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RESEARCH ARTICLE

On a problem for a delay differential equation

Narkesh Iskakova[1,2] | Svetlana Temesheva*[1,3] | Roza Uteshova[1,4]

[1]Department of Differential Equations, Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
[2]Department of Mathematics and Mathematical Modeling, Abai Kazakh National Pedagogical University, Almaty, Kazakhstan
[3]Department of Mathematics, Al-Farabi Kazakh National University, Almaty, Kazakhstan
[4]Department of Mathematical and Computer Modeling, International IT University, Almaty, Kazakhstan

Correspondence
*Svetlana Temesheva, Institute of Mathematics and Mathematical Modeling, Pushkin str, 125, Almaty 050010, Kazakhstan. Email: temeshevasvetlana@gmail.com

Abstract

In this paper, we propose an algorithm for solving a two-point boundary-value problem for a linear differential equation with constant delay subject to a nonlinear boundary condition. We derive sufficient conditions for the convergence of the algorithm and for the existence of an isolated solution to the problem under study. A numerical example is provided.

KEYWORDS:
delay differential equation; nonlinear boundary-value problem; isolated solution; algorithm

1 | INTRODUCTION

We consider the linear system of delay differential equations

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)x(t-\tau) + f(t), \quad x \in R^n, \quad t \in (0,T), \quad \tau > 0, \tag{1}$$

$$x(t) = \mathrm{diag}[x(0)] \cdot \varphi(t), \quad t \in [-\tau, 0], \tag{2}$$

subject to a nonlinear two-point boundary condition

$$g(x(0), x(T)) = 0. \tag{3}$$

Here the $n \times n$ matrices $A(t)$, $B(t)$ and the vector function $f(t)$ are continuous on $[0,T]$, $\varphi : [-\tau, 0] \to R^n$ is a continuously differentiable vector function such that $\varphi_i(0) = 1$, $i = 1 : n$; τ is a constant delay; $\|A(t)\| = \max\limits_{i=\overline{1,n}} \sum\limits_{j=1}^{n} \|a_{ij}(t)\| \leq \alpha$, $\|B(t)\| = \max\limits_{i=\overline{1,n}} \sum\limits_{j=1}^{n} \|b_{ij}(t)\| \leq \beta$, where α and β are some constants.

A solution to the boundary-value problem (1)-(3) is a continuous on $[-\tau, T]$, continuously differentiable on $(-\tau, 0) \bigcup (0, T)$ vector function $x^*(t)$, satisfying Eq. (1) and conditions (2), (3).

When modeling the processes of mass, energy and information transfer (in particular, hereditary information), it is necessary to consider differential equations with deviating arguments, including delay differential equations[1,2,3,4,5,6,7]. Basic concepts and definitions of the theory of differential equations with deviating arguments, as well as existence theorems and approximate methods of their solution, are covered in[8,9,10,11]. Boundary value problems for delay differential equations have been studied by many authors, e.g.,[4,9,12,13,14,15,16,17,18,19,20].

This paper aims to obtain sufficient solvability conditions of problem (1)-(3) and to develop algorithms to find its solution.

To this end, we use the idea of Dzhumabayev's parameterization method[21], which is one of constructive methods that allow to simultaneously investigate the existence and construct solutions of boundary value problems for differential equations. This method was originally developed to investigate the linear two-point boundary value problem for the system of ordinary differential equations

$$\frac{dx}{dt} = A(t)x + f(t), \quad t \in (0,T), \quad x \in R^n, \tag{4}$$

$$Bx(0) + Cx(T) = d, \quad d \in R^n. \tag{5}$$

The solution procedure for the boundary value problem reduces to the following two main problems: solving a system of linear algebraic equations in parameters, introduced as the values of the desired solution at the partition points, and solving Cauchy problems on the partition subintervals. This made it possible to derive necessary and sufficient conditions for the well-posedness of problem (4), (5) only in terms of the input data $A(t)$, B, C, $f(t)$, d, and T.

The parameterization method has been successfully applied to study and solve periodic boundary value problems for linear delayed differential equations[22,23,24] and nonlinear two-point boundary value problems for ordinary differential equations[25,26,27]. In[28], a modified algorithm of the parameterization method was developed to solve problem (4), (5). The results obtained served as an impetus for the study of problem (1)-(3).

When solving problem (4), (5) by the parametrization method, Cauchy problems for ordinary differential equations are solved at each step of the algorithm. It is natural to assume that applying the parametrization method to problem (1)-(3) will lead to Cauchy problems for delay differential equations, which is not a simple problem in itself. However, the modification of the parameterization method proposed in this paper, as applied to problem (1)-(3), is content with Cauchy problems for ordinary differential equations. This approach facilitated the procedure of solving problem (1)-(3).

In this paper, sufficient conditions for the existence of an isolated solution of problem (1)-(3) are obtained. An illustrating example is given.

2 | AN ALGORITHM FOR SOLVING PROBLEM (1)-(3)

Let us suppose $T = N\tau$, where N is a positive integer. We will use the following notation:

Δ_ℓ is a partition of the interval $[-\tau, T) = \bigcup\limits_{r=-\ell+1}^{\ell N} [t_{r-1}, t_r)$ by points $t_s = s\tau/\ell$, $s = -\ell : \ell N$, $\ell = 1, 2, \ldots$;

$C([-\tau, T], R^n)$ is the space of continuous functions $x : [-\tau, T] \to R^n$ with the norm $\|x\|_1 = \max\limits_{t\in[-\tau,T]} \|x(t)\|$;

$C([0,T], \Delta_\ell, R^{n\ell N})$ is the space of function systems $x[t] = (x_1(t),\ x_2(t),\ \ldots,\ x_{\ell N}(t))$ whose elements $x_r(t) \in C([t_{r-1}, t_r))$ have finite one-sided limits $\lim\limits_{t\to t_r-0} x_r(t)$, $r = 1 : (\ell N - 1)$, equipped with with the norm $\|x[\cdot]\|_2 = \max\limits_{r=1:\ell N} \sup\limits_{t\in[t_{r-1},t_r)} \|x_r(t)\|$;

$C([-\tau, T], \Delta_\ell, R^{n\ell(N+1)})$ is the space of function systems $x[t] = (x_{-\ell+1}(t), x_{-\ell+2}(t), \ldots, x_{-1}(t), x_0(t), x_1(t), x_2(t), \ldots, x_{\ell N}(t))$ whose elements $x_r(t) \in C[t_{r-1}, t_r)$, $\forall r = (-\ell+1) : 0$, and $x_r(t) \in C^1[t_{r-1}, t_r)$, $\forall r = 1 : \ell N$, have finite one-sided limits $\lim\limits_{t\to t_r-0} x_r(t) \in R^n$, $r = (-\ell+1) : \ell N$, equipped with the norm $\|x[\cdot]\|_3 = \max\limits_{r=-\ell:\ell N} \sup\limits_{t\in[t_{r-1},t_r)} \|x_r(t)\|$;

$C([0, (N-1)\tau], \Delta_\ell, R^{n\ell N})$ is the space of function systems $x[t] = (x_{-\ell+1}(t-\tau), x_{-\ell+2}(t-\tau), \ldots, x_0(t-\tau), x_1(t-\tau), \ldots, x_{\ell(N-1)}(t-\tau))$ with elements $x_p(t) \in C[t_{p-1}, t_p)$, $p = (-\ell+1) : \ell(N-1)$, equipped with the norm $\|x[\cdot]\|_4 = \max\limits_{r=1:\ell N} \sup\limits_{t\in[t_{r-1},t_r)} \|x_{-\ell+r}(t-\tau)\|$.

By introducing the parameters $\lambda_r = x(t_{r-1})$, $r = (-\ell+1) : \ell N$, $\lambda_{\ell N+1} = \lim\limits_{t\to T-0} x(t)$, and substituting $u_r(t) = x(t) - \lambda_r$, $t \in [t_{r-1}, t_r)$, $r = (-\ell+1) : \ell N$, we transform the boundary value problem (1)-(3) to the equivalent multi-point boundary value problem with parameters

$$\frac{du_r(t)}{dt} = A(t)(\lambda_r + u_r(t)) + B(t)(\lambda_{-\ell+r} + u_{-\ell+r}(t-\tau)) + f(t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : \ell N, \tag{6}$$

$$u_r(t_{r-1}) = 0, \quad r = 1 : \ell N, \tag{7}$$

$$\lambda_p + u_p(t) = \Phi(t)\cdot\lambda_1, \quad t\in[t_{p-1},t_p), \quad p=(-\ell+1):0, \tag{8}$$

$$-\lambda_p + \Phi(t_{p-1})\cdot\lambda_1 = 0, \quad p=(-\ell+1):0, \tag{9}$$

$$g\left(\lambda_1,\lambda_{\ell N+1}\right)=0, \tag{10}$$

$$\lambda_r + \lim_{t\to t_r-0} u_r(t) = \lambda_{r+1}, \quad r=1:\ell N, \tag{11}$$

Here $\Phi(t)=\mathrm{diag}[\varphi(t)]$, $t\in[t_{-\ell},t_0]$.

A solution to problem (6)-(11) is a pair $\left(\lambda^*,u^*[t]\right)$ with elements

$$\lambda^* = (\lambda^*_{-\ell+1},\dots,\lambda^*_{-1},\lambda^*_0,\lambda^*_1,\dots,\lambda^*_{\ell N},\lambda^*_{\ell N+1})\in R^{n(\ell(N+1)+1)},$$

$$u^*[t]=(u^*_{-\ell+1}(t),\dots,u^*_{-1}(t),u^*_0(t),u^*_1(t),\dots,u^*_{\ell N}(t))\in C([-\tau,T],\Delta_\ell,R^{n\ell(N+1)}),$$

that satisfy (6)-(11).

Let us introduce the notation

$$a_r(P,t)=X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\xi)P(\xi)d\xi, \quad t\in[t_{r-1},t_r), \quad r=1:\ell N,$$

$$b_r(u_{-\ell+r}(\cdot),\tau,t)=X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\xi)B(\xi)u_{-\ell+r}(\xi-\tau)d\xi, \quad t\in[t_{r-1},t_r), \quad r=1:\ell.$$

Here $X_r(t)$ is the linear operator[29, p. 61]

$$X_r(t)=I+\int_{t_{r-1}}^{t}A(s_1)ds_1+\sum_{j=2}^{\infty}\int_{t_{r-1}}^{t}A(s_1)\int_{t_{r-1}}^{s_1}A(s_2)\dots\int_{t_{r-1}}^{s_{j-1}}A(s_j)ds_j\dots ds_2ds_1, \quad t\in[t_{r-1},t_r), \quad r=1:\ell N,$$

I is the identity matrix of order n.

For fixed λ_r, $\lambda_{-\ell+r}$, and $u_{-\ell+r}(t-\tau)$, problem (6), (7) is the Cauchy problem for an ordinary differential equation, the solution of which is

$$u_r(t)=a_r(A,t)\lambda_r+a_r(B,t)\lambda_{-\ell+r}+b_r(u_{-\ell+r}(\cdot),\tau,t)+a_r(f,t), \quad t\in[t_{r-1},t_r), \quad r=1:\ell N. \tag{12}$$

Let us compose the function system

$$u[t]=(u_{-\ell+1}(t),u_{-\ell+2}(t),\dots,u_{-1}(t),u_0(t),u_1(t),u_2(t),\dots,u_{\ell N}(t)).$$

From (12) we determine $\lim_{t\to t_r-0}u_r(t)$, $r=1:\ell N$. Based on (9)-(11), we construct the system of nonlinear equations for unknown parameters

$$Q_{\Delta_\ell}(\lambda,u)=0,$$

here $\lambda=(\lambda_{-\ell+1},\lambda_{-\ell+2},\dots,\lambda_{-1},\lambda_0,\lambda_1,\lambda_2,\dots,\lambda_{\ell N},\lambda_{\ell N+1})\in R^{n(\ell(N+1)+1)}$ and

$$Q_{\Delta_\ell}(\lambda,u)=\begin{pmatrix} -\frac{\tau}{\ell}\cdot\lambda_{-\ell+1}+\frac{\tau}{\ell}\cdot\Phi(t_{-\ell})\cdot\lambda_1 \\ -\frac{\tau}{\ell}\cdot\lambda_{-\ell+2}+\frac{\tau}{\ell}\cdot\Phi(t_{-\ell+1})\cdot\lambda_1 \\ \dots \\ -\frac{\tau}{\ell}\cdot\lambda_0+\frac{\tau}{\ell}\cdot\Phi(t_{-1})\cdot\lambda_1 \\ \frac{\tau}{2\ell}\cdot g\left(\lambda_1,\lambda_{\ell N+1}\right)+\frac{\tau}{2\ell}\cdot g\left(\lambda_1,\lambda_{\ell N+1}\right) \\ a_1(B,t_1)\lambda_{-\ell+1}+(I+a_1(A,t_1))\lambda_1-\lambda_2+b_1(u_{-\ell+1}(\cdot),\tau,t_1)+a_1(f,t_1) \\ a_2(B,t_2)\lambda_{-\ell+2}+(I+a_2(A,t_2))\lambda_2-\lambda_3+b_2(u_{-\ell+2}(\cdot),\tau,t_2)+a_2(f,t_2) \\ \dots \\ a_{\ell N}(B,t_{\ell N})\lambda_{\ell(N-1)}+(I+a_{\ell N}(A,t_{\ell N}))\lambda_{\ell N}-\lambda_{\ell N+1}+b_{\ell N}(u_{\ell(N-1)}(\cdot),\tau,t_{\ell N})+a_{\ell N}(f,t_{\ell N}) \end{pmatrix}.$$

Let us assume that the conditions of Theorem 1[25] are met in a ball $S(\lambda^0,\rho)$ with $\lambda^{(0)} \in R^{n(\ell(N+1)+1)}$. In what follows, we also suppose the following condition to be fulfilled.

Condition 1. The system of nonlinear equations

$$Q_{\Delta_\ell}(\lambda, 0) = 0 \tag{13}$$

has a solution $\lambda^{(0)} = \left(\lambda^{(0)}_{-\ell+1}, \dots, \lambda^{(0)}_0, \lambda^{(0)}_1, \dots, \lambda^{(0)}_{\ell N+1}\right) \in R^{n(\ell(N+1)+1)}$ in $S(\lambda^0,\rho)$.

Let us compose the function system

$$u^{(0)}[t] = \left(u^{(0)}_{-\ell+1}(t), \dots, u^{(0)}_0(t), u^{(0)}_1(t), \dots, u^{(0)}_{\ell N}(t)\right) \in C([-\tau, T], \Delta_\ell, R^{n\ell(N+1)}), \tag{14}$$

where

$$u^{(0)}_p(t) = -\lambda^{(0)}_p + \Phi(t)\cdot\lambda^{(0)}_1, \quad t \in [t_{p-1}, t_p), \quad p = (-\ell+1) : 0;$$

$$u^{(0)}_r(t) = a_r(A,t)\lambda^{(0)}_r + a_r(B,t)\lambda^{(0)}_{-\ell+r} + a_r(f,t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : \ell N.$$

Then, using $\lambda^{(0)}$ and $u^{(0)}[t]$, we define the function $x^{(0)}(t)$, piecewise continuous on $[-\tau, T]$, as

$$x^{(0)}(t) = \begin{cases} \lambda^{(0)}_r + u^{(0)}_r(t) & \text{if } t \in [t_{r-1}, t_r), \quad r = (-\ell+1) : \ell N, \\ \lambda^{(0)}_{\ell N+1}, & \text{if } t = T. \end{cases} \tag{15}$$

Let us choose numbers $0 < \rho_\lambda < \rho$, $\rho_u > 0$, $\rho_x > 0$ and construct the sets:

$S(\lambda^{(0)}, \rho_\lambda) = \left\{ \lambda \in R^{n(\ell(N+1)+1)} : \|\lambda - \lambda^{(0)}\| = \max\left\{\|\lambda_r - \lambda^{(0)}_r\|, r = (-\ell+1) : (\ell N+1)\right\} < \rho_\lambda \right\}$,

$S(u^{(0)}[t], \rho_u) = \{u(t) \in C([-\tau, T], \Delta_\ell, R^{n\ell(N+1)}) : \|(u - u^{(0)})[\cdot]\|_3 < \rho_u\}$,

$S(x^{(0)}(t), \rho_x) = \{x(t) \in C([-\tau, T], R^n) : \|x - x^{(0)}\|_0 < \rho_x\}$,

$G_0(\rho_\lambda, \rho_x) = \{(w_1, w_2) \in R^{2n} : \|w_1 - x^{(0)}(0)\| < \rho_\lambda, \|w_2 - x^{(0)}(T)\| < \rho_x\}$.

Condition 2. The function $g(w_1, w_2)$ is continuous in $G_0(\rho_\lambda, \rho_x)$ and has uniformly continuous partial derivatives $g'_{w_1}(w_1, w_2)$ and $g'_{w_2}(w_1, w_2)$ that satisfy the following inequalities:

$$\sup\left\{ \|g'_{w_1}(w_1, w_2)\| : (w_1, w_2) \in G_0(\rho_\lambda, \rho_x) \right\} \le L_1,$$

$$\sup\left\{ \|g'_{w_2}(w_1, w_2)\| : (w_1, w_2) \in G_0(\rho_\lambda, \rho_x) \right\} \le L_2,$$

where L_1, L_2 are some constants.

Given $\lambda^{(0)} \in \mathbb{R}^{n(\ell(N+1)+1)}$ and $u^{(0)}[t] \in C([-\tau, T], \Delta_\ell, R^{n\ell(N+1)})$, we generate the sequence $(\lambda^{(k)}, u^{(k)}[t])$, $k = 1, 2, \dots$, according to the following algorithm.

Step 1.

(a) From the equation $Q_{\Delta_\ell}(\lambda, u^{(0)}) = 0$ find $\lambda^{(1)} = \left(\lambda^{(1)}_{-\ell+1}, \dots, \lambda^{(1)}_0, \lambda^{(1)}_1, \dots, \lambda^{(1)}_{\ell N+1}\right) \in R^{n(\ell(N+1)+1)}$.

(b) Compose the function system $u^{(1)}[t] = \left(u^{(1)}_{-\ell+1}(t), \dots, u^{(1)}_0(t), u^{(1)}_1(t), \dots, u^{(1)}_{\ell N}(t)\right) \in C([-\tau, T], \Delta_\ell, R^{n\ell(N+1)})$, where

$$u^{(1)}_p(t) = -\lambda^{(1)}_p + \Phi(t)\cdot\lambda^{(1)}_1, \quad t \in [t_{p-1}, t_p), \quad p = (-\ell+1) : 0,$$

$$u^{(1)}_r(t) = a_r(A,t)\lambda^{(1)}_r + a_r(B,t)\lambda^{(1)}_{-\ell+r} + b_r(u^{(0)}_{-\ell+r}(\cdot), \tau, t) + a_r(f,t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : \ell N.$$

Step 2.

(a) From the equation $Q_{\Delta_\ell}(\lambda, u^{(1)}) = 0$ find $\lambda^{(2)} = \left(\lambda^{(2)}_{-\ell+1}, \dots, \lambda^{(2)}_0, \lambda^{(2)}_1, \dots, \lambda^{(2)}_{\ell N+1}\right) \in R^{n(\ell(N+1)+1)}$.

(b) Compose the function system $u^{(2)}[t] = \left(u^{(2)}_{-\ell+1}(t), \dots, u^{(2)}_0(t), u^{(2)}_1(t), \dots, u^{(2)}_{\ell N}(t)\right) \in C([-\tau, T], \Delta_\ell, R^{n\ell(N+1)})$, where

$$u^{(2)}_p(t) = -\lambda^{(2)}_p + \Phi(t)\cdot\lambda^{(2)}_1, \quad t \in [t_{p-1}, t_p), \quad p = (-\ell+1) : 0,$$

$$u^{(2)}_r(t) = a_r(A,t)\lambda^{(2)}_r + a_r(B,t)\lambda^{(2)}_{-\ell+r} + b_r(u^{(1)}_{-\ell+r}(\cdot), \tau, t) + a_r(f,t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : \ell N.$$

Continuing the process, at Step k of the algorithm we obtain the pair $(\lambda^{(k)}, u^{(k)}[t])$.

3 | MAIN RESULT

In this section, we establish sufficient conditions for the convergence of the algorithm proposed in Section 2 to an isolated solution of problem (6)-(11).

Theorem 1. Suppose that for some $\ell(\ell) \in \mathbb{N}$, $\rho_\lambda > 0$, $\rho_u > 0$, and $\rho_x > 0$ the conditions 1 and 2 are satisfied, the Jacobi matrix $\dfrac{\partial Q_{\Delta_\ell}(\lambda, u)}{\partial \lambda}$ is invertible for all $(\lambda, u[t])$ $\big(\lambda \in S(\lambda^{(0)}, \rho_\lambda),\ u[t] \in S(u^{(0)}[t], \rho_u)\big)$, and the following inequalities hold:

(A1) $\Big\|\Big(\dfrac{\partial Q_{\Delta_\ell}(\lambda, u)}{\partial \lambda}\Big)^{-1}\Big\| \leq \gamma(\Delta_\ell)$,

(A2) $q(\Delta_\ell) = \dfrac{\beta\tau}{\ell}\gamma(\Delta_\ell)e^{\alpha\tau/\ell} \max\left\{1, \max\limits_{t\in[-\tau,0]}\|\Phi(t)\|, \dfrac{1}{\gamma(\Delta_\ell)}, e^{\alpha\tau/\ell} - 1, \dfrac{\beta\tau}{\ell}e^{\alpha\tau/\ell}\right\} < 1$,

(A3) $\dfrac{\beta\tau}{\ell} \cdot \dfrac{\gamma(\Delta_\ell)e^{\alpha\tau/\ell}}{1 - q(\Delta_\ell)}\|u^{(0)}[\cdot]\|_4 < \rho_\lambda$,

(A4) $\dfrac{q(\Delta_\ell)}{1 - q(\Delta_\ell)}\|u^{(0)}[\cdot]\|_4 < \rho_u$,

(A5) $\rho_\lambda + \rho_u < \rho_x$.

Then the sequence $(\lambda^{(k)}, u^{(k)}[t])$, $k \in \mathbb{N}$, with $\lambda^{(k)} \in S(\lambda^{(0)}, \rho_\lambda)$, $u^{(k)} \in S(u^{(0)}[t], \rho_u)$, determined by the algorithm, converges to $(\lambda^*, u^*[t])$ with $\lambda^* \in S(\lambda^{(0)}, \rho_\lambda)$, $u^{\}*}[t] \in S(u^{(0)}[t], \rho_u)$, an isolated solution to problem (6)-(11), and the following estimates hold true:

$$\|(u^* - u^{(k)})[\cdot]\|_3 \leqslant \frac{q(\Delta_\ell)}{1 - q(\Delta_\ell)}\|(u^{(k)} - u^{(k-1)})[\cdot]\|_4, \tag{16}$$

$$\|\lambda^* - \lambda^{(k)}\| \leq \frac{\gamma(\Delta_\ell)\beta\tau e^{\alpha\tau/\ell}}{\ell - \gamma(\Delta_\ell)\beta\tau e^{\alpha\tau/\ell}}\|(u^{(k)} - u^{(k-1)})[\cdot]\|_4. \tag{17}$$

Proof. Under Condition 1, we compose the function system (14) and choose $(\lambda^{(0)}, u^{(0)}[t])$ as an initial approximation to the solution of problem (6)-(11).

We then find the parameter $\lambda^{(1)}$ from the equation

$$Q_{\Delta_\ell}(\lambda, u^{(0)}) = 0, \qquad \lambda \in R^{n(\ell(N+1)+1)}.$$

The operator $Q_{\Delta_\ell}(\lambda, u^{(0)})$ satisfies all assumptions of Theorem 1 [25] in $S(\lambda^{(0)}, \rho_\lambda)$. Choosing a number $\varepsilon_0 > 0$, satisfying the inequalities

$$\varepsilon_0\gamma(\Delta_\ell) \leq \frac{1}{2}, \qquad \frac{\gamma(\Delta_\ell)}{1 - \varepsilon_0\gamma(\Delta_\ell)}\|Q_{\Delta_\ell}(\lambda^{(0)}, u^{(0)})\| < \rho_\lambda$$

and using the uniform continuity of the Jacobian matrix $\dfrac{\partial Q_{\Delta_\ell}(\lambda, u^{(0)})}{\partial \lambda}$ in $S(\lambda^{(0)}, \rho_\lambda)$, we find $\delta_0 \in \left(0, \frac{1}{2}\rho_\lambda\right]$ such that for any $\lambda, \tilde{\lambda} \in S(\lambda^{(0)}, \rho_\lambda)$ the inequality

$$\Big\|\frac{\partial Q_{\Delta_\ell}(\lambda, u^{(0)})}{\partial \lambda} - \frac{\partial Q_{\Delta_\ell}(\tilde{\lambda}, u^{(0)})}{\partial \lambda}\Big\| < \varepsilon_0$$

holds for $\|\lambda - \tilde{\lambda}\| < \delta_0$.

Choosing $\alpha_1 \geqslant \max\left\{1, \frac{\gamma(\Delta_\ell)}{\delta_0}\|Q_{\Delta_\ell}(\lambda^{(0)}, u^{(0)})\|\right\}$, we build the iterative process:

$$\begin{aligned} \lambda^{(1,0)} &= \lambda^{(0)}, \\ \lambda^{(1,m+1)} &= \lambda^{(1,m)} - \frac{1}{\alpha_1}\Big(\frac{\partial Q_{\Delta_\ell}(\lambda^{(1,m)}, u^{(0)})}{\partial \lambda}\Big)^{-1} Q_{\Delta_\ell}(\lambda^{(1,m)}, u^{(0)}), \qquad m = 0, 1, 2, \ldots. \end{aligned} \tag{18}$$

By Theorem 1 [25], the iterative process (18) converges to $\lambda^{(1)} \in S(\lambda^{(0)}, \rho_\lambda)$, an isolated solution to the equation $Q_{\Delta_\ell}(\lambda, u^{(0)}) = 0$. Taking into account $Q_{\Delta_\ell}(\lambda^{(0)}, 0) = 0$, we get

$$\begin{aligned} \|\lambda^{(1)} - \lambda^{(0)}\| &\leq \gamma(\Delta_\ell)\|Q_{\Delta_\ell}(\lambda^{(0)}, u^{(0)})\| = \gamma(\Delta_\ell)\|Q_{\Delta_\ell}(\lambda^{(0)}, u^{(0)}) - Q_{\Delta_\ell}(\lambda^{(0)}, 0)\| \leq \gamma(\Delta_\ell) \max_{r=1:\ell N}\|b_r(u^{(0)}_{-\ell+r}(\cdot), \tau, t_r)\| \leq \\ &\leq \frac{\beta\tau}{\ell}\gamma(\Delta_\ell)e^{\alpha\tau/\ell} \max_{r=1:\ell N} \sup_{t\in[t_{r-1}, t_r)} \|u^{(0)}_{-\ell+r}(t - \tau)\| = \frac{\beta\tau}{\ell}\gamma(\Delta_\ell)e^{\alpha\tau/\ell}\|u^{(0)}[\cdot]\|_4 < \rho_\lambda. \end{aligned} \tag{19}$$

The components of the function system

$$u^{(0)}[t] = \left(u^{(0)}_{-\ell+1}(t), \dots, u^{(0)}_0(t), u^{(0)}_1(t), \dots, u^{(0)}_{\ell N}(t)\right)$$

satisfy the following inequalities:

$$\|u^{(0)}_p(t)\| \le \|\lambda^{(0)}_p\| + \|\Phi(t)\| \cdot \|\lambda^{(0)}_1\|, \quad t \in [t_{p-1}, t_p], \quad p = (-\ell+1) : 0,$$
$$\|u^{(0)}_r(t)\| \le \max\left\{e^{\alpha\tau/\ell} - 1, \frac{\beta\tau}{\ell} e^{\alpha\tau/\ell}\right\} \|\lambda^{(0)}\| + \frac{\tau}{\ell} e^{\alpha\tau/\ell} \sup_{t\in[t_{r-1},t_r)} \|f(t)\|, \quad t \in [t_{r-1}, t_r), \quad r = 1 : \ell N.$$

Under our assumptions, the Cauchy problems

$$\frac{du_r(t)}{dt} = A(t)\left(\lambda^{(1)}_r + u_r(t)\right) + B(t)\left(\lambda^{(1)}_{-\ell+r} + u^{(0)}_{-\ell+r}(t-\tau)\right) + f(t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : \ell N,$$
$$u_r(t_{r-1}) = 0, \quad r = 1 : \ell N,$$

have the unique solutions determined by

$$u^{(1)}_r(t) = a_r(A,t)\lambda^{(1)}_r + a_r(B,t)\lambda^{(1)}_{-\ell+r} + b_r(u^{(0)}_{-\ell+r}(\cdot), \tau, t) + a_r(f,t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : \ell N,$$

and

$$u^{(1)}_p(t) = -\lambda^{(1)}_p + \Phi(t) \cdot \lambda^{(1)}_1, \quad t \in [t_{p-1}, t_p), \quad p = (-\ell+1) : 0.$$

It follows from (19) that

$$\|u^{(1)}_p(t) - u^{(0)}_p(t)\| \le \|\lambda^{(1)}_p - \lambda^{(0)}_p\| + \|\Phi(t)\| \cdot \|\lambda^{(1)}_1 - \lambda^{(0)}_1\| \le \max\left\{1, \max_{t\in[-\tau,0]} \|\Phi(t)\|\right\} \|\lambda^{(1)} - \lambda^{(0)}\| \le$$
$$\le \frac{\beta\tau}{\ell} \max\left\{1, \max_{t\in[-\tau,0]} \|\Phi(t)\|\right\} \gamma(\Delta_\ell) e^{\alpha\tau/\ell} \|u^{(0)}[\cdot]\|_4, \quad t \in [t_{p-1}, t_p], \quad p = (-\ell+1) : 0, \tag{20}$$

$$\|u^{(1)}_r(t) - u^{(0)}_r(t)\| \le \|a_r(A,t)\| \cdot \|\lambda^{(1)}_r - \lambda^{(0)}_r\| + \|a_r(B,t)\| \cdot \|\lambda^{(1)}_{-\ell+r} - \lambda^{(0)}_{-\ell+r}\| + \|b_r(u^{(0)}_{-\ell+r}(\cdot), \tau, t)\| \le$$
$$\le \max\left\{e^{\alpha\tau/\ell} - 1, \frac{\beta\tau}{\ell} e^{\alpha\tau/\ell}\right\} \cdot \|\lambda^{(1)} - \lambda^{(0)}\| + \frac{\beta\tau}{\ell} e^{\alpha\tau/\ell} \max_{r=1:\ell N} \sup_{t\in[t_{r-1},t_r)} \|u^{(0)}_{-\ell+r}(t-\tau)\| \le$$
$$\le \frac{\beta\tau}{\ell} \gamma(\Delta_\ell) e^{\alpha\tau/\ell} \max\left\{\frac{1}{\gamma(\Delta_\ell)}, e^{\alpha\tau/\ell} - 1, \frac{\beta\tau}{\ell} e^{\alpha\tau/\ell}\right\} \|u^{(0)}[\cdot]\|_4, \quad t \in [t_{r-1}, t_r), \quad r = 1 : \ell N. \tag{21}$$

Thus, we have the estimate

$$\|(u^{(1)} - u^{(0)})[\cdot]\|_3 \le \max_{r=(-\ell+1):\ell N} \sup_{t\in[t_{r-1},t_r)} \|u^{(1)}_r(t) - u^{(0)}_r(t)\| \le q(\Delta_\ell) \|u^{(0)}[\cdot]\|_4 < \rho_u,$$

i.e. $u^{(1)}[t] \in S(u^{(0)}[t], \rho_u)$.

The structure of the operator $Q_{\Delta_\ell}(\lambda, u)$ and the equality $Q_{\Delta_\ell}(\lambda^{(1)}, u^{(0)}) = 0$ imply

$$\gamma(\Delta_\ell)\|Q_{\Delta_\ell}(\lambda^{(1)}, u^{(1)})\| = \gamma(\Delta_\ell)\|Q_{\Delta_\ell}(\lambda^{(1)}, u^{(1)}) - Q_{\Delta_\ell}(\lambda^{(1)}, u^{(0)})\| \le \gamma(\Delta_\ell) \max_{r=1:\ell N} \|b_r((u^{(1)}_{-\ell+r} - u^{(0)}_{-\ell+r})(\cdot), \tau, t_r)\| \le$$
$$\le \frac{\beta\tau}{\ell} \gamma(\Delta_\ell) e^{\alpha\tau/\ell} \max_{r=1:\ell N} \sup_{t\in[t_{r-1},t_r)} \|u^{(1)}_{-\ell+r}(t-\tau) - u^{(0)}_{-\ell+r}(t-\tau)\| \le \frac{\beta\tau}{\ell} \gamma(\Delta_\ell) e^{\alpha\tau/\ell} \|(u^{(1)} - u^{(0)})[\cdot]\|_4, \tag{22}$$

where $u^{(1)}[t] \in C([-\tau, T], \Delta_\ell, R^{n\ell(N+1)})$.

If $\lambda \in S(\lambda^{(1)}, \rho_1)$, where $\rho_1 = \gamma(\Delta_\ell)\|Q_{\Delta_\ell}(\lambda^{(1)}, u^{(1)})\|$, then by (A2), (A3), (19), and (20), the estimate

$$\|\lambda - \lambda^{(0)}\| \leqslant \|\lambda - \lambda^{(1)}\| + \|\lambda^{(1)} - \lambda^{(0)}\| < \gamma(\Delta_\ell)\|Q_{\Delta_\ell}(\lambda^{(1)}, u^{(1)})\| + \|\lambda^{(1)} - \lambda^{(0)}\| \leqslant$$
$$\leqslant \frac{\beta\tau}{\ell} \gamma(\Delta_\ell) e^{\alpha\tau/\ell} (q(\Delta_\ell) + 1) \|u^{(0)}[\cdot]\|_4 < \frac{\beta\tau}{\ell} \cdot \frac{\gamma(\Delta_\ell) e^{\alpha\tau/\ell}}{1 - q(\Delta_\ell)} \|u^{(0)}[\cdot]\|_4 < \rho_\lambda,$$

holds, i.e $S(\lambda^{(1)}, \rho_1) \subset S(\lambda^{(0)}, \rho_\lambda)$.

The operator $Q_{\Delta_\ell}(\lambda, u^{(1)})$ in $S(\lambda^{(1)}, \rho_1)$ satisfies all conditions of Theorem 1 [25]. Therefore, the iterative process

$$\lambda^{(2,0)} = \lambda^{(1)},$$
$$\lambda^{(2,m+1)} = \lambda^{(2,m)} - \frac{1}{\alpha_1}\left(\frac{\partial Q_{\Delta_\ell}(\lambda^{(2,m)}, u^{(1)})}{\partial \lambda}\right)^{-1} Q_{\Delta_\ell}(\lambda^{(2,m)}, u^{(1)}), \quad m = 0, 1, 2, \dots.$$

converges to $\lambda^{(2)} = (\lambda^{(2)}_{-\ell+1}, \ldots, \lambda^{(2)}_{0}, \lambda^{(2)}_{1}, \ldots, \lambda^{(2)}_{\ell N+1}) \in S(\lambda^{(1)}, \rho_1)$, an isolated solution to the equation $Q_{\Delta_\ell}(\lambda, u^{(1)}) = 0$, and

$$\|\lambda^{(2)} - \lambda^{(1)}\| \le \gamma(\Delta_\ell)\|Q_{\Delta_\ell}(\lambda^{(1)}, u^{(1)})\|.$$

The last estimate and (22) imply

$$\|\lambda^{(2)} - \lambda^{(1)}\| \leqslant \frac{\beta\tau}{\ell}\gamma(\Delta_\ell)e^{\alpha\tau/\ell}\|(u^{(1)} - u^{(0)})[\cdot]\|_4. \tag{23}$$

The Cauchy problems

$$\frac{du_r(t)}{dt} = A(t)\left(\lambda^{(2)}_r + u_r(t)\right) + B(t)\left(\lambda^{(2)}_{-\ell+r} + u^{(1)}_{-\ell+r}(t-\tau)\right) + f(t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : \ell N,$$

$$u_r(t_{r-1}) = 0, \quad r = 1 : \ell N,$$

have the unique solutions determined by

$$u^{(2)}_r(t) = a_r(A,t)\lambda^{(2)}_r + a_r(B,t)\lambda^{(2)}_{-\ell+r} + b_r(u^{(1)}_{-\ell+r}(\cdot), \tau, t) + a_r(f,t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : \ell N,$$

and

$$u^{(2)}_p(t) = -\lambda^{(2)}_p + \Phi(t)\cdot\lambda^{(2)}_1 \quad \text{if} \quad t \in [t_{p-1}, t_p), \quad p = (-\ell+1) : 0.$$

Taking into account (23), we obtain

$$\begin{aligned}\|u^{(2)}_p(t) - u^{(1)}_p(t)\| &\le \|\lambda^{(2)}_p - \lambda^{(1)}_p\| + \|\Phi(t)\|\cdot\|\lambda^{(2)}_1 - \lambda^{(1)}_1\| \le \max\left\{1, \max_{t\in[-\tau,0]}\|\Phi(t)\|\right\}\|\lambda^{(2)} - \lambda^{(1)}\| \le \\ &\le \frac{\beta\tau}{\ell}\max\left\{1, \max_{t\in[-\tau,0]}\|\Phi(t)\|\right\}\gamma(\Delta_\ell)e^{\alpha\tau/\ell}\|(u^{(1)} - u^{(0)})[\cdot]\|_4, \quad t \in [t_{p-1}, t_p], \quad p = (-\ell+1) : 0,\end{aligned} \tag{24}$$

$$\begin{aligned}\|u^{(2)}_r(t) - u^{(1)}_r(t)\| &\le \|a_r(A,t)\|\cdot\|\lambda^{(2)}_r - \lambda^{(1)}_r\| + \|a_r(B,t)\|\cdot\|\lambda^{(2)}_{-\ell+r} - \lambda^{(1)}_{-\ell+r}\| + \|b_r(u^{(1)}_{-\ell+r}(\cdot) - u^{(0)}_{-\ell+r}(\cdot), \tau, t)\| \le \\ &\le \max\left\{e^{\alpha\tau/\ell} - 1, \frac{\beta\tau}{\ell}e^{\alpha\tau/\ell}\right\}\cdot\|\lambda^{(2)} - \lambda^{(1)}\| + \frac{\beta\tau}{\ell}e^{\alpha\tau/\ell}\max_{r=1:\ell N}\sup_{t\in[t_{r-1},t_r)}\|u^{(1)}_{-\ell+r}(t-\tau) - u^{(0)}_{-\ell+r}(t-\tau)\| \le \\ &\le \frac{\beta\tau}{\ell}\gamma(\Delta_\ell)e^{\alpha\tau/\ell}\max\left\{\frac{1}{\gamma(\Delta_\ell)}, e^{\alpha\tau/\ell} - 1, \frac{\beta\tau}{\ell}e^{\alpha\tau/\ell}\right\}\|(u^{(1)} - u^{(0)})[\cdot]\|_4, \quad t \in [t_{r-1}, t_r),\ r = 1 : \ell N.\end{aligned} \tag{25}$$

Thus, we have the estimates

$$\|(u^{(2)} - u^{(1)})[\cdot]\|_3 \le \max_{r=(-\ell+1):\ell N}\sup_{t\in[t_{r-1},t_r)}\|u^{(2)}_r(t) - u^{(1)}_r(t)\| \le q(\Delta_\ell)\|(u^{(1)} - u^{(0)})[\cdot]\|_4 < \rho_u,$$

i.e. $u^{(1)}[t] \in S(u^{(0)}[t], \rho_u)$, and

$$\|u^{(2)} - u^{(1)}\|_3 \leqslant q(\Delta_\ell)\|(u^{(1)} - u^{(0)})[\cdot]\|_4 \le q(\Delta_\ell)\|(u^{(1)} - u^{(0)})[\cdot]\|_3 \le q^2(\Delta_\ell)\|u^{(0)}[\cdot]\|_4,$$

$$\|u^{(2)} - u^{(0)}\|_3 \leqslant \|u^{(2)} - u^{(1)}\|_3 + \|u^{(1)} - u^{(0)}\|_3 \le (q^2(\Delta_\ell) + q(\Delta_\ell))\|u^{(0)}[\cdot]\|_4 < \frac{q(\Delta_\ell)}{1 - q(\Delta_\ell)}\|u^{(0)}[\cdot]\|_4 < \rho_u,$$

i.e. $u^{(2)}[t] \in S(u^{(0)}[t], \rho_u)$.

Assume that $(\lambda^{(k-1)}, u^{(k-1)}[t])$, where $\lambda^{(k-1)} \in S(\lambda^{(0)}, \rho_\lambda)$, $u^{(k-1)}[t] \in C([-\tau, T], \Delta_\ell, R^{n\ell(N+1)})$, is determined, and

$$\|\lambda^{(k-1)} - \lambda^{(k-2)}\| \le \frac{\beta\tau}{\ell}\gamma(\Delta_\ell)e^{\alpha\tau/\ell}\|(u^{(k-2)} - u^{(k-3)})[\cdot]\|_4, \tag{26}$$

$$\|u^{(k-1)} - u^{(k-2)}\|_3 \leqslant q(\Delta_\ell)\|(u^{(k-2)} - u^{(k-3)})[\cdot]\|_4. \tag{27}$$

Let us find the k-th approximation $\lambda^{(k)}$ of the parameter λ from the equation $Q_{\Delta_\ell}(\lambda, u^{(k-1)}) = 0$. Similarly to (20), using (26), (27) and the equality $Q_{\Delta_\ell}(\lambda^{(k-1)}, u^{(k-2)}) = 0$, we obtain

$$\gamma(\Delta_\ell)\|Q_{\Delta_\ell}(\lambda^{(k-1)}, u^{(k-1)})\| \leqslant \frac{\beta\tau}{\ell}\gamma(\Delta_\ell)e^{\alpha\tau/\ell}\|(u^{(k-1)} - u^{(k-2)})[\cdot]\|_4, \tag{28}$$

Let us now choose $\rho_{k-1} = \gamma(\Delta_\ell)\|Q_{\Delta_\ell}(\lambda^{(k-1)}, u^{(k-1)})\|$ and show that $S(\lambda^{(k-1)}, \rho_{k-1}) \subset S(\lambda^{(0)}, \rho_\lambda)$. Indeed, in view of (26)-(28) and (A3) we have

$$\|\lambda - \lambda^{(0)}\| \leqslant \|\lambda - \lambda^{(k-1)}\| + \|\lambda^{(k-1)} - \lambda^{(k-2)}\| + \|\lambda^{(k-2)} - \lambda^{(k-3)}\| + \ldots + \|\lambda^{(1)} - \lambda^{(0)}\| <$$
$$< \rho_{k-1} + \left(\frac{\beta\tau}{\ell}\gamma(\Delta_\ell)e^{\alpha\tau/\ell}\right)^{k-2}\|\lambda^{(1)} - \lambda^{(0)}\| + \left(\frac{\beta\tau}{\ell}\gamma(\Delta_\ell)e^{\alpha\tau/\ell}\right)^{k-3}\|\lambda^{(1)} - \lambda^{(0)}\| + \ldots + \|\lambda^{(1)} - \lambda^{(0)}\| \leqslant$$
$$\leq \left(\left(\frac{\beta\tau}{\ell}\gamma(\Delta_\ell)e^{\alpha\tau/\ell}\right)^{k-1} + \left(\frac{\beta\tau}{\ell}\gamma(\Delta_\ell)e^{\alpha\tau/\ell}\right)^{k-2} + \left(\frac{\beta\tau}{\ell}\gamma(\Delta_\ell)e^{\alpha\tau/\ell}\right)^{k-3} + \ldots + 1\right)\|\lambda^{(1)} - \lambda^{(0)}\| <$$
$$< \frac{\gamma(\Delta_\ell)(\beta\tau/\ell)e^{\alpha\tau/\ell}}{1 - \gamma(\Delta_\ell)(\beta\tau/\ell)e^{\alpha\tau/\ell}}\|u^{(0)}[\cdot]\|_4 \leq \frac{\beta\tau}{\ell}\cdot\frac{\gamma(\Delta_\ell)e^{\alpha\tau/\ell}}{1 - q(\Delta_\ell)}\|u^{(0)}[\cdot]\|_4 < \rho_\lambda.$$

Since $Q_{\Delta_\ell}(\lambda, u^{(k-1)})$ satisfies all conditions of Theorem 1 [25] in $S(\lambda^{(k-1)}, \rho_{k-1})$, there exists $\lambda^{(k)} \in S(\lambda^{(k-1)}, \rho_{k-1})$, a solution to the equation $Q_{\Delta_\ell}(\lambda, u^{(k-1)}) = 0$, and

$$\|\lambda^{(k)} - \lambda^{(k-1)}\| \leqslant \gamma(\Delta_\ell)\|Q_{\Delta_\ell}(\lambda^{(k-1)}, u^{(k-1)})\|. \tag{29}$$

The Cauchy problems

$$\frac{du_r(t)}{dt} = A(t)(\lambda_r^{(k)} + u_r(t)) + B(t)(\lambda_{-\ell+r}^{(k)} + u_{-\ell+r}^{(k-1)}(t-\tau)) + f(t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : \ell N,$$
$$u_r(t_{r-1}) = 0, \quad r = 1 : \ell N,$$

have the unique solutions determined by

$$u_r^{(k)}(t) = a_r(A,t)\lambda_r^{(k)} + a_r(B,t)\lambda_{-\ell+r}^{(k)} + b_r(u_{-\ell+r}^{(k-1)}(\cdot), \tau, t) + a_r(f,t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : \ell N,$$

and

$$u_p^{(k)}(t) = -\lambda_p^{(k)} + \Phi(t)\cdot\lambda_1^{(k)}, \quad t \in [t_{p-1}, t_p), \quad p = (-\ell+1) : 0.$$

If $\rho_k = \gamma(\Delta_\ell)\|Q_{\Delta_\ell}(\lambda^{(k)}, u^{(k)})\| = 0$, then $Q_{\Delta_\ell}(\lambda^{(k)}, u^{(k)}) = 0$, which implies

$$-\lambda_p^{(k)} + \Phi(t_{p-1})\cdot\lambda_1^{(k)} = 0, \quad p = (-\ell+1) : 0,$$
$$g\left(\lambda_1^{(k)}, \lambda_{\ell N+1}^{(k)}\right) = 0,$$
$$\lambda_r^{(k)} + \lim_{t\to t_r - 0} u_r^{(k)}(t) - \lambda_{r+1}^{(k)} = 0, \qquad r = 1 : \ell N,$$

i.e. $(\lambda^{(k)}, u^{(k)}[t])$ is a solution to problem (6)-(11).

From (28) and (29) we obtain the estimates

$$\|\lambda^{(k)} - \lambda^{(k-1)}\| \leq \frac{\beta\tau}{\ell}\gamma(\Delta_\ell)e^{\alpha\tau/\ell}\|(u^{(k-1)} - u^{(k-2)})[\cdot]\|_4, \tag{30}$$

$$\|u^{(k)} - u^{(k-1)}\|_3 \leqslant q(\Delta_\ell)\|(u^{(k-1)} - u^{(k-2)})[\cdot]\|_4 \leqslant q(\Delta_\ell)\|(u^{(k-1)} - u^{(k-2)})[\cdot]\|_3, \tag{31}$$
$$\|u^{(k)} - u^{(0)}\|_3 \leqslant \|u^{(k)} - u^{(k-1)}\|_3 + \ldots + \|u^{(1)} - u^{(0)}\|_3 \leq$$
$$\leq (q^k(\Delta_\ell) + \ldots + q(\Delta_\ell))\|u^{(0)}[\cdot]\|_4 < \frac{q(\Delta_\ell)}{1 - q(\Delta_\ell)}\|u^{(0)}[\cdot]\|_4 < \rho_u, \qquad u^{(k)}[t] \in S(u^{(0)}[t], \rho_u).$$

It follows from (30), (31) and $q(\Delta_\ell) < 1$ that the sequence of pairs $(\lambda^{(k)}, u^{(k)}[t])$ converges to $(\lambda^*, u^*[t])$, the solution to problem (6)-(11), as $k \to \infty$. Moreover, (A3) and (A4) imply that $\lambda^{(k)}, \lambda^* \in S(\lambda^{(0)}, \rho_\lambda)$, $k \in \mathbb{N}$, and $u^{(k)}[t], u^*[t] \in S(u^{(0)}[t], \rho_u)$. Letting $m \to \infty$ in the inequalities

$$\|\lambda^{(k+m)} - \lambda^{(k)}\| < \frac{\gamma(\Delta_\ell)(\beta\tau/\ell)e^{\alpha\tau/\ell}}{1 - \gamma(\Delta_\ell)(\beta\tau/\ell)e^{\alpha\tau/\ell}}\left(1 - \left(\frac{\beta\tau}{\ell}\gamma(\Delta_\ell)e^{\alpha\tau/\ell}\right)^m\right)\|(u^{(k)} - u^{(k-1)})[\cdot]\|_4,$$

$$\|(u^{(k+m)} - u^{(k)})[\cdot]\|_3 \leqslant \frac{q(\Delta_\ell)}{1 - q(\Delta_\ell)}(1 - q^m(\Delta_\ell))\|(u^{(k)} - u^{(k-1)})[\cdot]\|_4,$$

we obtain estimates (16) and (17).

Let us now show that the solution to problem (6)-(11) is isolated. Suppose $(\widetilde{\lambda}, \widetilde{u}[t])$ is a solution of problem (6)-(11), where $\widetilde{\lambda} \in S(\lambda^{(0)}, \rho_\lambda)$, $\widetilde{u}[t] \in S(u^{(0)}[t], \rho_u)$. Then there exist numbers $\widetilde{\delta}_1 > 0$ and $\widetilde{\delta}_2 > 0$ such that

$$\|\widetilde{\lambda} - \lambda^{(0)}\| + \widetilde{\delta}_1 < \rho_\lambda, \qquad \|(\widetilde{u} - u^{(0)})[\cdot]\|_3 + \widetilde{\delta}_2 < \rho_u.$$

If $\lambda \in S(\widetilde{\lambda}, \widetilde{\delta}_1)$ and $u[t] \in S(\widetilde{u}[t], \widetilde{\delta}_2)$, then the inequalities

$$\|\lambda - \lambda^{(0)}\| \leqslant \|\lambda - \widetilde{\lambda}\| + \|\widetilde{\lambda} - \lambda^{(0)}\| \leqslant \widetilde{\delta}_1 + \|\widetilde{\lambda} - \lambda^{(0)}\| < \rho_\lambda,$$

$$\|(u - u^{(0)})[\cdot]\|_3 \leqslant \|(u - \widetilde{u})[\cdot]\|_3 + \|(\widetilde{u} - u^{(0)})[\cdot]\|_3 \leqslant \widetilde{\delta}_2 + \|(\widetilde{u} - u^{(0)})[\cdot]\|_3 < \rho_u,$$

imply that $\lambda \in S(\lambda^{(0)}, \rho_\lambda)$ and $u[t] \in S(u^{(0)}[t], \rho_u)$, i.e. $S(\widetilde{\lambda}, \widetilde{\delta}_1) \subset S(\lambda^{(0)}, \rho_\lambda)$ and $S(\widetilde{u}[t], \widetilde{\delta}_2) \subset S(u^{(0)}[t], \rho_u)$.

We choose a number $\varepsilon > 0$ such that

$$\varepsilon\gamma(\Delta_\ell) < 1,$$

$$\frac{\beta\tau}{\ell} \cdot \frac{\gamma(\Delta_\ell)e^{\alpha\tau/\ell}}{1 - \varepsilon\gamma(\Delta_\ell)} \max\left\{1, \max_{t\in[-\tau,0]} \|\Phi(t)\|, \frac{1 - \varepsilon\gamma(\Delta_\ell)}{\gamma(\Delta_\ell)}, e^{\alpha\tau/\ell} - 1, \frac{\beta\tau}{\ell} \cdot e^{\alpha\tau/\ell}\right\} < 1.$$

Condition 2 and the structure of the Jacobi matrix $\dfrac{\partial Q_{\Delta_\ell}(\lambda, u)}{\partial\lambda}$ imply its uniform continuity for all $\lambda \in S(\widetilde{\lambda}, \widetilde{\delta}_1)$ and $u[t] \in S(\widetilde{u}[t], \widetilde{\delta}_2)$. Hence there exists $\delta \in (0, \min\{\widetilde{\delta}_1, \widetilde{\delta}_2\}]$ such that

$$\left\|\frac{\partial Q_{\Delta_\ell}(\lambda, u)}{\partial\lambda} - \frac{\partial Q_{\Delta_\ell}(\widetilde{\lambda}, \widetilde{u})}{\partial\lambda}\right\| < \varepsilon, \quad \lambda \in S(\widetilde{\lambda}, \delta), \quad u[t] \in S(\widetilde{u}[t], \delta).$$

Note that if $(\widetilde{\lambda}, \widetilde{u}[t])$ is a solution to problem (6)-(11), then the equality $Q_{\Delta_\ell}(\widetilde{\lambda}, \widetilde{u}) = 0$ holds.

Let $(\widehat{\lambda}, \widehat{u}[t])$, with $\widehat{\lambda} = \left(\widehat{\lambda}_{-\ell+1}, \widehat{\lambda}_{-\ell+2}, \ldots, \widehat{\lambda}_0, \widehat{\lambda}_1, \ldots, \widehat{\lambda}_{\ell N+1}\right) \in S(\widetilde{\lambda}, \delta)$ and $\widehat{u}[t] = \left(\widehat{u}_{-\ell+1}(t), \widehat{u}_{-\ell+2}(t), \ldots, \widehat{u}_0(t), \widehat{u}_1(t), \widehat{u}_2(t), \ldots, \widehat{u}_{\ell N}(t)\right) \in S(\widetilde{u}[t], \delta)$, be another solution to problem (6)-(11).

Since $Q_{\Delta_\ell}(\widetilde{\lambda}, \widetilde{u}) = 0$ and $Q_{\Delta_\ell}(\widehat{\lambda}, \widehat{u}) = 0$, the equalities

$$\widetilde{\lambda} = \widetilde{\lambda} - \left(\frac{\partial Q_{\Delta_\ell}(\widetilde{\lambda}, \widetilde{u})}{\partial\lambda}\right)^{-1} Q_{\Delta_\ell}(\widetilde{\lambda}, \widetilde{u}), \qquad \widehat{\lambda} = \widehat{\lambda} - \left(\frac{\partial Q_{\Delta_\ell}(\widetilde{\lambda}, \widetilde{u})}{\partial\lambda}\right)^{-1} Q_{\Delta_\ell}(\widehat{\lambda}, \widehat{u})$$

imply

$$\widetilde{\lambda} - \widehat{\lambda} = -\left(\frac{\partial Q_{\Delta_\ell}(\widetilde{\lambda}, \widetilde{u})}{\partial\lambda}\right)^{-1} \int_0^1 \left(\frac{\partial Q_{\Delta_\ell}(\widehat{\lambda} + \theta(\widetilde{\lambda} - \widehat{\lambda}), \widetilde{u})}{\partial\lambda} - \frac{\partial Q_{\Delta_\ell}(\widetilde{\lambda}, \widetilde{u})}{\partial\lambda}\right) d\theta \cdot (\widetilde{\lambda} - \widehat{\lambda}) -$$

$$- \left(\frac{\partial Q_{\Delta_\ell}(\widetilde{\lambda}, \widetilde{u})}{\partial\lambda}\right)^{-1} (Q_{\Delta_\ell}(\widehat{\lambda}, \widetilde{u}) - Q_{\Delta_\ell}(\widehat{\lambda}, \widehat{u})),$$

and

$$\|\widetilde{\lambda} - \widehat{\lambda}\| \leq \frac{\gamma(\Delta_\ell)}{1 - \varepsilon\gamma(\Delta_\ell)} \|Q_{\Delta_\ell}(\widehat{\lambda}, \widetilde{u}) - Q_{\Delta_\ell}(\widehat{\lambda}, \widehat{u})\| \leq \frac{\gamma(\Delta_\ell)}{1 - \varepsilon\gamma(\Delta_\ell)} \max_{r=1:\ell N} \left\{ \|b_r(\widetilde{u}_{-\ell+r} - \widehat{u}_{-\ell+r}, \tau, t_r)\| \right\} \leq$$

$$\leq \frac{\beta\tau}{\ell} \cdot \frac{\gamma(\Delta_\ell)e^{\alpha\tau/\ell}}{1 - \varepsilon\gamma(\Delta_\ell)} \|(\widetilde{u} - \widehat{u})[\cdot]\|_4 \leq \frac{\beta\tau}{\ell} \cdot \frac{\gamma(\Delta_\ell)e^{\alpha\tau/\ell}}{1 - \varepsilon\gamma(\Delta_\ell)} \|(\widetilde{u} - \widehat{u})[\cdot]\|_3. \quad (32)$$

The elements $\widetilde{\lambda} \in S(\lambda^{(0)}, \rho_\lambda)$, $\widetilde{u}[t] \in S(u^{(0)}[t], \rho_u)$ and $\widehat{\lambda} \in S(\widetilde{\lambda}, \delta)$, $\widehat{u}[t] \in S(\widetilde{u}[t], \delta)$ satisfy (6)-(11); that is, the components of the function system $\widetilde{u}[t]$ are of the form

$$\widetilde{u}_p(t) = -\widetilde{\lambda}_p + \Phi(t) \cdot \widetilde{\lambda}_1, \quad t \in [t_{p-1}, t_p], \quad \widetilde{u}_p(t_{p-1}) = 0, \quad p = (-\ell + 1) : 0,$$

$$\widetilde{u}_r(t) = a_r(A, t)\widetilde{\lambda}_r + a_r(B, t)\widetilde{\lambda}_{-\ell+r} + b_r(\widetilde{u}_{-\ell+r}(\cdot), \tau, t) + a_r(f, t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : \ell N,$$

and the components of the function system $\widehat{u}[t]$ are of the form

$$\widehat{u}_p(t) = -\widehat{\lambda}_p + \Phi(t) \cdot \widehat{\lambda}_1, \quad t \in [t_{p-1}, t_p], \quad \widehat{u}_p(t_{p-1}) = 0, \quad p = (-\ell + 1) : 0,$$

$$\widehat{u}_r(t) = a_r(A, t)\widehat{\lambda}_r + a_r(B, t)\widehat{\lambda}_{-\ell+r} + b_r(\widehat{u}_{-\ell+r}(\cdot), \tau, t) + a_r(f, t), \quad t \in [t_{r-1}, t_r), \quad r = 1 : \ell N.$$

For their difference, the following estimates hold:

$$\|\widetilde{u}_p(t) - \widehat{u}_p(t)\| \leq \|\widetilde{\lambda}_p - \widehat{\lambda}_p\| + \max_{t\in[-\tau,0]} \|\Phi(t)\| \cdot \|\widetilde{\lambda}_1 - \widehat{\lambda}_1\| \leq \max\left\{1, \max_{t\in[-\tau,0]} \|\Phi(t)\|\right\} \|\widetilde{\lambda} - \widehat{\lambda}\| \leq$$

$$\leq \frac{\beta\tau}{\ell} \cdot \frac{\gamma(\Delta_\ell)e^{\alpha\tau/\ell}}{1 - \varepsilon\gamma(\Delta_\ell)} \max\left\{1, \max_{t\in[-\tau,0]} \|\Phi(t)\|\right\} \|(\widetilde{u} - \widehat{u})[\cdot]\|_3, \quad t \in [t_{p-1}, t_p], \quad p = (-\ell + 1) : 0.$$

$$\|\widetilde{u}_r(t) - \widehat{u}_r(t)\| \leq \|a_r(A,t)\| \cdot \|\widetilde{\lambda}_r - \widehat{\lambda}_r\| + \|a_r(B,t)\| \cdot \|\widetilde{\lambda}_{-\ell+r} - \widehat{\lambda}_{-\ell+r}\| + \|b_r(\widetilde{u}_{-\ell+r}(\cdot) - \widehat{u}_{-\ell+r}(\cdot),\tau,t)\| \leq$$
$$\leq \max\left\{e^{\alpha\tau/\ell} - 1, \frac{\beta\tau}{\ell}e^{\alpha\tau/\ell}\right\} \cdot \|\widetilde{\lambda} - \widehat{\lambda}\| + \frac{\beta\tau}{\ell}e^{\alpha\tau/\ell} \max_{r=1:\ell N} \sup_{t\in[t_{r-1},t_r)} \|\widetilde{u}_{-\ell+r}(t-\tau) - \widehat{u}_{-\ell+r}(t-\tau)\| \leq$$
$$\leq \frac{\beta\tau}{\ell}\gamma(\Delta_\ell)e^{\alpha\tau/\ell} \max\left\{\frac{1}{\gamma(\Delta_\ell)}, e^{\alpha\tau/\ell} - 1, \frac{\beta\tau}{\ell}e^{\alpha\tau/\ell}\right\} \|(\widetilde{u} - \widehat{u})[\cdot]\|_4, \quad t \in [t_{r-1}, t_r),\ r = 1 : \ell N. \tag{33}$$

Thus, we obtain the estimate

$$\|(\widetilde{u} - \widehat{u})[\cdot]\|_3 \leq \max_{r=(-\ell+1):\ell N} \sup_{t\in[t_{r-1},t_r)} \|\widetilde{u}_r(t) - \widehat{u}_r(t)\| \leq q(\Delta_\ell)\|(\widetilde{u} - \widehat{u})[\cdot]\|_4.$$

It follows from (A2) that $\widetilde{u}[t] = \widehat{u}[t]$. Taking into account (32), we obtain that $\widetilde{\lambda} = \widehat{\lambda}$. □

Using $\lambda^{(k)} \in R^{n(\ell(N+1)+1)}$ and $u^{(k)}[t] \in S(u^{(0)}[t], \rho_u)$ $(k = 1, 2, \ldots)$, we determine the function

$$x^{(k)}(t) = \begin{cases} \lambda_r^{(k)} + u_r^{(k)}(t) & \text{if } t \in [t_{r-1}, t_r), \quad r = (-\ell+1) : \ell N, \\ \lambda_{\ell N+1}^{(k)} & \text{if } t = T \end{cases}$$

that is continuous on $[-\tau, T]$ and continuously differentiable on $[0, T]$.

Since problem (1)-(3) is equivalent to problem (6)-(11), Theorem 1 implies the following statement.

Theorem 2. Under assumptions of Theorem 1, the sequence $(x^{(k)}(t))$, $k \in \mathbb{N}$, is contained in $S(x^{(0)}(t), \rho_x)$, converges to $x^*(t) \in S(x^{(0)}(t), \rho_x)$, an isolated solution to problem (1) - (3), and the estimate

$$\max_{t\in[0,T]} \|x^*(t) - x^{(k)}(t)\| \leq \frac{2q(\Delta_\ell)}{1 - q(\Delta_\ell)} \|(u^{(k)} - u^{(k-1)})[\cdot]\|_3, \qquad k = 1, 2, \ldots$$

holds true.

4 | EXAMPLE

We consider the problem

$$\frac{dx(t)}{dt} = \frac{1}{4}\begin{pmatrix} 2t^2 & -t^3 \\ -t^3 & 2t^2 \end{pmatrix} x(t) + \frac{1}{8000}\begin{pmatrix} 0 & 5t \\ -10t^2 & 2 \end{pmatrix} x(t-0.1) + f(t), \quad x \in R^2, \quad t \in (0,1), \tag{34}$$

$$x(t) = \mathrm{diag}[x(0)] \cdot \begin{pmatrix} \cos(0.2t) \\ \exp(0.5t) \end{pmatrix}, \quad t \in [-0.1, 0], \tag{35}$$

$$\begin{pmatrix} \sqrt{\dfrac{3x_1(1) - 2x_2(1)}{209}} - \dfrac{4}{\pi}\tan^{-1}\left(\frac{x_1(0)}{x_2(0)}\right) - \sqrt{\dfrac{23}{2090}} + \dfrac{4}{\pi}\tan^{-1}\left(1.1\right) \\ \dfrac{4x_1(1) + 8x_2(1)}{x_1(0) \cdot x_2^2(0)} - \dfrac{124}{11} \end{pmatrix} = 0, \tag{36}$$

here

$$f(t) = \begin{cases} \begin{pmatrix} -0.00625t \cdot \left(0.1\exp(0.5t - 0.05) + 5t^5 + 20t^3 - 40t^2 + 88t - 80\right) \\ 0.025\left(0.055 \cdot t^2\cos(0.2t - 0.02) - 0.01\exp(0.5t - 0.05) + 5t^5 + 11t^3 - 35t^2\right) \end{pmatrix} & \text{if } t \in [0,\ 0.1), \\ \begin{pmatrix} -0.0078125t\left(4t^5 + 15.99t^3 - 31.997t^2 + 70.3997t - 63.91999\right) \\ 0.03125\left(4t^5 + 0.01t^4 + 8.799t^3 - 27.9562t^2 + 0.00003t - 0.008001\right) \end{pmatrix} & \text{if } t \in [0.1,\ 1]. \end{cases}$$

For $N = 10$ and $\ell = 1$, we make the partition Δ_1 of the interval:

$$[0, 1) = [-0.1,\ 0) \cup [0,\ 0.1) \cup [0.1,\ 0.2) \cup \ldots \cup [0.9,\ 1).$$

Let us take $\rho = 0.91$, $\rho_\lambda = 0.000277$, $\rho_u = 0.000278$, $\rho_x = 0.0006$ and

$$\lambda^0 = (\lambda_0^0, \lambda_1^0, \dots, \lambda_{10}^0, \lambda_{11}^0) = \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \in R^{24},$$

and construct system (13) for problem (34)-(36). We solve this system of nonlinear algebraic equations in the unknown parameter λ by using the iterative process

$$\begin{aligned} \lambda^{(0,0)} &= \lambda^0, \\ \lambda^{(0,m+1)} &= \lambda^{(0,m)} - \frac{1}{2}\left(\frac{\partial Q_{\Delta_1}(\lambda^{(0,m)}, 0)}{\partial \lambda} \right)^{-1} Q_{\Delta_1}(\lambda^{(0,m)}, 0), \quad m = 0, 1, 2, \dots . \end{aligned}$$

Since $\|Q_{\Delta_1}\left(\lambda^{(0,55)}, 0\right)\| < 10^{-15}$, we can set $\lambda^{(0)} = \lambda^{(0,55)}$. The parameter $\lambda^{(0)}$ is $\left(\lambda_0^{(0)}, \lambda_1^{(0)}, \dots, \lambda_{10}^{(0)}, \lambda_{11}^{(0)}\right)$ then have the components

$$\lambda_0^{(0)} = \begin{pmatrix} 1.0997831106403650 \\ 0.9512323129323107 \end{pmatrix}, \quad \lambda_1^{(0)} = \begin{pmatrix} 1.1000031039278944 \\ 1.0000030365246515 \end{pmatrix}, \quad \lambda_2^{(0)} = \begin{pmatrix} 1.1025030034601186 \\ 0.9998774325528766 \end{pmatrix},$$

$$\lambda_3^{(0)} = \begin{pmatrix} 1.1100030071178857 \\ 0.9990024391897256 \end{pmatrix}, \quad \lambda_4^{(0)} = \begin{pmatrix} 1.1225030215202900 \\ 0.9966274846858628 \end{pmatrix}, \quad \lambda_5^{(0)} = \begin{pmatrix} 1.1400030613205100 \\ 0.9920026218449720 \end{pmatrix},$$

$$\lambda_6^{(0)} = \begin{pmatrix} 1.1625031478414525 \\ 0.9843779229934498 \end{pmatrix}, \quad \lambda_7^{(0)} = \begin{pmatrix} 1.1900033088364850 \\ 0.9730034816057488 \end{pmatrix}, \quad \lambda_8^{(0)} = \begin{pmatrix} 1.2225035777346023 \\ 0.9571294149119083 \end{pmatrix},$$

$$\lambda_9^{(0)} = \begin{pmatrix} 1.2600039919001174 \\ 0.9360058677982158 \end{pmatrix}, \quad \lambda_{10}^{(0)} = \begin{pmatrix} 1.3025045891583442 \\ 0.9088830184287469 \end{pmatrix}, \quad \lambda_{11}^{(0)} = \begin{pmatrix} 1.3500054014062932 \\ 0.8750110862807049 \end{pmatrix}.$$

Further, we determine an approximate solution to problem (34)-(36). The numerical results are provided in Table 1 and Figure 1 (for the initial approximation) and in Table 2 (for the third step of the algorithm).

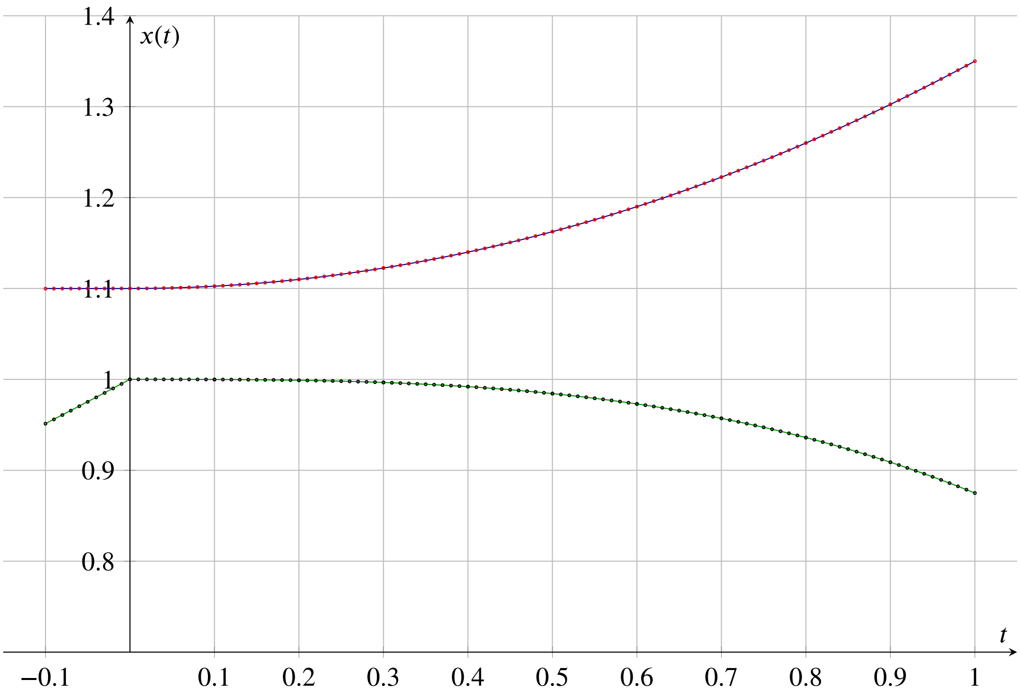

FIGURE 1 Exact solution $x_1^*(t)$ ——, approx. solution $x_1^{(0)}(t)$ ····, exact solution $x_2^*(t)$ ——, approx. solution $x_2^{(0)}(t)$ ····

TABLE 1 The values of the exact solution $x^*(t) = \begin{pmatrix} x_1^*(t) \\ x_2^*(t) \end{pmatrix}$, the approximate solution $x^{(0)}(t) = \begin{pmatrix} {}^{(0)}_{1}(t) \\ x^{(0)}(t) \end{pmatrix}$ and the error ($\mathrm{err}_1 = |x_1^{(0)}(t) - x_1^*(t)|$, $\mathrm{err}_2 = |x_2^{(0)}(t) - x_2^*(t)|$).

t	$x_1^*(t)$	$x_1^{(0)}(t)$	err_1	$x_2^*(t)$	$x_2^{(0)}(t)$	err_2
-0.10	1.0997800073332356	1.0997831106403650	3.103e-06	0.9512294245007140	0.9512323129323107	2.888e-06
-0.05	1.0999450004583318	1.0999481042310310	3.104e-06	0.9753099120283327	0.9753128735809234	2.962e-06
0.00	1.1000000000000000	1.1000031039278944	3.104e-06	1.0000000000000000	1.0000030365246515	3.037e-06
0.05	1.1006250000000002	1.1006280914908437	3.091e-06	0.9999843750000000	0.9999872617517296	2.887e-06
0.10	1.1025000000000000	1.1025030034601186	3.003e-06	0.9998750000000000	0.9998774325528766	2.433e-06
0.15	1.1056250000000000	1.1056280046163758	3.005e-06	0.9995781250000000	0.9995805587709788	2.434e-06
0.20	1.1100000000000000	1.1100030071178857	3.007e-06	0.9990000000000000	0.9990024391897256	2.439e-06
0.25	1.1156250000000000	1.1156280115518250	3.012e-06	0.9980468750000000	0.9980493234928042	2.448e-06
0.30	1.1225000000000000	1.1225030215202900	3.022e-06	0.9966250000000000	0.9966274846858628	2.485e-06
0.35	1.1306250000000000	1.1306280330823730	3.033e-06	0.9946406250000000	0.9946431393780809	2.514e-06
0.40	1.1400000000000001	1.1400030613205100	3.061e-06	0.9920000000000000	0.9920026218449720	2.622e-06
0.45	1.1506250000000000	1.1506280854512374	3.085e-06	0.9886093750000000	0.9886120641509687	2.689e-06
0.50	1.1625000000000000	1.1625031478414525	3.148e-06	0.9843750000000000	0.9843779229934498	2.923e-06
0.55	1.1756250000000001	1.1756281914940954	3.191e-06	0.9792031250000000	0.9792061757610511	3.051e-06
0.60	1.1900000000000002	1.1900033088364850	3.309e-06	0.9730000000000000	0.9730034816057488	3.482e-06
0.65	1.2056250000000002	1.2056283801894372	3.380e-06	0.9656718750000000	0.9656755744678882	3.699e-06
0.70	1.2225000000000001	1.2225035777346023	3.578e-06	0.9571250000000000	0.9571294149119083	4.415e-06
0.75	1.2406250000000000	1.2406286854849837	3.685e-06	0.9472656250000000	0.9472703860525825	4.761e-06
0.80	1.2600000000000002	1.2600039919001174	3.992e-06	0.9359999999999999	0.9360058677982158	5.868e-06
0.85	1.2806250000000001	1.2806291438062700	4.144e-06	0.9232343749999999	0.9232407665135098	6.392e-06
0.90	1.3025000000000002	1.3025045891583442	4.589e-06	0.9088750000000000	0.9088830184287469	8.018e-06
0.95	1.3256250000000000	1.3256297893082490	4.789e-06	0.8928281250000000	0.8928369087846616	8.784e-06
1.00	1.3500000000000000	1.3500054014062932	5.401e-06	0.8750000000000000	0.8750110862807049	1.109e-05

5 | ACKNOWLEDGEMENTS

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TABLE 2 The values of the exact solution $x^*(t) = \begin{pmatrix} x_1^*(t) \\ x_2^*(t) \end{pmatrix}$, the approximate solution $x^{(3)}(t) = \begin{pmatrix} x_1^{(3)}(t) \\ x_2^{(3)}(t) \end{pmatrix}$ and the error ($\mathrm{err}_1 = |x_1^{(3)}(t) - x_1^*(t)|$, $\mathrm{err}_2 = |x_2^{(3)}(t) - x_2^*(t)|$).

t	$x_1^*(t)$	$x_1^{(3)}(t)$	err_1	$x_2^*(t)$	$x_2^{(3)}(t)$	err_2
-0.10	1.0997800073332356	1.0997800806249047	7.329e-08	0.9512294245007140	0.9512294934360170	6.894e-08
-0.05	1.0999450004583318	1.0999450737609966	7.330e-08	0.9753099120283327	0.9753099827087413	7.068e-08
0.00	1.1000000000000000	1.1000000733063300	7.331e-08	1.0000000000000000	1.0000000724696916	7.247e-08
0.05	1.1006250000000002	1.1006250733078828	7.331e-08	0.9999843750000000	0.9999844474720773	7.247e-08
0.10	1.1025000000000000	1.1025000733183120	7.332e-08	0.9998750000000000	0.9998750724831861	7.248e-08
0.15	1.1056250000000000	1.1056250733457682	7.335e-08	0.9995781250000000	0.9995781975110192	7.251e-08
0.20	1.1100000000000000	1.1100000734237820	7.342e-08	0.9990000000000000	0.9990000728233261	7.282e-08
0.25	1.1156250000000000	1.1156250735071176	7.351e-08	0.9980468750000000	0.9980469479062948	7.291e-08
0.30	1.1225000000000000	1.1225000739014825	7.390e-08	0.9966250000000000	0.9966250747970745	7.480e-08
0.35	1.1306250000000000	1.1306250740656012	7.407e-08	0.9946406250000000	0.9946406999640680	7.496e-08
0.40	1.1400000000000001	1.1400000752674342	7.527e-08	0.9920000000000000	0.9920000803362015	8.034e-08
0.45	1.1506250000000000	1.1506250755317673	7.553e-08	0.9886093750000000	0.9886094556279711	8.063e-08
0.50	1.1625000000000000	1.1625000782605228	7.826e-08	0.9843750000000000	0.9843750920244814	9.202e-08
0.55	1.1756250000000001	1.1756250786355662	7.864e-08	0.9792031250000000	0.9792032175186408	9.252e-08
0.60	1.1900000000000002	1.1900000790683485	7.907e-08	0.9730000000000000	0.9730000930982746	9.310e-08
0.65	1.2056250000000002	1.2056250795597285	7.956e-08	0.9656718750000000	0.9656719687682068	9.377e-08
0.70	1.2225000000000001	1.2225000884036314	8.840e-08	0.9571250000000000	0.9571251273452722	1.273e-07
0.75	1.2406250000000000	1.2406250889617436	8.896e-08	0.9472656250000000	0.9472657536030776	1.286e-07
0.80	1.2600000000000002	1.2600001027814849	1.028e-07	0.9359999999999999	0.9360001799712908	1.800e-07
0.85	1.2806250000000001	1.2806251032667224	1.033e-07	0.9232343749999999	0.9232345573279291	1.823e-07
0.90	1.3025000000000002	1.3025001235186278	1.235e-07	0.9088750000000000	0.9088752570563178	2.571e-07
0.95	1.3256250000000000	1.3256251236030177	1.236e-07	0.8928281250000000	0.8928283863743098	2.614e-07
1.00	1.3500000000000000	1.3500001234896320	1.235e-07	0.8750000000000000	0.8750002662065343	2.662e-07

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