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Global existence for semi-linear time fractional σ-evolution equations†

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We investigate the global existence of small data solutions to the Cauchy problem for the semi-linear time fractional σ-evolution equation models with nonlinearity of derivative type and memory. Based on the $L^r - L^q$ estimates obtained in linear problem, and combined with the global iteration method, the global existence of small data solutions is proved under certain conditions of power p. Furthermore, we find that in the low-dimensional case, the limit of our conclusion can match the critical exponent in classical results.

KEYWORDS:
Semi-linear time fractional σ-evolution, Cauchy problem, Small data solutions, Global existence

1 | INTRODUCTION

In this paper, we consider the Cauchy problem for the semi-linear time fractional $\sigma-$evolution equations with nonlinearity of derivative type and memory in the following form

$$\begin{cases} \partial_t^{1+\alpha} u + (-\Delta)^\sigma u = F(u_t), \quad t \geq 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0, \quad u_t(0,x) = u_1, \quad x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $\alpha \in (0,1)$, $\sigma > 1,\ p > 1$. The function $F(u_t)$ represents the power nonlinearities

$$|u_t|^p \quad \text{or} \quad \int_0^t (t-\tau)^{-\mu} |u_t(\tau,x)|^p d\tau$$

for some $\mu \in (0,1)$. Here $\partial_t^{1+\alpha} u$ is the $1+\alpha$ order Caputo fractional derivative of $u(x,t)$ with respect to t, defined by

$$\partial_t^{j+\alpha} u(t,x) = J_{0|t}^{1-\alpha} (\partial_t^{j+1} u)(t,x), \tag{2}$$

for any $j \in \mathbb{N}$, where

$$J_{0|t}^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau) d\tau, \quad t > 0$$

is the Riemann-Liouville fractional integral, defined for $\beta > 0$, and $\Gamma(\beta)$ represents the Gamma function. The operator $(-\Delta)^\sigma$, is defined as follows

$$(-\Delta)^\sigma u = \mathcal{F}^{-1}(|\xi|^{2\sigma} \hat{u}), \quad \hat{u}(\xi) = \mathcal{F}(u)(\xi) := \int_{R^n} e^{-i\xi x} u dx. \tag{3}$$

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Fractional calculus has attracted a lot of attention and achieved considerable results over the past decades, it can describe the memory and inheritance of various material evolution processes, and reveal these properties neglected in integer calculus. This kind of equation (1) is evolved from the classical wave equation, which is obtained by substituting the second-order time derivative with the fractional derivative of $1+\alpha \in (1,2)$ in form, it is usually used to describe the propagation of mechanical waves in viscoelastic media with power-law characteristics and can describe the phenomenon between diffusion and wave propagation models. In recent years, many researchers[1–4] have paid attention to the well-posedness of solutions of this kind of problems. However, as one might expect, when α decreases to zero in (1), the existence of the second data u_1 becomes unnatural, therefore we cannot let the range of α be too close to zero in this paper.

The Cauchy problem for classical wave equation with nonlinearity of derivative type has been studied by many scholars, and there are abundant results for the following form

$$\begin{cases} \partial_t^2 u - \Delta u = |u_t|^p, \quad t \geq 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0, \quad u_t(0,x) = u_1, \quad x \in \mathbb{R}^n. \end{cases} \tag{4}$$

Specifically, in the 1980s, Glassey[5] proposed the conjecture that

$$p_{Gla}(n) = 1 + \frac{2}{n-1}$$

is the critical exponent for the Cauchy problem (4), it means that in the supercritical case $p > p_{Gla}$, small data solutions exist globally, meanwhile the global solutions dose not exist under proper assumptions about initial data in the case of $p \leq p_{Gla}$. Kunio[6] and Tzvetkov[7] researched global existence and asymptotic behavior of solutions for general data as $p > p_{Gla}$, in the case of space dimension $n = 2, 3$, respectively. Sideris and Thomas[8] studied global behavior of solutions in $n = 3$ for radial data and Hidano et al[9] proved existence of global solutions for all spatial dimensions. For higher dimension case $n \geq 4$, there are some results for blow up and estimation of life span in Zhou[10] as $p \leq p_{Gla}$.

Recently, D'Abbicco et al[4] studied the Cauchy problem for the following semi-linear time fractional diffusion-wave equation

$$\begin{cases} \partial_t^{1+\alpha} u - \Delta u = |u|^p, \quad t \geq 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0, \quad u_t(0,x) = u_1, \quad x \in \mathbb{R}^n, \end{cases} \tag{5}$$

where $\alpha \in (0,1)$, $p > 1$, they found two critical exponents corresponding to the equations when the second data $u_1 = 0$ and $u_1 \neq 0$, the two critical exponents are

$$\tilde{p} = 1 + \frac{2}{n-2+2(1+\alpha)^{-1}} \quad \text{and} \quad \bar{p} = 1 + \frac{2}{n-2(1+\alpha)^{-1}},$$

as we can see if $\alpha \to 0^+$, then $\tilde{p}$ tends to $1 + 2/n$ which is the Fujita critical exponent.[11] In addition, $\bar{p}$ tends to $1 + 2/(n-1)$ found by Kato[12] as $\alpha \to 1^-$, it is interesting that the problem (5) can be related to the conclusion of the classical problems.

For the Cauchy problem of the semi-linear time fractional σ-evolution equation

$$\begin{cases} \partial_t^{1+\alpha} u + (-\Delta)^\sigma u + m^2 u = |u|^p, \quad t \geq 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0, \quad u_t(0,x) = 0, \quad x \in \mathbb{R}^n, \end{cases} \tag{6}$$

where $\alpha \in (0,1)$, $\sigma \geq 1$, $m \geq 0$ and $p > 1$, Abdelatif and Michael[2] considered the effect of the mass term on the global small data solutions, then Abdelatif[1] took the case where the nonlinear term is

$$\int_0^t (t-\tau)^{-\mu} |u(\tau,\cdot)|^p d\tau$$

for some $\mu \in (0,1)$, and also studied the influence of the mass term on this model.

Inspired by the works mentioned above,[1,2,4,5] here we attempt to study semi-linear time fractional $\sigma-$evolution equation models with new nonlinearities $|u_t|^p$ and $\int_0^t (t-\tau)^{-\mu}|u_t(\tau,x)|^p d\tau$. Note that the second power nonlinearity is the Riemann-Liouville fractional integral of $|u_t|^p$, and we remark that

$$\int_0^t (t-\tau)^{-\mu} |u_t(\tau,\cdot)|^p d\tau = \Gamma(1-\mu) J_{0|t}^{1-\mu} |u_t(t,\cdot)|^p \tag{7}$$

and

$$\lim_{\mu\to 1^-}\int_0^t (t-\tau)^{-\mu}|u_t(\tau,\cdot)|^p d\tau = |u_t(t,\cdot)|^p, \tag{8}$$

thus, the Cauchy problem (1) with nonlinear memory term studied in this paper can be transformed into a classical problem of nonlinear term $|u_t|^p$ as $\mu\to 1^-$. It is of great significance to discuss the relationship between the properties of solutions of these two models.

Our main goal is to investigate the influence of the nonlinear terms on the global small data solutions and then compare their differences in the range of power p. The existence and uniqueness of global small data solutions to (1) may be proved by using global iterative methods based on the contraction mapping principle under certain conditions.

Here are some notations that will be used frequently.

Notation. We denote by $H^{s,p}$ the generalized Sobolev space

$$H^{s,p} = \{u\in L^p : \|\mathcal{F}^{-1}\big((1+|\xi|^2)^{\frac{s}{2}}\hat{u}\big)\|_{L^p} < \infty\} \tag{9}$$

for any $s\ge 0$ and $p\in[1,\infty]$. Recall that $H^{s,p}=W^{s,p}$ if $s\in\mathbb{N}$, and denote the norm of homogeneous Sobolev space

$$\|u\|_{\dot{H}^{s,p}} = \|\nabla^s u\|_{L^p} = \|\mathcal{F}^{-1}(|\xi|^s\hat{u})\|_{L^p}.$$

We write $f\lesssim g$, implying that there exists a constant $C>0$, such that $f\le Cg$.

In addition, the notation

$$\beta(r,q,\alpha,\sigma,n) = \frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r}-\frac{1}{q}\Big) := \beta_{r,q} \tag{10}$$

will be used throughout this paper.

Now our main results can be stated as follows.

Theorem 1.1. *Assume that $\alpha\in(0,1)$ and $\sigma>\frac{1+\alpha}{2\alpha}$. If $1\le n<\frac{2\sigma\alpha}{1+\alpha}$ and the exponent p satisfies the condition*

$$p>\tilde{p} := 1+\frac{2(1+\lambda)\sigma}{n(1+\alpha)-2\lambda\sigma}, \tag{11}$$

where $\lambda\in(0,\frac{1+\alpha}{2\sigma})$. Then there exists a positive enough small constant ε such that for any

$$(u_0,u_1)\in\mathcal{A} \doteq (H^{k,1}\cap H^{k,\infty})\times(L^1\cap L^\infty) \tag{12}$$

satisfy

$$\|(u_0,u_1)\|_{\mathcal{A}} = \|u_0\|_{H^{k,1}\cap H^{k,\infty}} + \|u_1\|_{L^1\cap L^\infty} \le \varepsilon,$$

where $k=\frac{2\sigma}{1+\alpha}$, that there exists a unique global solution

$$u\in C\big([0,\infty),\dot{H}^{k,1}\cap\dot{H}^{k,\infty}\big)\cap C^1\big([0,\infty),L^1\cap L^\infty\big) \tag{13}$$

to the Cauchy problem (1) with the nonlinear term $F(u_t)=|u_t|^p$. Moreover, for any $t\ge 0$, the solution satisfies the following estimates

$$\|u_t\|_{L^q} \lesssim (1+t)^{\lambda-\beta_{1,q}}\|(u_0,u_1)\|_{\mathcal{A}}, \quad q\in[1,\infty], \tag{14}$$

$$\|u\|_{\dot{H}^{k,q}} \lesssim (1+t)^{\lambda-\beta_{1,q}}\|(u_0,u_1)\|_{\mathcal{A}}, \quad q\in[1,\infty]. \tag{15}$$

Theorem 1.2. *Assume that $\alpha\in(0,1)$ and $\sigma>\frac{1+\alpha}{2\alpha}$. If $n\ge\frac{2\sigma\alpha}{1+\alpha}$ and the exponent p satisfies the condition*

$$p>\bar{p} := 1+\frac{n(1+\alpha)}{2\sigma\alpha}. \tag{16}$$

Then there exists a positive enough small constant ε such that for any

$$(u_0,u_1)\in\mathcal{M} \doteq (H^{k,m}\cap H^{k,\infty})\times(L^m\cap L^\infty) \tag{17}$$

satisfy

$$\|(u_0,u_1)\|_{\mathcal{M}} = \|u_0\|_{H^{k,m}\cap H^{k,\infty}} + \|u_m\|_{L^1\cap L^\infty} \le \varepsilon,$$

the parameter $m>1$ is chosen as the solution to

$$\frac{n}{2\sigma m} = \frac{\alpha}{1+\alpha}-\delta$$

with a sufficiently small constant $\delta > 0$, that there exists a unique global solution

$$u \in C\big([0,\infty), \dot{H}^{k,m} \cap \dot{H}^{k,\infty}\big) \cap C^1\big([0,\infty), L^m \cap L^\infty\big) \tag{18}$$

to the Cauchy problem (1) with the nonlinear term $F(u_t) = |u_t|^p$. Moreover, for any $t \geq 0$, the solution satisfies the following estimates

$$\|u_t\|_{L^q} \lesssim (1+t)^{\alpha-\eta-\beta_{m,q}} \|(u_0,u_1)\|_{\mathcal{M}}, \quad q \in [m,\infty], \tag{19}$$

$$\|u\|_{\dot{H}^{k,q}} \lesssim (1+t)^{\alpha-\eta-\beta_{m,q}} \|(u_0,u_1)\|_{\mathcal{M}}, \quad q \in [m,\infty], \tag{20}$$

where

$$0 < \eta = \frac{\alpha p}{p-1} - \frac{n(1+\alpha)(p-m)}{2\sigma m(p-1)} < \alpha.$$

Remark 1.1. The condition $\sigma > \frac{1+\alpha}{2\alpha}$ determines that α can't be too small, otherwise problem (1) will be meaningless. If $\alpha \to 1^-$, $\sigma \to 1^+$, we can see that $\tilde{p} \to p_{Gla}(1) = \infty$ as $\lambda \to 1^-$ and $n = 1$, while $\bar{p} \to p_{Gla}(2) = 3$ as $n = 2$, when $n \geq 3$, $\bar{p} > p_{Gla}(n)$. In other words, the limit of our conclusion matches the critical exponent in the classical problem (4) for the low-dimensional case.

Theorem 1.3. *Assume that $\alpha \in (0,1)$, $\mu \in (\alpha - \frac{1+\alpha}{2\sigma}, \alpha)$ and $\sigma > \frac{1+\alpha}{2\alpha}$. If $1 \leq n < \frac{2\sigma\alpha}{1+\alpha}$ and the exponent p satisfies the condition*

$$p > \tilde{\tilde{p}} := 1 + \frac{2(1+\alpha-\mu)\sigma}{n(1+\alpha) - 2(\alpha-\mu)\sigma}. \tag{21}$$

Then there exists a positive enough small constant ε such that for any

$$(u_0,u_1) \in \mathcal{A} \doteq (H^{k,1} \cap H^{k,\infty}) \times (L^1 \cap L^\infty) \tag{22}$$

satisfy

$$\|(u_0,u_1)\|_{\mathcal{A}} = \|u_0\|_{H^{k,1}\cap H^{k,\infty}} + \|u_1\|_{L^1\cap L^\infty} \leq \varepsilon,$$

that there exists a unique global solution

$$u \in C\big([0,\infty), \dot{H}^{k,1} \cap \dot{H}^{k,\infty}\big) \cap C^1\big([0,\infty), L^1 \cap L^\infty\big) \tag{23}$$

to the Cauchy problem (1) with the nonlinear term $F(u_t) = \int_0^t (t-\tau)^{-\mu} |u_t(\tau,x)|^p d\tau$. Moreover, for any $t \geq 0$, the solution satisfies the following estimates

$$\|u_t\|_{L^q} \lesssim (1+t)^{\alpha-\mu-\beta_{1,q}} \|(u_0,u_1)\|_{\mathcal{A}}, \quad q \in [1,\infty], \tag{24}$$

$$\|u\|_{\dot{H}^{k,q}} \lesssim (1+t)^{\alpha-\mu-\beta_{1,q}} \|(u_0,u_1)\|_{\mathcal{A}}, \quad q \in [1,\infty]. \tag{25}$$

Theorem 1.4. *Assume that $\alpha \in (0,1)$, $\mu \in (\alpha, 1)$ and $\sigma > \frac{1+\alpha}{2\alpha}$. If $n \geq \frac{2\sigma\alpha}{1+\alpha}$ and the exponent p satisfies the condition*

$$p > \bar{\bar{p}} := 1 + \frac{2\sigma(1-\mu) + n(1+\alpha)}{2\sigma\alpha}. \tag{26}$$

Then there exists a positive enough small constant ε such that for any

$$(u_0,u_1) \in \mathcal{M} \doteq (H^{k,m} \cap H^{k,\infty}) \times (L^m \cap L^\infty) \tag{27}$$

satisfy

$$\|(u_0,u_1)\|_{\mathcal{M}} = \|u_0\|_{H^{k,m}\cap H^{k,\infty}} + \|u_m\|_{L^1\cap L^\infty} \leq \varepsilon,$$

the parameter $m > 1$ is chosen as the solution to

$$\frac{n}{2\sigma m} = \frac{\alpha}{1+\alpha} - \delta$$

with a sufficiently small constant $\delta > 0$, that there exists a unique global solution

$$u \in C\big([0,\infty), \dot{H}^{k,m} \cap \dot{H}^{k,\infty}\big) \cap C^1\big([0,\infty), L^m \cap L^\infty\big) \tag{28}$$

to the Cauchy problem (1) with the nonlinear term $F(u_t) = \int_0^t (t-\tau)^{-\mu} |u_t(\tau,x)|^p d\tau$. Moreover, for any $t \geq 0$, the solution satisfies the following estimates

$$\|u_t\|_{L^q} \lesssim (1+t)^{\alpha+1-\mu-\xi-\beta_{m,q}} \|(u_0,u_1)\|_{\mathcal{M}}, \quad q \in [m,\infty], \tag{29}$$

$$\|u\|_{\dot{H}^{k,q}} \lesssim (1+t)^{\alpha+1-\mu-\xi-\beta_{m,q}} \|(u_0,u_1)\|_{\mathcal{M}}, \quad q \in [m,\infty], \tag{30}$$

where

$$0 < \xi = \frac{(\alpha+1-\mu)p}{p-1} - \frac{n(1+\alpha)(p-m)}{2\sigma m(p-1)} < \alpha+1-\mu.$$

Remark 1.2. Compared with the homogeneous problem (see Remark 2.1) corresponding to problem (1), a loss of decay $(1+t)^{\alpha-\mu}$ occurs in regard to the estimates (24)-(25). In addition, $\bar{\bar{p}} \to \bar{p}$ as $\mu \to 1^-$, at this time the decay rate in (29)-(30) is the same as that in (19)-(20).

2 | PRELIMINARIES

2.1 | Linear estimates

We first consider the linear problem corresponding to problem (1)

$$\begin{cases} \partial_t^{1+\alpha} u + (-\Delta)^\sigma u = f(t,x), \quad t>0, \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0, \quad u_t(0,x) = u_1, \quad x \in \mathbb{R}^n, \end{cases} \tag{31}$$

and introduce some useful lemmas before deriving the $L^r - L^q$ $(1 \le r \le q \le \infty)$ estimate of the linear problem (31).
Lemma 2.1 ([13]). *Let $\alpha \in (0,1)$, $b_1, b_2, \lambda \in \mathbb{R}$, then the following problem*

$$\begin{cases} \partial_t^{1+\alpha} y = \lambda y + f(t), \quad t \ge 0, \\ y(0) = b_1, \quad y'(0) = b_2 \end{cases} \tag{32}$$

has a unique solution

$$y(t) = b_1 E_{1+\alpha,1}(\lambda t^{1+\alpha}) + b_2 t E_{1+\alpha,2}(\lambda t^{1+\alpha}) + \int_0^t (t-\tau)^\alpha E_{1+\alpha,1+\alpha}(\lambda(t-\tau)^{1+\alpha}) f(\tau) d\tau, \tag{33}$$

where $E_{1+\alpha,\beta}$, $\beta = 1, 2, 1+\alpha$, are the Mittag-Leffler functions

$$E_{1+\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+\alpha k+\beta)}.$$

For the linear problem (31), by the Fourier transform in regard to x, we obtain

$$\begin{cases} \partial_t^{1+\alpha} \hat{u} + |\xi|^{2\sigma} \hat{u} = \hat{f}(t,\xi), \quad t>0, \quad x \in \mathbb{R}^n, \\ \hat{u}(0,\xi) = \hat{u_0}(\xi), \quad \hat{u}_t(0,\xi) = \hat{u_1}(\xi), \quad x \in \mathbb{R}^n. \end{cases} \tag{34}$$

Thanks to Lemma 2.1, the solution to (31) is given by

$$u(t,\cdot) = u^{\mathrm{lin}}(t,\cdot) + Nf, \tag{35}$$

where the homogeneous and nonhomogeneous parts of the solution are defined by

$$u^{\mathrm{lin}}(t,\cdot) = G_{1+\alpha,1}(t,\cdot) * u_0 + tG_{1+\alpha,2}(t,\cdot) * u_1 \tag{36}$$

and

$$Nf = \int_0^t (t-\tau)^\alpha G_{1+\alpha,1+\alpha}(t-\tau,\cdot) * f(\tau,\cdot) d\tau, \tag{37}$$

here

$$G_{1+\alpha,\beta}(t,x) := \mathcal{F}^{-1}\big(E_{1+\alpha,\beta}(-t^{1+\alpha}|\xi|^{2\sigma})\big). \tag{38}$$

And $G_{1+\alpha,\beta}(t,x)$, $\beta = 1, 2, 1+\alpha$, have the following property.
Lemma 2.2 ([4]). *The following estimate holds for any $t > 0$*

$$\|\nabla^\gamma G_{1+\alpha,\beta}(t,\cdot)\|_{L^p} \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p}) - \frac{(1+\alpha)\gamma}{2\sigma}}, \tag{39}$$

where $\gamma \geq 0$, *provided that*

$$\frac{n}{2\sigma}(1-\frac{1}{p})+\frac{\gamma}{2\sigma}<\begin{cases}1, & \beta=1,2,\\ 2, & \beta=1+\alpha.\end{cases} \tag{40}$$

By virtue of Young's inequality we can easily obtain

$$\|u^{\rm lin}\|_{\dot H^{\gamma,q}} \lesssim t^{-\beta_{r,q}}\|u_0\|_{\dot H^{\gamma,r}}+t^{1-\beta_{r,q}-\frac{(1+\alpha)\gamma}{2\sigma}}\|u_1\|_{L^r}, \quad t>0 \tag{41}$$

for any $1\leq r\leq q\leq\infty$, where $\frac{n}{2\sigma}(\frac{1}{r}-\frac{1}{q})+\frac{\gamma}{2\sigma}<1$. Now let's give an estimate of the solution to the linear problem (31).

Theorem 2.1. *Let* $n\geq 1,\ \alpha\in(0,1)$ *and* $\sigma>1$, *if* $u_0\in H^{k,r},\ u_1\in L^r$, *satisfying*

$$\frac{n}{2\sigma}(\frac{1}{r}-\frac{1}{q})<\frac{\alpha}{1+\alpha}, \quad 1\leq r\leq q\leq\infty, \tag{42}$$

where $k=\frac{2\sigma}{1+\alpha}$. *Assume that*

$$\|f(\tau,\cdot)\|_{L^r}\leq K(1+t)^{-\eta}, \quad \forall t\geq 0 \tag{43}$$

for some $K>0$ *and* $\eta\in\mathbb{R}$. *Then for* $\forall t>0$, *the solution to (31) verifies the following estimate*

$$\|(u_t,\nabla^k u)\|_{L^q}\lesssim t^{-\beta_{r,q}}(\|u_0\|_{H^{k,r}}+\|u_1\|_{L^r})+\begin{cases}K(1+t)^{\alpha-1-\beta_{r,q}}, & if\ \eta>1,\\ K(1+t)^{\alpha-1-\beta_{r,q}}\log(1+t), & if\ \eta=1,\\ K(1+t)^{\alpha-\eta-\beta_{r,q}}, & if\ \eta<1.\end{cases} \tag{44}$$

Remark 2.1. It's obviously that in the case of lower dimensions $1\leq n<\frac{2\sigma\alpha}{1+\alpha}$, the condition (42) holds for any $1\leq r\leq q\leq\infty$ and $\sigma>\frac{1+\alpha}{2\alpha}$, at this time we consider the solution of the homogeneous problem

$$u=u^{\rm lin}(t,x)=G_{1+\alpha,1}(t,x)*u_0+tG_{1+\alpha,2}(t,x)*u_1. \tag{45}$$

If $(u_0,u_1)\in\mathcal{A}$ (as defined in (12)), let $r=q$ for $t\in[0,1]$ and $r=1$ for $t\geq 1$ in (44) respectively, so that

$$\|(u_t,\nabla^k u)\|_{L^q}\lesssim\begin{cases}\|u_0\|_{H^{k,q}}+\|u_1\|_{L^q}, & t\in[0,1],\\ t^{-\beta_{1,q}}(\|u_0\|_{H^{k,1}}+\|u_1\|_{L^1}), & t\geq 1,\end{cases} \tag{46}$$

combining the two kinds of situations in (46) one can obtain the following estimate

$$\|(u_t,\nabla^k u)\|_{L^q}\lesssim(1+t)^{-\beta_{1,q}}(\|u_0\|_{H^{k,1}\cap H^{k,q}}+\|u_1\|_{L^1\cap L^q}), \quad \forall q\in[1,\infty], \quad t\geq 0. \tag{47}$$

Remark 2.2. In the case of higher dimensions $n\geq\frac{2\sigma\alpha}{1+\alpha}$, the condition (42) holds for any $1<m\leq r\leq q\leq\infty$ and $\sigma>\frac{1+\alpha}{2\alpha}$, where the parameter m is chosen as the solution to

$$\frac{n}{2\sigma m}=\frac{\alpha}{1+\alpha}-\delta \tag{48}$$

with a sufficiently small constant $\delta>0$. We now also consider the solution of the homogeneous problem (45), if $(u_0,u_1)\in\mathcal{M}$ (as defined in (17)), let $r=q$ for $t\in[0,1]$ and $r=m$ for $t\geq 1$ in (31) respectively, so that

$$\|(u_t,\nabla^k u)\|_{L^q}\lesssim\begin{cases}\|u_0\|_{H^{k,q}}+\|u_1\|_{L^q}, & t\in[0,1],\\ t^{-\beta_{m,q}}(\|u_0\|_{H^{k,m}}+\|u_1\|_{L^m}), & t\geq 1,\end{cases} \tag{49}$$

and this leads to the estimate

$$\|(u_t,\nabla^k u)\|_{L^q}\lesssim(1+t)^{-\beta_{m,q}}(\|u_0\|_{H^{k,m}\cap H^{k,q}}+\|u_1\|_{L^m\cap L^q}), \quad \forall q\in[m,\infty], \quad t\geq 0. \tag{50}$$

Here the assumption $u_0\in H^{k,r}$ in Theorem 2.1 is to ensure that estimates (47) and (50) are non-singular at $t=0$ when $r=q$.

2.2 | Proof of Theorem 2.1.

The following two properties of Mittag-Leffler are important in our proof.

Lemma 2.3 ([13]). *For the Mittag-Leffler function, the following formula holds*

$$\partial_z^n(z^{\beta-1}E_{1+\alpha,\beta}(\lambda z^{\alpha+1}))=z^{\beta-n-1}E_{1+\alpha,\beta-n}(\lambda z^{\alpha+1})$$

for any $z > 0,\ n \in \mathbb{N}$.

Lemma 2.4 ([14]). *If $\rho \in (1/3, 1),\ \beta \in \mathbb{R}$, and $m \in \mathbb{N}$ with $m \geq \rho\beta - 1$, then*

$$\begin{aligned} E_{1/\rho,\beta}(-z^{1/\rho}) = 2\rho z^{1-\beta} e^{z\cos(\pi\rho)} \cos\left(z\sin(\pi\rho) - \pi\rho(\beta-1)\right) \\ + \sum_{k=1}^{m} \frac{(-1)^{k-1}}{\Gamma(\beta - k/\rho)} z^{-k/\rho} + \Omega_m \end{aligned} \tag{51}$$

holds for any $z > 0$, where

$$\Omega_m(z) = \frac{(-1)^m z^{1-\beta}}{\pi} \left(I_{1,m} \sin\left(\pi(\beta - (m+1)/\rho)\right) + I_{2,m} \sin\left(\pi(\beta - m/\rho)\right) \right)$$

and

$$I_{j,m}(z) = \int_0^\infty \frac{s^{(m+j)/\rho - \beta}}{s^{2/\rho} + 2\cos(\pi/\rho) s^{1/\rho} + 1} e^{-zs} ds.$$

Here we note that for any $z \in (0, \infty)$, $I_{j,m}(z)$ is uniformly bounded

$$\int_0^\infty \frac{s^{(m+j)/\rho - \beta}}{s^{2/\rho} + 2\cos(\pi/\rho) s^{1/\rho} + 1} ds < \infty \tag{52}$$

if and only if

$$-1 < m + j - 1 + \rho(1 - \beta) < 1. \tag{53}$$

From Lemma 2.3 and u in (35) is the solution to (31), we can estimate the time-derivatives of the solution

$$u_t = u_t^{\text{lin}}(t, \cdot) + \int_0^t (t-\tau)^{\alpha-1} G_{1+\alpha,\alpha}(t-\tau, \cdot) * f(\tau, \cdot) d\tau, \tag{54}$$

where

$$u_t^{\text{lin}}(t, \cdot) = t^{-1} G_{1+\alpha,0}(t, \cdot) * u_0 + G_{1+\alpha,1}(t, \cdot) * u_1. \tag{55}$$

The estimate for $G_{1+\alpha,1}(t, \cdot)$ can be derived from (39) provided that $\frac{n}{2\sigma}(1 - \frac{1}{p}) < 1$, now we are going to perform the $L^r - L^q$ estimate of u_t for any $1 \leq r \leq q \leq \infty$.

Lemma 2.5 ([15]). *Let $f \in L^p(\mathbb{R}^n)$ is a radial function for some $p \in [1, 2]$. Then the inverse Fourier transform*

$$\mathcal{F}^{-1}(f)(x) = \int_0^\infty g(r) r^{n-1} \tilde{J}_{\frac{n}{2}-1}(r|x|) dr, \quad g(|x|) := f(x) \tag{56}$$

is also a radial function, where the modified Bessel function $\tilde{J}_\mu(s)$ is defined by $\tilde{J}_\mu(s) := \frac{J_\mu(s)}{s^\mu}$ and the Bessel function J_μ has the following form

$$J_\mu(s) = \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+\mu+1)} (\frac{s}{2})^{2k+\mu},$$

where μ cannot be a negative integer.

We introduce the following two estimates of the oscillating integrals.

Proposition 2.1 ([16]). *For all dimensions n, the estimate*

$$\|\mathcal{F}^{-1}(|\xi|^a e^{-c|\xi|^{2m}t})\|_{L^p(\mathbb{R}^n)} \lesssim (ct)^{-\frac{a}{2m} - \frac{n}{2m}(1-\frac{1}{p})} \tag{57}$$

holds for any $m \in (0, \infty),\ p \in [1, \infty],\ t > 0$. The parameters a and c are required to be non-negative and positive, respectively.

Proposition 2.2 ([16]). *For all dimensions n, let $a \geq 0,\ c_1 > 0$ and real $c_2 \neq 0$, then the estimate*

$$\|\mathcal{F}^{-1}\left(|\xi|^a e^{-c_1|\xi|^{2m}t} \cos(c_2|\xi|^{2m}t)\right)\|_{L^p(\mathbb{R}^n)} \lesssim t^{-\frac{a}{2m} - \frac{n}{2m}(1-\frac{1}{p})} \tag{58}$$

holds for any $m \in (0, \infty),\ p \in [1, \infty],\ t > 0$.

Taking $\rho=\frac{1}{1+\alpha}\in(1/3,1),\ z=t|\xi|^{2\sigma\rho},\ m=0$, applying Lemma 2.4, we have

$$\begin{aligned}\|G_{1+\alpha,\alpha}(t,x)\|_{L^p}&=\|\mathcal{F}^{-1}\big(E_{1+\alpha,\alpha}(-t^{1+\alpha}|\xi|^{2\sigma})\big)\|_{L^p}\\&\lesssim\|\mathcal{F}^{-1}\big((t|\xi|^{2\sigma\rho})^{1-\alpha}e^{t|\xi|^{2\sigma\rho}\cos(\pi\rho)}\cos\big(t|\xi|^{2\sigma\rho}\sin(\pi\rho)-\pi\rho(\alpha-1)\big)\big)\|_{L^p}\\&+\|\mathcal{F}^{-1}\Big((t|\xi|^{2\sigma\rho})^{1-\alpha}\int_0^\infty\frac{s^{2/\rho-\alpha}}{s^{2/\rho}+2\cos(\pi/\rho)s^{1/\rho}+1}e^{-ts|\xi|^{2\sigma\rho}}ds\Big)\|_{L^p}.\end{aligned}\tag{59}$$

From Proposition 2.2, the following polynomial type decay estimate holds

$$\|\mathcal{F}^{-1}\big((t|\xi|^{2\sigma\rho})^{1-\alpha}e^{t|\xi|^{2\sigma\rho}\cos(\pi\rho)}\cos\big(t|\xi|^{2\sigma\rho}\sin(\pi\rho)-\pi\rho(\alpha-1)\big)\big)\|_{L^p}\lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}.\tag{60}$$

According to Lemma 2.5, we get

$$\begin{aligned}&\mathcal{F}^{-1}\Big(\int_0^\infty\frac{s^{2/\rho-\alpha}}{s^{2/\rho}+2\cos(\pi/\rho)s^{1/\rho}+1}(t|\xi|^{2\sigma\rho})^{1-\alpha}e^{-ts|\xi|^{2\sigma\rho}}ds\Big)(t,x)\\&=\int_0^\infty\Big(\int_0^\infty\frac{s^{2/\rho-\alpha}}{s^{2/\rho}+2\cos(\pi/\rho)s^{1/\rho}+1}(tr^{2\sigma\rho})^{1-\alpha}e^{-tsr^{2\sigma\rho}}r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)dr\Big)ds\\&=\int_0^\infty\frac{s^{2/\rho-\alpha}}{s^{2/\rho}+2\cos(\pi/\rho)s^{1/\rho}+1}\Big(\int_0^\infty(tr^{2\sigma\rho})^{1-\alpha}e^{-tsr^{2\sigma\rho}}r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)dr\Big)ds\\&=\int_0^\infty\frac{s^{2/\rho-\alpha}}{s^{2/\rho}+2\cos(\pi/\rho)s^{1/\rho}+1}\Big(\mathcal{F}^{-1}\big((t|\xi|^{2\sigma\rho})^{1-\alpha}e^{-ts|\xi|^{2\sigma\rho}}\big)(t,x)\Big)ds.\end{aligned}\tag{61}$$

And using Proposition 2.1, the estimate

$$\|\mathcal{F}^{-1}\big((t|\xi|^{2\sigma\rho})^{1-\alpha}e^{-ts|\xi|^{2\sigma\rho}}\big)\|_{L^p}\lesssim s^{-1+\alpha-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}\tag{62}$$

holds, (61) and (62) imply that

$$\begin{aligned}&\|\mathcal{F}^{-1}\Big((t|\xi|^{2\sigma\rho})^{1-\alpha}\int_0^\infty\frac{s^{2/\rho-\alpha}}{s^{2/\rho}+2\cos(\pi/\rho)s^{1/\rho}+1}e^{-ts|\xi|^{2\sigma\rho}}ds\Big)\|_{L^p}\\&\lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}\int_0^\infty\frac{s^{2/\rho-1-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}}{s^{2/\rho}+2\cos(\pi/\rho)s^{1/\rho}+1}ds\lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})},\end{aligned}\tag{63}$$

where

$$\frac{n}{2\sigma}(1-\frac{1}{p})<2.$$

Thus, from (59), (60) and (63) we can obtain

$$\|G_{1+\alpha,\alpha}(t,x)\|_{L^p}\lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}.\tag{64}$$

For the operator $G_{1+\alpha,0}(t,\cdot)$, in particular, since $(-\Delta)^{\frac{\sigma}{1+\alpha}}u_0\in L^r$ for some $r\in[1,q]$, we have

$$\begin{aligned}\|(-\Delta)^{-\frac{\sigma}{1+\alpha}}G_{1+\alpha,0}(t,x)\|_{L^p}&=\|\mathcal{F}^{-1}\big(|\xi|^{-\frac{2\sigma}{1+\alpha}}E_{1+\alpha,0}(-t^{1+\alpha}|\xi|^{2\sigma})\big)\|_{L^p}\\&\lesssim\|\mathcal{F}^{-1}\big(te^{t|\xi|^{2\sigma\rho}\cos(\pi\rho)}\cos\big(t|\xi|^{2\sigma\rho}\sin(\pi\rho)+\pi\rho\big)\big)\|_{L^p}\\&+\|\mathcal{F}^{-1}\Big(t\int_0^\infty\frac{s^{1/\rho}}{s^{2/\rho}+2\cos(\pi/\rho)s^{1/\rho}+1}e^{-ts|\xi|^{2\sigma\rho}}ds\Big)\|_{L^p},\end{aligned}\tag{65}$$

by taking $m=0$. From Proposition 2.1 and Proposition 2.2, the following estimates hold

$$\|\mathcal{F}^{-1}\big(te^{t|\xi|^{2\sigma\rho}\cos(\pi\rho)}\cos\big(t|\xi|^{2\sigma\rho}\sin(\pi\rho)+\pi\rho\big)\big)\|_{L^p}\lesssim t^{1-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})},\tag{66}$$

and

$$\|\mathcal{F}^{-1}\big(te^{-ts|\xi|^{2\sigma\rho}}\big)\|_{L^p}\lesssim s^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}t^{1-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}.\tag{67}$$

Combining lemma 2.5 and (67), we get

$$\begin{aligned}&\|\mathcal{F}^{-1}\Big(t\int_0^\infty \frac{s^{1/\rho}}{s^{2/\rho}+2\cos(\pi/\rho)s^{1/\rho}+1}e^{-ts|\xi|^{2\sigma\rho}}ds\Big)\|_{L^p}\\&\lesssim t^{1-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}\int_0^\infty \frac{s^{1/\rho-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}}{s^{2/\rho}+2\cos(\pi/\rho)s^{1/\rho}+1}ds\lesssim t^{1-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})},\end{aligned}\tag{68}$$

where

$$\frac{n}{2\sigma}(1-\frac{1}{p})<1+\frac{1}{1+\alpha}.$$

Thus, from (65), (66) and (68) we have

$$\|G_{1+\alpha,0}(t,x)\|_{L^p}\lesssim t^{1-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}.\tag{69}$$

In virtue of Young's inequality, the following estimate of the homogeneous problem holds

$$\|u_t^{\mathrm{lin}}\|_{L^q}\lesssim t^{-\beta_{r,q}}(\|(-\Delta)^{\frac{\sigma}{1+\alpha}}u_0\|_{L^r}+\|u_1\|_{L^r})\lesssim t^{-\beta_{r,q}}(\|u_0\|_{H^{k,r}}+\|u_1\|_{L^r}),\tag{70}$$

for any $1\le r\le q\le\infty,\ t>0$, provided that

$$\frac{n}{2\sigma}(\frac{1}{r}-\frac{1}{q})<1.$$

Next we consider the estimate of

$$(Nf)_t:=\int_0^t (t-\tau)^{\alpha-1}G_{1+\alpha,\alpha}(t-\tau,\cdot)*f(\tau,\cdot)d\tau,\tag{71}$$

and

$$\nabla^\gamma Nf:=\int_0^t (t-\tau)^{\alpha}\nabla^\gamma G_{1+\alpha,1+\alpha}(t-\tau,\cdot)*f(\tau,\cdot)d\tau,\quad \gamma\ge 0.\tag{72}$$

The following estimate will be used frequently later in our proofs.

Lemma 2.6 ([4])**.** *Let $a<1$ and $b\in\mathbb{R}$. Then there exists a constant $C=C(a,b)>0$, the following estimate holds*

$$\int_0^t (t-\tau)^{-a}(1+\tau)^{-b}d\tau\le\begin{cases}C(1+t)^{-a}, & if\ a<1<b,\\ C(1+t)^{-a}\log(1+t), & if\ a<1=b,\\ C(1+t)^{-a+1-b}, & if\ a,\ b<1\end{cases}\tag{73}$$

for all $t>0$.

Thanks to (64) and Lemma 2.2, if $f(\tau,\cdot)\in L^r$, we have

$$\|(Nf)_t\|_{L^q}\lesssim\int_0^t (t-\tau)^{\alpha-1-\beta_{r,q}}\|f(\tau,\cdot)\|_{L^r}d\tau,\tag{74}$$

and

$$\|\nabla^\gamma Nf\|_{L^q}\lesssim\int_0^t (t-\tau)^{\alpha-\frac{(1+\alpha)\gamma}{2\sigma}-\beta_{r,q}}\|f(\tau,\cdot)\|_{L^r}d\tau.\tag{75}$$

Due to

$$\|f(\tau,\cdot)\|_{L^r}\le K(1+t)^{-\eta},\quad \forall t\ge 0\tag{76}$$

for some $K>0$ and $\eta\in\mathbb{R}$, then the following estimate holds

$$\|\big((Nf)_t,\nabla^k Nf\big)\|_{L^q}\lesssim\begin{cases}K(1+t)^{\alpha-1-\beta_{r,q}}, & if\ \eta>1,\\ K(1+t)^{\alpha-1-\beta_{r,q}}\log(1+t), & if\ \eta=1,\\ K(1+t)^{\alpha-\eta-\beta_{r,q}}, & if\ \eta<1.\end{cases}\tag{77}$$

From Lemma 2.6, it is not difficult to see that the inequality (77) is only holds if the condition

$$\frac{n}{2\sigma}(\frac{1}{r}-\frac{1}{q})<\frac{\alpha}{1+\alpha},\quad 1\le r\le q\le\infty\tag{78}$$

is satisfied. Thus, we obtain the L^r-L^q $(1\le r\le q\le\infty)$ estimate (44) of the linear problem (31).

3 | PROOF OF THE GLOBAL EXISTENCE RESULTS

We first give a brief description of the global iteration method introduced by Li.[17]

For a given space X, it is equipped with a finite norm $\|\cdot\|_X$ related to the decay rates for the solution to (31), then the global iteration method will be carried out through the following framework.

By (35), a solution $u \in X$ to the linear problem (31) satisfies the equality

$$u(t,x) := u^{\mathrm{lin}}(t,x) + Nu \tag{79}$$

in X, where we select $f(\tau,x) = F(u_t)$ in (37). In particular, if we get

$$\|u\|_X \lesssim \varepsilon + \|u\|_X^p, \tag{80}$$

and

$$\|Nu - Nv\|_X \lesssim \|u-v\|_X(\|u\|_X^{p-1} + \|v\|_X^{p-1}). \tag{81}$$

By Banach fixed point theory, and from (80) one can see that $u^{\mathrm{lin}} + Nu$ is a mapping of X to itself for small data as $u^{\mathrm{lin}} \in X$ and $p > 1$, and that estimates (80)-(81) determine a unique solution u to (79) exists globally. Then local and global existence results are obtained simultaneously.

3.1 | Proof of Theorem 1.1.

We define the space

$$X = \{u \mid u \in C\big([0,\infty), \dot{H}^{k,1} \cap \dot{H}^{k,\infty}\big) \cap C^1\big([0,\infty), L^1 \cap L^\infty\big),\ \|u\|_X < M\} \tag{82}$$

with the norm

$$\|u\|_X = \sup_{t\geq 0}(1+t)^{-\lambda}\{\|u_t\|_{L^1} + (1+t)^{\beta_{1,\infty}}\|u_t\|_{L^\infty} + \|u\|_{\dot{H}^{k,1}} + (1+t)^{\beta_{1,\infty}}\|u\|_{\dot{H}^{k,\infty}}\}, \tag{83}$$

where $\beta_{1,\infty} = \frac{n(1+\alpha)}{2\sigma},\ M > 0$.

For any $u \in X$, consider the following mapping

$$\phi : X \to X, \quad \phi u := u^{\mathrm{lin}}(t,x) + Nu.$$

We shall prove that

$$\|\phi u\|_X \lesssim \|(u_0,u_1)\|_{\mathcal{A}} + \|u\|_X^p, \tag{84}$$

and

$$\|\phi u - \phi v\|_X \lesssim \|u-v\|_X(\|u\|_X^{p-1} + \|v\|_X^{p-1}). \tag{85}$$

For the proof of (84), after taking into consideration the estimate (47), we have

$$\begin{aligned} \|u^{\mathrm{lin}}\|_X = \sup_{t\geq 0}(1+t)^{-\lambda}\{&\|u_t^{\mathrm{lin}}\|_{L^1} + (1+t)^{\beta_{1,\infty}}\|u_t^{\mathrm{lin}}\|_{L^\infty} \\ &+ \|u^{\mathrm{lin}}\|_{\dot{H}^{k,1}} + (1+t)^{\beta_{1,\infty}}\|u^{\mathrm{lin}}\|_{\dot{H}^{k,\infty}}\} \\ &\lesssim \|(u_0,u_1)\|_{\mathcal{A}}. \end{aligned} \tag{86}$$

It remains to prove that $\|Nu\|_X \lesssim \|u\|_X^p$. If $u \in X$, then by interpolation one can get the following estimate

$$\|u_t\|_{L^q} \lesssim (1+t)^{\lambda-\beta_{1,q}}\|u\|_X, \quad q \in [1,\infty]. \tag{87}$$

Consequently,

$$\||u_t|^p\|_{L^1} \lesssim \|u_t\|_{L^p}^p \lesssim (1+t)^{-(\beta_{1,p}-\lambda)p}\|u\|_X^p, \tag{88}$$

notice that $(\beta_{1,p} - \lambda)p > 1$ if and only if

$$p > 1 + \frac{2(1+\lambda)\sigma}{n(1+\alpha) - 2\lambda\sigma}.$$

We now apply Theorem 2.1 to the nonlinear part of the solution. Taking $r = 1$, $f(\tau,x) = |u_t|^p$ and $K = c\|u\|_X^p$, for some $c > 0$, and thanks to (88),

$$\|\big((Nu)_t, \nabla^k Nu\big)\|_{L^q} \lesssim (1+t)^{\alpha-1-\beta_{1,q}}\|u\|_X^p \lesssim (1+t)^{\alpha-1-\beta_{1,q}+\lambda}\|u\|_X^p, \tag{89}$$

let $q = 1, \infty$, respectively, thus

$$\|Nu\|_X \lesssim \|u\|_X^p. \tag{90}$$

Finally, it remains to show (85). By Hölder's inequality, for $u, v \in X$, then

$$\begin{aligned}\||u_t|^p - |v_t|^p\|_{L^1} &\lesssim \||u_t - v_t|(|u_t|^{p-1} + |v_t|^{p-1})\|_{L^1} \\ &\lesssim \|u_t - v_t\|_{L^p}\||u_t|^{p-1} + |v_t|^{p-1}\|_{L^{\frac{p}{p-1}}} \\ &\lesssim \|u_t - v_t\|_{L^p}(\|u_t\|_{L^{mp}}^{p-1} + \|v_t\|_{L^{mp}}^{p-1}) \\ &\lesssim (1+t)^{-(\beta_{1,p}-\lambda)p}\|u - v\|_X(\|u\|_X^{p-1} + \|v\|_X^{p-1}).\end{aligned} \tag{91}$$

Hence,

$$\|(Nu)_t - (Nv)_t\|_{L^q} \lesssim (1+t)^{\alpha-1-\beta_{1,q}+\lambda}\|u - v\|_X(\|u\|_X^{p-1} + \|v\|_X^{p-1}), \tag{92}$$

$$\|\nabla^k(Nu - Nv)\|_{L^q} \lesssim (1+t)^{\alpha-1-\beta_{1,q}+\lambda}\|u - v\|_X(\|u\|_X^{p-1} + \|v\|_X^{p-1}), \tag{93}$$

let $q = 1, \infty$, respectively, (92) and (93) lead to

$$\|\phi u - \phi v\|_X = \|Nu - Nv\|_X \lesssim \|u - v\|_X(\|u\|_X^{p-1} + \|v\|_X^{p-1}). \tag{94}$$

Then we may conclude a unique global solution

$$u \in C\big([0,\infty), \dot H^{k,1} \cap \dot H^{k,\infty}\big) \cap C^1\big([0,\infty), L^1 \cap L^\infty\big). \tag{95}$$

The proof is completed.

3.2 | Proof of Theorem 1.2.

We now define the space

$$X = \{u \mid u \in C\big([0,\infty), \dot H^{k,m} \cap \dot H^{k,\infty}\big) \cap C^1\big([0,\infty), L^m \cap L^\infty\big),\ \|u\|_X < \tilde M\} \tag{96}$$

with the norm

$$\|u\|_X = \sup_{t\ge 0}(1+t)^{\eta-\alpha}\{\|u_t\|_{L^m} + (1+t)^{\beta_{m,\infty}}\|u_t\|_{L^\infty} + \|u\|_{\dot H^{k,m}} + (1+t)^{\beta_{m,\infty}}\|u\|_{\dot H^{k,\infty}}\}, \tag{97}$$

where $\beta_{m,\infty} = \frac{n(1+\alpha)}{2\sigma m}$, $\eta = \frac{\alpha p}{p-1} - \frac{n(1+\alpha)(p-m)}{2\sigma m(p-1)} > 0$, $\tilde M > 0$. Notice that $\eta < \alpha$ if and only if

$$p > 1 + \frac{n(1+\alpha)}{2\sigma\alpha}$$

for sufficiently small constant $\delta \to 0^+$ in (48).

For any $u \in X$, consider the mapping

$$\psi : X \to X, \quad \psi u := u^{\mathrm{lin}}(t,x) + Nu.$$

We shall prove that

$$\|\psi u\|_X \lesssim \|(u_0,u_1)\|_{\mathcal M} + \|u\|_X^p, \tag{98}$$

and

$$\|\psi u - \psi v\|_X \lesssim \|u - v\|_X(\|u\|_X^{p-1} + \|v\|_X^{p-1}). \tag{99}$$

For the proof of (98), by means of the estimate (50), we have

$$\begin{aligned}\|u^{\mathrm{lin}}\|_X &= \sup_{t\ge 0}(1+t)^{\eta-\alpha}\{\|u_t^{\mathrm{lin}}\|_{L^m} + (1+t)^{\beta_{m,\infty}}\|u_t^{\mathrm{lin}}\|_{L^\infty} \\ &\qquad + \|u^{\mathrm{lin}}\|_{\dot H^{k,m}} + (1+t)^{\beta_{m,\infty}}\|u^{\mathrm{lin}}\|_{\dot H^{k,\infty}}\} \\ &\lesssim \|(u_0,u_1)\|_{\mathcal M}.\end{aligned} \tag{100}$$

It remains to prove that $\|Nu\|_X \lesssim \|u\|_X^p$. If $u \in X$, then by interpolation one can get the following estimate

$$\|u_t\|_{L^q} \lesssim (1+t)^{\alpha-\eta-\beta_{m,q}}\|u\|_X, \quad q \in [m,\infty]. \tag{101}$$

Consequently,

$$\begin{aligned}\||u_t|^p\|_{L^m} &\lesssim \|u_t\|_{L^{mp}}^p \lesssim (1+t)^{-(\eta-\alpha+\beta_{m,mp})p}\|u\|_X^p \\ &\lesssim (1+t)^{-(\eta-\alpha+\beta_{m,p})p}\|u\|_X^p\end{aligned} \tag{102}$$

due to $\beta_{m,mp} > \beta_{m,p}$, note that

$$0 < (\eta - \alpha + \beta_{m,p})p = \eta < \alpha < 1. \tag{103}$$

For the nonlinear part of the solution, similar to the proof in Theorem 1.1 except that set $r = m$, from (102),

$$\|\big((Nu)_t, \nabla^k Nu\big)\|_{L^q} \lesssim (1+t)^{\alpha-\eta-\beta_{m,q}} \|u\|_X^p, \tag{104}$$

let $q = m, \infty$, respectively, thus

$$\|Nu\|_X \lesssim \|u\|_X^p. \tag{105}$$

Finally, it remains to show (99). By Hölder's inequality, for $u, v \in X$, then

$$\begin{aligned}\||u_t|^p - |v_t|^p\|_{L^m} &\lesssim \||u_t - v_t|(|u_t|^{p-1} + |v_t|^{p-1})\|_{L^m} \\ &\lesssim \|u_t - v_t\|_{L^{mp}} \||u_t|^{p-1} + |v_t|^{p-1}\|_{L^{\frac{mp}{p-1}}} \\ &\lesssim \|u_t - v_t\|_{L^{mp}} (\|u_t\|_{L^{mp}}^{p-1} + \|v_t\|_{L^{mp}}^{p-1}) \\ &\lesssim (1+t)^{-(\eta-\alpha+\beta_{m,mp})p} \|u - v\|_X (\|u\|_X^{p-1} + \|v\|_X^{p-1}) \\ &\lesssim (1+t)^{-(\eta-\alpha+\beta_{m,p})p} \|u - v\|_X (\|u\|_X^{p-1} + \|v\|_X^{p-1}).\end{aligned} \tag{106}$$

Hence,

$$\|(Nu)_t - (Nv)_t\|_{L^q} \lesssim (1+t)^{\alpha-\eta-\beta_{m,q}} \|u - v\|_X (\|u\|_X^{p-1} + \|v\|_X^{p-1}), \tag{107}$$

$$\|\nabla^k (Nu - Nv)\|_{L^q} \lesssim (1+t)^{\alpha-\eta-\beta_{m,q}} \|u - v\|_X (\|u\|_X^{p-1} + \|v\|_X^{p-1}), \tag{108}$$

let $q = m, \infty$, respectively, (107) and (108) lead to

$$\|\Phi u - \Phi v\|_X = \|Nu - Nv\|_X \lesssim \|u - v\|_X (\|u\|_X^{p-1} + \|v\|_X^{p-1}). \tag{109}$$

Then we may conclude a unique global solution

$$u \in C\big([0,\infty), \dot H^{k,m} \cap \dot H^{k,\infty}\big) \cap C^1\big([0,\infty), L^m \cap L^\infty\big). \tag{110}$$

The proof is completed.

3.3 | Proof of Theorem 1.3.

We define the space

$$X = \{u \mid u \in C\big([0,\infty), \dot H^{k,1} \cap \dot H^{k,\infty}\big) \cap C^1\big([0,\infty), L^1 \cap L^\infty\big),\ \|u\|_X < M\} \tag{111}$$

with the norm

$$\|u\|_X = \sup_{t\ge 0} (1+t)^{\mu-\alpha} \{\|u_t\|_{L^1} + (1+t)^{\beta_{1,\infty}} \|u_t\|_{L^\infty} + \|u\|_{\dot H^{k,1}} + (1+t)^{\beta_{1,\infty}} \|u\|_{\dot H^{k,\infty}}\}. \tag{112}$$

For any $u \in X$, consider the mapping

$$\Phi : X \to X, \quad \Phi u := u^{\mathrm{lin}}(t,x) + Nu.$$

We shall prove that

$$\|\Phi u\|_X \lesssim \|(u_0, u_1)\|_{\mathcal{A}} + \|u\|_X^p, \tag{113}$$

and

$$\|\Phi u - \Phi v\|_X \lesssim \|u - v\|_X (\|u\|_X^{p-1} + \|v\|_X^{p-1}). \tag{114}$$

For the proof of (113), from the estimate (47), we have

$$\begin{aligned}\|u^{\mathrm{lin}}\|_X = \sup_{t\ge 0} &(1+t)^{\mu-\alpha} \{\|u_t^{\mathrm{lin}}\|_{L^1} + (1+t)^{\beta_{1,\infty}} \|u_t^{\mathrm{lin}}\|_{L^\infty} \\ &+ \|u^{\mathrm{lin}}\|_{\dot H^{k,1}} + (1+t)^{\beta_{1,\infty}} \|u^{\mathrm{lin}}\|_{\dot H^{k,\infty}}\} \\ &\lesssim \|(u_0, u_1)\|_{\mathcal{A}}.\end{aligned} \tag{115}$$

It remains to prove that $\|Nu\|_X \lesssim \|u\|_X^p$. If $u \in X$, then by interpolation one can get the following estimate

$$\|u_t\|_{L^q} \lesssim (1+t)^{\alpha-\mu-\beta_{1,q}} \|u\|_X, \quad q \in [1,\infty]. \tag{116}$$

Consequently,

$$\||u_t(t,x)|^p\|_{L^1} \lesssim \|u_t(t,x)\|_{L^p}^p \lesssim (1+t)^{-(\mu-\alpha+\beta_{1,p})p} \|u\|_X^p, \tag{117}$$

notice that $(\mu-\alpha+\beta_{1,p})p>1$ if and only if

$$p>1+\frac{2(1+\alpha-\mu)\sigma}{n(1+\alpha)-2(\alpha-\mu)\sigma}.$$

We now apply Lemma 2.6 twice to the nonlinear part of the solution, for (74) and (75). Taking $r=1$, $f(\tau,x)=\int_0^\tau(\tau-s)^{-\mu}|u_t(s,x)|^p ds$ and $K=c\|u\|_X^p$, for some $c>0$. Combine with (117), then

$$\begin{aligned}\|\big((Nu)_t,\nabla^k Nu\big)\|_{L^q}&\lesssim\int_0^t(t-\tau)^{\alpha-1-\beta_{1,q}}\|f(\tau,x)\|_{L^1}d\tau\\&\lesssim\int_0^t(t-\tau)^{\alpha-1-\beta_{1,q}}\int_0^\tau(\tau-s)^{-\mu}\||u_t(s,x)|^p\|_{L^1}dsd\tau\\&\lesssim\int_0^t(t-\tau)^{\alpha-1-\beta_{1,q}}\int_0^\tau(\tau-s)^{-\mu}(1+s)^{-(\mu-\alpha+\beta_{1,p})p}\|u\|_X^p dsd\tau\\&\lesssim\int_0^t(t-\tau)^{\alpha-1-\beta_{1,q}}(1+\tau)^{-\mu}d\tau\|u\|_X^p\\&\lesssim(1+t)^{\alpha-\mu-\beta_{1,q}}\|u\|_X^p,\end{aligned}\tag{118}$$

let $q=1,\infty$, respectively, thus

$$\|Nu\|_X\lesssim\|u\|_X^p.\tag{119}$$

Finally, it remains to show (114). By Hölder's inequality, for $u,v\in X$,

$$\begin{aligned}\||u_t|^p-|v_t|^p\|_{L^1}&\lesssim\||u_t-v_t|(|u_t|^{p-1}+|v_t|^{p-1})\|_{L^1}\\&\lesssim\|u_t-v_t\|_{L^p}\||u_t|^{p-1}+|v_t|^{p-1}\|_{L^{\frac{p}{p-1}}}\\&\lesssim\|u_t-v_t\|_{L^p}(\|u_t\|_{L^p}^{p-1}+\|v_t\|_{L^p}^{p-1})\\&\lesssim(1+t)^{-(\mu-\alpha+\beta_{1,p})p}\|u-v\|_X(\|u\|_X^{p-1}+\|v\|_X^{p-1}).\end{aligned}\tag{120}$$

Hence,

$$\|(Nu)_t-(Nv)_t\|_{L^q}\lesssim(1+t)^{\alpha-\mu-\beta_{1,q}}\|u-v\|_X(\|u\|_X^{p-1}+\|v\|_X^{p-1}),\tag{121}$$

$$\|\nabla^k(Nu-Nv)\|_{L^q}\lesssim(1+t)^{\alpha-\mu-\beta_{1,q}}\|u-v\|_X(\|u\|_X^{p-1}+\|v\|_X^{p-1}),\tag{122}$$

let $q=1,\infty$, respectively, (121) and (122) lead to

$$\|\Phi u-\Phi v\|_X=\|Nu-Nv\|_X\lesssim\|u-v\|_X(\|u\|_X^{p-1}+\|v\|_X^{p-1}).\tag{123}$$

Then we may conclude a unique global solution

$$u\in C\big([0,\infty),\dot H^{k,1}\cap\dot H^{k,\infty}\big)\cap C^1\big([0,\infty),L^1\cap L^\infty\big).\tag{124}$$

The proof is completed.

3.4 | Proof of Theorem 1.4.

We now define the space

$$X=\{u\mid u\in C\big([0,\infty),\dot H^{k,m}\cap\dot H^{k,\infty}\big)\cap C^1\big([0,\infty),L^m\cap L^\infty\big),\ \|u\|_X<\tilde M\}\tag{125}$$

with the norm

$$\begin{aligned}\|u\|_X=\sup_{t\ge0}(1+t)^{\mu-\alpha+\xi-1}\{&\|u_t\|_{L^m}+(1+t)^{\beta_{m,\infty}}\|u_t\|_{L^\infty}\\&+\|u\|_{\dot H^{k,m}}+(1+t)^{\beta_{m,\infty}}\|u\|_{\dot H^{k,\infty}}\},\end{aligned}\tag{126}$$

where $\xi = \frac{(\alpha+1-\mu)p}{p-1} - \frac{n(1+\alpha)(p-m)}{2\sigma m(p-1)} > 0$, notice that $\xi < \alpha + 1 - \mu$ if and only if

$$p > 1 + \frac{2\sigma(1-\mu) + n(1+\alpha)}{2\sigma\alpha}$$

for sufficiently small $\delta \to 0^+$ in (48).

For any $u \in X$, consider the following mapping

$$\Psi : X \to X, \quad \Psi u := u^{\mathrm{lin}}(t,x) + Nu.$$

We shall prove that

$$\|\Psi u\|_X \lesssim \|(u_0,u_1)\|_{\mathcal{M}} + \|u\|_X^p, \tag{127}$$

and

$$\|\Psi u - \Psi v\|_X \lesssim \|u-v\|_X(\|u\|_X^{p-1} + \|v\|_X^{p-1}). \tag{128}$$

For the proof of (127), according to the the estimate (50), we have

$$\begin{aligned} \|u^{\mathrm{lin}}\|_X = \sup_{t\ge 0}(1+t)^{\mu-\alpha+\xi-1}\{&\|u_t^{\mathrm{lin}}\|_{L^m} + (1+t)^{\beta_{m,\infty}}\|u_t^{\mathrm{lin}}\|_{L^\infty} \\ &+ \|u^{\mathrm{lin}}\|_{\dot H^{k,m}} + (1+t)^{\beta_{m,\infty}}\|u^{\mathrm{lin}}\|_{\dot H^{k,\infty}}\} \\ \lesssim{}& \|(u_0,u_1)\|_{\mathcal{M}}. \end{aligned} \tag{129}$$

It remains to prove that $\|Nu\|_X \lesssim \|u\|_X^p$. If $u \in X$, then by interpolation one can get the following estimate

$$\|u_t\|_{L^q} \lesssim (1+t)^{\alpha+1-\mu-\xi-\beta_{m,q}}\|u\|_X, \quad q \in [m,\infty]. \tag{130}$$

Consequently,

$$\begin{aligned} \||u_t(t,x)|^p\|_{L^m} \lesssim \|u_t(t,x)\|_{L^{mp}}^p &\lesssim (1+t)^{-(\mu-\alpha+\xi-1+\beta_{m,mp})p}\|u\|_X^p \\ &\lesssim (1+t)^{-(\mu-\alpha+\xi-1+\beta_{m,p})p}\|u\|_X^p. \end{aligned} \tag{131}$$

due to $\beta_{m,mp} > \beta_{m,p}$, note that

$$0 < (\mu-\alpha+\xi-1+\beta_{m,p})p = \xi < \alpha+1-\mu < 1, \tag{132}$$

since $\mu > \alpha$.

For the nonlinear part of the solution, similar to the proof in Theorem 1.3, by taking $r = m$ in (74) and (75) and combine with (131), then

$$\begin{aligned} \|\big((Nu)_t, \nabla^k Nu\big)\|_{L^q} &\lesssim \int_0^t (t-\tau)^{\alpha-1-\beta_{m,q}}\|f(\tau,\cdot)\|_{L^m}d\tau \\ &\lesssim \int_0^t (t-\tau)^{\alpha-1-\beta_{m,q}}\int_0^\tau (\tau-s)^{-\mu}\||u_t(s,x)|^p\|_{L^m}ds\,d\tau \\ &\lesssim \int_0^t (t-\tau)^{\alpha-1-\beta_{m,q}}\int_0^\tau (\tau-s)^{-\mu}(1+s)^{-(\mu-\alpha+\xi-1+\beta_{m,p})p}\|u\|_X^p ds\,d\tau \\ &\lesssim \int_0^t (t-\tau)^{\alpha-1-\beta_{m,q}}(1+\tau)^{-\mu+1-\xi}d\tau\|u\|_X^p \\ &\lesssim (1+t)^{\alpha+1-\mu-\xi-\beta_{m,q}}\|u\|_X^p, \end{aligned} \tag{133}$$

let $q = m, \infty$, respectively, thus

$$\|Nu\|_X \lesssim \|u\|_X^p. \tag{134}$$

Finally, it remains to show (128). By Hölder's inequality, for $u, v \in X$,

$$\begin{aligned} \||u_t|^p - |v_t|^p\|_{L^m} &\lesssim \||u_t - v_t|(|u_t|^{p-1} + |v_t|^{p-1})\|_{L^m} \\ &\lesssim \|u_t - v_t\|_{L^{mp}}\||u_t|^{p-1} + |v_t|^{p-1}\|_{L^{\frac{mp}{p-1}}} \\ &\lesssim \|u_t - v_t\|_{L^{mp}}(\|u_t\|_{L^{mp}}^{p-1} + \|v_t\|_{L^{mp}}^{p-1}) \\ &\lesssim (1+t)^{-(\mu-\alpha+\xi-1+\beta_{m,mp})p}\|u-v\|_X(\|u\|_X^{p-1} + \|v\|_X^{p-1}) \\ &\lesssim (1+t)^{-(\mu-\alpha+\xi-1+\beta_{m,p})p}\|u-v\|_X(\|u\|_X^{p-1} + \|v\|_X^{p-1}). \end{aligned} \tag{135}$$

Hence,

$$\|(Nu)_t - (Nv)_t\|_{L^q} \lesssim (1+t)^{\alpha+1-\mu-\xi-\beta_{m,q}} \|u-v\|_X (\|u\|_X^{p-1} + \|v\|_X^{p-1}), \tag{136}$$

$$\|\nabla^k (Nu - Nv)\|_{L^q} \lesssim (1+t)^{\alpha+1-\mu-\xi-\beta_{m,q}} \|u-v\|_X (\|u\|_X^{p-1} + \|v\|_X^{p-1}), \tag{137}$$

let $q = 1, \infty$, respectively, (136) and (137) lead to

$$\|\Phi u - \Phi v\|_X = \|Nu - Nv\|_X \lesssim \|u-v\|_X (\|u\|_X^{p-1} + \|v\|_X^{p-1}). \tag{138}$$

Then we may conclude a unique global solution

$$u \in C\big([0,\infty), \dot{H}^{k,1} \cap \dot{H}^{k,\infty}\big) \cap C^1\big([0,\infty), L^1 \cap L^\infty\big). \tag{139}$$

The proof is completed.

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