On extended partial (fuzzy) strong k-metric spaces

Yaoqiang Wu*

School of Sciences and Arts, Suqian University, Suqian, 223800, Jiangsu, P.R. China

Abstract

In this paper, we introduce a new notion of extended partial strong k-metric spaces with controlled operators and provide some examples to show that is different from extended k-metric spaces which initiated by Kamran et al. Furthermore, we introduce the concept of extended partial fuzzy strong k-metric spaces with controlled operators, which is a generalization of extended k-metric in the sense of Mehmood given. Finally, we establish fixed point theorems for self-mappings which satisfy Banach contraction principle on extended partial fuzzy strong k-metric spaces. Also, we provide some examples to illustrate our results.

Keywords: extended k-metric, controlled partial metric type, fuzzy metric space, extended partial (fuzzy) strong k-metric space, fixed point theorem.

2010 MSC: 54E35, 54D35, 54A40

1. Introduction

Since Bourbaki [2] initiated the notion of b-metrics in 1989, which was introduced formally by Czerwik [3] in 1993 (also called quasimetrics by Bakhtin [1]). Many researchers expanded their work by replacing the triangle inequality $d(x, z) \leq s[d(x, y) + d(y, z)]$ (see, e.g., partial b-metrics [21], quasi-partial b-metrics [5], strong b-metrics [8], strong partial b-metrics [15] etc). Recently, Kamran et al.[7] introduced the concept of extended b-metric spaces, which

*Corresponding author

Email address: 23036@squ.edu.cn (Yaoqiang Wu)

modified the constant s in the triangle inequality by a function $\theta(x, z)$, namely $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$. Following this view of extension, many authors presented other generalized extended metric spaces (see, e.g., controlled metric type spaces [14], controlled partial metric type spaces [22] etc).

In addition, since Kramosi and Michálek [10] introduced the notion of fuzzy metric spaces in 1975, basing on the concept of statistical metric spaces which was initiated by Menger [13]. A number of literatures about fuzzy metric spaces appeared, by constructing from metric spaces (see, e.g., fuzzy pseudo-metric spaces [4], fuzzy b-metric spaces [16], fuzzy metric type spaces [19], fuzzy strong b-metric spaces [17] etc). In 2017, Mehmood et al. [12] generalized the concept of fuzzy b-metric spaces by introducing a type of extended fuzzy b-metric spaces, which is an extension by a function $\alpha(x, z)$ depending on the parameters of left-hand side of the triangle inequality. Later, some researchers established other extensions fuzzy spaces different from extended fuzzy b-metric spaces and Banach-type fixed point results in these spaces. For instance, μ-extended fuzzy b-metric spaces [18], controlled fuzzy metric spaces [20] etc.

In this paper, we present the idea of extended partial strong k-metric spaces and extended partial fuzzy strong k-metric spaces with controlled operators $k(x, y)$, which generalizes the concepts of extended b-metric spaces and extended fuzzy b-metric spaces, respectively. Also, some examples of extended partial (fuzzy) strong k-metric spaces with controlled operators are given to show that our extension is different. Finally, we prove some fixed point results by Banach contraction in extended partial fuzzy strong k-metric spaces.

We recall some basic notions and results that will be used in the following sections (see more details in [4, 9–11]. Throughout this paper, the letters $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}^+$ always denote the set of real numbers, of positive real numbers and of positive integers, respectively.

2. Extended partial strong k-metric space

Definition 2.1. Let X be a nonempty set, $k : X \times X \to [1, +\infty)$ and the mapping $d_{ek} : X \times X \to [0, +\infty)$ satisfying the following conditions: $\forall x, y, z \in X$,

(EDK1) $x = y \Leftrightarrow d_{ek}(x, y) = 0$;

(EDK2) $d_{ek}(x, y) = d_{ek}(y, x)$;

(EDK3) $d_{ek}(x, z) \leq k(x, z)[d_{ek}(x, y) + d_{ek}(y, z)]$;

(EDK4) $d_{ek}(x, z) \leq k(x, y)d_{ek}(x, y) + k(y, z)d_{ek}(y, z)$;

(EDK5) $d_{ek}(x, z) \leq d_{ek}(x, y) + k(y, z)d_{ek}(y, z)$.

If d_{ek} satisfies the conditions (EDK1)-(EDK3), then d_{ek} is called an *extended k-metric* [7]. If d_{ek} satisfies the conditions (EDK1), (EDK2) and (EDK4), then d_{ek} is called a *controlled metric type* [14]. If d_{ek} satisfies the conditions (EDK1), (EDK2) and (EDK5), then d_{ek} is called an *extended strong k-metric* with a controlled operator k.

An extended strong k-metric space with a controlled operator k is a pair (X, d_{ek}) such that d_{ek} is an extended strong k-metric on X.

Particularly, when $k(x, y) = s$ for all $x, y \in X$ and some number $s \geq 1$, an extended k-metric (controlled metric type) space (X, d_{ek}) is a k-metric space and an extended strong k-metric space (X, d_{ek}) is a strong k-metric space [8], respectively.

Definition 2.2. Let X be a nonempty set, $k : X \times X \to [1, +\infty)$ and the mapping $p_{ek} : X \times X \to [0, +\infty)$ satisfying the following conditions: $\forall x, y, z \in X$,

(EPK1) $x = y \Leftrightarrow p_{ek}(x, x) = p_{ek}(x, y) = p_{ek}(y, y)$;

(EPK2) $p_{ek}(x, x) \leq p_{ek}(x, y)$;

(EPK3) $p_{ek}(x, y) = p_{ek}(y, x)$;

(EPK4) $p_{ek}(x, z) \leq p_{ek}(x, y) + k(y, z)p_{ek}(y, z) - p_{ek}(y, y)$;

(EPK5) $p_{ek}(x, z) \leq k(x, y)p_{ek}(x, y) + k(y, z)p_{ek}(y, z)$.

If p_{ek} satisfies the conditions (EPK1)-(EPK4), then p_{ek} is called an *extended partial strong k-metric* with a controlled operator k. If p_{ek} satisfies the conditions (EPK1)-(EPK3) and (EPK5), then p_{ek} is called a *controlled partial metric* *type* [22]. An extended partial strong k-metric space with a controlled operator k is a pair (X, p_{ek}) such that p_{ek} is an extended partial strong k-metric on X.

Particularly, when $k(x, y) = s$ for all $x, y \in X$ and some number $s \geq 1$, an extended partial strong k-metric space is called a *partial k-metric space* [21]. Apparently, when $s = 1$, an extended partial strong k-metric space (X, p_{ek}) is a partial metric space [11].

Example 2.3. Let $X = \{1, 2, 3\}$. Define $k : X \times X \to [1, +\infty)$ by $k(x, y) = 2 + x + y$ for all $x, y \in X$ and p_{ek}: $X \times X \to [0, +\infty)$ as follows:
$p_{ek}(1, 1) = p_{ek}(1, 2) = p_{ek}(2, 1) = p_{ek}(3, 3) = 8$, $p_{ek}(2, 2) = 0$, $p_{ek}(1, 3) = p_{ek}(3, 1) = 100$, $p_{ek}(2, 3) = p_{ek}(3, 2) = 60$. Then (X, p_{ek}) is an extended partial strong k-metric space with a controlled operator k.

It is trivial to verify the conditions (EPK1)-(EPK3) one by one. We will verify the condition (EPK4) in the following cases:

Case 1: Set $x = 1, y = 3, z = 2$. It follows that $p_{ek}(1, 2) = 8, p_{ek}(1, 3) = 100, p_{ek}(3, 2) = 60, p_{ek}(3, 3) = 8$ and $k(3, 2) = 7$. It is clear that $p_{ek}(1, 2) \leq p_{ek}(1, 3) + k(3, 2)p_{ek}(3, 2) - p_{ek}(3, 3)$. Similarly, we can deduce that $p_{ek}(2, 1) \leq p_{ek}(2, 3) + k(3, 1)p_{ek}(3, 1) - p_{ek}(3, 3)$.

Case 2: Set $x = 1, y = 2, z = 3$. It follows that $p_{ek}(1, 3) = 100, p_{ek}(1, 2) = 8, p_{ek}(2, 3) = 60, p_{ek}(2, 2) = 0$ and $k(2, 3) = 7$. It is clear that $p_{ek}(1, 3) \leq p_{ek}(1, 2) + k(2, 3)p_{ek}(2, 3) - p_{ek}(2, 2)$. Similarly, we can deduce that $p_{ek}(3, 1) \leq p_{ek}(3, 2) + k(2, 1)p_{ek}(2, 1) - p_{ek}(2, 2)$.

Case 3: Set $x = 2, y = 1, z = 3$. It follows that $p_{ek}(2, 3) = 60, p_{ek}(2, 1) = 8, p_{ek}(1, 3) = 100, p_{ek}(1, 1) = 8$ and $k(1, 3) = 6$. It is clear that $p_{ek}(2, 3) \leq p_{ek}(2, 1) + k(1, 3)p_{ek}(1, 3) - p_{ek}(1, 1)$. Similarly, we can deduce that $p_{ek}(3, 2) \leq p_{ek}(3, 1) + k(1, 2)p_{ek}(1, 2) - p_{ek}(1, 1)$.

Therefore, p_{ek} is an extended partial strong k-metric. However, it is not an extended k-metric since $p_{ek}(1, 1) = 8 \neq 0$.

Furthermore, we can claim that each extended k-metric space may not be an extended partial strong k-metric space as follows:

Example 2.4. Let $X = \{1, 2, 3\}$. Define $k : X \times X \to [1, +\infty)$ by $k(x, y) =$

$1+x+y$ for all $x,y \in X$ and p_{ek}: $X \times X \to [0,+\infty)$ as follows: $p_{ek}(1,1) = p_{ek}(2,2) = p_{ek}(3,3) = 0, p_{ek}(1,2) = p_{ek}(2,1) = 80$, $p_{ek}(1,3) = p_{ek}(3,1) = 1000$, $p_{ek}(2,3) = p_{ek}(3,2) = 600$.

In [8], the authors proved that (X, p_{ek}) is an extended k-metric space, but it is not an extended partial strong k-metric space with the same controlled operator k. Indeed, for $x = 3, y = 2, z = 1$, we have $p_{ek}(1,1) = 0, p_{ek}(3,1) = 1000, p_{ek}(3,2) = 600, p_{ek}(2,1) = 80$ and $k(2,1) = 4$. It implies that $p_{ek}(3,1) = 1000 > 920 = p_{ek}(3,2) + k(2,1)p_{ek}(2,1) - p_{ek}(1,1)$.

Remark 2.5. By Definition 2.2, it is obvious that $p_{ek}(x,z) \leq p_{ek}(x,y) + k(y,z)p_{ek}(y,z) - p_{ek}(y,y) \leq k(x,y)p_{ek}(x,y) + k(y,z)p_{ek}(y,z)$ for $k(x,y) \geq 1$ $\forall x,y \in X$. Then, each extended partial strong k-metric space is a controlled partial metric type space, but the converse is not true.

Example 2.6. Let $X = \{0,1,2\}$. Define $k : X \times X \to [1,+\infty)$ by $k(x,y) = 1+x+y$ for all $x,y \in X$ and p_{ek}: $X \times X \to [0,+\infty)$ as follows: $p_{ek}(x,y) = \max\{x,y\} + d(x,y)$ for all $x,y \in X$, where $d(0,0) = d(1,1) = d(2,2) = 0, d(0,1) = d(1,0) = 2, d(1,2) = d(2,1) = 11, d(0,2) = d(2,0) = 1$.

It is not difficult to pove that (X, p_{ek}) is a controlled partial metric type space. However, it is not an extended partial strong k-metric space with the same controlled operator k. Indeed, for $x = 1, y = 0, z = 2$, we have $p_{ek}(0,0) = 0, p_{ek}(1,0) = 3, p_{ek}(0,2) = 3, p_{ek}(1,2) = 13$ and $k(0,2) = 3$. It implies that $p_{ek}(1,2) = 13 > 12 = p_{ek}(1,0) + k(0,2)p_{ek}(0,2) - p_{ek}(0,0)$.

With the following proposition, more examples of the extended partial strong k-metric can be constructed by partial metric and extended strong k-metric.

Proposition 2.7. *Let X be a nonempty set, (X,p) be a partial metric space and (X, d_{ek}) be an extended strong k-metric space with a controlled operator k. Define p_{ek}: $X \times X \to [0,+\infty)$ as follows:*

$$p_{ek}(x,y) = p(x,y) + d_{ek}(x,y), \forall x,y \in X.$$

Then (X, p_{ek}) is an extended partial strong k-metric space with the controlled operator k.

Proof. By assumption, (X,p) be a partial metric space and (X,d_{ek}) be a extended strong k-metric space with a controlled operator k. Then (EPK1), (EPK2) and (EPK3) are obvious. We will verify condition (EPK4) in the following.

(EPK4): By Definition 2.1 and Definition 2.2, we have that

$$\begin{aligned}
p_{ek}(x,z) &\leq [p(x,y)+p(y,z)-p(y,y)]+[d_{ek}(x,y)+k(y,z)d_{ek}(y,z)] \\
&\leq [p(x,y)+d_{ek}(x,y)]+k(y,z)[p(y,z)+d_{ek}(y,z)]-[p(y,y)+d_{ek}(y,y)] \\
&= p_{ek}(x,y)+k(y,z)p_{ek}(y,z)-p_{ek}(y,y),
\end{aligned}$$

for all $x,y,z \in X$. □

Hence, (EPK4) holds.

3. Extended partial fuzzy strong k-metric space

Recently, Mehmood, Ali, Ionescu and Kamran [12] introduced the notion of extended fuzzy b-metric spaces. As a extension of fuzzy metric spaces, the authors replaced the triangularity axiom of the fuzzy metric spaces [10] by using a function $\alpha(x,z) \geq 1$ in the corresponding triangle inequality. Following their work, in this section, we introduce a concept of extended partial fuzzy strong k-metric spaces, which is a generalization of extended fuzzy b-metric spaces.

First, we recall some aspects on continuous t-norms, which will be used in the following section (see more details in [9]).

Definition 3.1. [21] A binary operation $*: [0,1]\times[0,1] \to [0,1]$ is called a *continuous triangular norm* (briefly t-norm) if it satisfies the following conditions:
(1) $*$ is associative and commutative;
(2) $*$ is continuous;
(3) $a * 1 = a$ for all $a \in [0,1]$;
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

The most commonly used t-norms are: minimum and usual product t-norm which are given as follows: $\forall a,b \in [0,1]$, $a *_M b = \min\{a,b\}$ and $a *_P b = a \cdot b$.

Definition 3.2. [16] Let X be a nonempty set, $\alpha, \mu : X \times X \to [1, +\infty)$, $*$ be a continuous t-norm , and M be a fuzzy set in $X \times X \times [0, +\infty)$ is called a *fuzzy k-metric* on X if $\forall x, y, z \in X$ and $t, s > 0$, the following conditions hold for some number $k \geq 1$:

(FKM1) $M(x, y, 0) = 0$;

(FKM2) $M(x, y, t) = 1$, $\forall t > 0$ if and only if $x = y$;

(FKM3) $M(x, y, t) = M(y, x, t)$;

(FKM4) $M(x, z, k(t + s)) \geq M(x, y, t) * M(y, z, s)$;

(FKM5) The function $M(x, y, \cdot) : [0, +\infty) \to [0, 1]$ is left-continuous;

(FKM6) $\lim_{t \to +\infty} M(x, y, t) = 1$.

Then $(X, M, *)$ is called a *fuzzy k-metric space.*

In [12], Mehmood et al. introduced the ideal of an extended fuzzy k-metric space by replacing the condition (FKM4) with the condition (FKMα4:)

(FKMα4) $M(x, z, \alpha(x, z)(t + s)) \geq M(x, y, t) * M(y, z, s)$.

Recently, Rome et al. [18] introduced the notion of μ-extended fuzzy k-metric space, which M satisfies the conditions (FKM1)-(FKM3),(FKM6), (FKM$\alpha\mu$4) and (FKM$\alpha\mu$5):

(FKM$\alpha\mu$4) $M(x, z, \alpha(x, z)t + \mu(x, z)s) \geq M(x, y, t) * M(y, z, s)$;

(FKM$\alpha\mu$5) The function $M(x, y, \cdot) : (0, +\infty) \to [0, 1]$ is continuous.

Apparently, when $\alpha(x, y) = \mu(x, y) = k$, for some number $k \geq 1$ and $\alpha(x, y) = \mu(x, y)$, μ-extended fuzzy k-metric is fuzzy k-metric and extended fuzzy k-metric, respectively.

Definition 3.3. Let X be a nonempty set, $k : X \times X \to [1, +\infty)$, $*$ is a continuous t-norm, and M_{pesk} is a fuzzy set on $X \times X \times [0, +\infty)$, satisfying the following conditions: $\forall x, y, z \in X$ and $t, s > 0$:

(EPFSK1) $M_{pesk}(x, y, 0) = 0$;

(EPFSK2) $M_{pesk}(x, x, t) \geq M_{pesk}(x, y, t)$;

(EPFSK3) $M_{pesk}(x, y, t) = M_{pesk}(y, x, t)$;

(EPFSK4) $M_{pesk}(x, x, t) = M_{pesk}(x, y, t) = M_{pesk}(y, y, t)$ if and only if $x = y$;

(EPFSK5) $M_{pesk}(x, z, t + k(x, z)s) \geq M_{pesk}(x, y, t) * M_{pesk}(y, z, s)$;

(EPFSK6) The function $M_{pesk}(x, y, \cdot) : [0, +\infty) \to [0, 1]$ is left-continuous.

If M_{pesk} satisfies the conditions (EPFSK1)-(EPFSK6), then M_{pesk} is called an *extended partial fuzzy strong k-metric*, the mapping k is called a *controlled operator*. A 3-triple $(X, M_{psk}, *)$ is called an *extended partial fuzzy strong k-metric space* with a controlled operator k if M_{pesk} is an *extended partial fuzzy strong k-metric.*

Example 3.4. Let $X = [0, 1]$ and define $k : X \times X \to [1, +\infty)$ by $k(x, y) = 1 + x + y$. Define a fuzzy set on $X \times X \times [0, +\infty)$ as follows:

$$M_{pesk}(x, y, t) = e^{-\frac{|x-y|}{t}}, \forall x, y \in X, t > 0$$

and $M_{pesk}(x, y, t) = 0$ when $t = 0$. Then $(X, M_{pesk}, *_P)$ is an extended partial fuzzy strong k-metric space with a controlled operator k.

It is trivial to check that $(X, M_{pesk}, *_P)$ satisfies (EPFSK1)-(EPFSK4) and (EPFSK6). We check the condition (EPFSK5) in the following:

(EPFSK5): We claim that $\frac{c+d}{a+b} \leq \frac{c}{a} + \frac{d}{b}$ for all $a, b > 0$ and $c, d \geq 0$.

Note that $M_{pesk}(x, y, t) = e^{-\frac{|x-y|}{t}}$, $M_{pesk}(y, z, s) = e^{-\frac{|y-z|}{s}}$, and $M_{pesk}(x, z, t+k(x, z)s) = e^{-\frac{|x-z|}{t+(1+x+z)s}}$. Since

$$\frac{|x-z|}{t+(1+x+z)s} \leq \frac{|x-y|+|y-z|}{t+(1+x+z)s}$$

and

$$\frac{|x-y|+|y-z|}{t+(1+x+z)s} \leq \frac{|x-y|+|y-z|}{t+s} \leq \frac{|x-y|}{t} + \frac{|y-z|}{s}$$

for all $x, y, z \in X$, $t, s > 0$, this implies that $e^{-\frac{|x-z|}{t+(1+x+z)s}} \geq e^{-\frac{|x-y|}{t} - \frac{|y-z|}{s}}$. Thus, we have that $M_{pesk}(x, z, t + k(x, z)s) \geq M_{pesk}(x, y, t) *_P M_{pesk}(y, z, s)$.

Hence, (EPFSK5) holds.

Example 3.5. Let $X = \{10, 20, 30\}$ and define $k : X \times X \to [1, +\infty)$ by $k(x, y) = 1 + x + y$. Define a fuzzy set on $X \times X \times [0, +\infty)$ as follows:

$$M_{pesk}(x, y, t) = \frac{t}{\max\{x, y\} + (x-y)^2 + t},$$

for all $x, y \in X$, $t > 0$ and $M_{pesk}(x, y, t) = 0$ when $t = 0$. Then $(X, M_{pesk}, *_M)$ is an extended partial fuzzy strong k-metric space with a controlled operator k.

Since (EPFSK1)-(EPFSK4) and (EPFSK6) are straightforward, we prove only (EPFSK5). Note that $k(10,10) = 21, k(20,20) = 41, k(30,30) = 61$, $k(10,20) = k(20,10) = 31$, $k(10,30) = k(30,10) = 41, k(20,30) = k(30,20) = 51$, $M_{pesk}(x,y,t) = \frac{t}{\max\{x,y\}+(x-y)^2+t}$, $M_{pesk}(y,z,s) = \frac{s}{\max\{y,z\}+(y-z)^2+s}$, and $M_{pesk}(x,z,t+k(x,z)s) = \frac{t+(1+x+z)s}{\max\{x,z\}+(x-z)^2+t+(1+x+z)s}$.

To verify that M_{pesk} satisfies (EPFSK5), we will distinguish in the following cases:

Case 1: Set $x = 10, y = 30, z = 20$. We have that $M_{pesk}(10,30,t) = \frac{t}{430+t} = 1 - \frac{430}{430+t}$, $M_{pesk}(30,20,s) = \frac{s}{130+s} = 1 - \frac{130}{130+s}$, and $M_{pesk}(10,20,t+k(10,20)s) = \frac{t+31s}{120+t+31s} = 1 - \frac{120}{120+t+31s}$. Since

$$1 - \frac{120}{120+t+31s} \geq 1 - \frac{120}{120+t} \geq 1 - \frac{430}{430+t}, \forall t,s > 0,$$

this implies that

$$1 - \frac{120}{120+t+31s} \geq \min\{1 - \frac{430}{430+t}, 1 - \frac{130}{130+s}\}.$$

Therefore, $M_{pesk}(10,20,t+k(10,20)s) \geq M_{pesk}(10,30,t) *_M M_{pesk}(30,20,s)$.

Similarly, we can deduce that $M_{pesk}(20,10,t+k(20,10)s) \geq M_{pesk}(20,30,t) *_M M_{pesk}(30,10,s)$.

Case 2: Set $x = 10, y = 20, z = 30$. We have that $M_{pesk}(10,20,t) = 1 - \frac{120}{120+t}$, $M_{pesk}(20,30,s) = 1 - \frac{130}{130+s}$, and $M_{pesk}(10,30,t+k(10,30)s) = 1 - \frac{430}{430+t+41s}$. Since

$$1 - \frac{430}{430+t+41s} \geq 1 - \frac{430}{430+41s} \geq 1 - \frac{130}{130+s}, \forall t,s > 0,$$

this implies that

$$1 - \frac{430}{430+t+41s} \geq \min\{1 - \frac{120}{120+t}, 1 - \frac{130}{130+s}\}.$$

Therefore, $M_{pesk}(10,30,t+k(10,30)s) \geq M_{pesk}(10,20,t) *_M M_{pesk}(20,30,s)$.

Similarly, we can deduce that $M_{pesk}(30,10,t+k(30,10)s) \geq M_{pesk}(30,20,t) *_M M_{pesk}(20,10,s)$.

Case 3: Set $x = 20, y = 10, z = 30$. We have that $M_{pesk}(20,10,t) = 1 - \frac{120}{120+t}$, $M_{pesk}(10,30,s) = 1 - \frac{430}{430+s}$, and $M_{pesk}(20,30,t+k(20,30)s) =$

$1-\frac{130}{130+t+51s}$. Since

$$1-\frac{130}{130+t+51s} \geq 1-\frac{130}{130+51s} \geq 1-\frac{430}{430+s}, \forall t, s > 0,$$

this implies that

$$1-\frac{130}{130+t+51s} \geq \min\{1-\frac{120}{120+t}, 1-\frac{430}{430+s}\}.$$

Therefore, $M_{pesk}(20,30,t+k(20,30)s) \geq M_{pesk}(20,10,t) *_M M_{pesk}(10,30,s)$.

Similarly, we can deduce that $M_{pesk}(30,20,t+k(30,20)s) \geq M_{pesk}(30,10,t) *_M M_{pesk}(10,20,s)$.

Hence, (EPFSK5) holds.

However, it is not a fuzzy metric space. Indeed, consider $x = 10, y = 20, z = 30$, and $t = s = 1$. We have $M_{pesk}(10,30,1+1) = \frac{1}{216}$, $M_{pesk}(10,20,1) = \frac{1}{121}$, $M_{pesk}(20,30,1) = \frac{1}{131}$. Thus, we have that $M_{pesk}(10,30,1+1) < M_{pesk}(10,20,1) *_M M_{pesk}(20,30,1)$.

Remark 3.6. By Definition 3.3, if (X, M_{pesk}) is an extended partial fuzzy strong k-metric space and satisfies $M_{pesk}(x,y,t) = 1 \Rightarrow x = y$ for all $x, y \in X, t > 0$, then (X, M_{pesk}) is an extended fuzzy k-metric space.

Example 3.7. Let $X = \{1,2,3\}$ and define $k : X \times X \to [1,+\infty)$ by $k(x,y) = 2+x+y$. Define a fuzzy set on $X \times X \times [0,+\infty)$ as follows:

$$M_{pesk}(x,y,t) = \frac{t}{t+p_{ek}(x,y)},$$

for all $x, y \in X$, $t > 0$ and $M_{pesk}(x,y,t) = 0$ when $t = 0$, where p_{ek} is defined in Example 2.3 i,e., $p_{ek}(1,1) = p_{ek}(1,2) = p_{ek}(2,1) = p_{ek}(3,3) = 8$, $p_{ek}(2,2) = 0$, $p_{ek}(1,3) = p_{ek}(3,1) = 100$, $p_{ek}(2,3) = p_{ek}(3,2) = 60$.

It is not difficult to prove that $(X, M_{pesk}, *_P)$ is an extended partial fuzzy strong k-metric space with a controlled operator k. However, it is not a μ-extended fuzzy k-metric space. Indeed, we have $M_{pesk}(x,x,t) = \frac{t}{t+p_{ek}(x,x)}$. Thus, it implies that $M_{pesk}(1,1,t) = \frac{t}{t+8} \neq 1$ for all $t > 0$.

Proposition 3.8. *Let X be a nonempty set and $(X, M_{pesk}, *_P)$ be an extended partial strong fuzzy k-metric space with a controlled operator k. Define a fuzzy set on $X \times X \times [0, +\infty)$ as follows:*

$$M(x, y, t) = \frac{t \cdot M_{pesk}(x, y, t)}{t + 1},$$

for all $x, y \in X$ and $t \geq 0$. Then $(X, M, *_P)$ is an extended partial strong fuzzy k-metric space with the same controlled operator k.

Proof. It is trivial to prove that $(X, M, *_P)$ satisfies (EPFSK1)-(EPFSK4) and (EPFSK6). We verify condition (EPFSK5) in the following.

(EPFSK5): Since $(X, M_{pesk}, *_P)$ is an extended partial strong fuzzy k-metric space. By Definition 3.3, we have

$$\begin{aligned}
M(x, z, t + k(x, z)s) &= \frac{t + k(x, z)s}{t + k(x, z)s + 1} \cdot M_{pesk}(x, z, t + k(x, z)s) \\
&\geq \frac{t + k(x, z)s}{t + k(x, z)s + 1} \cdot M_{pesk}(x, y, t) \cdot M_{pesk}(y, z, s) \\
&\geq \frac{t}{t + 1} \cdot \frac{s}{s + 1} M_{pesk}(x, y, t) \cdot M_{pesk}(y, z, s) \\
&= [\frac{t}{t + 1} \cdot M_{pesk}(x, y, t)] \cdot [\frac{s}{s + 1} \cdot M_{pesk}(y, z, s)] \\
&= M(x, y, t) *_P M(y, z, s)
\end{aligned}$$

for all $x, y, z \in X$, $t > 0$. □

We know that each fuzzy metric M on X generates a topology $\mathcal{T}_M$ on X with the basis $\mathcal{B} = \{B(x, r, t) : x \in X, 0 < r < 1, t > 0\}$, where the open ball $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$ for all $0 < r < 1$ and $t > 0$. Also, we call that $\mathcal{T}_M$ is induced by the fuzzy metric M (see more details in [6]).

Theorem 3.9. *Let X be a nonempty set and $(X, M_{pesk}, *)$ be an extended partial fuzzy strong k-metric space with a controlled operator k. For any $x \in X$, $0 < r < 1$ and $t > 0$, we define the open ball as follows:*

$$B(x, r, t) = \{y \in X : M_{pesk}(x, y, t) > 1 - r\}.$$

Then $\mathcal{T}_{M_{pesk}} = \{V \subset X :$ for each $x \in V$, there exist $0 < r < 1, t > 0$ such that $B(x, r, t) \subset V\}$ *is a topology on X.*

Proof. It is similar to the proof of Theorem 2.1 [16]. $\square$

Lemma 3.10. *Let X be a nonempty set, $(X, M_{pesk}, *)$ be an extended partial fuzzy strong k-metric space with a controlled operator k. If $M_{pesk}(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$, then $x = y$.*

Proof. By (EPFSK2), we have $M_{pesk}(x, x, t) \geq M_{pesk}(x, y, t)$ and $M_{pesk}(y, y, t) \geq M_{pesk}(x, y, t)$. Suppose that $M_{pesk}(x, y, t) = 1$. Thus, $M_{pesk}(x, x, t) \geq 1$ and $M_{pesk}(y, y, t) \geq 1$, which follows that $M_{pesk}(x, x, t) = M_{pesk}(y, y, t) = 1$. Namely, $M_{pesk}(x, x, t) = M_{pesk}(y, y, t) = M_{pesk}(x, y, t)$, and then we have $x = y$ by (EPFSK4). $\square$

4. Fixed point theorem on extended partial fuzzy strong k-metric space

In this section, we investigate fixed point theorems involving fuzzy Banach-type contractive mappings on extended partial fuzzy strong k-metric spaces.

Definition 4.1. Let X be a nonempty set and $(X, M_{pesk}, *)$ be an extended fuzzy partial strong k-metric space with a controlled operator k.

(1) A sequence $\{x_n\}_{n\in\mathbb{N}^+}$ in $(X, M_{pesk}, *)$ *converges to a point* $x \in X$ if for any $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}^+$ such that $M_{pesk}(x_n, x, t) > 1-\varepsilon$ for all $n > n_0$ (or equivalently for any open ball $B(x, r, t)$, there exists $n_0 \in \mathbb{N}^+$ such that $x_n \in B(x, r, t)$ for all $n \geq n_0$), we denote $\lim_{n\to+\infty} x_n = x$.

(2) A sequence $\{x_n\}_{n\in\mathbb{N}^+}$ is called a *Cauchy sequence* if for any $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}^+$ such that $M_{pesk}(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

(3) $(X, M_{pesk}, *)$ is said to be *complete* if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}^+}$ in X converges to a point $x \in X$.

Lemma 4.2. *Let X be a nonempty set, $(X, M_{pesk}, *)$ be an extended partial fuzzy strong k-metric space with a controlled operator k, and $\{x_n\}_{n\in\mathbb{N}^+}$ be a* *sequence in X. Then $\lim_{n\to+\infty} x_n = x$ if and only if $\lim_{n\to+\infty} M_{pesk}(x_n, x, t) = 1$ for all $t > 0$.*

Proof. ($\Rightarrow$) Suppose that $\lim_{n\to+\infty} x_n = x$. Then for any open ball $B(x, r, t)$, there exists $n_0 \in \mathbb{N}^+$ such that $x_n \in B(x, r, t)$ for all $n > n_0$. Thus $M_{pesk}(x_n, x, t) > 1 - r$ for all $n > n_0$ and $t, r > 0$, namely, $1 - M_{pesk}(x_n, x, t) < r$. Hence $\lim_{n\to+\infty} M_{pesk}(x_n, x, t) = 1$.

($\Leftarrow$) Suppose that $\lim_{n\to+\infty} M_{pesk}(x_n, x, t) = 1$. Then for each $t > 0$, there exists $n_0 \in \mathbb{N}^+$ such that $1 - M_{pesk}(x_n, x, t) < r$ for all $n \geq n_0$. Namely, $M_{pesk}(x_n, x, t) > 1 - r$ for all $n > n_0$. Therefore, $x_n \in B(x, r, t)$ for all $n > n_0$. Thus $\lim_{n\to+\infty} x_n = x$. $\square$

Theorem 4.3. *Let X be a nonempty set, $(X, M_{pesk}, *)$ be a complete extended partial fuzzy strong k-metric space with a controlled operator k and satisfies (EPFSK7): $\lim_{t\to+\infty} M_{pesk}(x, y, t) = 1$.*

Let $T : X \to X$ be a function satisfies the following conditions for some number $\lambda \in (0, 1)$:

(1) $M_{pesk}(Tx, Ty, \lambda t) \geq M_{pesk}(x, y, t)$, $\forall x, y \in X$, $t > 0$;

(2) $k(x_n, x_{n+m}) < \frac{1}{\lambda}$ for all $x_0 \in X$ and $n, m \in \mathbb{N}^+$, where $x_{n+1} = Tx_n$.

Then T has a unique fixed point.

Proof. By assumption, we define a sequence in the following way: $x_{n+1} = Tx_n$ for all $x_0 \in X$ and $n \in \mathbb{N}^+$.

Case 1: Suppose that $x_{n+1} = x_n$ for some $n \in \mathbb{N}^+$. It implies that $Tx_n = x_n$, this shows that x_n is a fixed point.

Case 2: Suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}^+$. We will prove the existence and uniqueness of fixed point in the following:

Step 1: Since $M_{pesk}(Tx, Ty, \lambda t) \geq M_{pesk}(x, y, t)$, we have

$$M_{pesk}(x_n, x_{n+1}, \lambda t) = M_{pesk}(Tx_{n-1}, Tx_n, \lambda t) \geq M_{pesk}(x_{n-1}, x_n, t),$$

for all $x_{n-1}, x_n, x_{n+1} \in X$, $t > 0$, and $n \in \mathbb{N}^+$. By repeating the above process, it follows that $M_{pesk}(x_n, x_{n+1}, \lambda t) \geq M_{pesk}(x_0, x_1, \frac{t}{\lambda^{n-1}})$.

On the other hand, by (EPFSK5), we have that

$$
\begin{aligned}
&M_{pesk}(x_n, x_{n+m}, t) \\
= &M_{pesk}(x_n, x_{n+m}, \frac{t}{m} + k(x_n, x_{n+m})\frac{mt-t}{mk(x_n, x_{n+m})}) \\
\geq &M_{pesk}(x_n, x_{n+1}, \frac{t}{m}) * M_{pesk}(x_{n+1}, x_{n+m}, \frac{mt-t}{mk(x_n, x_{n+m})}).
\end{aligned}
$$

Furthermore, we can deduce that

$$
\begin{aligned}
&M_{pesk}(x_{n+1}, x_{n+m}, \frac{mt-t}{mk(x_n, x_{n+m})}) \\
\geq &M_{pesk}(x_{n+1}, x_{n+2}, \frac{t}{mk(x_n, x_{n+m})}) \\
&* M_{pesk}(x_{n+2}, x_{n+m}, \frac{mt-2t}{mk(x_n, x_{n+m})k(x_{n+1}, x_{n+m})}),
\end{aligned}
$$

$\cdots$,

$$
\begin{aligned}
&M_{pesk}(x_{n+m-2}, x_{n+m}, \frac{mt-(m-2)t}{m\prod_{j=n}^{n+m-2} k(x_j, x_{n+m})}) \\
\geq &M_{pesk}(x_{n+m-2}, x_{n+m-1}, \frac{t}{m\prod_{j=n}^{n+m-2} k(x_j, x_{n+m})}) \\
&* M_{pesk}(x_{n+m-1}, x_{n+m}, \frac{t}{m\prod_{j=n}^{n+m-1} k(x_j, x_{n+m})}),
\end{aligned}
$$

for all $n, m \in \mathbb{N}^+$ and $t > 0$.

Therefore, it implies that

$$
\begin{aligned}
M_{pesk}(x_n, x_{n+m}, t) \geq &M_{pesk}(x_0, x_1, \tfrac{t}{m\lambda^n}) * M_{pesk}(x_0, x_1, \tfrac{t}{mk(x_n, x_{n+m})\lambda^{n+1}}) \\
&* \cdots * M_{pesk}(x_0, x_1, \tfrac{t}{m\prod_{j=n}^{n+m-1} k(x_j, x_{n+m})\lambda^{n+m-1}}).
\end{aligned}
$$

As $k(x_0, x_{n+m}) < \frac{1}{\lambda}$ for all $x_0 \in X$ and $n, m \in \mathbb{N}^+$, from condition (EPFSK7), we have $\lim_{n\to+\infty} M_{pesk}(x_n, x_{n+m}, t) = 1$. Hence, $\{x_n\}$ is a Cauchy sequence. From the completeness of $(X, M_{pesk}, *)$, there exists $x^* \in X$ such that $\lim_{n\to+\infty} x_n = x^*$.

Step 2: By (EPFSK5), we have that

$$
\begin{aligned}
M_{pesk}(Tx^*, x^*, t) &\geq M_{pesk}(Tx^*, x^*, \frac{t}{2} + k(Tx^*, x^*)\frac{t}{2k(Tx^*, x^*)}) \\
&\geq M_{pesk}(Tx^*, Tx_n, \frac{t}{2}) * M_{pesk}(Tx_n, x^*, \frac{t}{2k(Tx^*, x^*)}) \\
&\geq M_{pesk}(x^*, x_n, \frac{t}{2\lambda}) * M_{pesk}(Tx_n, x^*, \frac{t}{2k(Tx^*, x^*)}) \\
&= M_{pesk}(x^*, x_n, \frac{t}{2\lambda}) * M_{pesk}(x_{n+1}, x^*, \frac{t}{2k(Tx^*, x^*)}),
\end{aligned}
$$

for all $t > 0$. By Lemma 4.2, it follows that $\lim_{n\to+\infty} M_{pesk}(Tx^*, x^*, t) \geq 1*1 = 1$, namely $M_{pesk}(Tx^*, x^*, t) = 1$. From Lemma 3.10, we have $Tx^* = x^*$.

Step 3: Suppose that $x^* \neq y^*$, where $Ty^* = y^*$. We have $M_{pesk}(x^*, y^*, t) = M_{pesk}(Tx^*, Ty^*, t) \geq M_{pesk}(x^*, y^*, \frac{t}{\lambda}) \geq \cdots \geq M_{pesk}(x^*, y^*, \frac{t}{\lambda^n})$. By (EPF-SK7), it follows that $\lim_{n\to+\infty} M_{pesk}(x^*, y^*, t) = 1$, namely $M_{pesk}(x^*, y^*, t) = 1$. From Lemma 3.10, we have $x^* = y^*$. □

Lastly, we illustrate our result by Example 3.4. In fact, Define a mapping $T : X \to X$ by $Tx = 1 - \frac{x}{3}$ for all $x \in X$. It is easy to see that $(X, M_{pesk}, *_P)$ is a complete extended partial fuzzy strong k-metric space with the controlled operator $k(x, y) = 1 + x + y$ for all $x, y \in X$. In addition, we can verify that all the conditions of Theorem 4.3 are satisfied and $x = \frac{3}{4}$ is a fixed point of T (see more details in [18]).

Conclusions

In this paper, Firstly, we introduce the notion of extended partial strong k-metrics with a controlled operator k, which is an extension of extended k-metrics, by replacing the constant k with a controlled operator $k(x, y)$ in the left-hand side of the triangle inequality. In Section 2, we provide several examples for illustrating the relationships between the extended partial strong k-metrics and extended k-metrics (controlled partial metrics type). Additionally, the purpose of this paper is to obtain another extension of controlled (extended) fuzzy type metrics by replacing the $M(x, x, \alpha(x, z)(s + t)) \geq M(x, y, t) * M(y, z, s)$ with $M(x, z, t+k(x, z)s) \geq M(x, y, t)*M(y, z, s)$, we introduce a concept of extended partial fuzzy strong k-metrics. Finally, we present a fixed point theorem on extended partial fuzzy strong k-metric spaces.

Acknowledgments

The author thanks the editor and the referees for constructive and pertinent suggestions, which have improved the quality of the manuscript greatly. This work was supported by Program through the guiding science and technology plan project of Suqian No.Z2021131.

References

[1] Bakhtin IA. The contraction mapping principle in almost metric spaces. *Funct Anal.* 1989;30:26-37.

[2] Bourbaki N. *Topologie generale.* New York: Herman; 1974.

[3] Czerwik S. Contraction mappings in b-metric spaces. *Acta Math Inform Univ Ostra.* 1993;1:5-11.

[4] Deng Z. Fuzzy pseudo-metric spaces. *J Math Anal Appl.* 1982; 86:74-95.

[5] Gupta A. Some coupled fixed point theorems on quasi-partial b-metric spaces. *Int J Math Anal.* 2015;9:293-306.

[6] Kaleva O, Seikkala S. On fuzzy metric spaces. *Fuzzy Sets Syst.* 1984;12:215-229.

[7] kamran T, Samreen M, Ain QU. A generalization of b-mteric space and some fixed point theorems. *Mathematics.* 2017; 5:19.

[8] Kirk W, Shahzad N. *Fixed point theory in distance spaces.* New York: Springer Verlag; 2014.

[9] Klement EP, Mesiar R, Pap E. *Triangular norms.* London:Kluwer; 2000.

[10] Kramosil I, Michálek J. Fuzzy metrics and statistical metric spaces. *Kybernetika.* 1975;11:336-344.

[11] Matthews SG. Partial metric topology. *Ann New York Acad Sci.* 1994;728:183-197.

[12] Mehmood F, Ali R, Ionescu C, Kamran T. Extended fuzzy b-metric spaces. *J Math Anal.* 2017;8:124-131.

[13] Menger K. Statistical metrics. *Proc Natl Acad Sci*. 1942;28:535-537.

[14] Mlaiki N, Aydi H, Souayah N, Abdeljawad T. Controlled metric type spaces and the related contraction principle. *Mathematics*. 2018;6:194.

[15] Moshokoa S, Ncongwane F. On completeness in strong partial b-metric spaces, strong b-metric spaces and the 0-Cauchy completions. *Topol Appl*. 2020;275:107011.

[16] Nădăban S. Fuzzy b-metric spaces. *Int J Comput Commun Control*. 2016;11:273-281.

[17] Öner T. On topology of fuzzy strong b-metric spaces. *Journal of New Theory*. 2018;21:59-67.

[18] Rome B, Sarwar M, Abdeljawad T. μ-extended fuzzy b-metric spaces and related fixed results. *AIMS Mathematics*. 2020;5:5184-5192.

[19] Saadati R. On the topology of fuzzy metric type spaces. *Filomat*. 2015;29:133-141.

[20] Sezen MS. Controlled fuzzy metric spaces and some related fixed point rseults. *Numer Methods Partial Differ Eq*. 2021;37:583-593.

[21] Shukla S. Partial b-metric spaces and fixed point theorems. *Mediterr J Math*. 2014;11:703-711.

[22] Souayah N, Mrad M. On fixed-point results in controlled partial metric type spaces with a graph. *Mathematics*. 2020;8:33.