}$$

Taking the real part of (3.14) and using (3.15)-(3.16), together with (3.12), Lemma 2.2 and the trace theorems imply that

$$0 \geq C_1 \|\boldsymbol{u}\|^2_{\boldsymbol{H}_\alpha^1(D_b^-)} + C_2 \kappa^2 \|\nabla\varphi\|^2_{\boldsymbol{L}_\alpha^2(D_b^+)} - C_3 \kappa^2 \|\gamma_t \boldsymbol{H}_0\|^2_{\boldsymbol{H}_\alpha^{-\frac{1}{2}}(\Gamma)} - C_4 \kappa^2 \|e_3 \times \boldsymbol{H}_0\|^2_{\boldsymbol{H}_\alpha^{-\frac{1}{2}}(\Gamma_b^+)}, \tag{3.17}$$

when κ is sufficiently small. Then it follows from the trace theorems combined with (3.13) and (3.17) that $\boldsymbol{u} = \boldsymbol{0}$, $\varphi = 0$ and $\boldsymbol{H}_0 = \boldsymbol{0}$, which imply the uniqueness for sufficiently small frequencies and hence the uniqueness in the cases of all frequencies except a possible discrete set by the analytic Fredholm theorem. This completes the proof. □

3.3 Existence

In this subsection, the variational method will be applied to get the existence of the direct problem. To this end, we split the sesquilinear form $\mathcal{A}((\cdot,\cdot),(\cdot,\cdot))$ into two parts

$$\mathcal{A}((\boldsymbol{u}, \boldsymbol{H}), (\boldsymbol{v}, \boldsymbol{w})) = \boldsymbol{A}((\boldsymbol{u}, \boldsymbol{H}), (\boldsymbol{v}, \boldsymbol{w})) + \boldsymbol{K}((\boldsymbol{u}, \boldsymbol{H}), (\boldsymbol{v}, \boldsymbol{w})) \quad \forall (\boldsymbol{u}, \boldsymbol{H}), (\boldsymbol{v}, \boldsymbol{w}) \in Q \times X$$

where

$$
\begin{aligned}
\boldsymbol{A}((\boldsymbol{u},\boldsymbol{H}),(\boldsymbol{v},\boldsymbol{w})) :=& \frac{-\mathrm{i}\kappa}{b_1\bar{b}_2}\int_{D_b^-}(\mathcal{E}(\boldsymbol{u},\overline{\boldsymbol{v}})+\boldsymbol{u}\cdot\overline{\boldsymbol{v}})\,\mathrm{d}x+\frac{-\mathrm{i}\kappa}{b_1\bar{b}_2}\int_{\Gamma_b^-}\mathcal{T}_1\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}s\\
&-\Big(\int_{D_b^+}(\varepsilon_r^{-1}\mathrm{curl}\,\boldsymbol{H}\cdot\mathrm{curl}\,\overline{\boldsymbol{w}}-\kappa^2\mu_r\boldsymbol{H}\cdot\overline{\boldsymbol{w}})\,\mathrm{d}x-\int_{\Gamma_b^+}\mathcal{R}(\gamma_t^+\boldsymbol{H})\cdot\gamma_t^+\overline{\boldsymbol{w}}\,\mathrm{d}s\Big)\\
&+\frac{\mathrm{i}\kappa}{b_2}\langle\gamma_t\overline{\boldsymbol{w}},\gamma_T\boldsymbol{u}\rangle_{H_{\mathrm{Div}}^{-\frac12}\times H_{\mathrm{Curl}}^{-\frac12}}+\frac{\mathrm{i}\kappa}{\bar{b}_2}\langle\gamma_t\boldsymbol{H},\gamma_T\overline{\boldsymbol{v}}\rangle_{H_{\mathrm{Div}}^{-\frac12}\times H_{\mathrm{Curl}}^{-\frac12}},\\
\boldsymbol{K}((\boldsymbol{u},\boldsymbol{H}),(\boldsymbol{w},\boldsymbol{v})) :=& \frac{\mathrm{i}\kappa}{b_1\bar{b}_2}\int_D(\rho\omega^2+1)\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x+\frac{-\mathrm{i}\kappa}{b_1\bar{b}_2}\int_{\Gamma_b^-}\mathcal{T}_2\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}s.
\end{aligned}
$$

Further, the sesquilinear form $\boldsymbol{A}((\cdot,\cdot),(\cdot,\cdot))$ can be splitted into

$$
\begin{aligned}
\boldsymbol{A}((\boldsymbol{u},\boldsymbol{H}),(\boldsymbol{v},\boldsymbol{w})) =& \boldsymbol{A}_2(\boldsymbol{u},\boldsymbol{v})-\boldsymbol{A}_1(\boldsymbol{H},\boldsymbol{w})+\frac{\mathrm{i}\kappa}{\bar{b}_2}\langle\gamma_t\boldsymbol{H},\gamma_T\overline{\boldsymbol{v}}\rangle_{H_{\mathrm{Div}}^{-\frac12}\times H_{\mathrm{Curl}}^{-\frac12}}\\
&+\frac{\mathrm{i}\kappa}{b_2}\langle\gamma_t\overline{\boldsymbol{w}},\gamma_T\boldsymbol{u}\rangle_{H_{\mathrm{Div}}^{-\frac12}\times H_{\mathrm{Curl}}^{-\frac12}}\quad\forall\boldsymbol{H},\boldsymbol{w}\in X,\boldsymbol{u},\boldsymbol{v}\in Q
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{A}_1(\boldsymbol{H},\boldsymbol{w}) :=& \int_{D_b^+}(\epsilon_r^{-1}\mathrm{curl}\,\boldsymbol{H}\cdot\mathrm{curl}\,\overline{\boldsymbol{w}}-\kappa^2\mu_r\boldsymbol{H}\cdot\overline{\boldsymbol{w}})\,\mathrm{d}x-\int_{\Gamma_b^+}\mathcal{R}(e_3\times\boldsymbol{H})\cdot(e_3\times\overline{\boldsymbol{w}})\,\mathrm{d}s\\
\boldsymbol{A}_2(\boldsymbol{u},\boldsymbol{v}) :=& \frac{-\mathrm{i}\kappa}{b_1\bar{b}_2}\int_{D_b^-}(\mathcal{E}(\boldsymbol{u},\overline{\boldsymbol{v}})+\boldsymbol{u}\cdot\overline{\boldsymbol{v}})\,\mathrm{d}x+\frac{-\mathrm{i}\kappa}{b_1\bar{b}_2}\int_{\Gamma_b^-}\mathcal{T}_1\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}s.
\end{aligned}
$$

To overcome the difficulty that the embedding from X into $\boldsymbol{L}^2_\alpha(D_b^+)$ is not compact, we introduce the following Helmholtz decomposition for the sesquilinear form $\boldsymbol{A}_1(\cdot,\cdot)$. In the space S (see (3.6) for the definition), the sesquilinear form $\boldsymbol{A}_1(\cdot,\cdot)$ has the following property.

Lemma 3.3. *The sesquilinear form $-\boldsymbol{A}_1$ is bounded and coercive on ∇S.*

Proof. An obvious induction gives the boundedness of $\boldsymbol{A}_1$. From $\mathrm{Div}_{\Gamma_b^+}(e_3\times\nabla\varphi) = -e_3\cdot\mathrm{curl}\,(\nabla\varphi)=0$ and the definition of $\boldsymbol{A}_1(\cdot,\cdot)$, we have

$$
-\mathrm{Re}[\boldsymbol{A}_1(\nabla\varphi,\nabla\varphi)]\geq C\|\nabla\varphi\|_X^2,\quad \varphi\in S.
$$

The lemma is thus proved. □

Moreover, we define $\widetilde{X}_0$ by

$$
\begin{aligned}
\widetilde{X}_0 :=& \{\boldsymbol{H}_0\in X:-\kappa^2(\mu_r\boldsymbol{H}_0,\nabla\psi)-\int_{\Gamma_b^+}\mathcal{R}(e_3\times\boldsymbol{H}_0)\cdot(e_3\times\nabla\overline{\psi})\,\mathrm{d}s=0,\,\forall\psi\in S\}\\
=& \{\boldsymbol{H}_0\in X:\mathrm{div}(\mu_r\boldsymbol{H}_0)=0\text{ in }D_b^+,\ \nu\cdot(\mu_r\boldsymbol{H}_0)=0\text{ on }\Gamma\text{ and }e_3\cdot\boldsymbol{H}_0=\mathcal{D}(e_3\times\boldsymbol{H}_0)\text{ on }\Gamma_b^+\}
\end{aligned}
$$

where the operator $\mathcal{D}$: $H_\alpha^{-1/2}(\mathrm{Div},\Gamma_b^+)\to\boldsymbol{H}_{t,\alpha}^{-1/2}(\Gamma_b^+)$ is defined by

$$
\mathcal{D}(\boldsymbol{H})(x) := -\sum_{n\in\mathbb{Z}^2}\frac{1}{\eta_n}(e_3\times(\alpha_n^T,0)^T)\cdot\boldsymbol{H}_n e^{\mathrm{i}\alpha_n\cdot x'},\quad x\in\Gamma_b^+
$$

for $\boldsymbol{H} = \sum_{n\in\mathbb{Z}^2} \boldsymbol{H}_n e^{\mathrm{i}\alpha_n\cdot x'}$.

Analogous to the cases of bounded elastic body[48], we can follow the routine to obtain the properties of $\widetilde{X}_0$ and S without essential difficulties. Thus we only present the main results and omit the detailed proof.

Lemma 3.4. (cf. [29, 30, 48]). *The spaces ∇S and $\widetilde{X}_0$ are closed subspaces of X. The space X is the direct sum of the spaces ∇S and $\widetilde{X}_0$, that is, $X = \widetilde{X}_0 \bigoplus \nabla S$. Furthermore, the projections onto the subspaces are bounded, that is, there exists constants $c_1, c_2 > 0$ with*

$$c_1\|\boldsymbol{w} + \nabla\phi\|_X^2 \leq \|\boldsymbol{w}\|_X^2 + \|\nabla\phi\|_X^2 \leq c_2\|\boldsymbol{w} + \nabla\phi\|_X^2,$$

for all $\boldsymbol{w} \in \widetilde{X}_0$ and $\phi \in S$.

Lemma 3.5. (cf. [48]). *$\widetilde{X}_0$ is compactly imbedded in $\boldsymbol{L}^2_\alpha(D_b^+)$.*

Based on the Lemmas 3.4-3.5, the sesquilinear form $\boldsymbol{A}_1(\cdot,\cdot)$ in $\widetilde{X}_0$ can be splitted by

$$\boldsymbol{A}_1(\boldsymbol{H}_0, \boldsymbol{w}_0) = a_0(\boldsymbol{H}_0, \boldsymbol{w}_0) + b_0(\boldsymbol{H}_0, \boldsymbol{w}_0), \quad \forall \boldsymbol{H}_0, \boldsymbol{w}_0 \in \widetilde{X}_0,$$

where

$$a_0(\boldsymbol{H}_0, \boldsymbol{w}_0) := \int_{D_b^+} (\epsilon_r^{-1}\mathrm{curl}\,\boldsymbol{H}_0 \cdot \mathrm{curl}\,\overline{\boldsymbol{w}}_0 + M\boldsymbol{H}_0 \cdot \overline{\boldsymbol{w}}_0)\,\mathrm{d}x - \int_{\Gamma_b^+} \mathcal{R}(e_3 \times \boldsymbol{H}_0)\cdot(e_3 \times \overline{\boldsymbol{w}}_0)\,\mathrm{d}s$$
$$b_0(\boldsymbol{H}_0, \boldsymbol{w}_0) := -\int_{D_b^+} (\kappa^2\mu_r\boldsymbol{H}_0 \cdot \overline{\boldsymbol{w}}_0 + M\boldsymbol{H}_0 \cdot \overline{\boldsymbol{w}}_0)\,\mathrm{d}x$$

with M a positive constant to be determined.

Lemma 3.6. (cf. [29]). *If M is sufficiently large, a_0 is coercive on $\widetilde{X}_0$.*

Proof. By Lemma 2.1 and [29, (17)], it follows that

$$\begin{aligned}
-\mathrm{Re}\int_{\Gamma_b^+} \mathcal{R}(e_3 \times \boldsymbol{H}_0)\cdot(e_3 \times \overline{\boldsymbol{H}}_0)\,\mathrm{d}s \geq & C_1\|\mathrm{Div}_{\Gamma_b^+}(e_3 \times \boldsymbol{H}_0)\|^2_{H_\alpha^{-\frac{1}{2}}(\Gamma_b^+)} - C_2\|e_3 \times \boldsymbol{H}_0\|^2_{\boldsymbol{H}_\alpha^{-\frac{1}{2}}(\Gamma_b^+)} \\
\geq & C_1\|\mathrm{Div}_{\Gamma_b^+}(e_3 \times \boldsymbol{H}_0)\|^2_{H_\alpha^{-\frac{1}{2}}(\Gamma_b^+)} - C_3\eta^2\|\mathrm{curl}\,\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\alpha(D_b^+)} \\
& - C_3(1 + \frac{1}{\eta})^2\|\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\alpha(D_b^+)}
\end{aligned}$$

with every $\eta > 0$. Then we have

$$\begin{aligned}
\mathrm{Re}[a_0(\boldsymbol{H}_0, \boldsymbol{H}_0)] \geq & C_0\|\mathrm{curl}\,\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\alpha(D_b^+)} + M\|\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\alpha(D_b^+)} \\
& - C_3\eta^2\|\mathrm{curl}\,\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\alpha(D_b^+)} - C_3(1 + \frac{1}{\eta})^2\|\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\alpha(D_b^+)} \\
= & (C_0 - C_3\eta^2)\|\mathrm{curl}\,\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\alpha(D_b^+)} + (M - C_3(1 + \frac{1}{\eta})^2)\|\boldsymbol{H}_0\|^2_{\boldsymbol{L}^2_\alpha(D_b^+)} \\
\geq & C\|\boldsymbol{H}_0\|^2_X,
\end{aligned}$$

if η is sufficiently small and M is sufficiently large. The proof is thus complete. □

Combining the above Helmholtz decomposition, we split the the sesquilinear form $\boldsymbol{A}(\cdot,\cdot)$ as follows. For $\boldsymbol{H}$, $\boldsymbol{w} \in X$, there exist $\boldsymbol{H}_0$, $\boldsymbol{w}_0 \in \widetilde{X}_0$ and ϕ, $\psi \in S$ such that $\boldsymbol{H} = \boldsymbol{H}_0 + \nabla\phi$ and $\boldsymbol{w} = \boldsymbol{w}_0 + \nabla\psi$. Then we have

$$\boldsymbol{A}((\boldsymbol{u},\boldsymbol{H}),(\boldsymbol{v},\boldsymbol{w})) = \widetilde{\boldsymbol{A}}((\boldsymbol{u},\boldsymbol{H}),(\boldsymbol{v},\boldsymbol{w})) + \boldsymbol{K}_1((\boldsymbol{u},\boldsymbol{H}),(\boldsymbol{v},\boldsymbol{w}))$$

where

$$\begin{aligned}
\widetilde{\boldsymbol{A}}((\boldsymbol{u},\boldsymbol{H}),(\boldsymbol{v},\boldsymbol{w})) :=& \boldsymbol{A}_2(\boldsymbol{u},\boldsymbol{v}) - \boldsymbol{A}_1(\nabla\phi,\nabla\psi) + \frac{\mathrm{i}\kappa}{\overline{b}_2}\langle\gamma_t\nabla\phi,\gamma_T\overline{\boldsymbol{v}}\rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}}\times H_{\mathrm{Curl}}^{-\frac{1}{2}}} \\
&+ \frac{\mathrm{i}\kappa}{b_2}\langle\gamma_t\nabla\overline{\psi},\gamma_T\boldsymbol{u}\rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}}\times H_{\mathrm{Curl}}^{-\frac{1}{2}}} - a_0(\boldsymbol{H}_0,\boldsymbol{w}_0) - \boldsymbol{A}_1(\nabla\phi,\boldsymbol{w}_0), \\
\boldsymbol{K}_1((\boldsymbol{u},\boldsymbol{H}),(\boldsymbol{v},\boldsymbol{w})) :=& -b_0(\boldsymbol{H}_0,\boldsymbol{w}_0) + \frac{\mathrm{i}\kappa}{\overline{b}_2}\langle\gamma_t\boldsymbol{H}_0,\gamma_T\overline{\boldsymbol{v}}\rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}}\times H_{\mathrm{Curl}}^{-\frac{1}{2}}} + \frac{\mathrm{i}\kappa}{b_2}\langle\gamma_t\overline{\boldsymbol{w}}_0,\gamma_T\boldsymbol{u}\rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}}\times H_{\mathrm{Curl}}^{-\frac{1}{2}}}.
\end{aligned}$$

Besides, we define the following sesquilinear form

$$\widetilde{\boldsymbol{K}}((\boldsymbol{u},\boldsymbol{H}),(\boldsymbol{v},\boldsymbol{w})) := \boldsymbol{K}((\boldsymbol{u},\boldsymbol{H}),(\boldsymbol{v},\boldsymbol{w})) + \boldsymbol{K}_1((\boldsymbol{u},\phi,\boldsymbol{H}_0),(\boldsymbol{v},\psi,\boldsymbol{w}_0)).$$

Let $\widetilde{\boldsymbol{A}} : Q\times X \to (Q\times X)'$ and $\widetilde{\boldsymbol{K}} : Q\times X \to (Q\times X)'$ be the linear and bounded operators induced by the corresponding sesquilinear forms.

Theorem 3.7. *If* $\mathrm{Im}(b_1\overline{b}_2) < 0$*, then* $\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{K}}$ *is a Fredholm operator with zero index.*

Proof. By the definition of a Fredholm operator with zero index, we only need to show that $\widetilde{\boldsymbol{A}}$ is an isomorphism and $\widetilde{\boldsymbol{K}}$ is compact from $Q\times X$ to $(Q\times X)'$.

First, we establish the compactness of $\widetilde{\boldsymbol{K}}$. According to the compactness of $\mathcal{T}_2$ and the embedding from $\boldsymbol{H}^1(D)$ to $\boldsymbol{L}^2(D)$, we know that the sesquilinear form $\boldsymbol{K}(\cdot,\cdot)$ is compact. Lemma 3.5 shows that the sesquilinear form $b_0(\cdot,\cdot)$ is compact, which together with compactness of the embedding from $\boldsymbol{H}^1$ to $\boldsymbol{L}^2$ results in the compactness of the sesquilinear form $\boldsymbol{K}_1(\cdot,\cdot)$. So $\widetilde{\boldsymbol{K}}$ is compact.

We further verify that $\widetilde{\boldsymbol{A}}$ is an isomorphism from $Q\times X$ to $(Q\times X)'$. The hypothesis $\mathrm{Im}(b_1\overline{b}_2) < 0$, Lemma 2.2 and Korn's inequality ensure that

$$\mathrm{Re}[\boldsymbol{A}_2(\boldsymbol{u},\boldsymbol{u})] \geq C\|\boldsymbol{u}\|_Q^2 \quad \forall \boldsymbol{u}\in Q. \tag{3.18}$$

If we define

$$\begin{aligned}
\widetilde{\boldsymbol{A}}_1((\boldsymbol{u},\phi),(\boldsymbol{v},\psi)) :=& \boldsymbol{A}_2(\boldsymbol{u},\boldsymbol{v}) - \boldsymbol{A}_1(\nabla\phi,\nabla\psi) + \frac{\mathrm{i}\kappa}{\overline{b}_2}\langle\gamma_t\nabla\phi,\gamma_T\overline{\boldsymbol{v}}\rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}}\times H_{\mathrm{Curl}}^{-\frac{1}{2}}} \\
&+ \frac{\mathrm{i}\kappa}{b_2}\langle\gamma_t\nabla\overline{\psi},\gamma_T\boldsymbol{u}\rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}}\times H_{\mathrm{Curl}}^{-\frac{1}{2}}},
\end{aligned}$$

then (3.18) and Lemma 3.3 yield

$$\mathrm{Re}[\widetilde{\boldsymbol{A}}_1((\boldsymbol{u},\phi),(\boldsymbol{u},\phi))] \geq C(\|\boldsymbol{u}\|_Q^2 + \|\nabla\phi\|_X^2) \quad \forall(\boldsymbol{u},\phi)\in Q\times S. \tag{3.19}$$

Consider the following variational equation: given $f_1 \in Q'$ and $f_2 \in X'$, find $(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{H}}) \in Q\times X$ such that

$$\widetilde{\boldsymbol{A}}((\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{H}}),(\boldsymbol{v},\boldsymbol{w})) = f_1(\boldsymbol{v}) + f_2(\boldsymbol{w}) \quad \forall(\boldsymbol{v},\boldsymbol{w})\in Q\times X.$$

It follows from (3.19) and Lax-Milgram theorem that there exists $(\widetilde{\boldsymbol{u}}, \widetilde{\phi}) \in Q \times S$ such that

$$\widetilde{\boldsymbol{A}}_1((\widetilde{\boldsymbol{u}}, \widetilde{\phi}), (\boldsymbol{v}, \psi)) = f_1(\boldsymbol{v}) + f_2(\nabla\psi) \quad \forall(\boldsymbol{v}, \psi) \in Q \times S$$

and

$$\|\widetilde{\boldsymbol{u}}\|_Q + \|\nabla\widetilde{\phi}\|_X \le C(\|f_1\|_{Q'} + \|f_2\|_{X'}), \tag{3.20}$$

where $C > 0$ is a constant. Moreover, from Lemma 3.6 it holds

$$\operatorname{Re}[a_0(\boldsymbol{H}_0, \boldsymbol{H}_0)] \ge C\|\boldsymbol{H}_0\|_X^2 \quad \forall \boldsymbol{H}_0 \in \widetilde{X}_0, \tag{3.21}$$

which indicates that there exists $\widetilde{\boldsymbol{H}_0} \in \widetilde{X}_0$ such that

$$a_0(\widetilde{\boldsymbol{H}_0}, \boldsymbol{w}_0) = \boldsymbol{A}_1(\nabla\widetilde{\phi}, \boldsymbol{w}_0) + f_2(\boldsymbol{w}_0) \quad \forall \boldsymbol{w}_0 \in \widetilde{X}_0$$

and

$$\|\widetilde{\boldsymbol{H}_0}\|_X \le C(\|\nabla\widetilde{\phi}\|_X + \|f_2\|_{X'}) \le C(\|f_1\|_{Q'} + \|f_2\|_{X'}), \tag{3.22}$$

where $C > 0$ is a constant. Then from (3.20) and (3.22), we derive that $\widetilde{\boldsymbol{A}}$ is an isomorphism from $Q \times X$ to $(Q \times X)'$. Therefore, the statement of the theorem is proved. □

Based on the Theorems 3.2 and 3.7, we get the well-posedness result of the variational problem (3.2), which can be concluded as the following Theorem.

Theorem 3.8. *If the one of conditions in Theorem 3.2 is satisfied, then the problem (3.5) has a unique solution $(\boldsymbol{u}, \boldsymbol{H}) \in Q \times X$ such that*

$$\|\boldsymbol{u}\|_Q + \|\boldsymbol{H}\|_X \le C\left(\|\boldsymbol{f}_1\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} + \|\boldsymbol{f}_2\|_{H_{\mathrm{Div}}^{-\frac{1}{2}}(\Gamma)} + \|\boldsymbol{f}_3\|_{H_{\mathrm{Curl}}^{-\frac{1}{2}}(\Gamma_b^+)}\right),$$

where $C > 0$ is a constant independent with $\boldsymbol{f}_j$ for $j = 1, 2, 3$.

4 PML

For simplicity, we consider the following problem in an homogeneous medium.

$$\begin{cases} \operatorname{curl}\operatorname{curl}\boldsymbol{H} - \kappa^2\boldsymbol{H} = \boldsymbol{0} & \text{in } D_b^+, \\ \triangle^*\boldsymbol{u} + \rho_0\omega^2\boldsymbol{u} = \boldsymbol{0} & \text{in } D_b^-, \\ T\boldsymbol{u} - b_1\nu \times \boldsymbol{H} = b_1\nu \times \boldsymbol{H}^{in} & \text{on } \Gamma, \\ \nu \times \operatorname{curl}\boldsymbol{H} + \dfrac{\mathrm{i}\kappa}{b_2}\nu \times \boldsymbol{u} = \mathrm{i}\kappa\nu \times \boldsymbol{E}^{in} & \text{on } \Gamma, \\ \gamma_T^+\operatorname{curl}\boldsymbol{H} = \mathcal{R}(e_3 \times \boldsymbol{H}) & \text{on } \Gamma_b^+, \\ T\boldsymbol{u} = \mathcal{T}\boldsymbol{u} & \text{on } \Gamma_b^-. \end{cases} \tag{4.1}$$

Based on the above well-posedness results i.e., Theorem 3.8, we can assume the problem (4.1) is well-posed. We will introduce PML for (4.1) and analyze the convergence of PML, which can be easily extended to the case of the inhomogeneous medium.

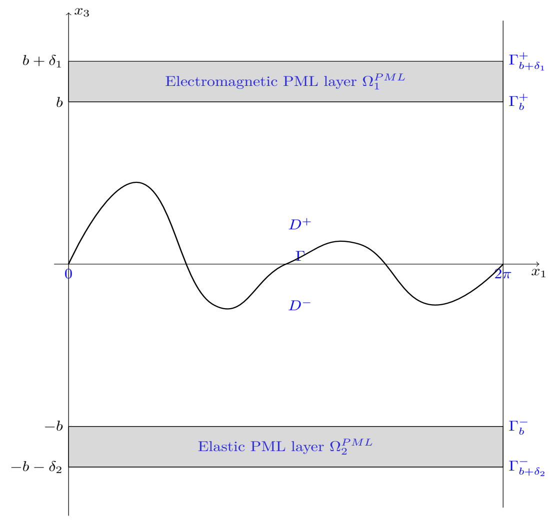

Figure 2: Electromagnetic and elastic PML layers

4.1 PML Equations

In this subsection, the complex coordinate stretching will be adopted to get the PML equations (cf. [20, 22]). Firstly, we introduce the PML regions Ω_1^{PML} and Ω_2^{PML} by

$$\Omega_1^{\mathrm{PML}} := \{x \in \mathbb{R}^3 : 0 < x_1, x_2 < 2\pi, b < x_3 < b + \delta_1\},$$
$$\Omega_2^{\mathrm{PML}} := \{x \in \mathbb{R}^3 : 0 < x_1, x_2 < 2\pi, -b - \delta_2 < x_3 < -b\},$$

which are corresponding to the electromagnetic and elastic domains, respectively. Then define the complex stretching $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)^T$ of $x = (x_1, x_2, x_3)^T$

$$\begin{aligned}
\widetilde{x}_1 &:= x_1, \\
\widetilde{x}_2 &:= x_2, \\
\widetilde{x}_3 &:= \begin{cases} \displaystyle\int_0^{x_3} d_1(t)\,\mathrm{d}t := \int_0^{x_3} [1 + (\zeta_1 + \mathrm{i})\sigma_1(t)]\,\mathrm{d}t & x_3 > b, \\ x_3 & -b \le x_3 \le b, \\ \displaystyle\int_0^{x_3} d_2(t)\,\mathrm{d}t := \int_0^{x_3} [1 + (\zeta_2 + \mathrm{i})\sigma_2(t)]\,\mathrm{d}t & x_3 < -b, \end{cases}
\end{aligned} \tag{4.2}$$

where $\sigma_j(t)$ $(j = 1, 2)$ denote the fictitious absorbing coefficients and the parameters ζ_j $(j = 1, 2)$ are non-negative constants, which are introduced to decrease the effect of the evanescent waves in [37]. In the following text, we always assume the fictitious absorbing coefficients $\sigma_j(t)$ $(j = 1, 2)$ satisfy the following property (cf. [20, (2.8) and (**H2**)]).

(**H1**) For $t \in \mathbb{R}$, $\sigma_j(t) \in C^1(\mathbb{R})$ $(j = 1, 2)$ is non-negative and even function such that $\sigma_j = 0$ for $|t| \le b$, $\sigma_j'(t) \ge 0$ for $|t| \ge b$ and $\sigma_j(t) = \hat{\sigma}_j$ for $t \ge b + \min\{\delta_1, \delta_2\}/2$.

To get the PML equations, we recall some knowledge about the potential theory. The Stratton-Chu formulation (see [39]) implies the scattering solution $\boldsymbol{H}^{sc}$ can be expressed by

$$\boldsymbol{H}^{sc} = SL^{\kappa}(\boldsymbol{\mu}_1) + DL^{\kappa}(\boldsymbol{\lambda}_1) \quad \text{for} \quad x_3 > b$$

where $\boldsymbol{\mu}_1 = e_3 \times \mathcal{R}(\boldsymbol{\lambda}_1)$ and the Maxwell single and double layer potentials are defined by

$$SL^\kappa(\boldsymbol{\mu}_1) := \int_{\Gamma_b^+} \mathbb{G}_\kappa^T(x,y)\boldsymbol{\mu}_1(y)\,\mathrm{d}s(y), \quad \boldsymbol{\mu}_1 \in H_\alpha^{-\frac{1}{2}}(\mathrm{Div}, \Gamma_b^+),$$

$$DL^\kappa(\boldsymbol{\lambda}_1) := \int_{\Gamma_b^+} (\nabla_y \times \mathbb{G}_\kappa)^T(x,y)\boldsymbol{\lambda}_1(y)\,\mathrm{d}s(y), \quad \boldsymbol{\lambda}_1 \in H_\alpha^{-\frac{1}{2}}(\mathrm{Div}, \Gamma_b^+)$$

where $\mathbb{G}_\kappa(x,y)$ is the quasi-periodic fundamental solution to the 3D Maxwell's equations and given by

$$\mathbb{G}_\kappa(x,y) := G_\kappa(x,y)\mathbb{I} + \frac{1}{\kappa^2}\nabla_y\nabla_y G_\kappa(x,y),$$

$$G_\kappa(x,y) := \frac{1}{8\pi^2}\sum_{n\in\mathbb{Z}^2}\frac{1}{\mathrm{i}\eta_n}e^{\mathrm{i}[\alpha_n\cdot(x'-y')+\eta_n|x_3-y_3|]}.$$

Similarly, the elastic scattering field can be expressed by (cf. [36])

$$\boldsymbol{u} = DL(\boldsymbol{\lambda}_2) - SL(\boldsymbol{\mu}_2) \quad \text{for} \quad x_3 < -b$$

where $\boldsymbol{\mu}_2 = \mathcal{T}(\boldsymbol{\lambda}_2)$ and the elastic single and double layer potentials are defined by

$$SL(\boldsymbol{\mu}_2) := \int_{\Gamma_b^-} \Psi(x,y)\boldsymbol{\mu}_2(y)\,\mathrm{d}s(y), \quad \boldsymbol{\mu}_2 \in \boldsymbol{H}_\alpha^{-\frac{1}{2}}(\Gamma_b^-),$$

$$DL(\boldsymbol{\lambda}_2) := \int_{\Gamma_b^-} [T_y\Psi(x,y)]^T\boldsymbol{\lambda}_2(y)\,\mathrm{d}s(y), \quad \boldsymbol{\lambda}_2 \in \boldsymbol{H}_\alpha^{\frac{1}{2}}(\Gamma_b^-)$$

where Ψ is the quasi-periodic fundamental solution to the 3D Navier equation and given by

$$\Psi(x,y) := \frac{1}{\mu^*}G_{\kappa_s}(x,y)\mathbb{I} + \frac{1}{\rho_0\omega^2}\nabla_y\nabla_y(G_{\kappa_s}(x,y) - G_{\kappa_p}(x,y))$$

Based on the complex stretching coordinates (4.2), we will derive the PML equations in Ω_1^{PML} and Ω_2^{PML}. For any $z \in \mathbb{C}\setminus[0,\infty)$, $z^{1/2}$ is the analytic branch of $\sqrt{z}$ such that $\operatorname{Im} z^{1/2} > 0$. Define the complex distance

$$d(\widetilde{x}_3, y_3) = [(\widetilde{x}_3 - y_3)^2]^{1/2}.$$

Accordingly, we can define the modified fundamental solutions

$$G_\kappa(\widetilde{x},y) := \frac{1}{8\pi^2}\sum_{n\in\mathbb{Z}^2}\frac{1}{\mathrm{i}\eta_n}e^{\mathrm{i}[\alpha_n\cdot(\widetilde{x}'-y')+\eta_n d(\widetilde{x}_3,y_3)]},$$

$$\mathbb{G}_\kappa(\widetilde{x},y) := G_\kappa(\widetilde{x},y)\mathbb{I} + \frac{1}{\kappa^2}\nabla_y\nabla_y G_\kappa(\widetilde{x},y),$$

$$\Psi(\widetilde{x},y) := \frac{1}{\mu^*}G_{\kappa_s}(\widetilde{x},y)\mathbb{I} + \frac{1}{\rho_0\omega^2}\nabla_y\nabla_y(G_{\kappa_s}(\widetilde{x},y) - G_{\kappa_p}(\widetilde{x},y)),$$

the modified Maxwell single and double layer potentials

$$\widetilde{SL}^\kappa(\boldsymbol{\mu}_1) := \int_{\Gamma_b^+} \mathbb{G}_\kappa^T(\widetilde{x},y)\boldsymbol{\mu}_1(y)\,\mathrm{d}s(y), \quad \boldsymbol{\mu}_1 \in H_\alpha^{-\frac{1}{2}}(\mathrm{Div}, \Gamma_b^+),$$

$$\widetilde{DL}^\kappa(\boldsymbol{\lambda}_1) := \int_{\Gamma_b^+} (\nabla_y \times \mathbb{G}_\kappa)^T(\widetilde{x},y)\boldsymbol{\lambda}_1(y)\,\mathrm{d}s(y), \quad \boldsymbol{\lambda}_1 \in H_\alpha^{-\frac{1}{2}}(\mathrm{Div}, \Gamma_b^+)$$

and the modified elastic single and double layer potentials

$$\widetilde{SL}(\boldsymbol{\mu}_2) := \int_{\Gamma_b^-} \Psi(\widetilde{x}, y)\boldsymbol{\mu}_2(y)\,\mathrm{d}s(y), \quad \boldsymbol{\mu}_2 \in \boldsymbol{H}_\alpha^{-\frac{1}{2}}(\Gamma_b^-),$$

$$\widetilde{DL}(\boldsymbol{\lambda}_2) := \int_{\Gamma_b^-} [T_y\Psi(\widetilde{x}, y)]^T \boldsymbol{\lambda}_2(y)\,\mathrm{d}s(y), \quad \boldsymbol{\lambda}_2 \in \boldsymbol{H}_\alpha^{\frac{1}{2}}(\Gamma_b^-).$$

For any $\boldsymbol{\lambda}_1 \in H_\alpha^{-1/2}(\mathrm{Div}, \Gamma_b^+)$, we define the electromagnetic PML extension $\mathbb{E}_1(\boldsymbol{\lambda}_1)$

$$\mathbb{E}_1(\boldsymbol{\lambda}_1) = \widetilde{SL}^\kappa(\boldsymbol{\mu}_1) + \widetilde{DL}^\kappa(\boldsymbol{\lambda}_1) \quad \text{for} \quad x_3 > b$$

where $\boldsymbol{\mu}_1 = e_3 \times \mathcal{R}(\boldsymbol{\lambda}_1)$ and for any $\boldsymbol{\lambda}_2 \in \boldsymbol{H}_\alpha^{1/2}(\Gamma_b^-)$, define the elastic PML extension $\mathbb{E}_2(\boldsymbol{\lambda}_2)$

$$\mathbb{E}_2(\boldsymbol{\lambda}_2) = \widetilde{DL}(\boldsymbol{\lambda}_2) - \widetilde{SL}(\mathcal{T}\boldsymbol{\lambda}_2) \quad \text{for} \quad x_3 < -b.$$

Assume $(\boldsymbol{H}, \boldsymbol{u})$ is the solution to (4.1). Let $\widetilde{\boldsymbol{H}} = \mathbb{E}_1(e_3 \times \boldsymbol{H}|_{\Gamma_b^+})$ and $\widetilde{\boldsymbol{u}} = \mathbb{E}_2(\boldsymbol{u}|_{\Gamma_b^-})$. It is easily verified that they satisfy

$$\widetilde{\nabla} \times \widetilde{\nabla} \times \widetilde{\boldsymbol{H}} - \kappa^2 \widetilde{\boldsymbol{H}} = \mathbf{0} \quad \text{in } x_3 > b, \tag{4.3}$$

$$\widetilde{\nabla} \cdot \widetilde{\tau}(\widetilde{\boldsymbol{u}}) + \rho_0\omega^2 \widetilde{\boldsymbol{u}} = \mathbf{0} \quad \text{in } x_3 < -b \tag{4.4}$$

where

$$\widetilde{\tau}(\widetilde{\boldsymbol{u}}) = 2\mu^*\widetilde{\varepsilon}(\widetilde{\boldsymbol{u}}) + \lambda tr(\widetilde{\varepsilon}(\widetilde{\boldsymbol{u}}))\mathbb{I}, \quad \widetilde{\varepsilon}(\widetilde{\boldsymbol{u}}) = \frac{1}{2}(\widetilde{\nabla}\widetilde{\boldsymbol{u}} + (\widetilde{\nabla}\widetilde{\boldsymbol{u}})^T),$$

$\widetilde{\nabla} = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial \widetilde{x}_3)$ denotes the modified gradient operator and $\widetilde{\nabla}\widetilde{\boldsymbol{u}}$ is denoted by $(\partial\widetilde{u}_i/\partial\widetilde{x}_j)_{i,j=1}^3$. It follows from the chain rule that the PML equations (4.3)-(4.4) become

$$\nabla \times A_1 \nabla \times (B_1\widetilde{\boldsymbol{H}}) - \kappa^2 A_1^{-1}(B_1\widetilde{\boldsymbol{H}}) = \mathbf{0} \quad \text{in } x_3 > b,$$

$$\nabla \cdot (\widetilde{\tau}(\widetilde{\boldsymbol{u}})A_2) + \rho_0\omega^2 J\widetilde{\boldsymbol{u}} = \mathbf{0} \quad \text{in } x_3 < -b$$

where

$$\widetilde{\tau}(\widetilde{\boldsymbol{u}}) = 2\mu^*\ \widetilde{\varepsilon}(\widetilde{\boldsymbol{u}}) + \lambda\ tr(\widetilde{\varepsilon}(\widetilde{\boldsymbol{u}}))\mathbb{I}, \quad \widetilde{\varepsilon}(\widetilde{\boldsymbol{u}}) = \frac{1}{2}(\nabla\widetilde{\boldsymbol{u}}B_2^T + B_2(\nabla\widetilde{\boldsymbol{u}})^T),$$

$$A_1 = \frac{1}{d_1(x_3)}\mathrm{diag}(1, 1, d_1^2(x_3)), \quad B_1 = \mathrm{diag}(1, 1, d_1(x_3)),$$

$$A_2 = d_2(x_3)\mathrm{diag}\left(1, 1, \frac{1}{d_2(x_3)}\right), \quad B_2 = \mathrm{diag}\left(1, 1, \frac{1}{d_2(x_3)}\right), \quad J = d_2(x_3).$$

Let $\Omega_1 := \{x \in D^+ : x_3 < b + \delta_1\}$ and $\Omega_2 := \{x \in D^- : x_3 > -b - \delta_2\}$ and define the solution spaces $\hat{X} := \{\boldsymbol{H} \in H_\alpha(\mathrm{curl}, \Omega_1) : e_3 \times \boldsymbol{H} = \mathbf{0} \text{ on } \Gamma_{h+\delta_1}^+\}$ and $\hat{Q} := \{\boldsymbol{u} \in \boldsymbol{H}_\alpha^1(\Omega_2) : \boldsymbol{u} = \mathbf{0} \text{ on } \Gamma_{h+\delta_2}^-\}$. Then the truncated PML problem is to find $(\hat{\boldsymbol{H}}, \hat{\boldsymbol{u}}) \in \hat{X} \times \hat{Q}$ such that

$$\begin{cases} \nabla \times A_1 \nabla \times \hat{\boldsymbol{H}} - \kappa^2 A_1^{-1}\hat{\boldsymbol{H}} = \mathbf{0} & \text{in } \Omega_1, \\ \nabla \cdot (\widetilde{\tau}(\hat{\boldsymbol{u}})A_2) + \rho_0\omega^2 J\hat{\boldsymbol{u}} = \mathbf{0} & \text{in } \Omega_2, \\ T\hat{\boldsymbol{u}} - b_1\nu \times \hat{\boldsymbol{H}} = b_1\nu \times \boldsymbol{H}^{in} & \text{on } \Gamma, \\ \nu \times \hat{\boldsymbol{u}} + b_2/(\mathrm{i}\kappa)\nu \times \mathrm{curl}\,\hat{\boldsymbol{H}} = b_2\nu \times \boldsymbol{E}^{in} & \text{on } \Gamma, \\ e_3 \times \hat{\boldsymbol{H}} = \mathbf{0} & \text{on } \Gamma_{b+\delta_1}^+, \\ \hat{\boldsymbol{u}} = \mathbf{0} & \text{on } \Gamma_{b+\delta_2}^-. \end{cases} \tag{4.5}$$

The variational formula of (4.5) is to find $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{H}}) \in \hat{Q} \times \hat{X}$ such that

$$\mathcal{A}_{\mathrm{PML}}((\hat{\boldsymbol{u}}, \hat{\boldsymbol{H}}), (\hat{\boldsymbol{v}}, \hat{\boldsymbol{w}})) = \hat{\mathcal{F}}((\hat{\boldsymbol{v}}, \hat{\boldsymbol{w}})) \quad \forall (\hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}) \in \hat{Q} \times \hat{X},$$

where

$$\begin{aligned}\mathcal{A}_{\mathrm{PML}}((\hat{\boldsymbol{u}}, \hat{\boldsymbol{H}}), (\hat{\boldsymbol{v}}, \hat{\boldsymbol{w}})) :=& \frac{-\mathrm{i}\kappa}{b_1 \bar{b}_2} \int_{\Omega_2^{\mathrm{PML}}} (\widetilde{\tau}(\hat{\boldsymbol{u}}) A_2 : \nabla \overline{\hat{\boldsymbol{v}}} - \rho\omega^2 J \hat{\boldsymbol{u}} \cdot \overline{\hat{\boldsymbol{v}}}) \,\mathrm{d}x \\ &- \Big(\int_{\Omega_1^{\mathrm{PML}}} (A_1 \mathrm{curl}\, \hat{\boldsymbol{H}} \cdot \mathrm{curl}\, \overline{\hat{\boldsymbol{w}}} - \kappa^2 A_1^{-1} \hat{\boldsymbol{H}} \cdot \overline{\hat{\boldsymbol{w}}}) \,\mathrm{d}x \Big) \\ &+ \frac{\mathrm{i}\kappa}{\bar{b}_2} \langle \gamma_t \hat{\boldsymbol{H}}, \gamma_T \overline{\hat{\boldsymbol{v}}} \rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}} \times H_{\mathrm{Curl}}^{-\frac{1}{2}}} + \frac{\mathrm{i}\kappa}{b_2} \langle \gamma_t \overline{\hat{\boldsymbol{w}}}, \gamma_T \hat{\boldsymbol{u}} \rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}} \times H_{\mathrm{Curl}}^{-\frac{1}{2}}}, \\ \hat{\mathcal{F}}((\hat{\boldsymbol{v}}, \hat{\boldsymbol{w}})) :=& \frac{-\mathrm{i}\kappa}{\bar{b}_2} \langle \gamma_t \boldsymbol{H}^{in}, \gamma_T \overline{\boldsymbol{v}} \rangle_{\boldsymbol{H}^{-\frac{1}{2}} \times \boldsymbol{H}^{\frac{1}{2}}} - \mathrm{i}\kappa \langle \gamma_t \boldsymbol{E}^{in}, \gamma_T \overline{\boldsymbol{w}} \rangle_{H_{\mathrm{Div}}^{-\frac{1}{2}} \times H_{\mathrm{Curl}}^{-\frac{1}{2}}}\end{aligned}$$

4.2 Convergence Analysis of PML

Before analyzing the convergence of PML, we consider the PML equations in the layers Ω_1^{PML}

$$\begin{cases} \nabla \times A_1 \nabla \times \boldsymbol{w} - \kappa^2 A_1^{-1} \boldsymbol{w} = \boldsymbol{0} & \text{in } \Omega_1^{\mathrm{PML}}, \\ e_3 \times \boldsymbol{w} = \boldsymbol{0} & \text{on } \Gamma_b^+, \\ e_3 \times \boldsymbol{w} = \boldsymbol{q}_1 & \text{on } \Gamma_{b+\delta_1}^+ \end{cases} \tag{4.6}$$

and Ω_2^{PML}

$$\begin{cases} \nabla \cdot (\widetilde{\tau}(\boldsymbol{v}) A_2) + \rho_0 \omega^2 J \boldsymbol{v} = \boldsymbol{0} & \text{in } \Omega_2^{\mathrm{PML}}, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \Gamma_b^-, \\ \boldsymbol{v} = \boldsymbol{q}_2 & \text{on } \Gamma_{b+\delta_2}^- \end{cases} \tag{4.7}$$

where $\boldsymbol{q}_1 \in H_\alpha^{-1/2}(\mathrm{Div}, \Gamma_{b+\delta_1}^+)$ and $\boldsymbol{q}_2 \in \boldsymbol{H}_\alpha^{1/2}(\Gamma_{b+\delta_2}^-)$.

The variational formula of (4.6) is to find $\boldsymbol{w} \in H_\alpha(\mathrm{curl}, \Omega_1^{\mathrm{PML}})$ such that $e_3 \times \boldsymbol{w} = \boldsymbol{0}$ on Γ_b^+, $e_3 \times \boldsymbol{w} = \boldsymbol{q}_1$ on $\Gamma_{b+\delta_1}^+$ and

$$c_1(\boldsymbol{w}, \boldsymbol{B}) = 0,$$

for any $\boldsymbol{B} \in H_{0,\alpha}(\mathrm{curl}, \Omega_1^{\mathrm{PML}}) := \{\boldsymbol{w} \in H_\alpha(\mathrm{curl}, \Omega_1^{\mathrm{PML}}) : e_3 \times \boldsymbol{w} = \boldsymbol{0} \text{ on } \Gamma_b^+ \text{ and } e_3 \times \boldsymbol{w} = \boldsymbol{0} \text{ on } \Gamma_{h+\delta_1}^+\}$, where

$$c_1(\boldsymbol{w}, \boldsymbol{B}) = \int_{\Omega_1^{\mathrm{PML}}} (A_1 \mathrm{curl}\, \boldsymbol{w} \cdot \mathrm{curl}\, \overline{\boldsymbol{B}} - \kappa^2 A_1^{-1} \boldsymbol{w} \cdot \overline{\boldsymbol{B}}) \,\mathrm{d}x.$$

Obviously,

$$\frac{1}{(1 + \zeta_1 \hat{\sigma}_1)^2 + \hat{\sigma}_1^2} |\xi|^2 \le \mathrm{Re}(A_1 \xi \cdot \overline{\xi}) \le (1 + \zeta_1 \hat{\sigma}_1) |\xi|^2$$

holds. If $\omega = \mathrm{i}$, the positive definiteness of A_1 and the Lax-Milgram theorem shows the problem (4.6) has a unique solution. This, combined with the Helmholtz decomposition and spectral theory of the compact operators imply the problem (4.6) has a unique solution for all real ω except a possible discrete set.

The variational formula of (4.7) is to find $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega_2^{\mathrm{PML}})$ such that $\boldsymbol{v} = \boldsymbol{0}$ on Γ_b^-, $\boldsymbol{v} = \boldsymbol{q}_2$ on $\Gamma_{b+\delta_2}^-$ and

$$c_2(\boldsymbol{v}, \phi) = 0,$$

for any $\phi \in \boldsymbol{H}_{0,\alpha}^1(\Omega_2^{\mathrm{PML}}) := \{\boldsymbol{v} \in \boldsymbol{H}_{\alpha}^1(\Omega_2^{\mathrm{PML}}) : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_b^- \text{ and } \Gamma_{b+\delta_2}^-\}$, where

$$c_2(\boldsymbol{v}, \phi) = \int_{\Omega_2^{\mathrm{PML}}} (\widetilde{\tau}(\boldsymbol{v}) A_2 : \nabla\overline{\phi} - \rho_0 \omega^2 J \boldsymbol{v} \cdot \overline{\phi}) \,\mathrm{d}x.$$

Note that for any $\boldsymbol{v} \in \boldsymbol{H}_{0,\alpha}^1(\Omega_2^{\mathrm{PML}})$

$$\begin{aligned}
&\mathrm{Re}\, c_2(\boldsymbol{v}, \boldsymbol{v}) \\
=& \int_{\Omega_2^{\mathrm{PML}}} \Big\{ \mu^* \Big[(1+\zeta_2\sigma_2)\big(2|\partial_1 v_1|^2 + |\partial_1 v_2|^2 + |\partial_1 v_3|^2 + |\partial_2 v_1|^2 + 2|\partial_2 v_2|^2 + |\partial_2 v_3|^2 \\
&+ 2\mathrm{Re}(\partial_2 v_1 \partial_1 \overline{v}_2)\big) + \Big(|\partial_3 v_1|^2 + 2|\partial_3 v_2|^2 + \frac{1+\zeta_2\sigma_2}{|d_2|^2}|\partial_3 v_3|^2 + 2\mathrm{Re}(\partial_1 v_3 \partial_3 \overline{v}_1) + 2\mathrm{Re}(\partial_2 v_3 \partial_3 \overline{v}_2)\Big)\Big] \\
&+ \lambda \Big[(1+\zeta_2\sigma_2)\big(|\partial_1 v_1|^2 + |\partial_2 v_2|^2 + 2\mathrm{Re}(\partial_1 v_1 \partial_2 \overline{v}_2)\big) + \Big(2\mathrm{Re}(\partial_1 v_1 \partial_3 \overline{v}_3) + 2\mathrm{Re}(\partial_3 v_3 \partial_2 \overline{v}_2) \\
&+ \frac{1+\zeta_2\sigma_2}{|d_2|^2}|\partial_3 v_3|^2\Big)\Big] - \rho_0\omega^2(1+\zeta_2\sigma_2)|\boldsymbol{v}|^2 \Big\} \,\mathrm{d}x.
\end{aligned}$$

The denseness of $\boldsymbol{C}_{0,\alpha}^{\infty}(\Omega_2^{\mathrm{PML}}) := \{\boldsymbol{v} \in \boldsymbol{C}_{\alpha}^{\infty}(\Omega_2^{\mathrm{PML}}) : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_b^- \text{ and } \Gamma_{b+\delta_2}^-\}$ in $\boldsymbol{H}_{0,\alpha}^1(\Omega_2^{\mathrm{PML}})$ shows $\mathrm{Re}\, c_2(\cdot,\cdot)$ satisfies Gárding inequality (cf. [20, Lemma 3.3]), if $\zeta_2 \geq \sqrt{(\lambda+2\mu^*)/\mu^*}$. In particular, $\mathrm{Re}\, c_2(\cdot,\cdot)$ is coercive if $\omega = \mathrm{i}$. Similarly, from this with spectral theory of compact operators it follows that the problem (4.7) has a unique solution for all real ω except a possible discrete set. For simplicity, we make the following assumption, which is similar with that in [17, 18].

(**H2**) The problem (4.6) and (4.7) have a unique solution.

Define

$$\overline{\sigma}_1 := \int_0^{b+\delta_1} \sigma_1(t) \,\mathrm{d}t, \quad \overline{\sigma}_2 := \int_0^{b+\delta_2} \sigma_2(t) \,\mathrm{d}t,$$

where $\overline{\sigma}_1$ and $\overline{\sigma}_2$ are two constants. Let

$$\Delta_1 = \min_{n\in\mathbb{Z}^2}\{|\eta_n|\}, \quad \Delta_2 = \min_{n\in\mathbb{Z}^2}\{|\beta_n|, |\gamma_n|\}.$$

Besides, we should make assumptions on ζ_1 and ζ_2, which can strengthen the ellipticity of PML operators [20] and the decay of the modified fundamental solutions such that the exponential convergence can be available for the PML as we will see below.

(**H3**) $\zeta_1 \geq 1$ and $\zeta_2 \geq \sqrt{(\lambda+2\mu^*)/\mu^*}$.

Theorem 4.1. *For any $x \in \Gamma^+_{b+\delta_1}$ and $y \in D^+_b$, it follows under the assumption* (**H3**) *that*

$$|G_\kappa(\widetilde{x}, y)| \le Cc_1(\delta_1)e^{-\Delta_1\overline{\sigma}_1}, \tag{4.8}$$

$$\Big|\frac{\partial G_\kappa(\widetilde{x}, y)}{\partial y_j}\Big| \le Cc_2(\delta_1)e^{-\Delta_1\overline{\sigma}_1}, \qquad \text{for} \quad j = 1, 2, 3, \tag{4.9}$$

$$\Big|\frac{\partial^2 G_\kappa(\widetilde{x}, y)}{\partial y_j \partial y_k}\Big| \le Cc_3(\delta_1)e^{-\Delta_1\overline{\sigma}_1}, \qquad \text{for} \quad j, k = 1, 2, 3, \tag{4.10}$$

$$\Big|\frac{\partial^3 G_\kappa(\widetilde{x}, y)}{\partial y_j \partial y_k \partial y_l}\Big| \le Cc_4(\delta_1)e^{-\Delta_1\overline{\sigma}_1}, \qquad \text{for} \quad j, k, l = 1, 2, 3, \tag{4.11}$$

$$\Big|\frac{\partial^3 G_\kappa(\widetilde{x}, y)}{\partial x_j \partial y_k \partial y_l}\Big| \le Cc_4(\delta_1)e^{-\Delta_1\overline{\sigma}_1}, \qquad \text{for} \quad j, k, l = 1, 2, 3, \tag{4.12}$$

$$\Big|\frac{\partial^4 G_\kappa(\widetilde{x}, y)}{\partial x_j \partial y_k \partial y_l \partial y_m}\Big| \le Cc_5(\delta_1)e^{-\Delta_1\overline{\sigma}_1}, \qquad \text{for} \quad j, k, l, m = 1, 2, 3, \tag{4.13}$$

where

$$c_j(\delta_1) := \sum_{|\alpha_n| < \kappa} \frac{1}{|\eta_n|} + \sum_{|\alpha_n| > \kappa} \frac{|\alpha_n|^{j-1}}{|\eta_n|} e^{-|\eta_n|(\delta_1 + (\zeta_1 - 1)\overline{\sigma}_1)}, \quad \text{for} \quad j = 1, 2, 3, 4, 5$$

and C may depend on κ and $\hat{\sigma}_1$. For $x \in \Gamma^-_{b+\delta_2}$ and $y \in D^-_b$,

$$|G_{\kappa_p}(\widetilde{x}, y)| + |G_{\kappa_s}(\widetilde{x}, y)| \le CC_1(\delta_2)e^{-\Delta_2\overline{\sigma}_2}, \tag{4.14}$$

$$\Big|\frac{\partial G_{\kappa_p}(\widetilde{x}, y)}{\partial y_j}\Big| + \Big|\frac{\partial G_{\kappa_s}(\widetilde{x}, y)}{\partial y_j}\Big| \le CC_2(\delta_2)e^{-\Delta_2\overline{\sigma}_2}, \qquad \text{for} \quad j = 1, 2, 3, \tag{4.15}$$

$$\Big|\frac{\partial^2 G_{\kappa_p}(\widetilde{x}, y)}{\partial y_j \partial y_k}\Big| + \Big|\frac{\partial^2 G_{\kappa_s}(\widetilde{x}, y)}{\partial y_j \partial y_k}\Big| \le CC_3(\delta_2)e^{-\Delta_2\overline{\sigma}_2}, \qquad \text{for} \quad j, k = 1, 2, 3, \tag{4.16}$$

$$\Big|\frac{\partial^3 G_{\kappa_p}(\widetilde{x}, y)}{\partial y_j \partial y_k \partial y_l}\Big| + \Big|\frac{\partial^3 G_{\kappa_s}(\widetilde{x}, y)}{\partial y_j \partial y_k \partial y_l}\Big| \le CC_4(\delta_2)e^{-\Delta_2\overline{\sigma}_2}, \qquad \text{for} \quad j, k, l = 1, 2, 3, \tag{4.17}$$

$$\Big|\frac{\partial^3 G_{\kappa_p}(\widetilde{x}, y)}{\partial x_j \partial y_k \partial y_l}\Big| + \Big|\frac{\partial^3 G_{\kappa_s}(\widetilde{x}, y)}{\partial x_j \partial y_k \partial y_l}\Big| \le CC_4(\delta_2)e^{-\Delta_2\overline{\sigma}_2}, \qquad \text{for} \quad j, k, l = 1, 2, 3, \tag{4.18}$$

$$\Big|\frac{\partial^4 G_{\kappa_p}(\widetilde{x}, y)}{\partial x_j \partial y_k \partial y_l \partial y_m}\Big| + \Big|\frac{\partial^4 G_{\kappa_s}(\widetilde{x}, y)}{\partial x_j \partial y_k \partial y_l \partial y_m}\Big| \le CC_5(\delta_2)e^{-\Delta_2\overline{\sigma}_2}, \quad \text{for} \quad j, k, l, m = 1, 2, 3, \tag{4.19}$$

where

$$\begin{aligned} C_j(\delta_2) := \max\Big\{ &\sum_{|\alpha_n| < \kappa_p} \frac{1}{|\beta_n|} + \sum_{|\alpha_n| > \kappa_p} \frac{|\alpha_n|^{j-1}}{|\beta_n|} e^{-|\beta_n|(\delta_2 + (\zeta_2 - 1)\overline{\sigma}_2)}, \\ &\sum_{|\alpha_n| < \kappa_s} \frac{1}{|\gamma_n|} + \sum_{|\alpha_n| > \kappa_s} \frac{|\alpha_n|^{j-1}}{|\gamma_n|} e^{-|\gamma_n|(\delta_2 + (\zeta_2 - 1)\overline{\sigma}_2)} \Big\}, \quad \text{for} \quad j = 1, 2, 3, 4, 5 \end{aligned}$$

and C may depend on κ_p, κ_s and $\hat{\sigma}_2$.

Proof. For any $x \in \Gamma^+_{b+\delta_1}$ and $y \in D^+_b$, $x_3 > y_3$. This yields $d(\widetilde{x}_3, y_3) = \widetilde{x}_3 - y_3$, which together with (**H3**) shows

$$
\begin{aligned}
|G_\kappa(\widetilde{x}, y)| &= \Big| \frac{1}{8\pi^2} \sum_{n\in\mathbb{Z}^2} \frac{1}{\mathrm{i}\eta_n} e^{\mathrm{i}[\alpha_n\cdot(x'-y')+\eta_n(\widetilde{x}_3-y_3)}\Big| \\
&\le \frac{1}{8\pi^2}\Big(\sum_{|\alpha_n|<\kappa} \frac{1}{|\eta_n|} e^{-|\eta_n|\overline{\sigma}_1} + \sum_{|\alpha_n|>\kappa} \frac{1}{|\eta_n|} e^{-|\eta_n|(\delta_1+\zeta_1\overline{\sigma}_1)}|\Big) \\
&\le \frac{1}{8\pi^2}\Big(\sum_{|\alpha_n|<\kappa} \frac{1}{|\eta_n|} + \sum_{|\alpha_n|>\kappa} \frac{1}{|\eta_n|} e^{-|\eta_n|(\delta_1+(\zeta_1-1)\overline{\sigma}_1)}|\Big) e^{-\Delta_1\overline{\sigma}_1} \\
&\le Cc_1(\delta_1) e^{-\Delta_1\overline{\sigma}_1}.
\end{aligned}
$$

Simple calculations show that

$$
\begin{aligned}
\frac{\partial G_\kappa(\widetilde{x}, y)}{\partial y_j} &= \frac{1}{8\pi^2} \sum_{n\in\mathbb{Z}^2} \frac{-\mathrm{i}\alpha_n^{(j)}}{\mathrm{i}\eta_n} e^{\mathrm{i}[\alpha_n\cdot(x'-y')+\eta_n(\widetilde{x}_3-y_3)]} \quad \text{for} \quad j = 1, 2, \\
\frac{\partial G_\kappa(\widetilde{x}, y)}{\partial y_j} &= \frac{1}{8\pi^2} \sum_{n\in\mathbb{Z}^2} \frac{-\mathrm{i}\eta_n}{\mathrm{i}\eta_n} e^{\mathrm{i}[\alpha_n\cdot(x'-y')+\eta_n(\widetilde{x}_3-y_3)]} \quad \text{for} \quad j = 3,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial G_\kappa(\widetilde{x}, y)}{\partial x_j} &= \frac{1}{8\pi^2} \sum_{n\in\mathbb{Z}^2} \frac{\mathrm{i}\alpha_n^{(j)}}{\mathrm{i}\eta_n} e^{\mathrm{i}[\alpha_n\cdot(x'-y')+\eta_n(\widetilde{x}_3-y_3)]} \quad \text{for} \quad j = 1, 2, \\
\frac{\partial G_\kappa(\widetilde{x}, y)}{\partial x_j} &= \frac{1}{8\pi^2} \sum_{n\in\mathbb{Z}^2} \frac{\mathrm{i}\eta_n d_1(x_3)}{\mathrm{i}\eta_n} e^{\mathrm{i}[\alpha_n\cdot(x'-y')+\eta_n(\widetilde{x}_3-y_3)]} \quad \text{for} \quad j = 3.
\end{aligned}
$$

On the other hand, the facts that $|\alpha_n^j| \le |\alpha_n| < \kappa$ for $|\alpha_n| < \kappa$ and $\eta_n \le |\alpha_n|$ for $|\alpha_n| > \kappa$ imply the estimates

$$
\begin{aligned}
\Big|\frac{\partial G_\kappa(\widetilde{x}, y)}{\partial y_j}\Big| &\le C\Big(\sum_{|\alpha_n|<\kappa} \frac{1}{|\eta_n|} e^{-|\eta_n|\overline{\sigma}_1} + \sum_{|\alpha_n|>\kappa} \frac{|\alpha_n|}{|\eta_n|} e^{-|\eta_n|(\delta_1+\zeta_1\overline{\sigma}_1)}\Big) \\
&\le C\Big(\sum_{|\alpha_n|<\kappa} \frac{1}{|\eta_n|} + \sum_{|\alpha_n|>\kappa} \frac{|\alpha_n|}{|\eta_n|} e^{-|\eta_n|(\delta_1+(\zeta_1-1)\overline{\sigma}_1)}\Big) e^{-\Delta_1\overline{\sigma}_1} \\
&\le Cc_2(\delta_1) e^{-\Delta_1\overline{\sigma}_1}
\end{aligned}
$$

and

$$
\begin{aligned}
\Big|\frac{\partial G_\kappa(\widetilde{x}, y)}{\partial x_j}\Big| &\le C\Big(\sum_{|\alpha_n|<\kappa} \frac{1}{|\eta_n|} e^{-|\eta_n|\overline{\sigma}_1} + \sum_{|\alpha_n|>\kappa} \frac{|\alpha_n|}{|\eta_n|} e^{-|\eta_n|(\delta_1+\zeta_1\overline{\sigma}_1)}\Big) \\
&\le C\Big(\sum_{|\alpha_n|<\kappa} \frac{1}{|\eta_n|} + \sum_{|\alpha_n|>\kappa} \frac{|\alpha_n|}{|\eta_n|} e^{-|\eta_n|(\delta_1+(\zeta_1-1)\overline{\sigma}_1)}\Big) e^{-\Delta_1\overline{\sigma}_1} \\
&\le Cc_2(\delta_1) e^{-\Delta_1\overline{\sigma}_1}.
\end{aligned}
$$

Similar arguments lead to (4.10)-(4.19). The proof is thus complete. $\square$

Theorem 4.2. *If the assumptions* (**H1**)-(**H3**) *hold, then*

$$\|\widetilde{SL}(\boldsymbol{\mu}_1)\|_{\boldsymbol{H}_\alpha^{\frac{1}{2}}(\Gamma^-_{b+\delta_2})} \le CC_5(\delta_2)e^{-\Delta_2\overline{\sigma}_2}\|\boldsymbol{\mu}_1\|_{\boldsymbol{H}_\alpha^{-\frac{1}{2}}(\Gamma^-_b)}, \tag{4.20}$$

$$\|\widetilde{DL}(\boldsymbol{\lambda}_1)\|_{\boldsymbol{H}_\alpha^{\frac{1}{2}}(\Gamma^-_{b+\delta_2})} \le CC_5(\delta_2)e^{-\Delta_2\overline{\sigma}_2}\|\boldsymbol{\lambda}_1\|_{\boldsymbol{H}_\alpha^{\frac{1}{2}}(\Gamma^-_b)}, \tag{4.21}$$

$$\|e_3 \times B_1\widetilde{SL}^\kappa(\boldsymbol{\mu}_2)\|_{H_\alpha^{-\frac{1}{2}}(\mathrm{Div},\Gamma^+_{b+\delta_1})} \le Cc_5(\delta_1)e^{-\Delta_1\overline{\sigma}_1}\|\boldsymbol{\mu}_2\|_{H_\alpha^{-\frac{1}{2}}(\mathrm{Div},\Gamma^+_b)}, \tag{4.22}$$

$$\|e_3 \times B_1\widetilde{DL}^\kappa(\boldsymbol{\lambda}_2)\|_{H_\alpha^{-\frac{1}{2}}(\mathrm{Div},\Gamma^+_{b+\delta_1})} \le Cc_4(\delta_1)e^{-\Delta_1\overline{\sigma}_1}\|\boldsymbol{\lambda}_2\|_{H_\alpha^{-\frac{1}{2}}(\mathrm{Div},\Gamma^+_b)}. \tag{4.23}$$

Proof. First we recall some Sobolev embedding results:

$$\begin{aligned}
\|f\|_{H_\alpha^{-1/2}(\Gamma_b^+)} &\le C\|f\|_{L_\alpha^\infty(\Gamma_b^+)} \quad \text{for } f \in L_\alpha^\infty(\Gamma_b^+),\\
\|f\|_{H_\alpha^{1/2}(\Gamma_b^+)} &\le C(\|f\|_{L_\alpha^\infty(\Gamma_b^+)} + \|\nabla f\|_{L_\alpha^\infty(\Gamma_b^+)}) \quad \text{for } f \in W_\alpha^{1,\infty}(\Gamma_b^+),
\end{aligned}$$

where C is a positive constant. Let $\boldsymbol{v}(x) = \widetilde{SL}(\boldsymbol{\mu}_1)(x)$ for $\boldsymbol{\mu}_1 \in \boldsymbol{H}_\alpha^{-1/2}(\Gamma_b^-)$. So the $\boldsymbol{H}_\alpha^{\frac{1}{2}}(\Gamma^-_{b+\delta_2})$ norm of $\boldsymbol{v}$ can be bounded by the $\boldsymbol{L}_\alpha^\infty(\Gamma^-_{b+\delta_2})$-norm of $\boldsymbol{v}$ and $\nabla_{x'}\boldsymbol{v}$. This, combined with the Theorem 4.1 gives

$$\begin{aligned}
\|\boldsymbol{v}\|_{\boldsymbol{L}_\alpha^\infty(\Gamma^-_{b+\delta_2})} &\le C \max_{x\in\Gamma^-_{b+\delta_2}} \|\Psi(\widetilde{x},\cdot)\|_{\boldsymbol{H}_{-\alpha}^{\frac{1}{2}}(\Gamma_b^-)}\|\boldsymbol{\mu}_1\|_{\boldsymbol{H}_\alpha^{-\frac{1}{2}}(\Gamma_b^-)}\\
&\le C \max_{x\in\Gamma^-_{b+\delta_2}} \left(\|\Psi(\widetilde{x},\cdot)\|_{\boldsymbol{L}_{-\alpha}^\infty(\Gamma_b^-)} + \|\nabla_{y'}\Psi(\widetilde{x},\cdot)\|_{\boldsymbol{L}_{-\alpha}^\infty(\Gamma_b^-)}\right)\|\boldsymbol{\mu}_1\|_{\boldsymbol{H}_\alpha^{-\frac{1}{2}}(\Gamma_b^-)}\\
&\le CC_4(\delta_2)e^{-\Delta_2\overline{\sigma}_2}\|\boldsymbol{\mu}_1\|_{\boldsymbol{H}_\alpha^{-\frac{1}{2}}(\Gamma_b^-)},
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla_{x'}\boldsymbol{v}\|_{\boldsymbol{L}_\alpha^\infty(\Gamma^-_{b+\delta_2})} &\le C \max_{x\in\Gamma^-_{b+\delta_2}} \|\nabla_x\Psi(\widetilde{x},\cdot)\|_{\boldsymbol{H}_{-\alpha}^{\frac{1}{2}}(\Gamma_b^-)}\|\boldsymbol{\mu}_1\|_{\boldsymbol{H}_\alpha^{-\frac{1}{2}}(\Gamma_b^-)}\\
&\le C \max_{x\in\Gamma^-_{b+\delta_2}} \left(\|\nabla_x\Psi(\widetilde{x},\cdot)\|_{\boldsymbol{L}_{-\alpha}^\infty(\Gamma_b^-)} + \|\nabla_{y'}\nabla_x\Psi(\widetilde{x},\cdot)\|_{\boldsymbol{L}_{-\alpha}^\infty(\Gamma_b^-)}\right)\|\boldsymbol{\mu}_1\|_{\boldsymbol{H}_\alpha^{-\frac{1}{2}}(\Gamma_b^-)}\\
&\le CC_5(\delta_2)e^{-\Delta_2\overline{\sigma}_2}\|\boldsymbol{\mu}_1\|_{\boldsymbol{H}_\alpha^{-\frac{1}{2}}(\Gamma_b^-)},
\end{aligned}$$

which yields the estimate (4.20).

Given $\boldsymbol{\mu}_2 \in H_\alpha^{-1/2}(\mathrm{Div},\Gamma_b^+)$, let $\boldsymbol{H}(x) = \widetilde{SL}^\kappa(\boldsymbol{\mu}_2)(x)$. By the definition of $H_\alpha^{-1/2}(\mathrm{Div},\Gamma^+_{b+\delta_1})$, we have

$$\begin{aligned}
&\|e_3 \times B_1\boldsymbol{H}\|_{H_\alpha^{-\frac{1}{2}}(\mathrm{Div},\Gamma^+_{b+\delta_1})}\\
\le& C\left(\|e_3 \times B_1\boldsymbol{H}\|_{\boldsymbol{H}_\alpha^{-\frac{1}{2}}(\Gamma^+_{b+\delta_1})} + \|e_3\cdot\nabla\times B_1\boldsymbol{H}\|_{H_\alpha^{-\frac{1}{2}}(\Gamma^+_{b+\delta_2})}\right)\\
\le& C\left(\|e_3 \times B_1\boldsymbol{H}\|_{\boldsymbol{L}_\alpha^\infty(\Gamma^+_{b+\delta_1})} + \|\nabla\times B_1\boldsymbol{H}\|_{\boldsymbol{L}_\alpha^\infty(\Gamma^+_{b+\delta_1})}\right).
\end{aligned}$$

Notice $B_1 = \mathrm{diag}(1,1,1+(\zeta_1+\mathrm{i})\hat{\sigma}_1)$ for $x_3 = b+\delta_1$. Then

$$
\begin{aligned}
&\|e_3 \times B_1 \boldsymbol{H}\|_{H_\alpha^{-\frac{1}{2}}(\mathrm{Div},\Gamma_{b+\delta_1}^+)} \\
\le & C\left(\|\boldsymbol{H}\|_{\boldsymbol{L}_\alpha^\infty(\Gamma_{b+\delta_1}^+)} + \|\nabla \boldsymbol{H}\|_{\boldsymbol{L}_\alpha^\infty(\Gamma_{b+\delta_1}^+)}\right) \\
\le & C \max_{x\in\Gamma_{b+\delta_1}^+} \Big(\|G_\kappa(\widetilde{x},\cdot)\|_{\boldsymbol{H}_{-\alpha}^{\frac{1}{2}}(\Gamma_b^+)} + \|\nabla_x G_\kappa(\widetilde{x},\cdot)\|_{\boldsymbol{H}_{-\alpha}^{\frac{1}{2}}(\Gamma_b^+)} \\
&\qquad + \|\nabla_y \nabla_y G_\kappa(\widetilde{x},\cdot)\|_{\boldsymbol{H}_{-\alpha}^{\frac{1}{2}}(\Gamma_b^+)} + \|\nabla_x\nabla_y \nabla_y G_\kappa(\widetilde{x},\cdot)\|_{\boldsymbol{H}_{-\alpha}^{\frac{1}{2}}(\Gamma_b^+)}\Big)\|\boldsymbol{\mu}_2\|_{\boldsymbol{H}_\alpha^{-\frac{1}{2}}(\Gamma_b^+)}.
\end{aligned}
$$

Besides, for $x \in \Gamma_{b+\delta_1}^+$ and $y \in \Gamma_b^+$, we have

$$
\begin{aligned}
\|G_\kappa(\widetilde{x},\cdot)\|_{\boldsymbol{H}_{-\alpha}^{\frac{1}{2}}(\Gamma_b^+)} &\le C\left(\|G_\kappa(\widetilde{x},\cdot)\|_{\boldsymbol{L}_{-\alpha}^\infty(\Gamma_b^+)} + \|\nabla_{y'} G_\kappa(\widetilde{x},\cdot)\|_{\boldsymbol{L}_{-\alpha}^\infty(\Gamma_b^+)}\right) \\
&\le C c_2(\delta_1) e^{-\Delta_1 \overline{\sigma}_1}, \qquad (4.24) \\
\|\nabla_x G_\kappa(\widetilde{x},\cdot)\|_{\boldsymbol{H}_{-\alpha}^{\frac{1}{2}}(\Gamma_b^+)} &\le C\left(\|\nabla_x G_\kappa(\widetilde{x},\cdot)\|_{\boldsymbol{L}_{-\alpha}^\infty(\Gamma_b^+)} + \|\nabla_{y'}\nabla_x G_\kappa(\widetilde{x},\cdot)\|_{\boldsymbol{L}_{-\alpha}^\infty(\Gamma_b^+)}\right) \\
&\le C c_3(\delta_1) e^{-\Delta_1 \overline{\sigma}_1}, \qquad (4.25) \\
\|\nabla_y\nabla_y G_\kappa(\widetilde{x},\cdot)\|_{\boldsymbol{H}_{-\alpha}^{\frac{1}{2}}(\Gamma_b^+)} &\le C\left(\|\nabla_y\nabla_y G_\kappa(\widetilde{x},\cdot)\|_{\boldsymbol{L}_{-\alpha}^\infty(\Gamma_b^+)} + \|\nabla_{y'}\nabla_y\nabla_y G_\kappa(\widetilde{x},\cdot)\|_{\boldsymbol{L}_{-\alpha}^\infty(\Gamma_b^+)}\right) \\
&\le C c_4(\delta_1) e^{-\Delta_1 \overline{\sigma}_1}, \qquad (4.26) \\
\|\nabla_x\nabla_y\nabla_y G_\kappa(\widetilde{x},\cdot)\|_{\boldsymbol{H}_{-\alpha}^{\frac{1}{2}}(\Gamma_b^+)} &\le C\left(\|\nabla_x\nabla_y\nabla_y G_\kappa(\widetilde{x},\cdot)\|_{\boldsymbol{L}_{-\alpha}^\infty(\Gamma_b^+)} + \|\nabla_{y'}\nabla_x\nabla_y\nabla_y G_\kappa(\widetilde{x},\cdot)\|_{\boldsymbol{L}_{-\alpha}^\infty(\Gamma_b^+)}\right) \\
&\le C c_5(\delta_1) e^{-\Delta_1 \overline{\sigma}_1}. \qquad (4.27)
\end{aligned}
$$

It is easy to see that (4.24)-(4.27) imply (4.22).

Similarly, (4.21) and (4.23) can be obtained by repeating the above procedure. The proof is thus complete. □

Next, by (**H2**) we can introduce the approximate DtN operator $\hat{\mathcal{R}} : H_\alpha^{-1/2}(\mathrm{Div},\Gamma_b^+) \to H_\alpha^{-1/2}(\mathrm{Curl},\Gamma_b^+)$ with respect to the electromagnetic field. For any $\boldsymbol{f} \in H_\alpha^{-1/2}(\mathrm{Div},\Gamma_b^+)$, let

$$\hat{\mathcal{R}}\boldsymbol{f} = e_3 \times (\mathrm{curl}\, \hat{\boldsymbol{w}}|_{\Gamma_b^+} \times e_3)$$

where $\hat{\boldsymbol{w}}$ satisfies

$$
\begin{cases}
\nabla \times A_1 \nabla \times \hat{\boldsymbol{w}} - \kappa^2 A_1^{-1}\hat{\boldsymbol{w}} = \boldsymbol{0} & \text{in } \Omega_1^{\mathrm{PML}}, \\
e_3 \times \hat{\boldsymbol{w}} = \boldsymbol{f} & \text{on } \Gamma_b^+, \\
e_3 \times \hat{\boldsymbol{w}} = \boldsymbol{0} & \text{on } \Gamma_{b+\delta_1}^+.
\end{cases}
$$

Similarly, by (**H2**) we also can introduce the approximate DtN operator $\hat{\mathcal{T}} : \boldsymbol{H}_\alpha^{1/2}(\Gamma_b^-) \to \boldsymbol{H}_\alpha^{-1/2}(\Gamma_b^-)$ with respect to the elastic field. For any $\boldsymbol{g} \in \boldsymbol{H}_\alpha^{1/2}(\Gamma_b^-)$, let $\hat{\mathcal{T}}\boldsymbol{g} = T\hat{\boldsymbol{v}}$, where

$$
\begin{cases}
\nabla \cdot (\widetilde{\tau}(\hat{\boldsymbol{v}})A_2) + \rho_0\omega^2 J\hat{\boldsymbol{v}} = \boldsymbol{0} & \text{in } \Omega_2^{\mathrm{PML}}, \\
\hat{\boldsymbol{v}} = \boldsymbol{g} & \text{on } \Gamma_b^-, \\
\hat{\boldsymbol{v}} = \boldsymbol{0} & \text{on } \Gamma_{b+\delta_2}^-.
\end{cases}
$$

Using the approximate DtN mappings $\hat{\mathcal{R}}$ and $\hat{\mathcal{T}}$, we get equivalent equations with (4.5):

$$\begin{cases} \operatorname{curl}\operatorname{curl}\hat{\boldsymbol{H}} - \kappa^2\hat{\boldsymbol{H}} = \boldsymbol{0} & \text{in } D_b^+, \\ \triangle^*\hat{\boldsymbol{u}} + \rho\omega^2\hat{\boldsymbol{u}} = \boldsymbol{0} & \text{in } D_b^-, \\ T\hat{\boldsymbol{u}} - b_1\nu\times\hat{\boldsymbol{H}} = b_1\nu\times\boldsymbol{H}^{in} & \text{on } \Gamma, \\ \nu\times\operatorname{curl}\hat{\boldsymbol{H}} + \dfrac{\mathrm{i}\kappa}{b_2}\nu\times\hat{\boldsymbol{u}} = \mathrm{i}\kappa\nu\times\boldsymbol{E}^{in} & \text{on } \Gamma, \\ \gamma_T^+\operatorname{curl}\hat{\boldsymbol{H}} = \hat{\mathcal{R}}(e_3\times\hat{\boldsymbol{H}}) & \text{on } \Gamma_b^+, \\ T\hat{\boldsymbol{u}} = \hat{\mathcal{T}}\boldsymbol{u} & \text{on } \Gamma_b^-. \end{cases} \tag{4.28}$$

The associate variational formula is to find $\hat{\boldsymbol{u}}\in\boldsymbol{H}_\alpha^1(D_b^-)$ and $\hat{\boldsymbol{H}}\in H_\alpha(\operatorname{curl}, D_b^+)$ satisfying

$$\hat{\mathcal{A}}((\hat{\boldsymbol{u}},\hat{\boldsymbol{H}}),(\hat{\boldsymbol{v}},\hat{\boldsymbol{w}})) = \hat{\mathcal{F}}((\hat{\boldsymbol{v}},\hat{\boldsymbol{w}})), \tag{4.29}$$

for any $(\hat{\boldsymbol{v}},\hat{\boldsymbol{w}})\in Q\times X$, where

$$\begin{aligned} \hat{\mathcal{A}}((\hat{\boldsymbol{u}},\hat{\boldsymbol{H}}),(\hat{\boldsymbol{v}},\hat{\boldsymbol{w}})) :=& \frac{-\mathrm{i}\kappa}{b_1\bar{b}_2}\int_{D_b^-}(\mathcal{E}(\hat{\boldsymbol{u}},\overline{\hat{\boldsymbol{v}}}) - \rho\omega^2\hat{\boldsymbol{u}}\cdot\overline{\hat{\boldsymbol{v}}})\,\mathrm{d}x - \frac{-\mathrm{i}\kappa}{b_1\bar{b}_2}\int_{\Gamma_b^-}\hat{\mathcal{T}}\hat{\boldsymbol{u}}\cdot\overline{\hat{\boldsymbol{v}}}\,\mathrm{d}s \\ &- \Big(\int_{D_b^+}(\operatorname{curl}\hat{\boldsymbol{H}}\cdot\operatorname{curl}\overline{\hat{\boldsymbol{w}}} - \kappa^2\hat{\boldsymbol{H}}\cdot\overline{\hat{\boldsymbol{w}}})\,\mathrm{d}x - \int_{\Gamma_b^+}\hat{\mathcal{R}}(e_3\times\hat{\boldsymbol{H}})\cdot(e_3\times\overline{\hat{\boldsymbol{w}}})\,\mathrm{d}s\Big) \\ &+ \frac{\mathrm{i}\kappa}{b_2}\langle\gamma_t\overline{\hat{\boldsymbol{w}}},\gamma_T\hat{\boldsymbol{u}}\rangle_{H_{\mathrm{Div}}^{-\frac12}\times H_{\mathrm{Curl}}^{-\frac12}} + \frac{\mathrm{i}\kappa}{\bar{b}_2}\langle\gamma_t\hat{\boldsymbol{H}},\gamma_T\overline{\hat{\boldsymbol{v}}}\rangle_{H_{\mathrm{Div}}^{-\frac12}\times H_{\mathrm{Curl}}^{-\frac12}}. \end{aligned}$$

Comparing with the variational formulas $\hat{\mathcal{A}}(\cdot,\cdot)$ of (4.28) and $\mathcal{A}(\cdot,\cdot)$ (see (3.2)) of (4.1), we know the difference between them mainly lies in the difference of the DtN mappings $\mathcal{R}$ and $\hat{\mathcal{R}}$ (resp. $\mathcal{T}$ and $\hat{\mathcal{T}}$). So, it is necessary to estimate the difference between the corresponding DtN mappings.

Theorem 4.3. *If* (**H1**)-(**H3**) *hold, then*

$$\begin{aligned} \|(\mathcal{T}-\hat{\mathcal{T}})\boldsymbol{f}\|_{\boldsymbol{H}_\alpha^{-\frac12}(\Gamma_b^-)} &\le CC_5(\delta_2)e^{-\Delta_2\bar{\sigma}_2}\|\boldsymbol{f}\|_{\boldsymbol{H}_\alpha^{\frac12}(\Gamma_b^-)}, \quad \boldsymbol{f}\in\boldsymbol{H}_\alpha^{\frac12}(\Gamma_b^-), \\ \|(\mathcal{R}-\hat{\mathcal{R}})\boldsymbol{g}\|_{H_\alpha^{-\frac12}(\mathrm{Curl},\Gamma_b^+)} &\le Cc_5(\delta_1)e^{-\Delta_1\bar{\sigma}_1}\|\boldsymbol{g}\|_{H_\alpha^{-\frac12}(\mathrm{Div},\Gamma_b^+)}, \quad \boldsymbol{g}\in H_\alpha^{-\frac12}(\mathrm{Div},\Gamma_b^+). \end{aligned}$$

Proof. Given $\boldsymbol{f}\in\boldsymbol{H}_\alpha^{1/2}(\Gamma_b^-)$ and $\boldsymbol{g}\in H_\alpha^{-1/2}(\mathrm{Div},\Gamma_b^+)$, the associated PML extension is $\mathbb{E}_2(\boldsymbol{f})$ and $\mathbb{E}_1(\boldsymbol{g})$. Obviously, we have

$$\begin{aligned} &e_3\times\mathbb{E}_1(\boldsymbol{g})|_{\Gamma_b^+} = \boldsymbol{g}, \quad e_3\times(\operatorname{curl}\mathbb{E}_1(\boldsymbol{g})\times e_3)|_{\Gamma_b^+} = \mathcal{R}(\boldsymbol{g}), \\ &\mathbb{E}_2(\boldsymbol{f})|_{\Gamma_b^-} = \boldsymbol{f}, \qquad T\mathbb{E}_2(\boldsymbol{f})|_{\Gamma_b^-} = \mathcal{T}(\boldsymbol{f}). \end{aligned}$$

It is easy to see that

$$(\mathcal{R}-\hat{\mathcal{R}})\boldsymbol{g} = e_3\times(\operatorname{curl}\boldsymbol{w}\times e_3)|_{\Gamma_b^+} \quad \text{and} \quad (\mathcal{T}-\hat{\mathcal{T}})\boldsymbol{f} = T\boldsymbol{v}|_{\Gamma_b^-},$$

where $\boldsymbol{w}$ satisfies

$$\begin{cases} \nabla\times A_1\nabla\times\boldsymbol{w} - \kappa^2 A_1^{-1}\boldsymbol{w} = \boldsymbol{0} & \text{in } \Omega_1^{\mathrm{PML}}, \\ e_3\times\boldsymbol{w} = \boldsymbol{0} & \text{on } \Gamma_b^+, \\ e_3\times\boldsymbol{w} = e_3\times B_1\mathbb{E}_1(\boldsymbol{g}) & \text{on } \Gamma_{b+\delta_1}^+, \end{cases} \tag{4.30}$$

and $\boldsymbol{v}$ satisfies

$$\begin{cases} \nabla \cdot (\widetilde{\tau}(\boldsymbol{v})A_2) + \rho_0\omega^2 J\boldsymbol{v} = \boldsymbol{0} & \text{in } \Omega_2^{\mathrm{PML}}, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \Gamma_b^-, \\ \boldsymbol{v} = \mathbb{E}_2(\boldsymbol{f}) & \text{on } \Gamma_{b+\delta_2}^-. \end{cases} \tag{4.31}$$

From the well-posedness results of (4.30) and (4.31) (see (**H2**)) together with the Theorem 4.2, it follows that

$$\begin{aligned} &\|\gamma_T \mathrm{curl}\, \boldsymbol{w}\|_{H_\alpha^{-\frac{1}{2}}(\mathrm{Curl},\Gamma_b^+)} \le C\|\gamma_t B_1 \mathbb{E}_1(\boldsymbol{g})\|_{H_\alpha^{-\frac{1}{2}}(\mathrm{Div},\Gamma_{b+\delta_1}^+)} \le Cc_5(\delta_1)e^{-\Delta_1\overline{\sigma}_1}\|\boldsymbol{g}\|_{H_\alpha^{-\frac{1}{2}}(\mathrm{Div},\Gamma_b^+)}, \\ &\|T\boldsymbol{v}\|_{\boldsymbol{H}_\alpha^{-\frac{1}{2}}(\Gamma_b^-)} \le C\|\mathbb{E}_2(\boldsymbol{f})\|_{\boldsymbol{H}_\alpha^{-\frac{1}{2}}(\Gamma_{b+\delta_2}^-)} \le CC_5(\delta_2)e^{-\Delta_2\overline{\sigma}_2}\|\boldsymbol{f}\|_{\boldsymbol{H}_\alpha^{\frac{1}{2}}(\Gamma_b^-)}. \end{aligned}$$

These complete the proof. □

Now we are in the position to get the main result, i.e., the exponential convergence of PML problem by the Theorem 4.3.

Theorem 4.4. *Assume the problem* (4.1) *is well-posedness and* (**H1**)*-*(**H3**) *hold. If* $\overline{\sigma}_1$ *and* $\overline{\sigma}_2$ *are are sufficiently large, the PML problem* (4.5) *has a unique solution* $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{H}}) \in \hat{Q} \times \hat{X}$*. Moreover, the following estimate*

$$\|\boldsymbol{u} - \hat{\boldsymbol{u}}\|_Q + \|\boldsymbol{H} - \hat{\boldsymbol{H}}\|_X \le C\Big(c_5(\delta_1)e^{-\Delta_1\overline{\sigma}_1}\|\gamma_t\boldsymbol{H}\|_{H_\alpha^{-\frac{1}{2}}(\mathrm{Div},\Gamma_b^+)} + C_5(\delta_2)e^{-\Delta_2\overline{\sigma}_2}\|\boldsymbol{u}\|_{\boldsymbol{H}_\alpha^{\frac{1}{2}}(\Gamma_b^-)}\Big)$$

holds, where $(\boldsymbol{u}, \boldsymbol{H})$ *is the solution to* (4.1)*.*

Proof. For any $(\hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}) \in Q \times X$,

$$\begin{aligned} \hat{\mathcal{A}}((\hat{\boldsymbol{u}}, \hat{\boldsymbol{H}}), (\hat{\boldsymbol{v}}, \hat{\boldsymbol{w}})) =& \mathcal{A}((\hat{\boldsymbol{u}}, \hat{\boldsymbol{H}}), (\hat{\boldsymbol{v}}, \hat{\boldsymbol{w}})) + \frac{-\mathrm{i}\kappa}{b_1\bar{b}_2}\int_{\Gamma_b^-} (\mathcal{T} - \hat{\mathcal{T}})\hat{\boldsymbol{u}} \cdot \overline{\hat{\boldsymbol{v}}}\,\mathrm{d}s \\ &+ \int_{\Gamma_b^+} (\hat{\mathcal{R}} - \mathcal{R})(e_3 \times \hat{\boldsymbol{H}}) \cdot (e_3 \times \overline{\hat{\boldsymbol{w}}})\,\mathrm{d}s. \end{aligned}$$

Combining with Theorem 4.3, we have

$$\begin{aligned} \sup_{\substack{(\hat{\boldsymbol{v}},\hat{\boldsymbol{w}}) \in Q\times X \\ (\hat{\boldsymbol{v}},\hat{\boldsymbol{w}}) \neq 0}} \frac{\hat{\mathcal{A}}((\hat{\boldsymbol{u}}, \hat{\boldsymbol{H}}), (\hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}))}{\|(\hat{\boldsymbol{v}}, \hat{\boldsymbol{w}})\|_{Q\times X}} \ge& C\|(\hat{\boldsymbol{u}}, \hat{\boldsymbol{H}})\|_{Q\times X} - CC_5(\delta_2)e^{-\Delta_2\overline{\sigma}_2}\|\hat{\boldsymbol{u}}\|_{\boldsymbol{H}_\alpha^{\frac{1}{2}}(\Gamma_b^-)} \\ &- Cc_5(\delta_1)e^{-\Delta_1\overline{\sigma}_1}\|\gamma_t\hat{\boldsymbol{H}}\|_{H_\alpha^{-\frac{1}{2}}(\mathrm{Div},\Gamma_b^+)} \\ \ge& C\|(\hat{\boldsymbol{u}}, \hat{\boldsymbol{H}})\|_{Q\times X}. \end{aligned} \tag{4.32}$$

Similar to (4.32), we can get

$$\sup_{\substack{(\hat{\boldsymbol{v}},\hat{\boldsymbol{w}}) \in Q\times X \\ (\hat{\boldsymbol{v}},\hat{\boldsymbol{w}}) \neq \boldsymbol{0}}} \frac{\hat{\mathcal{A}}((\hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}), (\hat{\boldsymbol{u}}, \hat{\boldsymbol{H}}))}{\|(\hat{\boldsymbol{v}}, \hat{\boldsymbol{w}})\|_{Q\times X}} \ge C\|(\hat{\boldsymbol{u}}, \hat{\boldsymbol{H}})\|_{Q\times X}.$$

Then the generalized Lax-Milgram theorem shows that the PML problem (4.5) has a unique solution $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{H}}) \in \hat{Q} \times \hat{X}$.

Simple calculations yields that

$$\begin{aligned}\hat{\mathcal{A}}((\hat{\boldsymbol{u}}-\boldsymbol{u},\hat{\boldsymbol{H}}-\boldsymbol{H}),(\hat{\boldsymbol{v}},\hat{\boldsymbol{w}})) &= \hat{\mathcal{A}}((\hat{\boldsymbol{u}},\hat{\boldsymbol{H}}),(\hat{\boldsymbol{v}},\hat{\boldsymbol{w}})) - \hat{\mathcal{A}}((\boldsymbol{u},\boldsymbol{H}),(\hat{\boldsymbol{v}},\hat{\boldsymbol{w}}))\\ &= \hat{\mathcal{F}}((\hat{\boldsymbol{v}},\hat{\boldsymbol{w}})) - \hat{\mathcal{A}}((\boldsymbol{u},\boldsymbol{H}),(\hat{\boldsymbol{v}},\hat{\boldsymbol{w}}))\\ &= \mathcal{A}((\boldsymbol{u},\boldsymbol{H}),(\hat{\boldsymbol{v}},\hat{\boldsymbol{w}})) - \hat{\mathcal{A}}((\boldsymbol{u},\boldsymbol{H}),(\hat{\boldsymbol{v}},\hat{\boldsymbol{w}}))\\ &= -\frac{-\mathrm{i}\kappa}{b_1\bar{b}_2}\int_{\Gamma_b^-}(\mathcal{T}-\hat{\mathcal{T}})\boldsymbol{u}\cdot\overline{\hat{\boldsymbol{v}}}\,\mathrm{d}s - \int_{\Gamma_b^+}(\hat{\mathcal{R}}-\mathcal{R})(e_3\times\boldsymbol{H})\cdot(e_3\times\overline{\hat{\boldsymbol{w}}})\,\mathrm{d}s.\end{aligned}$$

Therefore, this combined with the inf-sup condition (4.32) and the Theorem 4.3 leads to the desired error estimate. The proof is thus complete. □

5 Conclusions

We have studied the time-harmonic electromagnetic wave scattering by an unbounded elastic body with a periodic structure. By the variational method, we prove that the uniqueness results are available in the case of the lossy (or loseless) medium for small frequencies or all frequencies except a discrete set. Under some assumptions about the coefficients, the existence result is obtained by Fredholm alternative theorem. Further, we propose a PML for the interaction problem, where the PML extension is used to derive the PML equations, and certify that the PML problem exists a unique solution and the difference between the PML solution and that of the original problem vanishes exponentially with respect to the PML parameters.

Acknowledgements

This work was partially supported by the Talent Fund of Beijing Jiaotong University 2022XKRC001, and the NNSF of China grants 12122114 and 12201033.

References

[1] T. Arens, A new integral equation formulation for the scattering of plane elastic waves by diffraction gratings, *J. Integral Equ. Appl.,* **11** (1999), 275–297.

[2] G. Bao, Finite element approximation of time harmonic waves in periodic structures, *SIAM J. Numer. Anal.,* **32** (1995), 1155–1169.

[3] G. Bao, Variational approximation of Maxwell's equations in biperiodic structures, *SIAM J. Appl. Math.,* **57** (1997), 364–381.

[4] G. Bao, L. Cowsar, and W. Masters, eds., *Mathematical Modeling in Optical Sciences*, SIAM Frontiers in Applied Mathematics, (2001).

[5] G. Bao and D. Dobson, On the scattering by a biperiodic structure, *Proc. Amer. Math. Soc.,* **128** (2000), 2715–2723.

[6] G. Bao and H. Wu, Convergence analysis of the perfectly matched layer problems for time-harmonic Maxwell's equations, *SIAM J. Numer. Anal.,* **43** (2005), 2121–2143.

[7] A. Buffa and P. Ciarlet Jr., On traces for functional spaces related to Maxwell's equations Part I: An integration by parts formula in Lipschitz polyhedra, *Math. Methods Appl. Sci.,* **24** (2001), 9–30.

[8] A. Buffa and P. Ciarlet, Jr., On traces for functional spaces related to Maxwell's equations part II. Hodge decompositions on the boundary of Lipschitz polyhedra and applications, *Math. Methods Appl. Sci.,* **24** (2001), 31-48.

[9] A. Buffa, M. Costabel and C. Schwab, Boundary element methods for Maxwell's equations on non-smooth domains, *Numer. Math.,* **92** (2002), 679-710.

[10] A. Buffa, M. Costabel and D. Sheen, On traces for $H(curl, \Omega)$ in Lipschitz domains, *J. Math. Anal. Appl.,* **276** (2002), 845–867.

[11] A. Bernardo, A. Marquez and S. Meddahi, Analysis of an interaction problem between an electromagnetic field and an elastic body, *Int. J. Numer. Anal. Model.,* **7** (2010), 749–765.

[12] J.P. Bérenger, A perfectly matched layer for the absorption of electromagnetic waves, *J. Comput. Phys.,* **114** (1994), 185–200.

[13] J. Bramble and J. Pasciak, Analysis of a finite PML approximation for the three dimensional time-harmonic Maxwell and acoustic scattering problems, *Math. Comput.,* **76** (2007), 597–614.

[14] J. Bramble, J. Pasciak and D. Trenev, Analysis of a finite PML approximation to the three dimensional elastic wave scattering problem, *Math. Comput.,* **79** (2010), 2079–2101.

[15] F. Cakoni and G.C. Hsiao, Mathematical model of the interaction problem between electromagnetic field and elastic body, In: Acoustics, Mechanics, and the Related Topics of Mathematical Analysis (2002), 48–54.

[16] X. Chen and A. Friedman, Maxwell's equations in a periodic structure, *Trans. Am. Math. Soc.,* **323** (1991), 465–507.

[17] Z. Chen and X. Liu, An adaptive perfectly matched layer technique for time-harmonic scattering problems, *SIAM J. Numer. Anal.,* **43** (2005), 645–671.

[18] Z. Chen and H. Wu, An adaptive finite element method with perfectly matched absorbing layers for the wave scattering by periodic structures, *SIAM J. Numer. Anal.,* **41** (2003), 799–826.

[19] Z. Chen and X. Wu, Long-time stability and convergence of the uniaxial perfectly matched layer method for time-domain acoustic scattering problems, *SIAM J. Numer. Anal.,* **50** (2012), 2632–2655.

[20] Z. Chen, X. Xiang and X. Zhang, Convergence of the PML method for elastic wave scattering problems, *Math. Comp.,* **85** (2016), 2687–2714.

[21] Z. Chen and W. Zheng, PML method for electromagnetic scattering problem in a two-layer medium, *SIAM J. Numer. Anal.,* **55** (2017), 2050–2084.

[22] W.C. Chew and W. Weedon, A 3D perfectly matched medium from modified Maxwell's equations with stretched coordinates, *Microwave Opt. Tech. Lett.,* **7** (1994), 599–604.

[23] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 3nd Edition, Springer-Verlag (2013).

[24] D.C. Dobson, A variational method for electromagnetic diffraction in biperiodic structures, *ESAIM-Math. Model. Numer. Anal.,* **28** (1994), 419–439.

[25] J.W. Dunkin and A.C. Eringen, On the propagation of waves in an electromagnetic elastic solid, *Int. J. Eng. Sci.,* **1** (1963), 461–495.

[26] J. Elschner and G. Hu, Variational approach to scattering of plane elastic waves by diffraction gratings, *Math. Methods Appl. Sci.,* **33** (2010), 1924–1941.

[27] J. Elschner and G. Hu, Scattering of plane elastic waves by three-dimensional diffraction gratings, *Math. Models Methods Appl. Sci.,* **22** (2012), 1150019.

[28] G.N. Gatica, G.C.Hsiao and S.Meddahi, A coupled mixed finite element method for the interaction problem between an electromagnetic field and an elastic body, *SIAM J. Numer. Anal.,* **48** (2010), 1338–1368.

[29] G. Hu, F. Qu and B. Zhang, Direct and inverse problems for electromagnetic scattering by a doubly periodic structure with a partially coated dielectric, *Math. Methods Appl. Sci.,* **33** (2010), 147–156.

[30] G. Hu and A. Rathsfeld, Scattering of time-harmonic electromagnetic plane waves by perfectly conducting diffraction gratings, *IMA J. Appl. Math.,* **80** (2014), 508–532.

[31] G. Hu, A. Rathsfeld and T. Yin, Finite element method to fluid-solid interaction problems with unbounded periodic interfaces, *Numer. Meth. Part Differ. Equ.,* **32** (2016), 5–35.

[32] K. Hutter, A.A. Ven and A. Ursescu, *Electromagnetic Field Matter Interactions in Thermoelasic Solids and Viscous Fluids*, Springer, (2006).

[33] X. Jiang, P. Li, J. Lv and W. Zheng, Convergence of the PML solution for elastic wave scattering by biperiodic structures, *Commun. Math. Sci.,* **16** (2018), 987–1016.

[34] D.S. Jones, A uniqueness theorem in elastodynamics, *Q. J. Mech. Appl. Math.,* **37** (1984), 121–142.

[35] A. Kirsch, Diffraction by periodic structures, In: Inverse Problems in Mathematical Physics (1993), 87–102.

[36] V.K. Kupradze, *Potential Methods in Elasticity Theory*, Israel Program for Scientific Translations (1965).

[37] M. Kuzuoglu and R. Mittra, Frequency dependence of the constitutive parameters of causal perfectly matched anisotropic absorbers, *IEEE Microw. Guid. Wave Lett.,* **6** (1996), 447–449.

[38] G.A. Maugin, *Continuum Mechanics of Electromagnetic Solids*, North Holland, Amsterdam (1988).

[39] P. Monk, *Finite Element Method for Maxwell's Equations*, Oxford University Press, New York (2003).

[40] D. Natroshvili, S. Kharibegashvili and Z. Tediashrili, Direct and inverse fluid structure interaction problems, *Rend. Mat. Serie VII,* **7** (2000), 57–92.

[41] J.C. Nédélec and F. Starling, Integral equation methods in a quasi-periodic diffraction problem for the time-harmonic Maxwell's equations, *SIAM J. Math. Anal.,* **22** (1991), 1679–1701.

[42] T. Okaji, Strong unique continuation property for time harmonic Maxwell equations Dedicated to Professor Kiyoshi Mochizuki on the occasion of his sixtieth birthday, *J. Math. Soc. Jpn.,* **54** (2002) 89–122.

[43] H. Parkus, *Electromagnetic Interactions in Elastic Solids*, Springer-Verlag, New York (1979).

[44] G. Schmidt, Electromagnetic scattering by periodic structures, *J. Math. Sci.,* **124** (2004), 5390–5406.

[45] A.A. Ven, Interaction of electromagnetic and elastic fields in solids. Ph.D. Thesis, Techn. Univ. Eindhoven (mimeographed) Eindhoven, (1975).

[46] C. Wei, J. Yang and B. Zhang, A time-dependent interaction problem between an electromagnetic field and an elastic body, *Acta Math. Appl. Sin. Engl. Ser.,* **36** (2020), 95–118.

[47] C. Wei, J. Yang and B. Zhang, Convergence analysis of the PML method for time-domain electromagnetic scattering problems, *SIAM J. Numer. Anal.,* **58** (2020), 1918–1940.

[48] T. Zhu, J. Yang and B. Zhang, On recovery of a bounded elastic body by electromagnetic far-field measurements, *Inverse Probl. Imaging,* **16** (2022), 1063–1083.