INVERSE PROBLEMS FOR A BOUSSINESQ SYSTEM FOR INCOMPRESSIBLE VISCOELASTIC FLUIDS

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Abstract. In this paper, we study two inverse problems for the nonlinear Boussinesq system for incompressible viscoelastic non-isothermal Kelvin-Voigt fluids. The studying inverse problems consist of determining an intensities of density of external forces and heat source under given integral overdetermination conditions. Two types of boundary conditions for the velocity $\mathbf{v}$ are considered: sticking and sliding conditions on boundary. In both cases of these boundary conditions, the local and global in time existence and uniqueness of weak and strong solutions are established under suitable assumptions on the data. The large time behavior of weak solutions is also studied.

Keywords: Inverse problem; Boussinesq system; Kelvin-Voigt equations; viscoelastic incompressible fluids; unique solvability.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^d,\ d=2,3$, with a smooth boundary $\partial\Omega$, and $Q_T=\Omega\times(0,T)$ be a cylinder with a lateral $\Gamma_T=\partial\Omega\times[0,T]$. In Q_T, we consider the following inverse source problem of the Boussinesq system for an incompressible viscoelastic non-isothermal Kelvin-Voigt fluids

$$\mathbf{v}_t+(\mathbf{v}\cdot\nabla)\,\mathbf{v}-\varkappa\Delta\mathbf{v}_t-\nu\Delta\mathbf{v}+\boldsymbol{\nabla}\pi=\mathbf{g}(\mathbf{x},t)\theta(x,t)+f(t)\mathbf{h}(\mathbf{x},t),\quad (x,t)\in Q_T, \tag{1.1}$$

$$\operatorname{div}\mathbf{v}(\mathbf{x},t)=0,\quad (x,t)\in Q_T, \tag{1.2}$$

$$\theta_t+(\mathbf{v}\cdot\nabla)\,\theta-\lambda\Delta\theta=j(t)\phi(\mathbf{x},t),\quad (x,t)\in Q_T, \tag{1.3}$$

which is supplemented with the initial conditions

$$\mathbf{v}(\mathbf{x},0)=\mathbf{v}_0\,(\mathbf{x})\,,\ \theta(\mathbf{x},0)=\theta_0(\mathbf{x}),\quad \mathbf{x}\in\Omega, \tag{1.4}$$

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the dirichlet boundary condition for $\theta(\mathbf{x},t)$

$$\theta(\mathbf{x},t)=0,\ \ (\mathbf{x},t)\in\Gamma_T, \tag{1.5}$$

and the sticking-boundary condition for $\mathbf{v}$

$$\mathbf{v}(\mathbf{x},t)=0,\quad\ (\mathbf{x},t)\in\Gamma_T, \tag{1.6}$$

and with the integral overdetermination conditions

$$\int\limits_\Omega \mathbf{v}\boldsymbol{\sigma}(\mathbf{x})d\mathbf{x}=e(t),\ \ \int\limits_\Omega \theta\eta(\mathbf{x})d\mathbf{x}=\delta(t),\ \ t\geq 0,\text{where}\ \ \boldsymbol{\sigma}(\mathbf{x})=\boldsymbol{\omega}(\mathbf{x})-\varkappa\Delta\boldsymbol{\omega}(\mathbf{x}). \tag{1.7}$$

Instead of (1.6), the sliding-boundary condition (see [20], [23,24])

$$\mathbf{v_n}(\mathbf{x},t)=\mathbf{v}\cdot\mathbf{n}=0,\ \ \left(\mathbf{D}(\mathbf{v})\cdot\mathbf{n}\right)\mathbf{v}\times\mathbf{n}=0,\ \ \ (\mathbf{x},t)\in\Gamma_T \tag{1.8}$$

also will be considered. Thus, in this paper we will deal with the two inverse problems: the first inverse problem, which we will denote by PI for simplicity, consists of determining unknown functions $\mathbf{v}(\mathbf{x},t)$, $\pi(\mathbf{x},t)$, $\theta(\mathbf{x},t)$, $f(t)$, and $j(t)$ from (1.1)-(1.7); the second inverse problem consists of determining the unknown functions $\mathbf{v}(\mathbf{x},t)$, $\pi(\mathbf{x},t)$, $\theta(\mathbf{x},t)$, $f(t)$, $j(t)$ from (1.1)-(1.5), (1.8), and (1.7), and we will denote it by PII.

In (1.1)-(1.3), $\mathbf{v}(\mathbf{x},t)$, $\pi(x,t)$, and $\theta(\mathbf{x},t)$ are respectively a velocity field, a pressure and a temperature, and $\varkappa$, ν, λ are given constants and $\mathbf{v}_0(x)$, $\theta_0(x)$, $\mathbf{g}(\mathbf{x},t)$, $\mathbf{h}(\mathbf{x},t)$, $\phi(\mathbf{x},t)$, $e(t)$, $\delta(t)$, $\boldsymbol{\sigma}(\mathbf{x})$, and $\eta(\mathbf{x})$ are given functions. In specifically, ν, $\varkappa$, and $\lambda>0$ are coefficients of the kinematic viscosity, relaxation and heat conductivity of the fluids, respectively, $\mathbf{g}(\mathbf{x},t)$ is the acceleration due to gravity. The vector-functions $\mathbf{F}(\mathbf{x},t):=f(t)\mathbf{h}(x,t)$ and $G(\mathbf{x},t):=j(t)\phi(\mathbf{x},t)$ are the density of external forces and the heat source with unknown intensities $f(t)$ and $j(t)$, respectively. The scalar-valued functions $e(t)$ and $\delta(t)$ are the average value of the velocity and temperature over the entire area Ω by observing functions $\boldsymbol{\sigma}(\mathbf{x})$ and $\eta(\mathbf{x})$, respectively. In (1.8), $\mathbf{D}$ is the strain tensor, given by

$$\mathbf{D}(\mathbf{v})=\frac{1}{2}(\nabla\mathbf{v}+\nabla\mathbf{v}^T),$$

$\mathbf{v_n}$ is the normal component of $\mathbf{v}(\mathbf{x},t)$ on $\partial\Omega$, and $\mathbf{n}$ denotes the unit outward normal vector to $\partial\Omega$.

Note (see [18,23,24]) that the condition (1.8) is equivalent to

$$\mathbf{v_n}(\mathbf{x},t)=\mathbf{v}\cdot\mathbf{n}=0,\ \ \operatorname{rot}\mathbf{v}\times\mathbf{n}=0,\ \ (\mathbf{x},t)\in\Gamma_T \tag{1.9}$$

in the case $d=3$, and

$$\mathbf{v_n}(\mathbf{x},t)=\mathbf{v}\cdot\mathbf{n}=0,\ \ \ \operatorname{rot}\mathbf{v}\equiv\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}=0,\ \ \ (x_1,x_2,t)\in\Gamma_T \tag{1.10}$$

in the case $d=2$.

The system of equations (1.1)-(1.2) is called the Kelvin-Voigt system and it models the motion of a viscoelastic incompressible Kelvin-Voigt fluids, i.e. fluids with the properties of elasticity and viscosity, see e.g. [6], [29], [22], [27].

The Boussinesq system of hydrodynamics equations arises from a zero order approximation to the coupling between the momentum equations and the thermodynamic equation. For a detailed discussion on the Boussinesq approximation, see e.g. Joseph [14], Rajagopal et al. [25] and the references cited therein.

The existence and uniqueness of weak solutions to the direct problems (when the right-hand sides $F(\mathbf{x},t)$ and $G(\mathbf{x},t)$ are given) for the Boussinesq system (1.1)-(1.3) were established by Oskolkov [22] and Sukacheva [26] with the sticking-boundary condition (1.6), and by Khompysh [17] with the sliding-boundary condition (1.8).

In the study of direct problems, it is important know a significant amount of information of the physical parameters affecting to the processes such as the coefficients ν, $\varkappa$, and λ, and the external forces $F(\mathbf{x},t)$ and $G(\mathbf{x},t)$, and et al. However, there are problems requiring in addition to a solution of a direct problem, to determine some of such parameters, which are unknown or located in an unacceptable places for direct measurement, such as underground or in a high temperature media. Such problems are inverse problems, which the statements of them have to be supplemented with some additional information on the solutions due to the additional unknowns.

The investigating inverse problems here concerned to such type problems since they are consist of determining in addition to the velocity, the pressure, and the temperature, the unknown intensities of the density of external forces and heat source under given additional information (1.7).

On the other hand, if $\varkappa = 0$, the system (1.1)-(1.3) becomes a classical Boussinesq system which is connected to the Navier-Stokes equations. An inverse problems for this Boussinesq system have not been studied a lot, for instant, in [1], [8], [10], the results of existence and uniqueness of solutions of such inverse problems have been established in two dimensional case, by different methods. But, there are many works on inverse source problems of hydrodynamics, in particulary for Navier-Stokes system, we refer to [28], [7], [9], [11] and references there in. To our best knowledge, an inverse source problem for heat convection for Kelvin-Voigt system has not been studied, however there are several inverse problems for Kelvin-Voigt equations, which one can find in [5], [12], [15, 16], [21].

The aim of the present paper is to establish the local and global in time existence and uniqueness of a weak and also strong solutions to the inverse problems PI and PII.

The outline of the paper is the following. In Section 2, we introduce the functional spaces and some auxiliary materials related to the boundary conditions (1.6) and (1.8), and the main notation used throughout this paper. In Section 3, we define the weak and strong solutions to the inverse problems PI and PII and reduce them to an equivalent direct problems, which we handle further. The local in time existence of weak solutions of the equivalent direct problems corresponding to the inverse problems PI and PII is established in Section 4 and 5, respectively. Here, the Galerkin approximation method was used to prove the existence of solutions. Then a priori estimates and the convergence of the corresponding Galerkin approximations were obtained.

The Section 6 devoted to prove the existence of strong solutions of both inverse problems. In section 7, the uniqueness of weak and strong solutions of both PI and PII inverse problems is proved. The global in time existence and uniqueness of solutions for some modifications of the inverse problems PI and PII were established in Section 3. Finally, in Section 9, the asymptotic properties of the solutions of these inverse problems are proved.

2. Preliminaries

In this section, we introduce the main functional spaces and some useful inequalities related to the boundary conditions (1.6) and (1.8) from [18, 23].

We distinguish vectors from scalars by using boldface letters. For functions and function spaces we will use this distinction as well. The symbol C will denote a generic constant – generally a positive one, whose value will not be specified; it can change from one inequality to another.

We denote by $\mathbf{L}^2(\Omega)$ the usual Lebesgue space of square integrable vector-valued functions on Ω, and by $\mathbf{W}^{m,2}(\Omega)$ the Sobolev space of functions in $\mathbf{L}^2(\Omega)$ whose weak derivatives of order not greater than m are in $\mathbf{L}^2(\Omega)$.

Let us introduce the following functional spaces:

$$\begin{aligned}
&\mathcal{V}(\Omega) := \{\mathbf{v} \in \mathbf{C}_0^\infty(\Omega) : \operatorname{div}\mathbf{v} = 0\},\\
&\mathbf{H}(\Omega) := \text{closure of } \mathcal{V} \text{ in the norm of } \mathbf{L}^2(\Omega), \text{ and}\\
&\mathbf{H}^1(\Omega) := \text{closure of } \mathcal{V} \text{ in the norm of } \mathbf{W}^{1,2}(\Omega), \text{ in the case (1.6)};\\
&\mathbf{H}_{\mathbf{n}}(\Omega) := \text{closure of } \mathcal{V} \text{ in the norm of } \mathbf{L}^2(\Omega), \text{ and}\\
&\mathbf{H}^1_{\mathbf{n}}(\Omega) := \text{closure of } \mathcal{V} \text{ in the norm of } \mathbf{W}^{1,2}(\Omega), \text{ in the case (1.9) or (1.10)};\\
&\mathbf{H}^2(\Omega) := \{\mathbf{v} : \mathbf{v} \in \mathbf{W}^{2,2}(\Omega),\ \operatorname{div}\mathbf{v} = 0 \text{ and } \mathbf{v} = 0 \text{ on } \partial\Omega\};\\
&\mathbf{H}^2_{\mathbf{n}}(\Omega) := \{\mathbf{v} : \mathbf{v} \in \mathbf{W}^{2,2}(\Omega),\ \operatorname{div}\mathbf{v} = 0 \text{ and } \mathbf{v}_{\mathbf{n}} = 0 \text{ and } \operatorname{rot}\mathbf{v} \times \mathbf{n} = 0 \text{ on } \partial\Omega\};
\end{aligned}$$

and for the simplicity, we use the following common notation for both cases

$$\mathbf{H}^i := \begin{cases} \mathbf{H}^i(\Omega), \text{ in the case (1.6)};\\ \mathbf{H}^i_{\mathbf{n}}(\Omega), \text{ in the case (1.9) or (1.10)}, \ i = 0, 1, 2, \end{cases} \tag{2.1}$$

where $\mathbf{H}^0 \equiv \mathbf{H}$. According to [20], [18, 23] and the references cited in them (see for example [2, 13]), for any function $\mathbf{v} \in \mathbf{H}^1_{\mathbf{n}}(\Omega)$ (for $\mathbf{H}(\Omega)$ is well known from Navier-Stokes theory), the following inequalities are hold:
Poincare's inequality

$$\|\mathbf{v}\|_{2,\Omega} \le C_1(\Omega)\,\|\nabla\mathbf{v}\|_{2,\Omega}\,,\ \mathbf{v} \in \mathbf{H}^1_{\mathbf{n}}(\Omega); \tag{2.2}$$

$$N_1(\Omega)\,\|\mathbf{v}\|_{\mathbf{W}^{1,2}(\Omega)} \le \|\operatorname{rot}\mathbf{v}\|_{2,\Omega} \le N_2(\Omega)\,\|\mathbf{v}\|_{\mathbf{W}^{1,2}(\Omega)}\,, \quad \forall \mathbf{v} \in \mathbf{H}_{\mathbf{n}}(\Omega); \tag{2.3}$$

$$N_3(\Omega)\,\|\mathbf{v}\|_{\mathbf{W}^{2,2}(\Omega)} \le \|\Delta\mathbf{v}\|_{2,\Omega} = \|\operatorname{rotrot}\mathbf{v}\|_{2,\Omega} \le N_4(\Omega)\,\|\mathbf{v}\|_{\mathbf{W}^{2,2}(\Omega)}\,, \quad \forall \mathbf{v} \in \mathbf{H}^2_{\mathbf{n}}(\Omega); \tag{2.4}$$

Ladyzhenskaya's inequalities [19, 20]

$$\|\mathbf{v}\|^4_{4,\Omega} \le 2\,\|\mathbf{v}\|^2_{2,\Omega}\,\|\nabla\mathbf{v}\|^2_{2,\Omega}\,; \tag{2.5}$$

in case $d = 2$, and

$$\|\mathbf{v}\|^4_{4,\Omega} \le (4/3)^{\frac{3}{2}}\,\|\mathbf{v}\|_{2,\Omega}\,\|\nabla\mathbf{v}\|^3_{2,\Omega}\,; \tag{2.6}$$

in case $d = 3$, and

$$\|\mathbf{v}\|_{6,\Omega} \le (48)^{\frac{1}{6}}\,\|\nabla\mathbf{v}\|_{2,\Omega}\,,\ d = 3 \tag{2.7}$$

Let us introduce now the bilinear and continuous form $\mathbf{a}$ on $\mathbf{H}^1$, associated with the operator $-\Delta$:

$$\mathbf{a}(\mathbf{v},\mathbf{u}) = (\nabla\mathbf{v},\nabla\mathbf{u})_{2,\Omega}\,, \quad \forall \mathbf{v},\mathbf{u}\in\mathbf{H}^1(\Omega) \tag{2.8}$$

in the case (1.6), and

$$\mathbf{a}(\mathbf{v},\mathbf{u}) = (\operatorname{rot}\mathbf{v},\operatorname{rot}\mathbf{u})_{2,\Omega}\,, \quad \forall \mathbf{v},\mathbf{u}\in\mathbf{H}^1_{\mathbf{n}}(\Omega) \tag{2.9}$$

in the case (1.8) ((1.9) or (1.10)). It is clear that $\mathbf{a}(\mathbf{v},\mathbf{v})$ is a norm on $\mathbf{H}^1(\Omega)$, which is equivalent to $\mathbf{W}^{1,2}(\Omega)$-norm. In particular, due to (2.2), in $\mathbf{H}^1_{\mathbf{n}}$ the norm $\|\operatorname{rot}\mathbf{v}\|_{2,\Omega}$ is equivalent to the norm $\|\mathbf{v}\|_{\mathbf{W}^{1,2}(\Omega)}$, and therefore equivalent to the norm $\|\nabla\mathbf{v}\|_{2,\Omega}$.

Thus, $\mathbf{a}$ defines an isomorphism A from $\mathbf{H}^1(\Omega)$ to $\mathbf{H}^{-1}(\Omega)$,

$$\langle A\mathbf{v},\mathbf{u}\rangle \equiv \mathbf{a}(\mathbf{v},\mathbf{u}), \quad \forall \mathbf{v},\mathbf{u}\in\mathbf{H}^1(\Omega), \tag{2.10}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing of $\mathbf{H}^1$ and $\mathbf{H}^{-1}$. There hold the following continuous inclusions

$$\mathbf{H}^1(\Omega)\hookrightarrow\mathbf{L}^2(\Omega)\hookrightarrow\mathbf{H}^{-1}(\Omega), \tag{2.11}$$

where each of the first two spaces is dense in the next one.

It follows from (2.4) also that in $\mathbf{H}^2_{\mathbf{n}}$ the norm $\|\Delta\mathbf{v}\|_{2,\Omega} = \|\operatorname{rotrot}\mathbf{v}\|_{2,\Omega}$ is equivalent to the norm $\|\mathbf{v}\|_{\mathbf{W}^{2,2}(\Omega)}$.

Regarding to the sliding condition (1.8), we have the following Green formulas (see [20] and [18, 23]:

$$\begin{gathered}(-\Delta\mathbf{v},\mathbf{u})_{2,\Omega} = -\left(\nabla\operatorname{div}\mathbf{v},\mathbf{u}\right)_{2,\Omega} + \left(\operatorname{rot}^2\mathbf{v},\mathbf{u}\right)_{2,\Omega} = -\int_{\partial\Omega}\operatorname{div}\mathbf{v}\cdot\mathbf{u}_n\,dS+\\ (\operatorname{div}\mathbf{v},\operatorname{div}\mathbf{u})_{2,\Omega} + \int_{\partial\Omega}\mathbf{u}\cdot(\operatorname{rot}\mathbf{v}\times\mathbf{n})\;dS + (\operatorname{rot}\mathbf{v},\operatorname{rot}\mathbf{u})_{2,\Omega} = (\operatorname{rot}\mathbf{v},\operatorname{rot}\mathbf{u})_{2,\Omega}\end{gathered} \tag{2.12}$$

in the case $d=3$, and

$$\begin{gathered}(-\Delta\mathbf{v},\mathbf{u})_{2,\Omega} = (\operatorname{div}\mathbf{v},\operatorname{div}\mathbf{u})_{2,\Omega} + (\overline{\operatorname{rot}}(\operatorname{rot}\mathbf{v}),\mathbf{u})_{2,\Omega} =\\ \int_{\partial\Omega}(\operatorname{rot}\mathbf{v}\times\mathbf{n})\,\mathbf{u}\,dS + (\operatorname{rot}\mathbf{v},\operatorname{rot}\mathbf{u})_{2,\Omega} = (\operatorname{rot}\mathbf{v},\operatorname{rot}\mathbf{u})_{2,\Omega}\,,\end{gathered} \tag{2.13}$$

in the case $d=2$, where $\overline{\operatorname{rot}}\varphi$ is the vector $(\varphi_{x_2},-\varphi_{x_1})_{2,\Omega}$ for the scalar function φ.

Lemma 1. *Let $\Omega\subset\mathbb{R}^d$, $d\geq 2$, be a bounded domain and assume that $r\geq 1$. If the boundary $\partial\Omega$ is assumed to be of class $C^{0,1}$, then there exist positive constants C_1, C_2 and C_3, depending only on Ω and d, such that*

$$\|u\|_{L^{r^*}(\Omega)} \leq C_1\|\boldsymbol{\nabla}u\|_{\mathbf{L}^r(\Omega)} \qquad \forall\; u\in W^{1,r}_0(\Omega),\; r^*=\frac{dr}{d-r},\; r<d, \tag{2.14}$$

$$\|\boldsymbol{\nabla}u\|_{\mathbf{L}^{r^*}(\Omega)} \leq C_2\|D^2u\|_{L^r(\Omega)} \qquad \forall\; u\in W^{2,r}(\Omega)\cap W^{1,r}_0(\Omega), \tag{2.15}$$

$$\frac{1}{C_3}\|\Delta u\|_{L^r(\Omega)} \leq \|D^2u\|_{L^r(\Omega)} \leq C_3\|\Delta u\|_{L^r(\Omega)} \qquad \forall\; u\in W^{2,r}(\Omega)\cap W^{1,r}_0(\Omega). \tag{2.16}$$

The following nonlinear version of Gronwall's inequality will be used to establish the first and second estimates below.

Lemma 2. *If $y:\mathbb{R}^+\longrightarrow[0,\infty)$ is a continuous function such that*

$$y(t)\le C_1\int_0^t y^{\mu}(s)ds+C_2,\quad t\in\mathbb{R}^+,\quad \mu>1$$

for some positive constants C_1 and C_2, then

$$y(t)\le C_2\left(1-(\mu-1)C_1C_2^{\mu-1}t\right)^{-\frac{1}{\mu-1}}\quad for\ \ 0\le t<t_{\max}:=\frac{1}{(\mu-1)C_1C_2^{\mu-1}}.$$

Proof. See *e.g.* [3]. □

3. Weak formulation

The weak and strong solutions to the inverse problems PI and PII are understood as the following sense.

Definition 1. *The collection of functions $(\mathbf{v}(\mathbf{x},t),\theta(\mathbf{x},t),f(t),j(t))$ is called a weak solution to the inverse problem PI (and PII), if:*

(1) $\mathbf{v}\in\mathbf{L}^{\infty}(0,T;\mathbf{H}^1)\cap\mathbf{L}^2(0,T;\mathbf{H}^1)$, $\mathbf{v}_t\in\mathbf{L}^2(0,T;\mathbf{H}^1)$, $f(t)\in L^2[0,T]$;
(2) $\theta\in L^{\infty}(0,T;L^2)\cap L^2(0,T;W_0^{1,2})$, $\theta_t\in L^2(Q_T)$, $j(t)\in L^2[0,T]$;
(3) $\mathbf{v}(0)=\mathbf{v}_0$ *and* $\theta(0)=\theta_0$ *a.e. in* Ω;
(4) (1.7) *holds for all* $t\in[0,T]$;
(5) For every $\boldsymbol{\varphi}\in\mathbf{H}^1(\Omega)$

$$\begin{aligned}&\frac{d}{dt}\Big((\mathbf{v},\boldsymbol{\varphi})_{2,\Omega}+\varkappa\mathbf{a}\left(\mathbf{v},\boldsymbol{\varphi}\right)\Big)+\left(\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\boldsymbol{\varphi},\mathbf{v}\right)_{2,\Omega}+\nu\mathbf{a}\left(\mathbf{v},\boldsymbol{\varphi}\right)=\\&\qquad f(t)\left(\mathbf{h}(\mathbf{x},t),\boldsymbol{\varphi}\right)_{2,\Omega}+\left(\mathbf{g}(\mathbf{x},t)\theta,\boldsymbol{\varphi}\right)_{2,\Omega}\end{aligned}\tag{3.1}$$

holds in the distribution sense on $(0,T)$;
(6) For every $\psi\in W_0^{1,2}(\Omega)$

$$\frac{d}{dt}\left(\theta,\psi\right)_{2,\Omega}+\lambda\left(\nabla\theta,\nabla\psi\right)_{2,\Omega}+\left(\left(\mathbf{v}\cdot\nabla\right)\theta,\psi\right)_{2,\Omega}=j(t)\left(\phi(\mathbf{x},t),\psi\right)_{2,\Omega}\tag{3.2}$$

holds in the distribution sense on $(0,T)$.

Definition 2. *The collection of functions $(\mathbf{v}(\mathbf{x},t),\theta(\mathbf{x},t),f(t),j(t))$ is called a strong solution to the inverse problem PI (PII), if:*

(1) $\mathbf{v}\in\mathbf{L}^{\infty}(0,T;\mathbf{H}^1(\Omega)\cap\mathbf{H}^2(\Omega))\cap\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)\cap\mathbf{H}^2(\Omega))$, $\mathbf{v}_t\in\mathbf{L}^2(0,T;\mathbf{H}^2(\Omega))$, $f(t)\in L^2[0,T]$;
(2) $\theta\in L^{\infty}(0,T;L^2(\Omega)\cap W_0^{1,2}(\Omega))\cap L^2(0,T;W^{2,2}(\Omega))$, $\theta_t\in L^2(Q_T)$, $j(t)\in L^2[0,T]$;
(3) and each equation of (1.1)-(1.7) *holds in the distribution sense in the their corresponding domain.*

Remark 1. *As noted in previous section, in Definition 1 and 2, we use $\mathbf{a}\left(\mathbf{v},\boldsymbol{\varphi}\right)=\left(\boldsymbol{\nabla}\mathbf{v},\boldsymbol{\nabla}\boldsymbol{\varphi}\right)_{2,\Omega}$ and $\mathbf{H}^i:=\mathbf{H}^i(\Omega)$ for PI, and $\mathbf{a}\left(\mathbf{v},\boldsymbol{\varphi}\right)=\left(\operatorname{rot}\mathbf{v},\operatorname{rot}\boldsymbol{\varphi}\right)_{2,\Omega}$ and $\mathbf{H}^i:=\mathbf{H}^i_{\mathbf{n}}(\Omega)$ for PII, see* (2.8), (2.9), *and* (2.1).

Remark 2. *The pressure π, as usual, was not included in the definition of a weak solution. It can be uniquely recovered from equation (1.1) by using de Rhaam's lemma, after existence of $\mathbf{v}$, θ, f, j as in [4].*

Assume that data of the problem satisfy the following conditions

$$\mathbf{v}_0(\mathbf{x}) \in \mathbf{H}^1(\Omega) \text{ and } \theta_0(\mathbf{x}) \in W_0^{1,2}(\Omega); \tag{3.3}$$

$$\exists k_0 = const : 0 < k_0 < \infty, \text{ and } |h_0(t)| = \left|(\mathbf{h}(\mathbf{x},t), \boldsymbol{\omega}(\mathbf{x}))_{2,\Omega}\right| \geq k_0 > 0, \ \forall t \geq 0; \tag{3.4}$$

$$\exists k_1 = const : 0 < k_1 < \infty, \text{ and } |\phi_0(t)| = \left|(\phi(\mathbf{x},t), \eta(\mathbf{x}))_{2,\Omega}\right| \geq k_1 > 0, \ \forall t \geq 0; \tag{3.5}$$

$$\mathbf{h}(\mathbf{x},t) \in \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega)), \quad \phi(\mathbf{x},t) \in L^\infty(0,T;L^2(\Omega)); \tag{3.6}$$

$$\boldsymbol{\omega}(\mathbf{x}) \in \mathbf{H}^1(\Omega) \cap \mathbf{H}^2(\Omega), \ e(t) \in W_2^1([0,T]); \tag{3.7}$$

$$\int\limits_\Omega \mathbf{v}_0 \cdot \boldsymbol{\sigma} d\mathbf{x} = (\mathbf{v}_0, \boldsymbol{\omega})_{2,\Omega} + \varkappa\, \mathbf{a}(\mathbf{v}_0, \boldsymbol{\omega}) = e(0); \tag{3.8}$$

$$\eta(\mathbf{x}) \in W_0^{1,2}(\Omega), \quad \delta(t) \in W_2^1([0,T]), \quad \text{and} \quad \int\limits_\Omega \theta_0 \cdot \eta d\mathbf{x} = \delta(0); \tag{3.9}$$

$$\mathbf{g}(\mathbf{x},t) \in \mathbf{C}(Q_T) \text{ and } \exists g_0 = const : 0 < g_0 < \infty, \text{ such that } \max_{Q_T} |\mathbf{g}(\mathbf{x},t)| \leq g_0. \tag{3.10}$$

Next, we will show that both inverse problems can be reduced to equivalent direct problems but for equations (1.1),(1.5) with non-linear functionals that depend on $\mathbf{v}$ and θ.

Let us multiply the equations (1.1) and (1.3) by $\omega(\mathbf{x})$ and $\eta(\mathbf{x})$, respectively, and integrate over Ω. Integrating by parts and using the assumptions (3.4) and (3.5), we have

$$f(t) = \frac{1}{h_0(t)}\left(e'(t) - ((\mathbf{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{\omega}, \mathbf{v})_{2,\Omega} + \nu a(\mathbf{v}, \boldsymbol{\omega}) - (\mathbf{g}(\mathbf{x},t)\theta, \boldsymbol{\omega})_{2,\Omega}\right) := \Phi(\mathbf{v},\theta), \tag{3.11}$$

$$j(t) = \frac{1}{\phi_0(t)}\left(\delta'(t) + \lambda(\nabla\theta, \nabla\eta)_{2,\Omega} - ((\mathbf{v}\cdot\nabla)\,\eta, \theta)_{2,\Omega}\right) := J(\mathbf{v},\theta), \tag{3.12}$$

where $\mathbf{a}(\mathbf{v}, \boldsymbol{\omega})$ is defined at (2.8) and (2.9). Let us replace functions $f(t)$ and $j(t)$ in equations (1.1)-(1.3) with functions defined by expressions (3.11), (3.12):

$$\begin{gathered}\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \varkappa\Delta\mathbf{v}_t - \nu\Delta\mathbf{v} + \boldsymbol{\nabla}\pi = \mathbf{g}(\mathbf{x},t)\theta(x,t) + \Phi(\mathbf{v},\theta)\mathbf{h}(\mathbf{x},t), (x,t) \in Q_T, \\ \operatorname{div}\mathbf{v}(\mathbf{x},t) = 0, \quad (x,t) \in Q_T,\end{gathered} \tag{3.13}$$

$$\theta_t + (\mathbf{v}\cdot\nabla)\,\theta - \lambda\Delta\theta = J(\mathbf{v},\theta)\phi(\mathbf{x},t), \quad (x,t) \in Q_T. \tag{3.14}$$

The following lemma is valid.

Lemma 3. *Assume that the conditions (3.4)-(3.9) are fulfilled. Then an every solution to the inverse problem PI is a weak solution to the nonlocal problem (3.13)-(3.14), (1.4)-(1.6), which the functions $\Phi(\mathbf{v},\theta)$ and $J(\mathbf{v},\theta)$ are defined by the formulas (3.11) and (3.12), respectively, and vice versa every weak solution to the nonlocal problem (3.13)-(3.14), (1.4)-(1.6) is a solution of inverse problem PI, i.e. it satisfies the conditions (1.7).*
Moreover, the uniqueness of the solution to the problem PI implies the uniqueness of the solutions of the problem (3.13)-(3.14), (1.4)-(1.6).
This statement is also true for inverse problem PII, i.e. an every solution of PII is a

solution of the nonlocal problem (3.13)-(3.14), (1.4)-(1.5), (1.8), *which the functions* $\Phi(\mathbf{v},\theta)$ *and* $J(\mathbf{v},\theta)$ *are defined by* (3.11) *and* (3.12), *respectively, and vice versa, an every solution of* (3.13)-(3.14), (1.4)-(1.5), (1.8) *is a solution of inverse problem* PII, *i.e. it satisfies the conditions (1.7).*

Remark 3. *The statement of Lemma 3 means that if the collection* $(\mathbf{v}(\mathbf{x},t),\theta(\mathbf{x},t),f(t),j(t))$ *is a solution to the inverse problem* PI *(*PII*), then the pair of functions* $(\mathbf{v}(\mathbf{x},t),\theta(\mathbf{x},t))$ *is a solution to the nonlocal problem* (3.13)-(3.14), (1.4)-(1.6) *(*(3.13)-(3.14), (1.4)-(1.5), (1.8)*), which the functions* $\Phi(\mathbf{v},\theta)$ *and* $J(\mathbf{v},\theta)$ *are defined by the formulas* (3.11) *and* (3.12), *respectively, and vice versa, if the pair* $(\mathbf{v}(\mathbf{x},t),\theta(\mathbf{x},t))$ *is a solution to the direct problem* (3.13)-(3.14), (1.4)-(1.6) *(*(3.13)-(3.14), (1.4)-(1.5), (1.8)*), then these functions* $\mathbf{v}(\mathbf{x},t)$ *and* $\theta(\mathbf{x},t)$ *together with the functions* $f(t)$ *and* $j(t)$*, defined by the explicit formulas* (3.11)-(3.12), *give the solution to the inverse problem* PI *(*PII*), i.e. they satisfy the condition (1.7).*

Proof. 1. Let $(\mathbf{v}(\mathbf{x},t),\theta(\mathbf{x},t),f(t),j(t))$ be a solution to the inverse problem PI (PII). Multiplying the equations (1.1) and (1.3) by $\omega(\mathbf{x})$ and $\eta(\mathbf{x})$, and arguing as above, we derive $f(t)$ and $j(t)$ by the explicit formulas (3.11)-(3.12), respectively. Then substituting them into (1.1)-(1.3), we obtain the system (3.13)-(3.14). The conditions (1.4)-(1.6) ((1.4)-(1.5), (1.8) for PII) are same for both inverse and direct problems.

2. Let now $(\mathbf{v}(\mathbf{x},t),\theta(\mathbf{x},t))$ be a weak solution to the direct problem (3.13)-(3.14), (1.4)-(1.6) ((3.13)-(3.14), (1.4)-(1.5), (1.8)) with the right hand side $\Phi(\mathbf{v},\theta)\mathbf{h}(\mathbf{x},t):=f(t)\mathbf{h}(\mathbf{x},t)$ and $J(\mathbf{v},\theta)\phi(\mathbf{x},t):=j(t)\phi(\mathbf{x},t)$, where $f(t),j(t)$ defined by the formulas (3.11)-(3.12). In order to prove the collection $(\mathbf{v}(\mathbf{x},t),\theta(\mathbf{x},t),f(t),j(t))$ of these functions to be a solution to the inverse problem PI (PII), it is sufficient to prove that these functions are satisfied the overdetermination conditions in (1.7).

Let us assume that for contradiction, i.e. the overdetermination conditions (1.7) do not hold. Suppose that

$$\int_\Omega \mathbf{v}\boldsymbol{\sigma}d\mathbf{x}=e_1(t),\ \int_\Omega \theta\eta(\mathbf{x})d\mathbf{x}=\delta_1(t)\,t\geq 0. \tag{3.15}$$

where $e_1(t)\neq e(t)$ and $\delta_1(t)\neq d(t)$ for some $t\geq 0$. It follows from (3.15) that $e_1(t),\delta_1(t)\in W_2^1([0,T])$ and due to the compatibility conditions (3.7)-(3.9), we have

$$e_1(0)=\int_\Omega \mathbf{v}_0(\mathbf{x})\boldsymbol{\sigma}(\mathbf{x})d\mathbf{x}=e(0),\quad \delta_1(0)=\int_\Omega \theta_0(\mathbf{x})\eta(\mathbf{x})d\mathbf{x}=\delta(0).$$

Again, multiply the equation (3.13) by $\omega(\mathbf{x})$ and (3.14) by $\eta(\mathbf{x})$ and integrate over Ω. Integrating by parts and using (3.15), we get

$$e_1'(t)-\left(\left(\mathbf{v}\cdot\boldsymbol{\nabla}\right)\boldsymbol{\omega},\mathbf{v}\right)_{2,\Omega}+\nu\mathbf{a}\left(\mathbf{v},\boldsymbol{\omega}\right)_{2,\Omega}-\left(\mathbf{g}(\mathbf{x},t)\theta,\boldsymbol{\omega}\right)_{2,\Omega}=\Phi(\mathbf{v},\theta)h_0(t) \tag{3.16}$$

$$\delta_1'(t)+\lambda\left(\nabla\theta,\nabla\eta\right)_{2,\Omega}-\left(\left(\mathbf{v}\cdot\nabla\right)\eta,\theta\right)_{2,\Omega}=J(\mathbf{v},\theta)\phi_0(t) \tag{3.17}$$

Next, plugging (3.11) and (3.12) into (3.16) and (3.17) respectively, we obtain the following Cauchy problems for $E(t) = e_1(t) - e(t)$ and $D(t) = \delta_1(t) - \delta(t)$

$$\begin{cases} E'(t) = 0, \\ E(0) = e_1(0) - e(0) = 0, \end{cases} \quad \begin{cases} D'(t) = 0, \\ D(0) = \delta_1(0) - \delta(0) = 0 \end{cases} \tag{3.18}$$

which yield $e_1(t) \equiv e(t)$ and $\delta_1(t) = \delta(t)$ for all $t > 0$. □

Let $(\mathbf{v}, \theta, f, j)$ be a unique solution of the inverse problem PI. If the corresponding direct problem (3.13)-(3.14), (1.4)-(1.6) has two distinct solutions $(\mathbf{v}_1, \theta_1)$ and $(\mathbf{v}_2, \theta_2)$, then we see that $(\mathbf{v}_i, \theta_i, f_i, j_i)$, $i = 1, 2$, with the functions $f_i(t)$ and $j_i(t)$, uniquely defined by formulas (3.11) and (3.12), respectively, are two distinct solutions of PI, and it contraries to the above assumption.
Now, let $(\mathbf{v}, \theta)$ be a unique solution of (3.13)-(3.14), (1.4)-(1.6). Assume to the contrary that there are two distinct solutions $(\mathbf{v}_1, \theta_1, f_1, j_1)$ and $(\mathbf{v}_2, \theta_2, f_2, j_2)$ of the inverse problem PI. Then arguing as above, we see that $(\mathbf{v}_1, \theta_1)$ and $(\mathbf{v}_2, \theta_2)$ are two distinct solutions of the direct problem (3.13)-(3.14), (1.4)-(1.6), however, it fails to be true.

4. Existence of local in time weak solutions of PI

In this section we study the direct problem (3.13)-(3.14), (1.4)-(1.6), which by Lemma 3 is equivalent to the inverse problem PI. The direct problem associated to PII will be studied in the next section.

Theorem 1. *Let the conditions (3.3)-(3.10) be fulfilled. Then there exists $T_0 \in (0, T]$, such that the direct problem (3.13)-(3.14), (1.4)-(1.6) has at least a weak solution in the cylinder Q_{T_0}, where T_0 is defined at (4.22) below. Accordingly, the inverse problem PI has at least a weak solution. Moreover, for a weak solution to the inverse problem PI the following estimates are hold*

$$\begin{aligned} &\|\mathbf{v}\|^2_{\mathbf{L}^\infty(0,T_0;\mathbf{H}^1(\Omega))} + \|\mathbf{v}_t\|^2_{\mathbf{L}^2(0,T_0;\mathbf{H}^1(\Omega))} + \|\mathbf{v}\|^2_{\mathbf{L}^2(0,T_0;\mathbf{H}^1(\Omega))} + \|f(t)\|^2_{L^2([0,T_0])} \le C_1 < \infty, \\ &\|\theta\|^2_{L^\infty\left(0,T_0;W^{1,2}_0(\Omega)\right)} + \|\nabla\theta\|^2_{2,Q_{T_0}} + \|\theta_t\|^2_{2,Q_{T_0}} + \|\Delta\theta\|^2_{2,Q_{T_0}} + \|j(t)\|^2_{L^2([0,T_0])} \le C_2 < \infty. \end{aligned} \tag{4.1}$$

where C_1 and C_2 are positive constants depending on data of the problem.

Proof. The proof of this theorem consists of the steps: constructing Galerkin's approximations, obtain first and second energy estimates for Galerkin's approximations and passage to the limit. □

4.1. **Galerkin's approximations.** Let us construct a solution to the problem (3.13)-(3.14), (1.4)-(1.6) as a limit of the Galerkin approximations.

Let $\{\boldsymbol{\varphi}_k\}_{k\in\mathbb{N}}$ be an orthonormal family in $\mathbf{L}^2(\Omega)$ formed by functions of $\mathbf{H}$ whose linear combinations are dense in $\mathbf{H}^1(\Omega)$, and $\{\psi_k\}_{k\in\mathbb{N}}$ be a system of eigenfunctions of the following spectral problem for the Laplace operator such that $\psi_k(x) \in W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)$ and

$$\begin{aligned} &-\triangle\psi_k(x) = l_k\ \psi_k(x), \quad x \in \Omega, \\ &\psi_k(x)|_{\partial\Omega} = 0. \end{aligned} \tag{4.2}$$

It follows from the theory of spectral problems that the family $\{\psi_k\}_{k\in\mathbb{N}}$ can be made orthogonal in $W_0^{1,2}(\Omega)$ and orthonormal in $L^2(\Omega)$.

Given $n\in\mathbb{N}$, let us consider the n-dimensional spaces $\mathbf{X}^n$ and Y^n spanned by $\boldsymbol{\varphi}_k$ and ψ_k, $k=1,\dots,n$, respectively. For each $n\in\mathbb{N}$, we search for approximate solutions to the problem (3.13)-(3.14), (1.4)-(1.6) in the form

$$\begin{aligned}
\mathbf{v}^n(x,t)&=\sum_{j=1}^{n}c_j^n(t)\boldsymbol{\varphi}_j(x),\quad \boldsymbol{\varphi}_j\in\mathbf{X}^n,\\
\theta^n(\mathbf{x},t)&=\sum_{j=1}^{n}d_j^n(t)\psi_j(x),\quad \psi_j\in Y^n,
\end{aligned}\tag{4.3}$$

where unknown coefficients $c_j^n(t)$, $d_j^n(t)$, $j=1,...,n$ are defined as solutions of the following system of ordinary differential equations (ODE) derived from

$$\begin{aligned}
&\frac{d}{dt}\Big((\mathbf{v}^n,\boldsymbol{\varphi}_k)_{2,\Omega}+\varkappa(\boldsymbol{\nabla}\mathbf{v}^n,\boldsymbol{\nabla}\boldsymbol{\varphi}_k)_{2,\Omega}\Big)+((\mathbf{v}^n\cdot\boldsymbol{\nabla})\,\boldsymbol{\varphi}_k,\mathbf{v}^n)_{2,\Omega}+\nu(\boldsymbol{\nabla}\mathbf{v}^n,\boldsymbol{\nabla}\boldsymbol{\varphi}_k)_{2,\Omega}=\\
&\quad\Phi^n(\mathbf{v}^n,\theta^n)\,(\mathbf{h}(\mathbf{x},t),\boldsymbol{\varphi}_k)_{2,\Omega}+(\mathbf{g}(\mathbf{x},t)\theta^n,\boldsymbol{\varphi}_k)_{2,\Omega}\,,\\
&\frac{d}{dt}(\theta^n,\psi_k)_{2,\Omega}+\lambda(\nabla\theta^n,\nabla\psi_k)_{2,\Omega}+((\mathbf{v}^n\cdot\nabla)\,\theta^n,\psi_k)_{2,\Omega}=J^n(\mathbf{v}^n,\theta^n)\,(\phi(\mathbf{x},t),\psi_k)_{2,\Omega}\,.
\end{aligned}\tag{4.4}$$

for $k=1,\ 2,\ \dots,\ n$, where

$$\Phi^n(\mathbf{v}^n,\theta^n)=\frac{1}{h_0(t)}\Big(e'(t)-((\mathbf{v}^n\cdot\boldsymbol{\nabla})\,\boldsymbol{\omega},\mathbf{v}^n)_{2,\Omega}+\nu(\boldsymbol{\nabla}\mathbf{v}^n,\boldsymbol{\nabla}\boldsymbol{\omega})_{2,\Omega}-(\mathbf{g}(\mathbf{x},t)\theta^n,\boldsymbol{\omega})_{2,\Omega}\Big)\,,\tag{4.5}$$

$$J^n(\mathbf{v}^n,\theta^n)=\frac{1}{\phi_0(t)}\Big(\delta'(t)+\lambda(\nabla\theta^n,\nabla\eta)_{2,\Omega}-((\mathbf{v}^n\cdot\nabla)\,\eta,\theta^n)_{2,\Omega}\Big)\,,\tag{4.6}$$

The system (4.4) of ODEs is supplemented with the following Cauchy data

$$\mathbf{v}^n(0)=\mathbf{v}_0^n,\quad \theta^n(0)=\theta_0^n\quad\text{in }\Omega.\tag{4.7}$$

where

$$\mathbf{v}_0^n=\sum_{j=1}^{n}(\mathbf{v}_0,\boldsymbol{\varphi}_j)_{2,\Omega}\,\boldsymbol{\varphi}_j,\quad \theta_0^n=\sum_{j=1}^{n}(\theta_0,\psi_j)\,\psi_j$$

are sequences in $\mathbf{L}^2(\Omega)\cap\mathbf{H}^1(\Omega)$ and $L^2(\Omega)\cap W_0^{1,2}(\Omega)$ respectively such that

$$\mathbf{v}_0^n\to\ \mathbf{v}_0(x)\text{ strong as }n\to\infty\ \text{ in }\ \mathbf{L}^2(\Omega)\cap\mathbf{H}^1(\Omega).\tag{4.8}$$

$$\theta_0^n\to\ \theta_0(x)\text{ strong as }n\to\infty\ \text{ in }\ L^2(\Omega)\cap W_0^{1,2}(\Omega).\tag{4.9}$$

According to a general theory of ordinary differential equations, the system (4.4)-(4.7) has a solution $c_j^n(t),d_j^n(t)$ in $[0,t_0]$. By a priori estimates which we shall establish below, the solution can be extended to $[0,T_0]\subset[0,T]$, where $[0,T_0]$ is a maximal time interval, such that a priori estimates are hold.

4.2. **First a priori estimates.**

Lemma 4. *Let $d \leq 4$ be valid. Assume that*

$$\mathbf{v}_0(x) \in \mathbf{H}^1(\Omega), \quad \theta_0(\mathbf{x}) \in L^2(\Omega)$$

and the conditions (3.4)-(3.10) and (4.8), (4.9) are fulfilled. Then there exists a finite time $T_0 \in [0, T]$ such that the following a priori estimate is valid for all $t \in (0, T_0]$

$$\|\mathbf{v}^n\|^2_{\mathbf{L}^\infty(0,T_0;\mathbf{H}^1(\Omega))} + \|\theta^n\|^2_{L^\infty(0,T_0;L^2(\Omega))} + \|\mathbf{v}^n\|^2_{\mathbf{L}^2(0,T_0;\mathbf{H}^1(\Omega))} + \|\nabla\theta^n\|^2_{2,Q_{T_0}} \leq M_0 < \infty, \quad (4.10)$$

where M_0 and T_0 are a positive constants depending only on data of the problem.

Proof. Multiply the first equation of (4.4) by $c_k^n(t)$ and the second equation by $d_k^n(t)$ and summing with respect to k, from 1 to n, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{v}^n\|^2_{2,\Omega} + \varkappa\|\nabla\mathbf{v}^n\|^2_{2,\Omega}\right) + \nu\|\nabla\mathbf{v}^n\|^2_{2,\Omega} = \Phi^n(\mathbf{v}^n,\theta^n)\left(\mathbf{h}(\mathbf{x},t),\mathbf{v}^n\right)_{2,\Omega} + \left(\mathbf{g}(\mathbf{x},t)\theta^n,\mathbf{v}^n\right)_{2,\Omega}, \quad (4.11)$$

$$\frac{1}{2}\frac{d}{dt}\|\theta^n\|^2_{2,\Omega} + \lambda\|\nabla\theta^n\|^2_{2,\Omega} = J^n(\mathbf{v}^n,\theta^n)\left(\phi(\mathbf{x},t),\theta^n\right)_{2,\Omega}. \quad (4.12)$$

First, we apply Hölder's and Young's inequalities together with Ladyzhenskaya's inequality in the case $d = 2$, and the following Sobolev inequality in the case $d \leq 4$

$$\|\mathbf{u}\|_{4,\Omega} \leq C_s(\Omega)\|\nabla\mathbf{u}\|_{2,\Omega}, \quad 4 \leq \frac{2d}{d-2}, \ d > 2 \Leftrightarrow 2 < d \leq 4, \ \mathbf{u}(x) \in \mathbf{H}^1(\Omega) \quad (4.13)$$

to (4.5) and (4.6) to obtain

$$|\Phi^n(\mathbf{v}^n,\theta^n)| \leq \frac{1}{k_0}\left[|e'(t)| + \nu\|\nabla\mathbf{v}^n\|_{2,\Omega}\|\nabla\boldsymbol{\omega}\|_{2,\Omega} + g_0\|\theta^n\|_{2,\Omega}\|\boldsymbol{\omega}\|_{2,\Omega} + \|\mathbf{v}^n\|^2_{4,\Omega}\|\nabla\boldsymbol{\omega}\|_{2,\Omega}\right] \leq$$
$$\frac{1}{k_0}\left[|e'(t)| + \nu\|\nabla\mathbf{v}^n\|_{2,\Omega}\|\nabla\boldsymbol{\omega}\|_{2,\Omega} + g_0\|\theta^n\|_{2,\Omega}\|\boldsymbol{\omega}\|_{2,\Omega} + C_s^2\|\nabla\mathbf{v}^n\|^2_{2,\Omega}\|\nabla\boldsymbol{\omega}\|_{2,\Omega}\right], \quad (4.14)$$

and

$$|J^n(\mathbf{v}^n,\theta^n)| \leq \frac{1}{k_1}\left[|\delta'(t)| + \lambda\|\nabla\theta^n\|_{2,\Omega}\|\nabla\eta\|_{2,\Omega} + \|\mathbf{v}^n\|_{4,\Omega}\|\theta^n\|_{4,\Omega}\|\nabla\eta\|_{2,\Omega}\right] \leq$$
$$\frac{1}{k_1}\left[|\delta'(t)| + \lambda\|\nabla\theta^n\|_{2,\Omega}\|\nabla\eta\|_{2,\Omega} + C_s^2\|\nabla\mathbf{v}^n\|_{2,\Omega}\|\nabla\theta^n\|_{2,\Omega}\|\nabla\eta\|_{2,\Omega}\right], \quad (4.15)$$

respectively.

Next, we estimate the terms on the right-hand side of (4.11) and (4.12) by using Hölder's and Young's inequalities together with (4.14), (4.15)

$$\left|\left(\mathbf{g}(\mathbf{x},t)\theta^n,\mathbf{v}^n\right)_{2,\Omega}\right| \leq g_0\|\theta^n\|_{2,\Omega}\|\mathbf{v}^n\|_{2,\Omega} \leq \frac{g_0}{2}\left(\|\mathbf{v}^n\|^2_{2,\Omega} + \|\theta^n\|^2_{2,\Omega}\right), \quad (4.16)$$

$$
\begin{aligned}
&\left|\Phi^n(\mathbf{v}^n,\theta^n)\left(\mathbf{h}(\mathbf{x},t),\mathbf{v}^n\right)_{2,\Omega}\right| \le |\Phi^n(\mathbf{v}^n,\theta^n)(t)|\,\|\mathbf{h}\|_{2,\Omega}\,\|\mathbf{v}^n\|_{2,\Omega} \le\\
&\frac{1}{k_0}\|\mathbf{h}\|_{2,\Omega}\|\mathbf{v}^n\|_{2,\Omega}\left[|e'(t)|+\nu\|\nabla\mathbf{v}^n\|_{2,\Omega}\|\nabla\boldsymbol{\omega}\|_{2,\Omega}+g_0\|\theta^n\|_{2,\Omega}\|\boldsymbol{\omega}\|_{2,\Omega}+C_s^2\|\nabla\mathbf{v}^n\|_{2,\Omega}^2\|\nabla\boldsymbol{\omega}\|_{2,\Omega}\right]\le\\
&\frac{1}{6}\|\mathbf{v}^n\|_{2,\Omega}^2+\frac{3}{2k_0^2}\|\mathbf{h}\|_{2,\Omega}^2|e'(t)|^2+\frac{\nu}{2}\|\nabla\mathbf{v}^n\|_{2,\Omega}^2+\frac{\nu}{2k_0^2}\|\mathbf{h}\|_{2,\Omega}^2\|\nabla\boldsymbol{\omega}\|_{2,\Omega}^2\|\mathbf{v}^n\|_{2,\Omega}^2+\\
&\frac{1}{6}\|\mathbf{v}^n\|_{2,\Omega}^2+\frac{3g_0^2}{2k_0^2}\|\mathbf{h}\|_{2,\Omega}^2\|\boldsymbol{\omega}\|_{2,\Omega}^2\|\theta^n\|_{2,\Omega}^2+\frac{1}{6}\|\mathbf{v}^n\|_{2,\Omega}^2+\frac{3C_s^4}{2k_0^2}\|\mathbf{h}\|_{2,\Omega}^2\|\nabla\boldsymbol{\omega}\|_{2,\Omega}^2\|\nabla\mathbf{v}^n\|_{2,\Omega}^4\le\\
&\frac{1}{2}\|\mathbf{v}^n\|_{2,\Omega}^2+\frac{\nu}{2}\|\nabla\mathbf{v}^n\|_{2,\Omega}^2+\frac{3}{2k_0^2}\|\mathbf{h}\|_{2,\Omega}^2|e'(t)|^2+\frac{3g_0^2}{2k_0^2}\|\mathbf{h}\|_{2,\Omega}^2\|\boldsymbol{\omega}\|_{2,\Omega}^2\|\theta^n\|_{2,\Omega}^2+\\
&\frac{1}{2k_0^2}\|\mathbf{h}\|_{2,\Omega}^2\|\nabla\boldsymbol{\omega}\|_{2,\Omega}^2\left(\nu\|\mathbf{v}^n\|_{2,\Omega}^2+3C_s^4\|\nabla\mathbf{v}^n\|_{2,\Omega}^4\right)
\end{aligned}
\tag{4.17}
$$

$$
\begin{aligned}
&\left|J^n(\mathbf{v}^n,\theta^n)\left(\phi(\mathbf{x},t),\theta^n\right)_{2,\Omega}\right| \le \|\phi\|_{2,\Omega}\|\theta^n\|_{2,\Omega}\le\\
&\frac{1}{k_1}\|\phi\|_{2,\Omega}\|\theta^n\|_{2,\Omega}\left[|\delta'(t)|+\lambda\|\nabla\theta^n\|_{2,\Omega}\|\nabla\eta\|_{2,\Omega}+C_s^2\|\nabla\mathbf{v}^n\|_{2,\Omega}\|\nabla\theta^n\|_{2,\Omega}\|\nabla\eta\|_{2,\Omega}\right]\le\\
&\frac{1}{2}\|\theta^n\|_{2,\Omega}^2+\frac{1}{2k_1^2}\|\phi\|_{2,\Omega}^2|\delta'(t)|^2+\frac{\lambda}{4}\|\nabla\theta^n\|_{2,\Omega}^2+\frac{\lambda}{k_1^2}\|\phi\|_{2,\Omega}^2\|\nabla\eta\|_{2,\Omega}^2\|\theta^n\|_{2,\Omega}^2+\\
&\frac{\lambda}{4}\|\nabla\theta^n\|_{2,\Omega}^2+\frac{C_s^4}{\lambda k_1^2}\|\phi\|_{2,\Omega}^2\|\nabla\eta\|_{2,\Omega}^2\|\nabla\mathbf{v}^n\|_{2,\Omega}^2\|\theta^n\|_{2,\Omega}^2\le\\
&\frac{1}{2}\|\theta^n\|_{2,\Omega}^2+\frac{\lambda}{2}\|\nabla\theta^n\|_{2,\Omega}^2+\frac{1}{2k_1^2}\|\phi\|_{2,\Omega}^2|\delta'(t)|^2+\\
&\frac{1}{k_1^2}\|\phi\|_{2,\Omega}^2\|\nabla\eta\|_{2,\Omega}^2\left[\lambda\|\theta^n\|_{2,\Omega}^2+\frac{1}{\lambda}C_s^4\left(\|\nabla\mathbf{v}^n\|_{2,\Omega}^2+\|\theta^n\|_{2,\Omega}^2\right)^2\right]
\end{aligned}
\tag{4.18}
$$

Plugging the inequalities (4.16)-(4.17) into (4.11) and the inequality (4.18) into (4.12), and adding the results, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\mathbf{v}^n\|_{2,\Omega}^2+\varkappa\|\nabla\mathbf{v}^n\|_{2,\Omega}^2+\|\theta^n\|_{2,\Omega}^2\right)+\nu\|\nabla\mathbf{v}^n\|_{2,\Omega}^2+\lambda\|\nabla\theta^n\|_{2,\Omega}^2\le\\
&C_1'(t)\left(\|\mathbf{v}^n\|_{2,\Omega}^2+\|\theta^n\|_{2,\Omega}^2\right)+C_2'(t)\left(\varkappa\|\nabla\mathbf{v}^n\|_{2,\Omega}^2+\|\theta^n\|_{2,\Omega}^2\right)^2+C_3'(t),
\end{aligned}
\tag{4.19}
$$

where

$$
C_1'(t)=1+g_0+\max\left\{\frac{\nu}{k_0^2}\|\mathbf{h}\|_{2,\Omega}^2\|\nabla\boldsymbol{\omega}\|_{2,\Omega}^2,\ \frac{3g_0^2}{k_0^2}\|\mathbf{h}\|_{2,\Omega}^2\|\boldsymbol{\omega}\|_{2,\Omega}^2+\frac{2\lambda}{k_1^2}\|\phi\|_{2,\Omega}^2\|\nabla\eta\|_{2,\Omega}^2\right\},
$$

$$
C_2'(t)=\frac{C_s^4}{\varkappa}\left(\frac{3}{k_0^2}\|\mathbf{h}\|_{2,\Omega}^2\|\nabla\boldsymbol{\omega}\|_{2,\Omega}^2+\frac{2}{\lambda k_1^2}\|\phi\|_{2,\Omega}^2\|\nabla\eta\|_{2,\Omega}^2\right),
$$

and

$$
C_3'(t)=\frac{3}{k_0^2}\|\mathbf{h}\|_{2,\Omega}^2|e'(t)|^2+\frac{1}{k_1^2}\|\phi\|_{2,\Omega}^2|\delta'(t)|^2.
$$

Integrating last inequality by s from 0 to t and doing some elementary calculations, we obtain

$$\begin{aligned}
&\|\mathbf{v}^n\|_{2,\Omega}^2+\varkappa\|\nabla\mathbf{v}^n\|_{2,\Omega}^2+\|\theta^n\|_{2,\Omega}^2+\int_0^t\left(\nu\|\nabla\mathbf{v}^n\|_{2,\Omega}^2+\lambda\|\nabla\theta^n\|_{2,\Omega}^2\right)ds\leq\\
&\frac{1}{4}C_1T+C_1\int_0^t\left(\|\mathbf{v}^n\|_{2,\Omega}^2+\|\theta^n\|_{2,\Omega}^2\right)^2ds+C_2\int_0^t\left(\varkappa\|\nabla\mathbf{v}^n\|_{2,\Omega}^2+\|\theta^n\|_{2,\Omega}^2\right)^2ds+\\
&C_3\left(\|e'(t)\|_{2,[0,T]}^2+\|\delta'(t)\|_{2,[0,T]}^2\right)+\|\mathbf{v}_0\|_{2,\Omega}^2+\varkappa\|\nabla\mathbf{v}_0\|_{2,\Omega}^2+\|\theta_0\|_{2,\Omega}^2\leq\\
&C_4\int_0^t\left(\|\mathbf{v}^n\|_{2,\Omega}^2+\varkappa\|\nabla\mathbf{v}^n\|_{2,\Omega}^2+\|\theta^n\|_{2,\Omega}^2\right)^2ds+C_5
\end{aligned}\tag{4.20}$$

where $C_i, i=1,2,3,4,5$ are positive finite constants depending only on the data of the problem, i.e. $C_i=\sup\limits_{t\in[0,T]}C_i'(t)<\infty, i=1,2$, $C_4=C_1+C_2<\infty$, and

$$C_5=C_3+\|\mathbf{v}_0\|_{2,\Omega}^2+\varkappa\|\nabla\mathbf{v}_0\|_{2,\Omega}^2+\|\theta_0\|_{2,\Omega}^2<\infty,$$

and due to the assumptions (3.3)-(3.10) all these constants are finite.

Omitting the integrals on the left hand side of (4.20) we arrive at the following nonlinear integral inequality

$$y(t)\leq C_4\int_0^t y^2(s)ds+C_5$$

for $y(t)\equiv\|\mathbf{v}^n\|_{2,\Omega}^2+\varkappa\|\nabla\mathbf{v}^n\|_{2,\Omega}^2+\|\theta^n\|_{2,\Omega}^2$. Applying the generalized Gronwall's Lemma 2 with $\mu=2$, we obtain estimate

$$y(t)\leq\frac{C_5}{1-C_4C_5t}\equiv K<\infty\tag{4.21}$$

for

$$0\leq t\leq T_0<T_\star:=\frac{1}{C_4C_5}.\tag{4.22}$$

Thus, for all $t\leq T_0<T_\star$, (4.21) yields

$$\|\mathbf{v}^n\|_{2,\Omega}^2+\varkappa\|\nabla\mathbf{v}^n\|_{2,\Omega}^2+\|\theta^n\|_{2,\Omega}^2\leq K.\tag{4.23}$$

Applying the estimate (4.23) to the right hand side of (4.20) and taking the supremum by $t\in[0,T_0]$, we obtain from (4.20) the following estimate

$$\sup_{t\in(0,T_0]}\left(\|\mathbf{v}^n\|_{2,\Omega}^2+\|\nabla\mathbf{v}^n\|_{2,\Omega}^2+\|\theta^n\|_{2,\Omega}^2\right)+\|\nabla\mathbf{v}^n\|_{\mathbf{L}^2(Q_{T_0})}^2+\|\nabla\theta^n\|_{\mathbf{L}^2(Q_{T_0})}^2\leq K_0<\infty,\tag{4.24}$$

where $K_0=K_0(\mu,\lambda,\varkappa,T_0,C_1,C_2,C_3,C_5)$. □

4.3. **Second energy estimate.**

Lemma 5. *Assume that all conditions of Lemma 4 are fulfilled. Then for* $\mathbf{v}^n$ *the following estimate is valid*

$$\sup_{t\in[0,T_0]} \|\nabla \mathbf{v}^n\|^2_{2,\Omega} + \|\mathbf{v}^n_t\|^2_{2,Q_{T_0}} + \|\nabla \mathbf{v}^n_t\|^2_{2,Q_{T_0}} \leq M_1 < \infty, \quad \forall t \in [0, T_0], \tag{4.25}$$

where T_0 *is defined at* (4.22) *and* M_1 *is positive constant depending on data of the problem.*

Proof. Multiply both sides of the first equation of (4.4) by $\frac{d c^n_k}{dt}$ and sum up from $k = 1$ to $k = n$. Integrating the result by s in $[0, t]$, $t \leq T_0$, we have

$$\begin{aligned} &\frac{\nu}{2}\|\nabla \mathbf{v}^n\|^2_{2,\Omega} + \|\mathbf{v}^n_t(t)\|^2_{2,Q_t} + \varkappa \|\nabla \mathbf{v}^n_t(t)\|^2_{2,Q_t} = \frac{\nu}{2}\|\nabla \mathbf{v}^n(0)\|^2_{2,\Omega} + \\ &\int_0^t \left[\Phi^n(\mathbf{v}^n, \theta^n)\left(\mathbf{h}(s), \mathbf{v}^n_t(s)\right) + \left(\mathbf{g}(s)\theta^n(s), \mathbf{v}^n_t(s)\right) + \left(\left(\mathbf{v}^n(s)\cdot\nabla\right)\mathbf{v}^n_t(s), \mathbf{v}^n(s)\right)\right] ds. \end{aligned} \tag{4.26}$$

Now, by using Hölder and Young inequalities together with the estimate (4.10), we estimate each term on the right hand side of (4.26)

$$\begin{aligned} &\int_0^t |\Phi^n(\mathbf{v}^n, \theta^n)(s)\left(\mathbf{h}, \mathbf{v}^n_t\right)| \, ds \leq \int_0^t |\Phi^n(\mathbf{v}^n, \theta^n)(s)| \, \|\mathbf{h}\|_{2,\Omega} \|\mathbf{v}^n_t\|_{2,\Omega} \, ds \leq \\ &\frac{\varepsilon_1}{2}\|\mathbf{v}^n_t\|^2_{2,Q_t} + \frac{1}{2\varepsilon_1}\|\mathbf{h}\|^2_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))}\|\Phi^n(\mathbf{v}^n,\theta^n)\|^2_{L^2([0,T_0])} \leq \frac{\varepsilon_1}{2}\|\mathbf{v}^n_t\|^2_{2,Q_t} + \frac{M_0}{2\varepsilon_1}\|\mathbf{h}\|^2_{\mathbf{L}^\infty(0,T;\mathbf{L}(\Omega))}, \end{aligned} \tag{4.27}$$

$$\int_0^t |\left(\mathbf{g}(s)\theta^n(s), \mathbf{v}^n_t(s)\right) ds| \leq \frac{\varepsilon_2}{2}\|\mathbf{v}^n_t\|^2_{2,Q_t} + \frac{g_0^2}{2\varepsilon_2}\int_0^t \|\theta^n(s)\|^2_{2,\Omega} \, ds \leq \frac{\varepsilon_2}{2}\|\mathbf{v}^n_t\|^2_{2,Q_t} + \frac{g_0^2}{2\varepsilon_2}M_0 T, \tag{4.28}$$

Let be $d \leq 4$.

$$\begin{aligned} &\int_0^t |\left(\left(\mathbf{v}^n\cdot\nabla\right)\mathbf{v}^n_t, \mathbf{v}^n\right)| \, ds \leq \int_0^t \|\nabla \mathbf{v}^n_t\|_{2,\Omega}\|\mathbf{v}^n\|^2_{4,\Omega} \, ds \leq C(\Omega)\int_0^t \|\nabla \mathbf{v}^n_t\|_{2,\Omega}\|\nabla \mathbf{v}^n\|^2_{2,\Omega} \, ds \leq \\ &\frac{\varepsilon_3}{2}\|\nabla \mathbf{v}^n_t\|^2_{2,Q_t} + \frac{C^2(\Omega)}{2\varepsilon_3}\int_0^t \|\nabla \mathbf{v}^n\|^4_{2,\Omega} \, ds \leq \frac{\varepsilon_3}{2}\|\nabla \mathbf{v}^n_t\|^2_{2,Q_t} + \frac{C^2(\Omega)}{2\varepsilon_3}M_0^2 T, \end{aligned} \tag{4.29}$$

Substituting (4.27)-(4.29) into (4.26) with $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$ and $\varepsilon_3 = \varkappa$, and taking supremum by $t \in [0, T_0]$, we have

$$\nu \sup_{t\in[0,T_0]} \|\nabla \mathbf{v}^n\|^2_{\mathbf{L}^\infty(0,T_0;\mathbf{L}^2(\Omega))} + \|\mathbf{v}^n_t(t)\|^2_{2,Q_{T_0}} + \varkappa \|\nabla \mathbf{v}^n_t(t)\|^2_{2,Q_{T_0}} \leq M_1 < \infty, \tag{4.30}$$

where $M_1 = \nu \|\nabla \mathbf{v}_0\|_{2,\Omega}^2 + M_0 \left(\|\mathbf{h}\|_{\mathbf{L}^\infty(0,T;\mathbf{L}(\Omega))}^2 + g_0^2 T + \frac{C^2(\Omega)}{\varkappa} M_0 T \right)$. $\square$

Lemma 6. *Let be $d \leq 3$. Assume that all conditions of the Lemma 4 are valid and*

$$\theta_0(x) \in W_0^{1,2}(\Omega).$$

Then the following estimate is valid for θ^n

$$\sup_{t\in[0,T_0]} \|\nabla\theta^n\|_{2,\Omega}^2 + \|\theta_t^n\|_{2,Q_{T_0}}^2 + \|\Delta\theta^n\|_{2,Q_{T_0}}^2 \leq M_2 < \infty. \tag{4.31}$$

Proof. Multiply the second equation of (4.4) by $l_k d_k^n(t)$ and $\frac{d\,d_k^n}{d\,t}$, and sum up the resulting equation from $k = 1$ till $k = n$. Taking in account (4.2), we obtain the following equalities, respectively

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta^n\|_{2,\Omega}^2 + \lambda\|\Delta\theta^n\|_{2,\Omega}^2 = \left(\left(\mathbf{v}^n\cdot\nabla\right)\theta^n, \Delta\theta^n\right)_{2,\Omega} + J^n(\mathbf{v}^n,\theta^n)\left(\phi(\mathbf{x},t), -\Delta\theta^n\right)_{2,\Omega} \tag{4.32}$$

and

$$\frac{\lambda}{2}\frac{d}{dt}\|\nabla\theta^n\|_{2,\Omega}^2 + \|\theta_t^n\|_{2,\Omega}^2 = \left(\left(\mathbf{v}^n\cdot\nabla\right)\theta^n, \theta_t^n\right)_{2,\Omega} + J^n(\mathbf{v}^n,\theta^n)\left(\phi(\mathbf{x},t), \theta_t^n\right)_{2,\Omega}. \tag{4.33}$$

Let us first, estimate the terms on right hand side of (4.32). Applying the Hölder inequality together with (2.7) and (2.15)-(2.16) we get

$$\begin{aligned}
&|((\mathbf{v}^n\cdot\nabla)\theta^n, \triangle\theta^n)| \leq \|\triangle\theta^n\|_{2,\Omega}\|\mathbf{v}^n\|_{6,\Omega}\|\nabla\theta^n\|_{3,\Omega} \leq \\
&C(\Omega)\|\triangle\theta^n\|_{2,\Omega}\|\nabla\mathbf{v}^n\|_{2,\Omega}\|\nabla\theta^n\|_{2,\Omega}^{\frac{1}{2}}\|\nabla\theta^n\|_{6,\Omega}^{\frac{1}{2}} \leq \\
&C(\Omega)\|\triangle\theta^n\|_{2,\Omega}\|\nabla\mathbf{v}^n\|_{2,\Omega}\|\nabla\theta^n\|_{2,\Omega}^{\frac{1}{2}}\|\nabla\theta^n\|_{2^*,\Omega}^{\frac{1}{2}} \leq \\
&C(\Omega)\|\triangle\theta^n\|_{2,\Omega}^{\frac{3}{2}}\|\nabla\mathbf{v}^n\|_{2,\Omega}\|\nabla\theta^n\|_{2,\Omega}^{\frac{1}{2}}, \quad 6 \leq 2^* := \frac{2d}{d-2} \;\Leftrightarrow\; d \leq 3,
\end{aligned} \tag{4.34}$$

in the case $d = 3$, and with Ladyzhenskaya's inequality (2.5) and (2.15)-(2.16)

$$\begin{aligned}
&|((\mathbf{v}^n\cdot\nabla)\theta^n, \triangle\theta^n)| \leq \|\triangle\theta^n\|_{2,\Omega}\|\mathbf{v}^n\|_{4,\Omega}\|\nabla\theta^n\|_{4,\Omega} \leq \\
&C(\Omega)\|\triangle\theta^n\|_{2,\Omega}\|\nabla\mathbf{v}^n\|_{2,\Omega}^2\|\nabla\theta^n\|_{2,\Omega}^{\frac{1}{2}}\|\theta_{xx}^n\|_{2,\Omega}^{\frac{1}{2}} \leq C(\Omega)\|\triangle\theta^n\|_{2,\Omega}^{\frac{3}{2}}\|\nabla\mathbf{v}^n\|_{2,\Omega}^2\|\nabla\theta^n\|_{2,\Omega}^{\frac{1}{2}}
\end{aligned} \tag{4.35}$$

in the case $d = 2$. Thus, in any case $d = 2$ or $d = 3$, we have

$$|((\mathbf{v}^n\cdot\nabla)\theta^n, \triangle\theta^n)| \leq \frac{\varepsilon_4}{2}\|\triangle\theta^n\|_{2,\Omega}^2 + \frac{C^2(\Omega)}{2\varepsilon_4}\|\nabla\mathbf{v}^n\|_{2,\Omega}^4\|\nabla\theta^n\|_{2,\Omega}^2. \tag{4.36}$$

Likewise, we have

$$\begin{aligned}
&|J^n(\mathbf{v}^n,\theta^n)(t)\left(\phi(\mathbf{x},t), -\triangle\theta^n\right)_{2,\Omega}| \leq \\
&|J^n(\mathbf{v}^n,\theta^n)|\,\|\phi\|_{2,\Omega}\|\triangle\theta^n\|_{2,\Omega} \leq \frac{\varepsilon_5}{2}\|\triangle\theta^n\|_{2,\Omega}^2 + \frac{1}{2\varepsilon_5}\|\phi\|_{2,\Omega}^2|J^n(\mathbf{v}^n,\theta^n)(t)|^2.
\end{aligned} \tag{4.37}$$

Plugging last two inequalities with $\varepsilon_4 = \varepsilon_5 = \frac{\lambda}{2}$ into (4.32), and integrating the resulting inequality between 0 and $t \in [0, T_0]$, we can show that

$$\begin{aligned}&\|\nabla\theta^n\|_{2,\Omega}^2 + \lambda\|\Delta\theta^n\|_{2,Q_t}^2 \le \\ &\|\nabla\theta^n(0)\|_{2,\Omega}^2 + \frac{2}{\lambda}\int_0^t \left[\|\phi\|_{2,\Omega}^2 |J^n(\mathbf{v}^n,\theta^n)(t)|^2 + C^2(\Omega)\|\nabla\mathbf{v}^n\|_{2,\Omega}^4 \|\nabla\theta^n\|_{2,\Omega}^2\right] ds,\end{aligned} \tag{4.38}$$

in which by (4.15) and (4.24), yields that

$$\begin{aligned}&\|\nabla\theta^n\|_{L^\infty(0,T_0;L^2(\Omega))}^2 + \lambda\|\Delta\theta^n\|_{2,Q_{T_0}}^2 \le \|\nabla\theta_0\|_{2,\Omega}^2 + \\ &\frac{2}{\lambda}\left[\|\phi\|_{L^\infty(0,T_0;L^2(\Omega))}^2 \|j^n(t)\|_{L^2[0,T_0]}^2 + C^2(\Omega)\|\nabla\mathbf{v}^n\|_{L^\infty(0,T_0;L^2(\Omega))}^4 \|\nabla\theta^n\|_{2,Q_{T_0}}^2\right] \le \\ &\|\nabla\theta_0\|_{2,\Omega}^2 + \frac{2}{\lambda}\left[\|\phi\|_{L^\infty(0,T_0;L^2(\Omega))}^2 \|J^n(\mathbf{v}^n,\theta^n)\|_{L^2[0,T_0]}^2 + C^2K_0^3\right] := K_3 < \infty.\end{aligned} \tag{4.39}$$

Analogically we obtain the following estimates for the terms on the right hand side

$$\begin{aligned}&|J^n(\mathbf{v}^n,\theta^n)(t)\,(\phi(\mathbf{x},t),\theta_t^n)_{2,\Omega}| \le |J^n(\mathbf{v}^n,\theta^n)(t)|\,\|\phi\|_{2,\Omega}\|\theta_t^n\|_{2,\Omega} \le \\ &\frac{\varepsilon_6}{2}\|\theta_t^n\|_{2,\Omega}^2 + \frac{1}{2\varepsilon_6}\|\phi\|_{2,\Omega}^2 |J^n(\mathbf{v}^n,\theta^n)(t)|^2\end{aligned} \tag{4.40}$$

and

$$\begin{aligned}&|((\mathbf{v}^n\cdot\nabla)\,\theta^n,\theta_t^n)| \le \|\theta_t^n\|_{2,\Omega}\|\mathbf{v}^n\|_{4,\Omega}\|\nabla\theta^n\|_{4,\Omega} \le C^2(\Omega)\|\theta_t^n\|_{2,\Omega}\|\nabla\mathbf{v}^n\|_{2,\Omega}\|\triangle\theta^n\|_{2,\Omega} \le \\ &\frac{\varepsilon_7}{2}\|\theta_t^n\|_{2,\Omega}^2 + \frac{C^4(\Omega)}{2\varepsilon_7}\|\nabla\mathbf{v}^n\|_{2,\Omega}^2\|\triangle\theta^n\|_{2,\Omega}^2.\end{aligned} \tag{4.41}$$

Plugging the inequalities (4.27)-(4.28) with $\varepsilon_6 = \varepsilon_7 = \frac{1}{2}$ into (4.33) and integrating the resulting inequality between 0 and $t \in (0, T_0)$,and using the estimates (4.10) and (4.39), we have

$$\begin{aligned}&\lambda\|\nabla\theta^n\|_{L^\infty(0,T_0;L^2(\Omega))}^2 + \|\theta_t^n\|_{2,Q_{T_0}}^2 \le \lambda\|\nabla\theta_0\|_{2,\Omega}^2 + \\ &2\|\phi\|_{L^\infty(0,T_0;L^2(\Omega))}^2 \|J^n(\mathbf{v}^n,\theta^n)\|_{L^2[0,T_0]}^2 + \|\nabla\mathbf{v}^n\|_{L^\infty(0,T_0;L^2(\Omega))}^2 \|\triangle\theta^n\|_{2,Q_{T_0}}^2 = K_4 < \infty.\end{aligned} \tag{4.42}$$

The estimates (4.39) and (4.42) yield the estimate (4.31) with $M_2 = K_3 + K_4 < \infty$. □

4.4. **Passage to the limit as $n \to \infty$.** By means of reflexivity and up to some subsequences, the estimates (4.10), (4.25), and (4.31) imply that

$$\mathbf{v}^n \rightharpoonup \mathbf{v} \quad \text{weakly in } L^2(0,T_0;\mathbf{H}^1(\Omega)), \quad \text{as } n\to\infty, \tag{4.43}$$

$$\theta^n \rightharpoonup \theta \quad \text{weakly in } L^2(0,T_0;W_0^{1,2}(\Omega)), \quad \text{as } n\to\infty, \tag{4.44}$$

$$\mathbf{v}^n \rightharpoonup \mathbf{v} \quad \text{weakly-}* \text{ in } L^\infty(0,T_0;\mathbf{H}^1(\Omega)), \quad \text{as } n\to\infty, \tag{4.45}$$

$$\Phi^n(\mathbf{v}^n,\theta^n) \rightharpoonup \Phi(\mathbf{v},\theta) \quad \text{weakly in } L^2([0,T_0]), \quad \text{as } n\to\infty, \tag{4.46}$$

$$J^n(\mathbf{v}^n,\theta^n) \rightharpoonup J(\mathbf{v},\theta) \quad \text{weakly in } L^2([0,T_0]), \quad \text{as } n\to\infty, \tag{4.47}$$

$$\theta^n \rightharpoonup \theta \quad \text{weakly -}* \text{ in } L^\infty(0,T_0;W_0^{1,2}(\Omega)), \quad \text{as } n\to\infty, \tag{4.48}$$

$$\mathbf{v}_t^n \rightharpoonup \mathbf{v}_t \quad \text{weakly in } L^2(0,T_0;\mathbf{H}^1(\Omega)), \quad \text{as } n\to\infty, \tag{4.49}$$

$$\theta_t^n \rightharpoonup \theta_t \quad \text{weakly in } L^2(0,T_0;L^2(\Omega)), \quad \text{as } n\to\infty, \tag{4.50}$$

$$\theta^n \rightharpoonup \theta \quad \text{weakly in } L^2(0,T_0;W^{2,2}(\Omega)), \quad \text{as } n\to\infty, \tag{4.51}$$

where $\Phi(\mathbf{v},\theta)$ and $J(\mathbf{v},\theta)$ are the functional defined at (3.11) and (3.12), respectively.

On the other hand, due to the compact embedding $\mathbf{W}_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^2(\Omega)$ and the Aubin-Lions compactness lemma, it follows that

$$\mathbf{v}^n \longrightarrow \mathbf{v} \quad \text{strongly in } \mathbf{L}^2(0,T_0;\mathbf{L}^2(\Omega)) \quad \text{and a.e. } \ Q_{T_0} \quad \text{as } n\to\infty \tag{4.52}$$

and

$$\theta^n \longrightarrow \theta \quad \text{strongly in } L^2(0,T_0;L^2(\Omega)) \quad \text{and a.e. } \ Q_{T_0} \quad \text{as } n\to\infty. \tag{4.53}$$

Let be $\zeta(t),\xi(t)\in C_0^\infty([0,T_0])$. Multiplying the first equation of (4.4) by $\zeta(t)$ and second by $\xi(t)$, integrating the resulting equations between 0 and T_0, we obtain

$$\begin{aligned}
&\int_{Q_{T_0}} \mathbf{v}_t^n\cdot\boldsymbol{\varphi}_k\zeta\, d\mathbf{x}dt + \varkappa\int_{Q_{T_0}} \boldsymbol{\nabla}\mathbf{v}_t^n : \boldsymbol{\nabla}\boldsymbol{\varphi}_k\zeta\, d\mathbf{x}dt + \int_{Q_{T_0}} (\mathbf{v}^n\cdot\boldsymbol{\nabla})\mathbf{v}^n\cdot\boldsymbol{\varphi}_k\zeta\, d\mathbf{x}dt + \\
&\nu\int_{Q_{T_0}} \boldsymbol{\nabla}\mathbf{v}^n : \boldsymbol{\nabla}\boldsymbol{\varphi}_k\zeta\, d\mathbf{x}dt = \int_{Q_{T_0}} \mathbf{g}(\mathbf{x},t)\theta^n\boldsymbol{\varphi}_k\zeta\, d\mathbf{x}dt + \int_0^{T_0} \Phi^n(\mathbf{v}^n,\theta^n)(t)\int_\Omega \mathbf{h}(\mathbf{x},t)\boldsymbol{\varphi}_k\zeta\, d\mathbf{x}\, dt
\end{aligned} \tag{4.54}$$

$$\begin{aligned}
&\int_{Q_{T_0}} \theta_t^n\cdot\psi_k\xi\, d\mathbf{x}dt + \int_{Q_{T_0}} (\mathbf{v}^n\cdot\boldsymbol{\nabla})\theta^n\cdot\psi_k\xi\, d\mathbf{x}dt + \lambda\int_{Q_{T_0}} \nabla\theta^n\nabla\psi_k\xi\, d\mathbf{x}dt \\
&= \int_0^{T_0} J^n(\mathbf{v}^n,\theta^n)(t)\int_\Omega \phi(x,t)\psi_k\xi\, d\mathbf{x}dt
\end{aligned} \tag{4.55}$$

for $k\in\{1,\dots,n\}$.

Then, fixing k, we can pass in equations (4.54) and (4.55) to the limit $n\to\infty$, by using the convergence results (4.43)-(4.53). Then, we obtain

$$\begin{aligned}
&\int_{Q_{T_0}} \mathbf{v}_t\cdot\boldsymbol{\varphi}_k\zeta\, d\mathbf{x}dt + \varkappa\int_{Q_{T_0}} \boldsymbol{\nabla}\mathbf{v}_t : \boldsymbol{\nabla}\boldsymbol{\varphi}_k\zeta\, d\mathbf{x}dt + \int_{Q_{T_0}} (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\cdot\boldsymbol{\varphi}_k\zeta\, d\mathbf{x}dt \\
&+\nu\int_{Q_{T_0}} \boldsymbol{\nabla}\mathbf{v} : \boldsymbol{\nabla}\boldsymbol{\varphi}_k\zeta\, d\mathbf{x}dt = \int_{Q_{T_0}} \mathbf{g}(\mathbf{x},t)\theta\boldsymbol{\varphi}_k\zeta\, d\mathbf{x}dt + \int_0^{T_0} \Phi(\mathbf{v},\theta)\int_\Omega \mathbf{h}(\mathbf{x},t)\boldsymbol{\varphi}_k\zeta\, d\mathbf{x}\, dt
\end{aligned} \tag{4.56}$$

$$\begin{aligned}
&\int_{Q_{T_0}} \theta_t\cdot\psi_k\xi\, d\mathbf{x}dt + \int_{Q_{T_0}} (\mathbf{v}\cdot\boldsymbol{\nabla})\theta\cdot\psi_k\xi\, d\mathbf{x}dt + \lambda\int_{Q_{T_0}} \nabla\theta\nabla\psi_k\xi\, d\mathbf{x}dt \\
&= \int_0^{T_0} J(\mathbf{v},\theta)\int_\Omega \phi(x,t)\psi_k\xi\, d\mathbf{x}dt
\end{aligned} \tag{4.57}$$

for $k\in\{1,\dots,n\}$.

Here, for the convective terms, we passed to the limit by using the following convergence

$$(\mathbf{v}^n\cdot\boldsymbol{\nabla})\mathbf{v}^n \longrightarrow (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} \quad \text{strongly in } \ \mathbf{L}^1(Q_{T_0}), \quad \text{as } n\to\infty, \tag{4.58}$$

$$(\mathbf{v}^n\cdot\nabla)\theta^n \longrightarrow (\mathbf{v}\cdot\nabla)\theta \quad \text{strongly in } \ \mathbf{L}^1(Q_{T_0}), \quad \text{as } n\to\infty, \tag{4.59}$$

which can be proved under (4.10), (4.25), and (4.31). In fact, writing the corresponding integrals in (4.58) as

$$\int_{Q_{T_0}}[(\mathbf{v}^n\cdot\boldsymbol{\nabla})\mathbf{v}^n-(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}]\,d\mathbf{x}dt=\int_{Q_{T_0}}[(\mathbf{v}^n-\mathbf{v})\cdot\boldsymbol{\nabla}]\,\mathbf{v}^n\,d\mathbf{x}dt-\int_{Q_{T_0}}(\mathbf{v}\cdot\boldsymbol{\nabla})(\mathbf{v}^n-\mathbf{v})\,d\mathbf{x}dt,$$

we see that the first right-hand side integral converges to zero by application of Hölder's inequality together with (4.10) and (4.52):

$$\begin{aligned}\int_{Q_{T_0}}[(\mathbf{v}^n-\mathbf{v})\cdot\boldsymbol{\nabla}]\,\mathbf{v}^n\,d\mathbf{x}dt\leq&\|\mathbf{v}^n-\mathbf{v}\|_{\mathbf{L}^2(Q_T)}\|\boldsymbol{\nabla}\mathbf{v}^n\|_{\mathbf{L}^2(Q_{T_0})}\leq\\&\sqrt{M_1}\|\mathbf{v}^n-\mathbf{v}\|_{\mathbf{L}^2(Q_{T_0})}\longrightarrow 0,\quad \text{as } n\to\infty,\end{aligned}$$

The second integral converges to zero, due to (4.43) and because $\mathbf{v}\in\mathbf{L}^2(Q_{T_0})$.

Analogical way, the convergence (4.59) can be proved due to (4.10) and (4.31).

By linearity, the equations (4.54) and (4.55) hold for any finite linear combination of $\boldsymbol{\varphi}_1,\ldots,\boldsymbol{\varphi}_n$ and $\psi_1,\ldots,\psi_n$, respectively, and, by a continuity argument, they are still true for any $\boldsymbol{\varphi}\zeta\in\mathbf{L}^2(0,T_0;\mathbf{H}^1(\Omega))$ and $\psi\xi\in L^2(0,T_0;W_0^{1,2}(\Omega))$ with $\zeta,\xi\in C_0^\infty(0,T)$, respectively. Moreover, all terms in the equations (4.54) and (4.55) are absolutely continuous as functions of t defined by integrals over $[0,T_0]$. So we obtain the following equalities which hold for a.e $t\in[0,T_0]$ and for any $\boldsymbol{\varphi}\in\mathcal{V}$ and $\psi\in W_0^{1,2}$, respectively

$$\begin{aligned}&\int_\Omega\Big[\mathbf{v}_t(t)+\Big(\mathbf{v}(t)\cdot\boldsymbol{\nabla}\Big)\mathbf{v}(t)\Big]\cdot\boldsymbol{\psi}\,d\mathbf{x}+\nu\int_\Omega\boldsymbol{\nabla}\mathbf{v}(t):\boldsymbol{\nabla}\boldsymbol{\varphi}\,d\mathbf{x}+\varkappa\int_\Omega\boldsymbol{\nabla}\mathbf{v}_t(t):\boldsymbol{\nabla}\boldsymbol{\varphi}\,d\mathbf{x}=\\&\int_\Omega\mathbf{g}(\mathbf{x},t)\theta(t)\boldsymbol{\varphi}d\mathbf{x}+\Phi(\mathbf{v},\theta)\int_\Omega\mathbf{h}(\mathbf{x},t)\boldsymbol{\varphi}\,d\mathbf{x}\end{aligned}\tag{4.60}$$

and

$$\int_\Omega\theta_t(t)\cdot\psi\,d\mathbf{x}+\int_\Omega(\mathbf{v}(t)\cdot\boldsymbol{\nabla})\theta(t)\psi\,d\mathbf{x}+\lambda\int_\Omega\boldsymbol{\nabla}\theta(t)\boldsymbol{\nabla}\psi\,d\mathbf{x}=J(\mathbf{v},\theta)\int_\Omega\phi(x,t)\psi\,d\mathbf{x}dt.\tag{4.61}$$

Thus, the pair of limit functions $(\mathbf{v},\theta)$ is the weak solution to the direct problem of (3.13)-(3.14), (1.4)-(1.6), and by Lemma 3, it together with the limit functions $f(t)=\Phi(\mathbf{v},\theta)$ and $j(t)=J(\mathbf{v},\theta)$ (see (3.11) and (3.12)) gives the weak solution to the inverse problem PI.

Furthermore, due to the weakly lower semicontinuity of norms, we obtain the following estimate from (4.10), (4.25), and (4.31)

$$\begin{aligned}&\|\mathbf{v}\|^2_{\mathbf{L}^\infty(0,T_0;\mathbf{H}^1(\Omega))}+\|\theta\|^2_{L^\infty\left(0,T_0;W_0^{1,2}(\Omega)\right)}+\|\mathbf{v}\|^2_{\mathbf{L}^2(0,T_0;\mathbf{H}^1(\Omega))}+\|\nabla\theta\|^2_{2,Q_{T_0}}+\\&\|\mathbf{v}_t\|^2_{\mathbf{L}^2(0,T_0;\mathbf{H}^1(\Omega))}+\|\theta_t\|^2_{2,Q_{T_0}}+\|\Delta\theta\|^2_{2,Q_{T_0}}\leq C<\infty.\end{aligned}\tag{4.62}$$

and from (4.14) and (4.15)

$$\begin{aligned}&\|f(t)\|^2_{L^2([0,T_0])}=\|\Phi(\mathbf{v},\theta)\|^2_{L^2([0,T_0])}=\int_0^{T_0}|f(t)|^2\,dt\leq\frac{3}{k_0^2}\Big[\|e'(t)\|^2_{L^2([0,T_0])}+\\&(\nu+C_s^2)^2\,\|\mathbf{v}\|^2_{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega))}\|\nabla\boldsymbol{\omega}\|^2_{2,\Omega}+g_0T\,\|\theta\|^2_{L^\infty(0,T_0;L(\Omega))}\|\boldsymbol{\omega}\|^2_{2,\Omega}\Big]=K_1<\infty\end{aligned}\tag{4.63}$$

and

$$\|j(t)\|^2_{L^2([0,T_0])} = \|J(\mathbf{v},\theta)\|^2_{L^2([0,T_0])} = \int_0^{T_0} |j(t)|^2\, dt \le \frac{3}{k_1^2}\left[\|\delta'(t)\|^2_{L^2([0,T_0])} + \left(\lambda^2 + C_s^4 \|\mathbf{v}^n\|^2_{\mathbf{L}^\infty(0,T_0;\mathbf{H}^1(\Omega))}\right)\|\nabla\eta\|^2_{2,\Omega}\|\nabla\theta\|^2_{2,Q_{T_0}}\right] = K_2 < \infty \tag{4.64}$$

respectively. The set of estimates (4.62)-(4.64) gives (4.1).

5. Existence of local in time weak solutions of PII

In this section, we study the inverse problem PII, associated to the sliding condition (1.8), and therefore, by Lemma 3, the corresponding equivalent direct problem (3.13)-(3.14), (1.4)-(1.5), (1.8). For this problem the following is hold.

Theorem 2. *Let the conditions (3.3)-(3.10) be fulfilled. Then there exists $T_1 \in (0,T]$, such that the direct problem* (3.13)-(3.14), (1.4)-(1.5), (1.8) *has at least a weak solution $(\mathbf{v}(\mathbf{x},t),\theta(\mathbf{x},t))$ in the cylinder Q_{T_1}. Accordingly, the inverse problem PII has at least a weak solution $(\mathbf{v}(\mathbf{x},t),\theta(\mathbf{x},t),f(t),j(t))$ in Q_{T_1}, and for a weak solution to the inverse problem PII the estimate* (4.1) *is hold for all $t \in (0,T_1]$, where T_1 is defined at* (5.6) *below.*

Proof. In order to prove this theorem, it is enough to prove an alternative estimates of (4.10) and (4.25). Due to the equivalencies of norms $\|\operatorname{rot}\mathbf{v}\|_{2,\Omega}$ and $\|\mathbf{v}\|_{\mathbf{W}^{1,2}(\Omega)}$ or $\|\nabla\mathbf{v}\|_{2,\Omega}$ in $\mathbf{H}^1_{\mathbf{n}}$, many techniques are similar as in previous section, for instance, the estimate (4.31) is still true. Therefore, we will omit some details of proof.

Thus, in the case (1.8), due to the Green's formulas (2.12)-(2.13), the equalities (4.11) and (4.26) have the following form, respectively

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{v}^n\|^2_{2,\Omega} + \varkappa\|\operatorname{rot}\mathbf{v}^n\|^2_{2,\Omega}\right) + \nu\|\operatorname{rot}\mathbf{v}^n\|^2_{2,\Omega} = \Phi^n(\mathbf{v}^n,\theta^n)(t)\,(\mathbf{h},\mathbf{v}^n)_{2,\Omega} + (\mathbf{g}\theta^n,\mathbf{v}^n)_{2,\Omega}, \tag{5.1}$$

$$\frac{\nu}{2}\|\operatorname{rot}\mathbf{v}^n\|^2_{2,\Omega} + \|\mathbf{v}^n_t(t)\|^2_{2,Q_t} + \varkappa\|\operatorname{rot}\mathbf{v}^n_t(t)\|^2_{2,Q_t} = \frac{\nu}{2}\|\operatorname{rot}\mathbf{v}^n(0)\|^2_{2,\Omega} + \int_0^t \left[\Phi^n(\mathbf{v}^n,\theta^n)(s)\,(\mathbf{h}(s),\mathbf{v}^n_t(s)) + (\gamma g(s)\theta^n(s),\mathbf{v}^n_t(s)) + ((\mathbf{v}^n(s)\cdot\nabla)\,\mathbf{v}^n_t(s),\mathbf{v}^n(s))\right] ds, \tag{5.2}$$

where

$$\Phi^n(\mathbf{v}^n,\theta^n)(t) = \frac{1}{h_0(t)}\left(e'(t) - ((\mathbf{v}^n\cdot\boldsymbol{\nabla})\,\boldsymbol{\omega},\mathbf{v}^n)_{2,\Omega} + \nu\,(\operatorname{rot}\mathbf{v}^n,\operatorname{rot}\boldsymbol{\omega})_{2,\Omega} - (\mathbf{g}(\mathbf{x},t)\theta^n,\boldsymbol{\omega})_{2,\Omega}\right). \tag{5.3}$$

Estimating the terms on right-hand side of (5.1) and (4.12) as (4.16)-(4.18), and using the equivalence norms $\|\operatorname{rot}\mathbf{v}\|_{2,\Omega}$ and $\|\nabla\mathbf{v}\|_{2,\Omega}$ ((2.3)-(2.4)), we obtain from (5.1) and (4.12)

$$\frac{d}{dt}\left(\|\mathbf{v}^n\|^2_{2,\Omega} + \varkappa\|\operatorname{rot}\mathbf{v}^n\|^2_{2,\Omega} + \|\theta^n\|^2_{2,\Omega}\right) + \nu\|\operatorname{rot}\mathbf{v}^n\|^2_{2,\Omega} + \lambda\|\nabla\theta^n\|^2_{2,\Omega} \le C_6'(t)\left(\|\mathbf{v}^n\|^2_{2,\Omega} + \|\theta^n\|^2_{2,\Omega}\right) + C_7'(t)\left(\varkappa\|\operatorname{rot}\mathbf{v}^n\|^2_{2,\Omega} + \|\theta^n\|^2_{2,\Omega}\right)^2 + C_8'(t). \tag{5.4}$$

Integrating (5.4) by s from 0 to t and using Hölder inequality, we get

$$\begin{gathered}
\|\mathbf{v}^n\|_{2,\Omega}^2+\varkappa\|\operatorname{rot}\mathbf{v}^n\|_{2,\Omega}^2+\|\theta^n\|_{2,\Omega}^2+\int_0^t\left(\nu\|\operatorname{rot}\mathbf{v}^n\|_{2,\Omega}^2+\lambda\|\nabla\theta^n\|_{2,\Omega}^2\right)ds\leq\\
\frac{1}{4}C_6T+C_6\int_0^t\left(\|\mathbf{v}^n\|_{2,\Omega}^2+\|\theta^n\|_{2,\Omega}^2\right)^2ds+C_7\int_0^t\left(\varkappa\|\operatorname{rot}\mathbf{v}^n\|_{2,\Omega}^2+\|\theta^n\|_{2,\Omega}^2\right)^2ds+\\
C_8\left(\|e'(t)\|_{2,[0,T]}^2+\|\delta'(t)\|_{2,[0,T]}^2\right)+\|\mathbf{v}_0\|_{2,\Omega}^2+\varkappa\|\operatorname{rot}\mathbf{v}_0\|_{2,\Omega}^2+\|\theta_0\|_{2,\Omega}^2\leq\\
C_9\int_0^t\left(\|\mathbf{v}^n\|_{2,\Omega}^2+\varkappa\|\operatorname{rot}\mathbf{v}^n\|_{2,\Omega}^2+\|\theta^n\|_{2,\Omega}^2\right)^2ds+C_{10},
\end{gathered}\tag{5.5}$$

where $C_i, i=6,...,10$ are positive constants independent of n.

Analogical as we got (4.24) from (4.20), it follows from (5.5) that there exists a finite time T_1

$$T_1<T_{**}:=\frac{1}{C_9C_{10}},\tag{5.6}$$

such that for all $0<t\leq T_1$ the following estimate is hold

$$\sup_{t\in[0,T_1]}\left(\|\mathbf{v}^n\|_{2,\Omega}^2+\varkappa\|\operatorname{rot}\mathbf{v}^n\|_{2,\Omega}^2+\|\theta^n\|_{2,\Omega}^2\right)+\nu\|\operatorname{rot}\mathbf{v}^n\|_{\mathbf{L}^2(Q_{T_1})}^2+\lambda\|\nabla\theta^n\|_{\mathbf{L}^2(Q_{T_1})}^2\leq M_1',\tag{5.7}$$

which is an analog of the estimate (4.10) with a constant $M_0:=M_1'<\infty$.

Next, estimate the right-hand side of (5.2) by using Hölder, Young inequalities together with (2.3) and (5.7) as in (4.27)-(4.29). Then, we get

$$\sup_{t\in[0,T_1]}\|\operatorname{rot}\mathbf{v}^n\|_{\mathbf{L}^\infty(0,T_1;\mathbf{L}^2(\Omega))}^2+\|\mathbf{v}_t^n(t)\|_{2,Q_{T_1}}^2+\|\operatorname{rot}\mathbf{v}_t^n(t)\|_{2,Q_{T_1}}^2\leq M_2'<\infty.\tag{5.8}$$

□

6. Existence of local in time strong solutions of PI and PII

In this section, we establish the existence of the strong solution of PI and PII, defined in Definition 2.

Theorem 3. *Let the conditions (3.3)-(3.10) and (4.8), (4.9) be fulfilled. Assume that also*

$$\mathbf{v}_0\in\mathbf{H}^1(\Omega)\cap\mathbf{H}^2(\Omega).\tag{6.1}$$

Then there exists $T_2\in(0,T]$, such that the direct problem problems (3.13)-(3.14), (1.4)-(1.6) and (3.13)-(3.14), (1.4)-(1.5), (1.8) have at least a weak solution $(\mathbf{v}(\mathbf{x},t),\theta(\mathbf{x},t))$ in the cylinder Q_{T_3}. Therefore, corresponding inverse problems PI and PII have a strong solutions and for them the following estimates are hold

$$\begin{gathered}
\|\mathbf{v}\|_{\mathbf{L}^\infty(0,T_2;\mathbf{H}^1\cap\mathbf{H}^2(\Omega))}^2+\|\mathbf{v}_t\|_{\mathbf{L}^2(0,T_2;\mathbf{H}^1\cap\mathbf{H}^2(\Omega))}^2+\|f(t)\|_{L^2([0,T_2])}^2+\\
\|\theta\|_{\mathbf{L}^\infty\left(0,T_2;W_0^{1,2}(\Omega)\right)}^2+\|\theta\|_{L^2\left(0,T_2;W_0^{1,2}(\Omega)\cap W^{2,2}(\Omega)\right)}^2+\|\theta_t\|_{2,Q_{T_2}}^2+\|j(t)\|_{L^2([0,T_2])}^2\leq M_3<\infty.
\end{gathered}\tag{6.2}$$

where $T_3 = T_0$ in the case (1.6) and $T_3 = T_1$ in the case (1.8), and M_3 is positive constant depending on data of the problem.

Proof. To prove the existence of a strong solutions to these problems, we use the special basis, associated to the eigenfunctions of the spectral problem

$$A_1\boldsymbol{\varphi} := \widetilde{\Delta}\boldsymbol{\varphi}_k(x) = \mu_k\ \boldsymbol{\varphi}_k(x),\ \ \boldsymbol{\varphi}_k(x) \in \mathbf{H}^1(\Omega) \cap \mathbf{H}^2(\Omega), \tag{6.3}$$

in the case (1.6) and

$$A_2\boldsymbol{\varphi} := -\Delta\boldsymbol{\varphi}_k(x) = \mu_k\ \boldsymbol{\varphi}_k(x), \quad \boldsymbol{\varphi}_k \in \mathbf{H}^1_{\mathbf{n}}(\Omega) \cap \mathbf{H}^2_{\mathbf{n}}(\Omega) \tag{6.4}$$

in the case (1.8). The latter is due to the fact (see [20])

$$(\Delta\boldsymbol{\varphi}, \nabla\pi) = 0 \text{ for any } \ \boldsymbol{\varphi} \in \mathbf{H}^1_{\mathbf{n}} \cap \mathbf{H}^2_{\mathbf{n}}(\Omega),\ \pi \in W^{1,2}(\Omega),\ \text{and } \mathbf{L}^2(\Omega) = \mathbf{H}_{\mathbf{n}}(\Omega) \oplus \mathbf{G}(\Omega).$$

In (6.3), $\widetilde{\Delta}\boldsymbol{\varphi} = -\mathbb{P}\Delta\boldsymbol{\varphi}$, and $\mathbb{P} : \mathbf{L}^2(\Omega) \to \mathbf{H}(\Omega)$ is the Leray projector.

It is known from [19] and [20], that the system $\{\boldsymbol{\varphi}_k\}_{k\in\infty}$ of eigenfunctions of both spectral problems (6.3) and (6.4) are orthogonal in $\mathbf{H}$ and an orthonormal basis in the space $\mathbf{H}^1(\Omega)\cap \mathbf{W}^{2,2}(\Omega)$ and $\mathbf{H}^1_n(\Omega) \cap \mathbf{H}^2_n(\Omega)$, respectively.

Let us first consider the PI, the problem PII is similar. In this case, all first and second estimates are true for strong solution. Thus, in order to complete the proof this theorem, it is sufficient to get more strong estimates, i.e. estimate $\Delta\mathbf{v}^n$ and $\Delta\mathbf{v}^n_t$. Let us multiply the first equation of (4.4) by $-\mu_k c^n_k(t)$ and $-\mu_k \frac{dc^n_k(t)}{dt}$, and sum with respect to k, from 1 to n. Taking in account equality (6.3), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{v}^n\|^2_{\mathbf{H}^1(\Omega)} + \varkappa\left\|\widetilde{\Delta}\mathbf{v}^n\right\|^2_{2,\Omega}\right) + \nu\left\|\widetilde{\Delta}\mathbf{v}^n\right\|^2_{2,\Omega} = \\ &\Phi^n(\mathbf{v}^n,\theta^n)(t)\left(\mathbf{h}(\mathbf{x},t), -\widetilde{\Delta}\mathbf{v}^n\right)_{2,\Omega} + \left(\mathbf{g}(\mathbf{x},t)\theta^n, -\widetilde{\Delta}\mathbf{v}^n\right)_{2,\Omega} + \left(\left(\mathbf{v}^n\cdot\nabla\right)\mathbf{v}^n, -\widetilde{\Delta}\mathbf{v}^n\right),\end{aligned} \tag{6.5}$$

$$\begin{aligned}&\frac{\nu}{2}\frac{d}{dt}\left\|\widetilde{\Delta}\mathbf{v}^n\right\|^2_{2,\Omega} + \|\mathbf{v}^n_t\|^2_{\mathbf{H}^1(\Omega)} + \varkappa\left\|\widetilde{\Delta}\mathbf{v}^n_t\right\|^2_{2,\Omega} = \\ &\Phi^n(\mathbf{v}^n,\theta^n)(t)\left(\mathbf{h}(\mathbf{x},t), -\widetilde{\Delta}\mathbf{v}^n_t\right)_{2,\Omega} + \left(\mathbf{g}(\mathbf{x},t)\theta^n, -\widetilde{\Delta}\mathbf{v}^n_t\right)_{2,\Omega} + \left(\left(\mathbf{v}^n\cdot\nabla\right)\mathbf{v}^n, -\widetilde{\Delta}\mathbf{v}^n_t\right).\end{aligned} \tag{6.6}$$

Estimating the terms on right hand side by using Hölder and Cauchy inequalities together with first energy estimates, we obtain the following inequalities

$$\begin{aligned}&\left|\Phi^n(\mathbf{v}^n,\theta^n)\left(\mathbf{h}(\mathbf{x},t), -\widetilde{\Delta}\mathbf{v}^n\right)_{2,\Omega} + \left(\mathbf{g}(\mathbf{x},t)\theta^n, -\widetilde{\Delta}\mathbf{v}^n\right)_{2,\Omega} + \left(\left(\mathbf{v}^n\cdot\nabla\right)\mathbf{v}^n, -\widetilde{\Delta}\mathbf{v}^n\right)\right| \leq \\ &|\Phi^n(\mathbf{v}^n,\theta^n)|\,\|\mathbf{h}(t)\|_{2,\Omega}\left\|\widetilde{\Delta}\mathbf{v}^n\right\|_{2,\Omega} + g_0\|\theta^n\|_{2,\Omega}\left\|\widetilde{\Delta}\mathbf{v}^n\right\|_{2,\Omega} + \|\mathbf{v}^n\|_{4,\Omega}\cdot\|\nabla\mathbf{v}^n\|_{4,\Omega}\left\|\widetilde{\Delta}\mathbf{v}^n\right\|_{2,\Omega} \leq \\ &\frac{\nu}{2}\left\|\widetilde{\Delta}\mathbf{v}^n\right\|^2_{2,\Omega} + \frac{3}{2\nu}|\Phi^n(\mathbf{v}^n,\theta^n)|^2\|\mathbf{h}(t)\|^2_{2,\Omega} + \frac{3g_0^2}{2\nu}\|\theta^n\|^2_{2,\Omega} + \frac{3}{2\nu}C(\Omega)\|\mathbf{v}^n\|^2_{\mathbf{H}^1(\Omega)}\cdot\|\mathbf{v}^n\|^2_{\mathbf{W}^{2,2}(\Omega)}\end{aligned} \tag{6.7}$$

Likewise,

$$\begin{aligned}&\left|\Phi^n(\mathbf{v}^n,\theta^n)\left(\mathbf{h}(\mathbf{x},t), -\widetilde{\Delta}\mathbf{v}^n_t\right)_{2,\Omega} + \left(\mathbf{g}(\mathbf{x},t)\theta^n, -\widetilde{\Delta}\mathbf{v}^n_t\right)_{2,\Omega} + \left(\left(\mathbf{v}^n\cdot\nabla\right)\mathbf{v}^n, -\widetilde{\Delta}\mathbf{v}^n_t\right)\right| \leq \\ &\frac{\varkappa}{2}\left\|\widetilde{\Delta}\mathbf{v}^n_t\right\|^2_{2,\Omega} + \frac{3}{2\varkappa}|\Phi^n(\mathbf{v}^n,\theta^n)|^2\|\mathbf{h}(t)\|^2_{2,\Omega} + \frac{3g_0^2}{2\varkappa}\|\theta^n\|^2_{2,\Omega} + \frac{3}{2\varkappa}C(\Omega)\|\mathbf{v}^n\|^2_{\mathbf{H}^1(\Omega)}\cdot\|\mathbf{v}^n\|^2_{\mathbf{W}^{2,2}(\Omega)}\end{aligned} \tag{6.8}$$

Plugging (6.7) and (6.8) into (6.5) and (6.6), respectively, and integrating by τ from 0 to $t\in[0,T_0]$, we derive

$$\varkappa\left\|\widetilde{\Delta}\mathbf{v}^n\right\|_{2,\Omega}^2+\nu\int_0^t\left\|\widetilde{\Delta}\mathbf{v}^n(s)\right\|_{2,\Omega}^2ds\le\|\mathbf{v}_0\|_{\mathbf{H}^1(\Omega)}^2+\varkappa\left\|\widetilde{\Delta}\mathbf{v}_0\right\|_{2,\Omega}^2+\frac{3}{2\nu}g_0^2\int_0^t\|\theta^n(s)\|_{2,\Omega}^2ds+$$
$$\frac{3}{2\nu}\|\mathbf{h}(t)\|_{L^\infty(0,t;\mathbf{L}^2(\Omega))}^2\int_0^t|\Phi^n(\mathbf{v}^n,\theta^n)(s)|^2ds+\frac{3}{2\nu}C(\Omega)\int_0^t\|\mathbf{v}^n(s)\|_{\mathbf{H}^1(\Omega)}^2\cdot\|\mathbf{v}^n(s)\|_{\mathbf{W}^{2,2}(\Omega)}^2ds,\tag{6.9}$$

$$\nu\left\|\widetilde{\Delta}\mathbf{v}^n\right\|_{2,\Omega}^2+\varkappa\int_0^t\left\|\widetilde{\Delta}\mathbf{v}_t^n(s)\right\|_{2,\Omega}^2ds\le\nu\left\|\widetilde{\Delta}\mathbf{v}_0\right\|_{2,\Omega}^2+\frac{3g_0^2}{2\varkappa}\int_0^t\|\theta^n(s)\|_{2,\Omega}^2ds+$$
$$\frac{3}{2\varkappa}\|\mathbf{h}(t)\|_{L^\infty(0,t;\mathbf{L}^2(\Omega))}^2\int_0^t|\Phi^n(\mathbf{v}^n,\theta^n)(s)|^2ds+\frac{3}{2\varkappa}C(\Omega)\int_0^t\|\mathbf{v}^n(s)\|_{\mathbf{H}^1(\Omega)}^2\cdot\|\mathbf{v}^n(s)\|_{\mathbf{W}^{2,2}(\Omega)}^2ds\tag{6.10}$$

Adding (6.9) and (6.10), and applying the already obtained estimates for $f(t)$, θ, and $\nabla\mathbf{v}^n$, and using

$$\|\mathbf{v}^n\|_{\mathbf{W}^{2,2}(\Omega)}\le C(\Omega)\left\|\widetilde{\Delta}\mathbf{v}^n\right\|_{2,\Omega},\ \forall\mathbf{u}\in\mathbf{H}^1(\Omega)\cap\mathbf{W}^{2,2}(\Omega)\ (\text{and }\mathbf{H}^1(\Omega)\cap\mathbf{V}_n^2(\Omega)),$$

see (2.4) and Lemma 1, we obtain

$$\left\|\widetilde{\Delta}\mathbf{v}^n\right\|_{2,\Omega}^2+\int_0^t\left\|\widetilde{\Delta}\mathbf{v}^n(s)\right\|_{2,\Omega}^2ds+\int_0^t\left\|\widetilde{\Delta}\mathbf{v}_t^n(s)\right\|_{2,\Omega}^2ds\le C_6+C_7\int_0^t\left\|\widetilde{\Delta}\mathbf{v}^n(s)\right\|_{2,\Omega}^2ds,\tag{6.11}$$

where

$$C_6=C(\nu,\varkappa)\left(\|\nabla\mathbf{v}_0\|_{2,\Omega}^2+\|\Delta\mathbf{v}_0\|_{2,\Omega}^2+3M_0\|\mathbf{h}(t)\|_{L^\infty(0,t;\mathbf{L}^2(\Omega))}^2+3g_0^2M_1\right)=const<\infty,$$
$$C_7=C(\nu,\varkappa)3C(\Omega)C_s\|\mathbf{v}^n\|_{L^\infty(0,T_0;\mathbf{H}^1(\Omega))}^2=3C(\nu,\varkappa)C(\Omega)C_sM_0=const<\infty.$$

By standard techniques, it follows from (6.11) that

$$\|\Delta\mathbf{v}\|_{\mathbf{L}^\infty(0,T_2;\mathbf{L}^2(\Omega))}^2+\|\Delta\mathbf{v}\|_{\mathbf{L}^2(Q_{T_2})}^2+\|\Delta\mathbf{v}_t\|_{\mathbf{L}^2(Q_{T_2})}^2\le C_8<\infty.\tag{6.12}$$

Thus, the estimates (4.10), (4.25) (or (5.7), (5.8)) and (4.31) together (6.12) give (4.1).

The passing to the limit for a strong solution can be proved by using arguments similar to above, thus we omit the details of the corresponding proof. □

7. Uniqueness of weak solutions of PI and PII

In this section, we study the uniqueness of weak and strong solutions of the above inverse problems. In order to establish these, by Remark 3 and Lemma 3, it is enough to prove the the uniqueness of solutions of the corresponding an equivalent direct problems.

Theorem 4. *Let the assumptions (3.3)-(3.10) be fulfilled. Then the weak solution, fortiori a strong solution of the direct problems (3.13)-(3.14), (1.4)-(1.6) and (3.13)-(3.14), (1.4)-(1.5), (1.8) is unique in $Q_{T_{max}}$, where T_{max} is a maximal time such that the solutions of corresponding problems are exist.*

Proof. We prove for (3.13)-(3.14), (1.4)-(1.6), and (3.13)-(3.14), (1.4)-(1.5), (1.8) is a similar. Let $(\mathbf{v}_i, \theta_i)$, with $i = 1,\ 2$, be two different weak solutions of (3.13)-(3.14), (1.4)-(1.6), and set us $\mathbf{v} := \mathbf{v}_1 - \mathbf{v}_2$, $\theta = \theta_1 - \theta_2$. Then, arguing as proof of Lemma 3, we obtain the following equivalent nonlocal problem

$$\mathbf{v}_t - \varkappa\Delta\mathbf{v}_t - \nu\Delta\mathbf{v} + \boldsymbol{\nabla} p + (\mathbf{v}\cdot\nabla)\,\mathbf{v}_1 + (\mathbf{v}_2\cdot\nabla)\,\mathbf{v} = \mathbf{g}(\mathbf{x},t)\theta(x,t) + \Phi(\mathbf{v},\theta)\mathbf{h}(\mathbf{x},t), \quad Q_T, \tag{7.1}$$

$$\operatorname{div}\mathbf{v}(\mathbf{x},t) = 0, \quad (x,t)\in Q_T, \tag{7.2}$$

$$\theta_t + (\mathbf{v}\cdot\nabla)\,\theta_1 + (\mathbf{v}_2\cdot\nabla)\,\theta - \lambda\Delta\theta = J(\mathbf{v},\theta)\phi(\mathbf{x},t), \quad (x,t)\in Q_T, \tag{7.3}$$

$$\mathbf{v}(\mathbf{x},0) = 0, \quad \theta(\mathbf{x},0) = 0, \quad \mathbf{x}\in\Omega, \tag{7.4}$$

$$\theta(\mathbf{x},t) = 0, \ \ (\mathbf{x},t)\in\Gamma_T. \tag{7.5}$$

and

$$\mathbf{v}(\mathbf{x},t) = 0 \ \text{ or } \ \mathbf{v}_{\mathbf{n}}(\mathbf{x},t) = 0, \quad (D(\mathbf{v})\cdot\mathbf{n})\,\mathbf{v}\times\mathbf{n} = 0 \ \ (\mathbf{x},t)\in\Gamma_T \tag{7.6}$$

in the case (1.6) or (1.8), respectively, where

$$\Phi(\mathbf{v},\theta) = \frac{1}{h_0(t)}\left[\nu\mathbf{a}\left(\mathbf{v},\boldsymbol{\omega}\right) - \left((\mathbf{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{\omega},\mathbf{v}_1\right)_{2,\Omega} - \left((\mathbf{v}_2\cdot\boldsymbol{\nabla})\,\boldsymbol{\omega},\mathbf{v}\right)_{2,\Omega} - \left(\mathbf{g}\theta,\boldsymbol{\omega}\right)_{2,\Omega}\right] \tag{7.7}$$

$$J(\mathbf{v},\theta) = \frac{1}{\phi_0(t)}\left[\lambda\left(\nabla\theta,\nabla\eta\right)_{2,\Omega} - \left((\mathbf{v}\cdot\nabla)\,\eta,\theta_1\right)_{2,\Omega} - \left((\mathbf{v}_2\cdot\nabla)\,\eta,\theta\right)_{2,\Omega}\right]. \tag{7.8}$$

Multiplying (7.1) and (7.3) by $\mathbf{v}$ and θ, respectively, and integrating the result over Ω, we obtain

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{v}(t)\|^2_{\mathbf{L}^2(\Omega)} + \varkappa\|\mathbf{v}(t)\|^2_{\mathbf{H}^1(\Omega)}\right) + \nu\|\mathbf{v}(t)\|^2_{\mathbf{H}^1(\Omega)} = \\ \left(\mathbf{g}(\mathbf{x},t)\theta,\,\mathbf{v}\right)_{2,\Omega} + \Phi(\mathbf{v},\theta)\left(\mathbf{h},\,\mathbf{v}\right)_{2,\Omega} - \left((\mathbf{v}\cdot\nabla)\,\mathbf{v}_1,\,\mathbf{v}\right)_{2,\Omega},\end{gathered} \tag{7.9}$$

$$\frac{1}{2}\frac{d}{dt}\|\theta(t)\|^2_{\mathbf{L}^2(\Omega)} + \lambda\|\nabla\theta(t)\|^2_{\mathbf{L}^2(\Omega)} = J(\mathbf{v},\theta)\left(\phi,\,\theta\right)_{2,\Omega} - \left((\mathbf{v}\cdot\nabla)\,\theta_1,\,\theta\right)_{2,\Omega}. \tag{7.10}$$

Now, we estimate the terms on the right-hand side by using Hölder and Young inequalities together with the Ladyzhenskaya inequalities:

$$\left|\left(\mathbf{g}(\mathbf{x},t)\theta,\,\mathbf{v}\right)_{2,\Omega}\right| \le g_0\left\|\theta\right\|_{2,\Omega}\left\|\mathbf{v}(t)\right\|_{2,\Omega} \le \frac{g_0}{2}\left(\|\theta\|^2_{2,\Omega} + \|\mathbf{v}(t)\|^2_{2,\Omega}\right), \tag{7.11}$$

$$\begin{aligned}
&\left|\Phi(\mathbf{v},\theta)\left(\mathbf{h},\,\mathbf{v}\right)_{2,\Omega}\right| \le |\Phi(\mathbf{v},\theta)|\,\|\mathbf{h}\|_{2,\Omega}\,\|\mathbf{v}(t)\|_{2,\Omega} \le \frac{1}{k_0}\Big[\nu\,\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}\,\|\boldsymbol{\omega}\|_{\mathbf{H}^1(\Omega)}+\\
&\|\mathbf{v}_1\|_{\mathbf{H}^1(\Omega)}\,\|\boldsymbol{\omega}\|_{\mathbf{H}^1(\Omega)}\,\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}+\|\mathbf{v}_2\|_{\mathbf{H}^1(\Omega)}\,\|\boldsymbol{\omega}\|_{\mathbf{H}^1(\Omega)}\,\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}+\\
&g_0\,\|\theta\|_{2,\Omega}\,\|\boldsymbol{\omega}\|_{2,\Omega}\Big]\,\|\mathbf{h}\|_{2,\Omega}\,\|\mathbf{v}(t)\|_{2,\Omega}\le\\
&\frac{\nu}{4}\,\|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega)}+\frac{1}{\nu k_0^2}\left(\nu+\|\mathbf{v}_1\|_{\mathbf{H}^1(\Omega)}+\|\mathbf{v}_2\|_{\mathbf{H}^1(\Omega)}\right)^2\|\boldsymbol{\omega}\|^2_{\mathbf{H}^1(\Omega)}\,\|\mathbf{h}\|^2_{2,\Omega}\,\|\mathbf{v}\|^2_{2,\Omega}+\\
&\frac{g_0}{2k_0}\,\|\mathbf{h}\|_{2,\Omega}\,\|\boldsymbol{\omega}\|_{2,\Omega}\left(\|\theta\|^2_{2,\Omega}+\|\mathbf{v}\|^2_{2,\Omega}\right).
\end{aligned} \tag{7.12}$$

$$\left|-\left(\left(\mathbf{v}\cdot\nabla\right)\mathbf{v}_1,\,\mathbf{v}\right)_{2,\Omega}\right|\le\|\nabla\mathbf{v}_1\|_{2,\Omega}\,\|\mathbf{v}\|^2_{4,\Omega}\le C(\Omega)\,\|\mathbf{v}_1\|_{\mathbf{H}^1(\Omega)}\,\|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega)}\,, \tag{7.13}$$

$$\begin{aligned}
&\left|J(\mathbf{v},\theta)\left(\phi,\,\theta\right)_{2,\Omega}\right|\le|J(\mathbf{v},\theta)|\,\|\phi\|_{2,\Omega}\,\|\theta(t)\|_{2,\Omega}\le\frac{1}{k_1}\Big[\lambda\,\|\nabla\theta\|_{2,\Omega}\,\|\nabla\eta\|_{2,\Omega}+\\
&C(\Omega)\,\|\nabla\theta_1\|_{2,\Omega}\,\|\nabla\eta\|_{2,\Omega}\,\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}+C(\Omega)\,\|\mathbf{v}_2\|_{\mathbf{H}^1(\Omega)}\,\|\nabla\eta\|_{2,\Omega}\,\|\nabla\theta\|_{2,\Omega}\Big]\,\|\phi\|_{2,\Omega}\,\|\theta(t)\|_{2,\Omega}\le\\
&\frac{\lambda}{4}\,\|\nabla\theta\|^2_{2,\Omega}+\frac{\nu}{4}\,\|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega)}+a_0(t)\,\|\theta(t)\|^2_{2,\Omega}\,,
\end{aligned} \tag{7.14}$$

where

$$a_0(t)=\frac{1}{k_1^2}\left(\frac{2}{\lambda}+\frac{1}{\nu}C^2(\Omega)\,\|\nabla\theta_1\|^2_{2,\Omega}+\frac{2}{\lambda}C^2(\Omega)\,\|\mathbf{v}_2\|^2_{\mathbf{H}^1(\Omega)}\right)\|\nabla\eta\|^2_{2,\Omega}\,\|\phi(t)\|^2_{2,\Omega}\,.$$

$$\begin{aligned}
&\left|-\left(\left(\mathbf{v}\cdot\nabla\right)\theta_1,\,\theta\right)_{2,\Omega}\right|\le\|\nabla\theta_1\|_{2,\Omega}\,\|\mathbf{v}\|_{4,\Omega}\,\|\theta\|_{4,\Omega}\le\\
&C(\Omega)\,\|\nabla\theta_1\|_{2,\Omega}\,\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}\,\|\nabla\theta\|_{2,\Omega}\le\frac{\lambda}{4}\,\|\nabla\theta\|^2_{2,\Omega}+\frac{1}{\lambda}C^2(\Omega)\,\|\nabla\theta_1\|^2_{2,\Omega}\,\|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega)}
\end{aligned} \tag{7.15}$$

Plugging (7.11)-(7.13) into (7.9) and (7.14)-(7.15) into (7.10), and adding the results we get

$$\begin{aligned}
&\frac{d}{dt}\left(\|\mathbf{v}(t)\|^2_{2,\Omega}+\varkappa\|\mathbf{v}(t)\|^2_{\mathbf{H}^1(\Omega)}+\|\theta(t)\|^2_{2,\Omega}\right)+\nu\|\mathbf{v}(t)\|^2_{\mathbf{H}^1(\Omega)}+\lambda\|\nabla\theta(t)\|^2_{2,\Omega}\le\\
&a_1(t)\left(\|\mathbf{v}(t)\|^2_{2,\Omega}+\varkappa\|\mathbf{v}(t)\|^2_{\mathbf{H}^1(\Omega)}+\|\theta(t)\|^2_{2,\Omega}\right),
\end{aligned} \tag{7.16}$$

where

$$a_1(t)=\max\left\{a_2(t),\;a_3(t),\;\frac{2}{\varkappa}a_4(t),\right\},$$

and

$$a_2(t)=g_0\left(1+\frac{1}{k_0}\,\|\mathbf{h}\|_{2,\Omega}\,\|\boldsymbol{\omega}\|_{2,\Omega}\right)+\frac{2}{\nu k_0^2}\left(\nu+\|\mathbf{v}_1\|_{\mathbf{H}^1(\Omega)}+\|\mathbf{v}_2\|_{\mathbf{H}^1(\Omega)}\right)^2\|\boldsymbol{\omega}\|^2_{\mathbf{H}^1(\Omega)}\,\|\mathbf{h}\|^2_{2,\Omega}\,,$$

$$a_3(t)=g_0\left(1+\frac{1}{k_0}\,\|\mathbf{h}\|_{2,\Omega}\,\|\boldsymbol{\omega}\|_{2,\Omega}\right)+2a_0(t),$$

$$a_4(t)=C(\Omega)\,\|\mathbf{v}_1\|_{\mathbf{H}^1(\Omega)}+\frac{1}{\lambda}C^2(\Omega)\,\|\nabla\theta_1\|^2_{2,\Omega}\,.$$

Due to the conditions (3.4)-(3.10) and the first and second a priori estimates (4.10) and (4.31), $a_i(t)\in L^1[0,T_{max}]$, $i=0,1,2,3,4$ and then by Grönwall's lemma, it follows from (7.16) that

$$\|\mathbf{v}(t)\|^2_{2,\Omega}+\varkappa\|\mathbf{v}(t)\|^2_{\mathbf{H}^1(\Omega)}+\|\theta(t)\|^2_{2,\Omega}=0\quad\text{for all}\quad t\in[0,T_{max}],$$

which yields $\mathbf{v}_1 = \mathbf{v}_2$ and $\theta_1 = \theta_2$ for all $t \in [0, T_{max}]$, i.e. the weak and strong solution of (3.13)-(3.14), (1.4)-(1.6) is unique, where $T_{max} = T_0$ ($T_{max} = T_1$ for (3.13)-(3.14), (1.4)-(1.5), (1.8)). □

8. Modifications of inverse problems PI and PII allowing global in time solutions.

In this section we consider the questions of global in time existence and uniqueness of a weak solutions to the inverse problems PI and PII. The main difficulty in proving the existence of global in time solutions to the inverse problems PI and PII is associated with obtaining the first a priori estimate (4.1). This difficulty arises from the presence of a nonlinear convective member $(\mathbf{v}\cdot\nabla)\,\mathbf{v}$ in the functional $\Phi(\mathbf{v},\theta)$ defined by (3.11). However, the global solvability can be established under some additional restrictions on given functions or when the convective term is neglected.

8.1. **Global existence: in the case of special source terms.** Let us consider the problem PI (the inverse problem PII is similar) with the special right-hand sides $\mathbf{h}(x,t) := \boldsymbol{\sigma}(\mathbf{x})$ and $\phi(\mathbf{x}) := \eta(\mathbf{x})$, i.e. with the same functions $\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\omega}(\mathbf{x}) - \varkappa\Delta\boldsymbol{\omega}(\mathbf{x})$ and $\eta(\mathbf{x})$ included in the integral overdetermination conditions (1.7):

$$\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \varkappa\Delta\mathbf{v}_t - \nu\Delta\mathbf{v} + \boldsymbol{\nabla}\pi = \mathbf{g}(\mathbf{x},t)\theta(x,t) + f(t)\boldsymbol{\sigma}(\mathbf{x}), \quad (x,t)\in Q_T, \tag{8.1}$$

$$\operatorname{div}\mathbf{v}(\mathbf{x},t) = 0, \quad (x,t)\in Q_T, \tag{8.2}$$

$$\theta_t + (\mathbf{v}\cdot\nabla)\,\theta - \lambda\Delta\theta = j(t)\eta(\mathbf{x}), \quad (x,t)\in Q_T. \tag{8.3}$$

$$\mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}),\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}), \quad \mathbf{x}\in\Omega, \tag{8.4}$$

$$\theta(\mathbf{x},t) = 0,\ \mathbf{v}(\mathbf{x},t) = 0, \quad (\mathbf{x},t)\in\Gamma_T, \tag{8.5}$$

or

$$\theta(\mathbf{x},t) = 0,\ \mathbf{v}_{\mathbf{n}}(\mathbf{x},t) = \mathbf{v}\cdot\mathbf{n} = 0, \quad (\mathbf{D}(\mathbf{v})\cdot\mathbf{n})\,\mathbf{v}\times\mathbf{n} = 0, \quad (\mathbf{x},t)\in\Gamma_T \tag{8.6}$$

and

$$\int\limits_{\Omega}\mathbf{v}\boldsymbol{\sigma}(\mathbf{x})d\mathbf{x} = e(t),\ \int\limits_{\Omega}\theta\eta(\mathbf{x})d\mathbf{x} = \delta(t),\ t\geq 0, \text{where}\ \ \boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\omega}(\mathbf{x}) - \varkappa\Delta\boldsymbol{\omega}(\mathbf{x}). \tag{8.7}$$

Let us assume that in addition to (3.7)-(3.9) the following conditions are fulfilled

$$\boldsymbol{\omega}\neq 0\ \ \eta(\mathbf{x})\neq 0, \quad \forall\mathbf{x}\in\Omega \quad (\text{or}\ \ \|\boldsymbol{\omega}\|^2_{2,\Omega} + \varkappa\,\|\Delta\boldsymbol{\omega}\|^2_{2,\Omega}\neq 0, \quad \|\eta\|_{2,\Omega}\neq 0). \tag{8.8}$$

In this case, an equivalent direct problem corresponding to (8.1)-(8.5), (8.7): P_1I (or (8.1)-(8.4), (8.6), (8.7): P_1II) is the following initial-boundary value problem, which need to define a pair $(\mathbf{v},\theta)$ from (8.4), (8.6) (or 8.4, (8.6)) and

$$\begin{gathered}\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \varkappa\Delta\mathbf{v}_t - \nu\Delta\mathbf{v} + \boldsymbol{\nabla}\pi = \mathbf{g}(\mathbf{x},t)\theta(x,t) + \Phi_1(\mathbf{v},\theta)\boldsymbol{\sigma}(\mathbf{x}), (x,t)\in Q_T,\\ \operatorname{div}\mathbf{v}(\mathbf{x},t) = 0, \quad (x,t)\in Q_T,\end{gathered} \tag{8.9}$$

$$\theta_t + (\mathbf{v}\cdot\nabla)\,\theta - \lambda\Delta\theta = J_1(\mathbf{v},\theta)\eta(\mathbf{x}), \quad (x,t)\in Q_T, \tag{8.10}$$

with the nonlocal functionals

$$\Phi_1(\mathbf{v},\theta) := f(t) = \frac{1}{\omega_0}\left(e'(t) - ((\mathbf{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{\omega},\mathbf{v})_{2,\Omega} + \nu a\,(\mathbf{v},\boldsymbol{\omega}) - (\mathbf{g}(\mathbf{x},t)\theta,\boldsymbol{\omega})_{2,\Omega}\right), \tag{8.11}$$

$$J_1(\mathbf{v},\theta) := j(t) = \frac{1}{\eta_0}\left(\delta'(t) + \lambda\,(\nabla\theta,\nabla\eta)_{2,\Omega} - ((\mathbf{v}\cdot\nabla)\,\eta,\theta)_{2,\Omega}\right), \tag{8.12}$$

where $\omega_0 := \|\boldsymbol{\omega}\|^2 + \varkappa\,\|\Delta\boldsymbol{\omega}\|^2 > 0$ and $\eta_0 := \|\eta\| > 0$ are strictly positive numbers.

For this problem the following assertion is hold.

Theorem 5. *Assume that the conditions (3.3), (3.7)-(3.10), and (8.8) are fulfilled. Then the inverse problem P_1I (P_1II) has global in time a unique weak solution $(\mathbf{v}(\mathbf{x},t),\theta(\mathbf{x},t),f(t),j(t))$ in Q_T, and for a weak solution the estimate (4.1) is hold for all $t\in(0,T]$.*

Proof. Here we prove for the inverse problem P_1I, for P_1II is a similar. As we note above, in order to prove this, it is sufficient to establish the first a priori estimate (4.10) for any $t\in(0,T]$ for solutions of (8.9)-(8.12) and (8.4)-(8.5). Then repeat the next steps of the proof of Theorem 1 and 4.
In this case, the energy equalities (4.11) and (4.12) have the form

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{v}^n\|_{2,\Omega}^2 + \varkappa\,\|\nabla\mathbf{v}^n\|_{2,\Omega}^2\right) + \nu\,\|\nabla\mathbf{v}^n\|_{2,\Omega}^2 = \Phi_1^n(\mathbf{v}^n,\theta^n)e(t) + (\mathbf{g}(\mathbf{x},t)\theta^n,\mathbf{v}^n)_{2,\Omega}\,, \tag{8.13}$$

$$\frac{1}{2}\frac{d}{dt}\|\theta^n\|_{2,\Omega}^2 + \lambda\,\|\nabla\theta^n\|_{2,\Omega}^2 = J_1^n(\mathbf{v}^n,\theta^n)\delta(t). \tag{8.14}$$

where the functionals $\Phi_1^n(\mathbf{v}^n,\theta^n)$ and $J_1^n(\mathbf{v}^n,\theta^n)$ are defined by (8.11) and (8.12), and for them hold the estimates (4.14) and (4.15) with $k_0 := \omega_0$ and $k_1 := \eta_0$, respectively.

Next, estimate the terms on the right-hand side of (8.13) and (8.14) as (4.16)-(4.18)

$$\begin{aligned}
&|\Phi^n(\mathbf{v}^n,\theta^n)e(t)| \le \\
&\frac{1}{\omega_0}|e(t)|\left[|e'(t)| + \nu\,\|\nabla\mathbf{v}^n\|_{2,\Omega}\|\nabla\boldsymbol{\omega}\|_{2,\Omega} + g_0\,\|\theta^n\|_{2,\Omega}\|\boldsymbol{\omega}\|_{2,\Omega} + C_s^2\,\|\nabla\mathbf{v}^n\|_{2,\Omega}^2\|\nabla\boldsymbol{\omega}\|_{2,\Omega}\right] \le \\
&\frac{1}{2\omega_0}|e(t)|^2 + \frac{1}{2\omega_0}|e'(t)|^2 + \frac{\nu}{2}\|\nabla\mathbf{v}^n\|_{2,\Omega}^2 + \frac{\nu}{2\omega_0^2}\|\nabla\boldsymbol{\omega}\|_{2,\Omega}^2|e(t)|^2 + \frac{g_0^2}{2\omega_0^2}\|\boldsymbol{\omega}\|_{2,\Omega}^2|e(t)|^2 + \\
&\frac{1}{2}\|\theta^n\|_{2,\Omega}^2 + \frac{1}{2\omega_0}C_s^2\,|e(t)|\,\|\nabla\boldsymbol{\omega}\|_{2,\Omega}\|\nabla\mathbf{v}^n\|_{2,\Omega}^2 \le \\
&\frac{\nu}{2}\|\nabla\mathbf{v}^n\|_{2,\Omega}^2 + \frac{1}{2}\|\theta^n\|_{2,\Omega}^2 + C_0(t)\,\|\nabla\mathbf{v}^n\|_{2,\Omega}^2 + C_1(t) \\
&\text{where}\ \ C_0(t) = \frac{1}{2\omega_0}C_s^2\,|e(t)|\,\|\nabla\boldsymbol{\omega}\|_{2,\Omega}\,, \\
&C_0(t) = \frac{1}{2\omega_0}\left(|e(t)|^2 + |e'(t)|^2\right) + \frac{1}{2\omega_0^2}|e(t)|^2\left(\nu\,\|\nabla\boldsymbol{\omega}\|_{2,\Omega}^2 + g_0^2\,\|\boldsymbol{\omega}\|_{2,\Omega}^2\right).
\end{aligned} \tag{8.15}$$

$$
\begin{aligned}
&|J^n(\mathbf{v}^n,\theta^n)\delta(t)| \leq \\
&\frac{1}{\eta_0}|\delta(t)|\left[|\delta'(t)| + \lambda\|\nabla\theta^n\|_{2,\Omega}\|\nabla\eta\|_{2,\Omega} + C_s^2\|\nabla\mathbf{v}^n\|_{2,\Omega}\|\nabla\theta^n\|_{2,\Omega}\|\nabla\eta\|_{2,\Omega}\right] \leq \\
&\frac{1}{2\eta_0}\left(|\delta(t)|^2 + |\delta'(t)|^2\right) + \frac{\lambda}{4}\|\nabla\theta^n\|_{2,\Omega}^2 + \frac{\lambda}{\eta_0^2}\|\nabla\eta\|_{2,\Omega}^2|\delta(t)|^2 + \frac{\lambda}{4}\|\nabla\theta^n\|_{2,\Omega}^2 + \\
&\frac{C_s^4}{\lambda\eta_0^2}|\delta(t)|^2\|\nabla\eta\|_{2,\Omega}^2\|\nabla\mathbf{v}^n\|_{2,\Omega}^2 \leq \frac{\lambda}{2}\|\nabla\theta^n\|_{2,\Omega}^2 + C_2(t)\|\nabla\mathbf{v}^n\|_{2,\Omega}^2 + C_3(t),
\end{aligned}
\tag{8.16}
$$

where $C_2(t) = \dfrac{C_s^4}{\lambda\eta_0^2}|\delta(t)|^2\|\nabla\eta\|_{2,\Omega}^2,$

$$C_3(t) = \frac{1}{2\eta_0}\left(|\delta(t)|^2 + |\delta'(t)|^2\right) + \frac{\lambda}{\eta_0^2}\|\nabla\eta\|_{2,\Omega}^2|\delta(t)|^2$$

Plugging (4.16) and (8.15) into (8.13), and (8.16) into (8.14), and adding the results, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\mathbf{v}^n\|_{2,\Omega}^2 + \varkappa\|\nabla\mathbf{v}^n\|_{2,\Omega}^2 + \|\theta^n\|_{2,\Omega}^2\right) + \nu\|\nabla\mathbf{v}^n\|_{2,\Omega}^2 + \lambda\|\nabla\theta^n\|_{2,\Omega}^2 \leq \\
&C_4(t)\left(\|\mathbf{v}^n\|_{2,\Omega}^2 + \varkappa\|\nabla\mathbf{v}^n\|_{2,\Omega}^2 + \|\theta^n\|_{2,\Omega}^2\right) + C_5(t),
\end{aligned}
\tag{8.17}
$$

where

$$C_4(t) = \max\left\{g_0,\ \frac{2}{\varkappa}(C_0(t) + C_2(t))\right\},\quad C_5(t) = 2(C_1(t) + C_3(t)),$$

and $C_i(t) \in L^1([0,T]),\ i = 4,5$, due to the assumptions of the Theorem 5. It follows from (8.17) that the estimate (4.10), which is hold for any $t \in (0,T]$. □

Theorem 6. *Assume that the conditions (3.3), (3.7)-(3.10), (8.8), and (6.1) are fulfilled. Then the inverse problem P_1I (and P_1II) has a unique strong solution for all $t \in (0,T]$ and the estimate (6.2) is valid.*

8.2. **Global existence: without convective term.** Let us consider the problem PI (PII) without the convective term $(\mathbf{v}\cdot\nabla)\,\mathbf{v}$:

$$\mathbf{v}_t - \varkappa\Delta\mathbf{v}_t - \nu\Delta\mathbf{v} + \boldsymbol{\nabla}\pi = \mathbf{g}(\mathbf{x},t)\theta(x,t) + f(t)\mathbf{h}(\mathbf{x},t),\quad \operatorname{div}\mathbf{v}(\mathbf{x},t) = 0,\quad (x,t) \in Q_T, \tag{8.18}$$

$$\theta_t + (\mathbf{v}\cdot\nabla)\,\theta - \lambda\Delta\theta = j(t)\phi(\mathbf{x},t),\quad (x,t) \in Q_T, \tag{8.19}$$

$$\mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}),\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}),\quad \mathbf{x} \in \Omega, \tag{8.20}$$

$$\theta(\mathbf{x},t) = 0,\ \mathbf{v}(\mathbf{x},t) = 0,\quad (\mathbf{x},t) \in \Gamma_T \tag{8.21}$$

or

$$\theta(\mathbf{x},t) = 0,\ \mathbf{v}_{\mathbf{n}}(\mathbf{x},t) = \mathbf{v}\cdot\mathbf{n} = 0,\quad (\mathbf{D}(\mathbf{v})\cdot\mathbf{n})\,\mathbf{v}\times\mathbf{n} = 0,\quad (\mathbf{x},t) \in \Gamma_T, \tag{8.22}$$

$$\int_\Omega \mathbf{v}\boldsymbol{\sigma}(\mathbf{x})d\mathbf{x} = e(t),\ \int_\Omega \theta\eta(\mathbf{x})d\mathbf{x} = \delta(t),\ t \geq 0, \text{where}\ \ \boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\omega}(\mathbf{x}) - \varkappa\Delta\boldsymbol{\omega}(\mathbf{x}). \tag{8.23}$$

We denote (8.18)-(8.21) and (8.23) by P_2I, and (8.18)-(8.20) and (8.22)-(8.23) by P_2II. For these problems the following analogical results are valid for any $t \in (0,T]$, which their proofs are a very similar to the proofs of Theorems 1 - 2, and 4.

Theorem 7. *Let the conditions (3.3)-(3.10) be fulfilled. Then for all $t \in (0,T]$ the inverse problem P_2I (P_2II) has a unique weak solution and the estimate (4.1) is valid.*

Theorem 8. *Assume that the conditions (3.3)-(3.10) and (6.1) are fulfilled. Then for any $t \in (0,T]$ the inverse problem P_2I (and P_2II) has a unique strong solution and the estimate (6.2) is valid.*

9. Large time behavior

In this section, we study the asymptotic behavior of weak solutions of P_2I and P_2II. First we prove for the problem P_2I, for the problem P_2II it is a similar.

Let us consider the energy relations (8.13) and (8.14). Repeating the estimates (8.15) and (8.16) by using Poincare's inequality

$$\|u\|_{2,\Omega} \le C_p(\Omega)\,\|\nabla u\|_{2,\Omega}\,,\ \forall u \in W_0^{1,2}(\Omega),$$

we obtain

$$\begin{aligned} &|\Phi^n(\mathbf{v}^n,\theta^n)e(t)| \le \\ &\frac{|e(t)|}{\omega_0}\left[|e'(t)| + \nu\,\|\nabla\mathbf{v}^n\|_{2,\Omega}\,\|\nabla\boldsymbol{\omega}\|_{2,\Omega} + C_p g_0\,\|\nabla\theta^n\|_{2,\Omega}\,\|\boldsymbol{\omega}\|_{2,\Omega} + C_s^2\,\|\nabla\mathbf{v}^n\|_{2,\Omega}^2\,\|\nabla\boldsymbol{\omega}\|_{2,\Omega}\right] \\ &\le A\,\|\nabla\mathbf{v}^n\|_{2,\Omega}^2 + \epsilon\,\|\nabla\theta^n\|_{2,\Omega}^2 + B, \end{aligned} \tag{9.1}$$

where

$$A = \frac{e(t)}{\omega_0}\left(C_s^2 + \nu^2\right)\|\nabla\boldsymbol{\omega}\|_{2,\Omega}^2\,,\ B = \frac{|e'(t)||e(t)|}{\omega_0} + \frac{|e(t)|}{4\omega_0}\,\|\nabla\boldsymbol{\omega}\|_{2,\Omega}^2 + \frac{|e(t)|C_p^2 g_0^2}{4\epsilon\omega_0}\,\|\boldsymbol{\omega}\|_{2,\Omega}^2\,,\ \forall\epsilon > 0$$

and

$$|J^n(\mathbf{v}^n,\theta^n)\delta(t)| \le \epsilon\,\|\nabla\theta^n\|_{2,\Omega}^2 + D\,\|\nabla\mathbf{v}^n\|_{2,\Omega} + E, \tag{9.2}$$

where

$$D = \frac{|\delta(t)|}{2\epsilon\eta_0}C_s^4\,\|\nabla\eta\|_{2,\Omega}^2\,,\ E = \frac{|\delta(t)||\delta'|}{\eta_0} + \frac{\lambda^2|\delta(t)|^2}{2\epsilon\eta_0^2}\,\|\nabla\eta\|_{2,\Omega}^2\,,\ \forall\epsilon > 0$$

respectively. Next using the inequality

$$\|\mathbf{v}^n\|_{\frac{2d}{d-2},\Omega} \le C_{em}\,\|\nabla\mathbf{v}^n\|_{2,\Omega}\,,$$

we estimate

$$\begin{aligned} &|\,(\mathbf{g}(\mathbf{x},t)\theta^n,\mathbf{v}^n)_{2,\Omega}\,| \le \|\theta^n\|_{\frac{2d}{d-2},\Omega}\,\|\mathbf{v}^n\|_{\frac{2d}{d-2},\Omega}\,\|\mathbf{g}\|_{\frac{d}{2},\Omega} \le C_{em}^2\,\|\nabla\theta^n\|_{2,\Omega}\,\|\nabla\mathbf{v}^n\|_{2,\Omega}\,\|\mathbf{g}\|_{\frac{d}{2},\Omega} \le \\ &\qquad \epsilon\,\|\nabla\theta^n\|_{2,\Omega}^2 + \frac{C_{em}^4}{4\epsilon}\,\|\mathbf{g}(t)\|_{\frac{d}{2},\Omega}^2\,\|\nabla\mathbf{v}^n\|_{2,\Omega}^2\,. \end{aligned} \tag{9.3}$$

Let us introduce the energy function

$$Y(t) = \left(\|\mathbf{v}^n\|_{2,\Omega}^2 + \varkappa\,\|\nabla\mathbf{v}^n\|_{2,\Omega}^2 + \|\theta^n\|_{2,\Omega}^2\right).$$

Adding (8.13) and (8.14), and combining the result with (9.1), (9.2), and (9.3), we arrive at the inequality

$$\frac{1}{2}\frac{dY(t)}{dt} + \nu\,\|\nabla\mathbf{v}^n\|_{2,\Omega}^2 + \lambda\,\|\nabla\theta^n\|_{2,\Omega}^2 \le Q \tag{9.4}$$

where

$$Q = \left(A + D + \frac{C_{em}^4}{4\epsilon}\,\|\mathbf{g}(t)\|_{\frac{d}{2},\Omega}^2\right)\|\nabla\mathbf{v}^n\|_{2,\Omega}^2 + 3\epsilon\,\|\nabla\theta^n\|_{2,\Omega}^2 + B + E$$

First we choose ϵ such that $6\epsilon \leq \lambda$. Assume that the functions

$$\|\mathbf{g}(t)\|^2_{\frac{d}{2},\Omega},\ e(t),\ \delta(t) \tag{9.5}$$

are monotonically decreasing in time and tripping to zero. Then there is a finite time $T^* < \infty$ such that

$$\left|\left(A + D + \frac{C_{em}^2}{4\epsilon}\|\mathbf{g}(t)\|^2_{\frac{d}{2},\Omega}\right)\right| \leq \nu/2.$$

Then it follows from (9.4) that

$$\frac{dY(t)}{dt} + \nu\|\nabla\mathbf{v}^n\|^2_{2,\Omega} + \lambda\|\nabla\theta^n\|^2_{2,\Omega} \leq 2(B+E) := J, \tag{9.6}$$

where

$$J := 2|e(t)|\left(\frac{|e'(t)|}{\omega_0} + \frac{1}{4\omega_0}\|\nabla\boldsymbol{\omega}\|^2_{2,\Omega} + \frac{C_p^2 g_0^2}{4\epsilon\omega_0}\|\boldsymbol{\omega}\|^2_{2,\Omega}\right) + 2|\delta(t)|\left(\frac{|\delta'|}{\eta_0} + \frac{\lambda^2|\delta(t)|}{2\epsilon\eta_0^2}\|\nabla\eta\|^2_{2,\Omega}\right).$$

It is easy verify that

$$J \leq (|e(t)| + |\delta(t)|)\,K, \tag{9.7}$$

where

$$K = 2\sup_{t\in[0,\infty)}\left(\frac{1}{\omega_0}\left(|e'(t)| + \frac{1}{4}\|\nabla\boldsymbol{\omega}\|^2_{2,\Omega} + \frac{C_p^2 g_0^2}{4\epsilon}\|\boldsymbol{\omega}\|^2_{2,\Omega}\right) + \frac{1}{\eta_0}\left(|\delta'| + \frac{\lambda^2|\delta(t)|}{2\epsilon\eta_0}\|\nabla\eta\|^2_{2,\Omega}\right)\right).$$

On the other hand, taking into account Poincare's inequality we derive

$$\nu\|\nabla\mathbf{v}^n\|^2_{2,\Omega} + \lambda\|\nabla\theta^n\|^2_{2,\Omega} \geq \frac{\nu}{2}\|\nabla\mathbf{v}^n\|^2_{2,\Omega} + \frac{\nu}{2C_p^2}\|\mathbf{v}^n\|^2_{2,\Omega} + \frac{\lambda}{C_p^2}\|\theta^n\|^2_{2,\Omega} \geq \mu Y(t), \tag{9.8}$$

with

$$\mu = \min\left\{\frac{\nu}{2\varkappa}, \frac{\nu}{2C_p^2}, \frac{\lambda}{C_p^2}\right\}. \tag{9.9}$$

Finally, plugging (9.7) and (9.8) into (9.6), we get the ordinary differential inequality

$$\frac{dY(t)}{dt} + \mu Y(t) \leq (|e(t)| + |\delta(t)|)\,K.$$

Integrating last inequality, we obtain

$$Y(t) \leq e^{-\mu t}\left(K\int_{T_*}^t e^{\mu s}\left(|e(s)| + |\delta(s)|\right)ds + Y(T_*)e^{\mu T_*}\right),\ t \geq T_* \tag{9.10}$$

Assume that in addition (9.5) the following condition holds

$$\int_0^\infty e^{\mu s}\left(|e(s)| + |\delta(s)|\right)ds \leq C < \infty.$$

Then it follows from (9.10), that $Y(t) \to 0$ as $t \to \infty$.

Thus, we can to formulate the following assertion.

Theorem 9. *Let the functions* $\|\mathbf{g}(t)\|^2_{\frac{d}{2},\Omega}$, $e(t)$, *and* $\delta(t)$ *be monotonically decreasing in time and tripping to zero. Assume that also*

$$\int_0^\infty e^{\mu s}\left(|e(s)|+|\delta(s)|\right)ds<\infty, \tag{9.11}$$

where μ *is defined at* (9.9)*. Then there exits a positive constant* C *such that*

$$Y(t)\le Ce^{-\mu t},$$

i.e. the function $Y(t)$ *is exponential decay as* $t\to\infty$.

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