BLOW UP IN FINITE TIME OF SOLUTIONS TO A LESLIE-GOWER PREDATOR-PREY MODEL IN ABSCENCE OF THE MIDDLE PREDATOR.

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ABSTRACT. In order to study the asymptotic behavior, several authors claimed global existence in time of solutions to a tritrophic food chain models following a modified Leslie-Gower formulation considering the interactions between three species: a generalist top predator depredating on a middle predator, that in turn is depreds a prey. To the contrary it is shown finite time blow-up in such models can occur. We show in this work that blow up in finite time persists even when the intermediate (middle) predator is abscent to the contrary to what it is claimed by Kundu and Patra (2022, [13]). It is shown under some restrictions on the parameters, the model has bounded solutions for all positive initial conditions. We show that this is not true. Solutions to the model can blow up in finite time, for initial data sufficiently large, even under the restrictions derived by the authors. We can show same results even for small initial data but we concentrate our proofs for the first case. We also show similar results for the spatially extended system. We illustrate all our results through numerical simulations.

1. Introduction

Top predators have the potential to act as biological control agents. Biological control methods, which help in protecting the flora and fauna of an ecosystem, are used in many recovery plans [5].

After introducing the logistic delay term ρ to the system and changing variables following the authors in [13], we consider the following nondimensionalised reduced predator–prey model of two species food chain ODE model

$$\begin{aligned} \frac{du}{dt} &= u\left(1 - u\left(t-\rho\right)\right) - \frac{muv}{u^p + c} = f_1(u,v), \\ \frac{dv}{dt} &= \alpha_3\left(\gamma - \frac{\omega}{u+a}\right)v^2 = f_2\left(u,v\right), \end{aligned} \tag{1}$$

where $m,\ c,\ p,\ \omega,\ \gamma,\ a$ and α_3 are positive constants. System (1) represents a predator-prey model with one prey and one predator with maturation delay on the growth function of the prey population. Predator's functional response represents group defence in the prey species. The functional response of the generalist

1991 *Mathematics Subject Classification.* Primary: 34K12, 34D20 Secondary: 92D25, 92D40.
Key words and phrases. Delay, Group defence, Generalist predator, Finite time blow-up, Numerical analysis of blow-up.

predator species is modelled by the modified Leslie–Gower scheme, and it is of sexually reproductive type. Here u is the population density of the prey and v is the population density of the predator whose only food is that prey.

Authors in [13] took a dimensional system where predatory species are lost due to a severe shortage of species.

Predators will form groups to maximize their predation rate, while preys will form groups to reduce the predation rate. The term group defense is used for characterization Phenomenon due to the prey species' ability to defend or camouflage itself against attacking predatory species, so the predation rate reduced or sometimes prevented by a large number of prey items (see [12], [24], [11], [8], [25] and [3]).

There are many literatures which use different functional responses to depict the herd behavior in prey species (see [1], [4], [7], [9] and [14]).

A two-way competition model is considered and tested with the assumption that the prey species exhibits logistic growth retardation and that the predator style is generalistic. Switching food when prey is scarce is one of the most common behaviors of generalist predators. The predator's functional response is modelled by a functional response which shows the grouping behavior of a prey species in defence against predation.

In this regard the works of the authors in [2], [15], [16], [6] concerning tritrophic food chain models and [13] when the intermediate predator equation hasn't been considered which is the case of system (1) are very interesting, and such investigations are highly desirable. However, one must be very careful with the derivation of global existence results for such systems, given recent results that show finite time blow-up in such models ([17]; [18], [19], [20], [21] and [23]).

2. Blow up at finite time in the ODE

Given a system of ODE's, a solution cannot always exist globally in time and blow-up in finite time may occur. Recall,

Definition 2.1. *(Finite time blow-up for ODE) We say that a solution of a given ODE, with suitable initial conditions, blows-up at finite time if*

$$\lim_{t \to T^*} |v(t)| = +\infty,$$

where T^* is the blow-up time.

We state the following

Theorem 2.1. *Consider the system (1), $v(t)$ blows up at finite time for large initial data, that is*

$$\lim_{t \to T_{\max}} v(t) = +\infty,$$

where $0 < T_{\max} < +\infty$

Proof. Let $u_0 = u(0)$ and $v_0 = v(0)$ be the initial conditions in system (1). By integrating the second equation, we obtain

$$-\frac{1}{v} + \frac{1}{v_0} = \alpha_3 \left(\gamma t - \omega \int_0^t \frac{\gamma t}{u + a} \right),$$

whereupon

$$v = \frac{1}{\frac{1}{v_0} - \alpha_3 \left(\gamma t - \omega \int\limits_0^t \frac{\gamma t}{u+a} \right)}.$$

If we prove that the following function

$$t \to \psi(t) = \frac{1}{v_0} - \alpha_3 \left(\gamma t - \omega \int\limits_0^t \frac{\gamma t}{u+a} \right),$$

vanishes at a time $T > 0$, then the solution will blow up in finite time.
Since the reaction terms are continuous functions, then the solutions are classical and continuous and

$$\lim_{t \to 0} \left(\frac{1}{t} \int\limits_0^t \frac{t}{u+a} \right) = \frac{1}{u_0 + a}.$$

If u_0 is sufficiently large, then there exists $\delta > 0$ such that

$$\frac{1}{t} \int\limits_0^t \frac{\gamma t}{u+a} < \frac{\gamma}{2\omega}, \qquad \text{for all } t \in (0, \delta).$$

Then for all $t \in (0, \delta)$, we have

$$\begin{aligned} \psi(t) &= \frac{1}{v_0} - \alpha_3 \left(\gamma t - \omega \int\limits_0^t \frac{\gamma t}{u+a} \right) \\ &= \frac{1}{v_0} + \alpha_3 \left[-\gamma + \frac{\omega}{t} \int\limits_0^t \frac{\gamma t}{u+a} \right] t < \frac{1}{v_0} - \frac{\gamma \alpha_3}{2} t. \end{aligned}$$

If v_0 is sufficiently large, then we can find $T \in (0, \delta)$ such that

$$T = \frac{2}{v_0} . \frac{1}{\gamma \alpha_3} < \delta.$$

Then the solution of the ODE blows up in finite time for initial conditions u_0 and v_0 sufficiently large:

$$\frac{1}{u_0 + a} < \frac{1}{2\omega} \text{ and } \frac{1}{v_0} - \frac{\gamma \alpha_3}{2} \delta < 0.$$

□

3. Finite time blow-up in the associated PDE model

Definition 3.1. *Given a PDE, with suitable initial and boundary conditions, finite time blow-up occurs if*

$$\lim_{t \to T_{\max}} \| z(t,.) \|_\infty = +\infty,$$

where the norm $\|.\|_\infty$ *denotes the supremum norm on* Ω *and* $z(t,.)$ *is the solution to the PDE in question.*

We now consider the following PDE system

$$\begin{aligned} &\frac{\partial w}{\partial \nu} = d_1 \Delta w + (1 - w((t-\rho), x)) - \frac{mwz}{w^p + c} = f_1 w, z), \\ &\frac{\partial z}{\partial \nu} = d_2 \Delta z + \alpha_3 \left(\gamma - \frac{\omega}{w+a} \right) z^2 = f_2(w,z), \end{aligned} \qquad t > 0,\ x \in \Omega, \tag{2}$$

with homogeneous boundary conditions of Dirichlet type

$$w(t,x) = z(t,x) = 0,\ t > 0,\ x \in \partial\Omega, \tag{3}$$

or homogeneous boundary conditions of Neumann type

$$\frac{\partial w(t,x)}{\partial \nu} = \frac{\partial z(t,x)}{\partial \nu} = 0,\ t > 0,\ \ x \in \partial\Omega, \tag{4}$$

where Ω is an open bounded set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. The parameters $m,\ c,\ p,\ \omega,\ \gamma,\ a$ and α_3 are positive constants as described above. The constants of diffusion d_1 and d_2 are also positive. The initial data

$$w(0,x) = w_0(x) \ \text{ and } z(0,x) = z_0(x),\ x \in \Omega,$$

are assumed to be nonnegative and uniformly bounded on Ω.

The usual norms in spaces $\mathbb{L}^p(\Omega)$, $\mathbb{L}^\infty(\Omega)$ and $\mathbb{C}\left(\overline{\Omega}\right)$ are respectively denoted by

$$\|w\|_p^p = \frac{1}{|\Omega|} \int_\Omega |w(x)|^p \, dx, \tag{5}$$

$$\|w\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} |w(x)|. \tag{6}$$

Since the nonlinear right hand sides of (2) are continuously differentiable on $\mathbb{R}^+ \times \mathbb{R}^+$, then for any initial data in $\mathbb{C}\left(\overline{\Omega}\right)$ or $\mathbb{L}^p(\Omega),\ p \in (1, +\infty)$, it is easy to directly verify its Lipschitz continuity on bounded subsets of the domain of a fractional power of the operator

$$\begin{pmatrix} -d_1 \Delta & 0 \\ 0 & -d_2 \Delta \end{pmatrix}. \tag{7}$$

Under these assumptions, as it is well known, we have the following local existence result on PDE (see D. Henry [10])

Proposition 3.1. *The system (2) admits a unique, classical solution (w, z) on $[0, T_{\max}[\times\Omega$. Moreover if $T_{\max} < \infty$ then*

$$\lim_{t \nearrow T_{\max}} \left\{ \|w(t,.)\|_\infty + \|z(t,.)\|_\infty \right\} = \infty, \tag{8}$$

where $T_{\max}$ denotes the eventual blowing-up time in $\mathbb{L}^\infty(\Omega)$.

The non-negativity of the solutions is preserved by application of classical results on invariant regions since the reaction terms are quasi-positive, i.e.

$$f_1(0,z) \geq 0,\ f_2(w,0) \geq 0, \quad \text{for all } w \geq 0,\ z \geq 0.$$

We state the following

Corollary 3.2. *Consider the system (2) with Neumann homogeneous boundary conditions, then $z(t,.)$ blows up at finite time, in the $\mathbb{L}^p$-norm for all $1 \leq p \leq +\infty$ for large initial data, that is*

$$\lim_{t \to T_{\max}} \|z(t,.)\|_p = +\infty, \quad 1 \leq p \leq +\infty,$$

where $0 < T_{\max} < +\infty$.

Proof. (of Corollary 3.2) It suffices to prove that $z(t,.)$ blows up at finite time, in the $\mathbb{L}^1$-norm, then using the inequality

$$|\Omega|^{1-\frac{1}{p}} . \|z(t,.)\|_p \geq \|z(t,.)\|_1, \quad 1 < p \leq +\infty,$$

we will deduce the blow up at finite time, in the $\mathbb{L}^p$-norm for all $1 < p \leq +\infty$ for large initial data.

Integrating the two terms of the second equation of system (2) and using the homogeneous condition (4), we get

$$\frac{d}{dt}\left(\int_\Omega z dx\right) = \alpha_3 \int_\Omega \left(\gamma - \frac{\omega}{w+a}\right) z^2 dx. \tag{9}$$

For w_0 sufficiently large, we can find $\delta > 0$ such that

$$\frac{\omega}{w+a} \leq \frac{\gamma}{2}, \text{ for all } x \in \Omega, \quad \text{and all } 0 < t < \delta.$$

This implies

$$\frac{d}{dt}\left(\int_\Omega z dx\right) \geq \frac{\alpha_3}{2\gamma} \int_\Omega z^2 dx.$$

As

$$\left(\int_\Omega z dx\right)^2 \leq |\Omega| \int_\Omega z^2 dx.$$

By putting $Z(t) = \int_\Omega z(t,x)\,dx$, we get the following quadratic differential inequality

$$\frac{dZ}{dt} \geq \frac{\alpha_3}{2\gamma |\Omega|} Z^2,$$

which gives, after integration on the interval $(0,t)$,

$$-\frac{1}{Z} + \frac{1}{Z_0} \geq \frac{\alpha_3}{2\gamma |\Omega|} t, \text{ for all } 0 < t < \delta,$$

where $Z_0 = \int_\Omega z_0(x)\,dx$. That is

$$Z \geq \frac{1}{\frac{1}{Z_0} - \frac{\alpha_3}{2\gamma|\Omega|} t},$$

which gives

$$\lim_{t \to T_{\max}} Z = +\infty,$$

whenever $T_{\max} = \frac{2\gamma|\Omega|}{\alpha_3 Z_0}$. This ends the proof of the Theorem. □

For Dirichlet homogeneous boundary condition, using the positivity of z on Ω, we have

$$\int_\Omega \Delta z dx \leq 0,$$

and so we can't obtain an inequality analogous to (9). However using comparison theorems, we can prove the following.

Proposition 3.3. *Consider the system (2) with Dirichlet homogeneous boundary conditions, then $z(t,.)$ blows up at finite time, in the $\mathbb{L}^{\infty}$-norm for large initial data, that is*

$$\lim_{t\to T_{\max}} \|z(t,.)\|_{\infty} = +\infty, \ ,$$

where $0 < T_{\max} < +\infty$.

Proof. We follow the same reasoning as in the case of Neumann homogeneous boundary conditions, but without integration on Ω. We compare the second equation of the system (2) with initial condition $(w_0(x), z_0(x))$ and the corresponding equation of the system (1) with initial condition (u_0, v_0) sufficiently large ($w_0(x) \geq u_0 \gg 0$ and $z_0(x) \geq v_0 \gg 0$).

Let us choose in the system (1) u_0 sufficiently large, then we can find $\delta > 0$ such that

$$\frac{\omega}{u+a} \leq \frac{\gamma}{2}, \text{ for all } 0 < t < \delta.$$

Following the same reasonning as before, we can prove that v blows up at finite time $T^* = \frac{2}{\gamma\alpha v_0}$. Comparing the initial condition of the PDE (2)) with those of the ODE (1), i.e. $z_0(x) \geq v_0$ on Ω, then the standard comparison method gives

$$z(t,x) \geq v(t) \text{ for all } x \in \Omega, \text{ and all } 0 < t < T^*.$$

Consequently, we deduce blow up at finite time of the PDE system. This ends the proof of the proposition. □

4. Numerical illustration

In this section, we show by computer simulations, that the v component of the ODE system (1), as well as the z component of the PDE system (2), blow up in a finite time. In these illustrations we took $\rho = 0$, but the blowing up in a fine time of the ODE or PDE systems remains valid for $\rho \neq 0$.

4.1. **Illustration of Theorem 2.1.** We validate in this section the result stated in

Theorem 2.1. For this, we consider the ODE system (1) using the following choice of its parameters,

$$\begin{array}{llll} \rho = 0 \quad m = 10 & ; \quad p = \dfrac{6}{5} & ; \quad c = 10 & \\ \alpha_3 = 1.11 & ; \quad \gamma = 0.125 & ; \quad \omega = \dfrac{6}{5} & ; \quad a = 10 \end{array}$$

These values were chosen so that the solution $v(t)$ of system (1) blows up in a finite for initial conditions $(u_0, v_0) = (10, 157)$.

ODE system (1) is simulated with Matlab R2017a. The numerical resolution of this system is carried out using the `ode45` solver of ordinary differential equations. Figure 1 presents the evolution of $v(t)$ over the time interval $[0, 1]$.

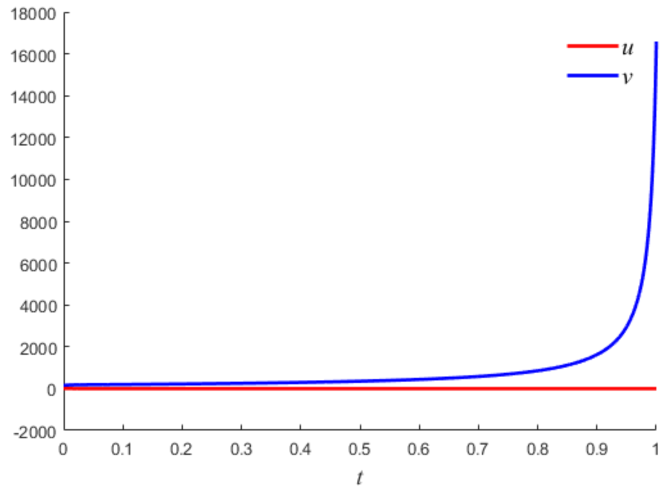

FIGURE 1. Blow-up in the ODE case. The initial conditions are $(u_0, v_0) = (10, 157)$.

4.2. **Illustration of Corollary 3.2 and Proposition 3.3.** We validate in this section the result stated in Corollary 3.2 and Proposition 3.3. For this, we consider the PDE system (2) using the following choice of its parameters,

$$
\begin{array}{llllllll}
d_1 = d_2 = 2.5 & ; & \rho = 0 & ; & m = 0.55 & ; & c = 10 \\
\alpha_3 = 1.1 & ; & \gamma = 0.0512 & ; & a = 20 & ; & \omega = 1.2
\end{array} \tag{10}
$$

These values were chosen so that the solution $z(t)$ of system (2) blows up in a finite for initial conditions $(w_0, z_0) = (1520, 18.1)$. PDE system (2) is simulated with MATLAB R2017a. The numerical resolution of this system is carried out using the `pdepe` solver of partial differential equations. Figure 2 presents the evolution of $z(t, x)$ over the domain $[0, 1] \times [0, 1]$.

5. CONCLUSION

In the present manuscript we study the solutions of the ODE system (1), modeling a reduced predator–prey model of two species food chain ODE with delay. The system is with a generalist predator and group defence in the prey species. We prove that for large initial data and under conditions stated in ([13], Theorem 3.2), the solutions of system (1) can blow up in a finite time. This is also true in the case of the spatially explicit model (2). Thus, even when the logistic delay term ρ is not zero, the blow up in finite time persists in the ODE and PDE cases. These results are confirmed by computer simulations.

6. DATA AVAILABILITY

The datasets generated during and/or analysed during the current study are available in the github repository, http://github.com/abdelhamidzaidi/BlowUp.

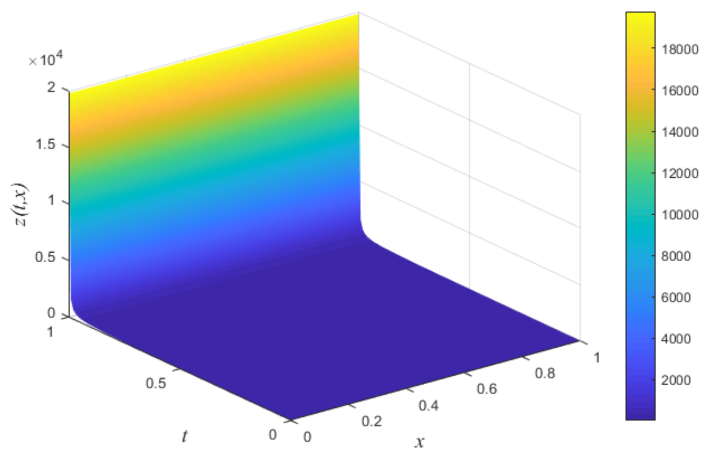

FIGURE 2. Blow-up in the 1-D PDE case. The initial conditions are $(w_0, z_0) = (1520, 18.1)$. The boundray condition are given by (3), and (4).

REFERENCES

[1] V. Ajraldi, M. Pittavino, E. Venturino, Modeling herd behavior in population systems. Nonlinear Anal. Real World Appl. 12(4), 2319–2338 (2011).

[2] Aziz-Alaoui MA. Study of a Leslie–Gower type tri-trophic population model. Chaos Solitons Fractals 2002;14(8):1275–93.

[3] B. Boon, H. Laudelout, Kinetics of nitrite oxidation by Nitrobacter winogradskyi. Biochem. J. 85(3), 440–447 (1962).

[4] P.A. Braza, Predator-prey dynamics with square root functional responses. Nonlinear Anal. Real World Appl. 13(4), 1837–1843 (2012).

[5] C.E. Causton, Dossier on Rodolia carthe second equation of the PDE systemdinalis Mulsant (Coccinellidae:Cocinellinae), a potential biological control agent for the cottony cushion scale, Icerya purchasi Maskell (Margarodidae) (Charles Darwin Research Station, Galápagos Islands, 2001).

[6] Debnath S, Ghosh U, Sarkar S. Global dynamics of a tritrophic food chain model subject to the Allee effects in the prey population with sexually reproductive generalized-type top predator. Comp Math Methods. 2019;e1079:1-23.

[7] S. Djilali, Impact of prey herd shape on the predator-prey interaction. Chaos Solitons Fract. 120, 139–148 (2019).

[8] V.H. Edwards, The influence of high substrate concentrations on microbial kinetics. Biotechnol. Bioeng. 12(5), 679–712 (1970).

[9] S.A.H. Geritz, M. Gyllenberg, Group defence and the predators functional response. J. Math. Biol. 66(4), 705–717 (2013).

[10] D. Henry, Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Mathematics 840, Springer-Verlag,New-York, 1984.

[11] J.C. Holmes, W.M. Bethel, Modification of inter.mediate host behaviour by parasites. Suppl. I Zool. J. Linnean Soc. 51, 123–149 (1972).

[12] C. C. Ioannou, Grouping and predation. Encycl. Evolut. Psychol. Sci. 1-6 (2017).

[13] S. Kundu and R. R. Patra, Effect of delay and control on a predator–prey ecosystem with generalist predator and group defence in the prey species" Eur. Phys. J. Plus (2022) 137:28, pp. 1-25.

[14] V. Kumar, N. Kumari, Bifurcation study and pattern formation analysis of a tritrophic food chain model with group defense and Ivlev-like nonmonotonic functional response. Chaos Solitons Fract. 147, 110964 (2021).

[15] Mishra P , Raw SN , Tiwari B . Study of a Leslie-Gower predator-prey model with prey defense and mutual interference of predators. Chaos Solitons Fractals 2019;120:1–16.

[16] Misra OP, Babu AR, Mathematical study of a Leslie-Gower type tritrophic population model in a polluted environment. Modeling earth systems and environment 2(1):1–11 (2016).

[17] R. D. Parshad, R. K. Abderrahmanne H, Upadhyay and N. Kumari, Finite time blowup in a realistic food chain model. ISRN Biomath 2013 (Article ID 424062).

[18] R. D. Parshad, N. Kumari N and S. Kouachi, A remark on "Study of a Leslie-Gower-type tritrophic population model [Chaos, Solitons and Fractals 14 (2002) 1275–1293]. Chaos Solitons Fractals 71:22–28 (2015).

[19] R. D. Parshad, S. Kouachi and N. Kumari , A comment on "Mathematical study of a Leslie-Gower type tritrophic population model in a polluted environment" [Modeling in Earth Systems and Environment 2 (2016) 1–11] Model. Earth Syst. Environ. (2016) 2:93.

[20] R. D. Parshad, E. Quansah, M. A. Beauregard and S. Kouachi, On "small" data blow-up in a three species food chain model, Computers and Mathematics with Applications 73 pp. 576-587(2017).

[21] R. D. Parshad, E. M. Takyi and S. Kouachi, A remark on "Study of a Leslie-Gower predator-prey model with prey defense and mutual interference of predators" [Chaos, Solitons & Fractals 120 (2019) 1–16], Chaos, Solitons and Fractals 123 (2019) 201–205.

[22] F. Roth, Global Solutions of Reaction-Diffusion Systems. Lecture Notes in Mathematics. 1072, Springer-Verlag, Berlin (1984).

[23] E. M. Takyi, K. Antwi-Fordjour, S. Kouachi and R. D. Parshad, A remark on " Global dynamics of a tritrophic food chain model subject to the Allee effects in the prey population with sexually reproductive generalized-type top predator" [Comp and Math Methods. 2019; E1079, pp. 1–23.], Comp and Math Methods. 2021.

[24] J.S. Tener, Muskoxen in Canada: A biological and taxonomic review (Queens Printer, Ottawa, 1965).

[25] Upadhyay RK , Rai V . Why chaos is rarely observed in natural populations? Chaos Solitons Fractals 1997;8(12):1933–9 .

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