# Cauchy problems of fractional evolution equations on an infinite interval

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**Abstract**

In this paper, we investigate the existence and attractivity of mild solutions to fractional evolution equations with Caputo fractional derivative on an infinite interval. Our methods are based on fractional calculus, semigroup theory, compactness methods and the measure of noncompactness. Several sufficient conditions for the existence of solutions to the given problem are proposed. Examples illustrating the main results are presented.

**Keywords:** Fractional evolution equations; Caputo derivative; Infinite interval; Existence; Attractivity.

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## 1 Introduction

Fractional differential equations have gained much attention in the recent years. Many applications in science and engineering have indicated that fractional differential equations can better describe some mathematical models than their integer order counterparts. For significant developments on fractional derivatives, see the monographs by Kiryakova [14], Podlubny [18], Kilbas et al. [12], Diethelm [8], Bajlekova [1], Zhou [23] and the references therein. One of the branches of fractional differential equations is the fractional evolution equations, where the standard abstract theory in Banach space can handle fractional partial differential equations, such as the fractional diffusion equations, fractional Rayleigh-Stokes equations and fractional Navier-Stokes equations and so on, for instance, see [4, 11, 21, 26, 27] and the references cited therein.

In recent years, the study of fractional evolution equations has mainly focused on the existence of solutions on a finite interval $[0, a]$, where $a \in (0, \infty)$ (for instance, see

[3, 15]). It is well-known that the solutions for a semilinear fractional diffusion equation will blow-up or just lie on a local time region, for example, see [13, 22], and the problem will be globally well-posed for small initial data, see for example [6, 11] or the behavior of solutions is uniformly convergent to zero at infinity [16]. Moreover, the long time behavior of solutions may depend on the global time region (infinite interval $[0,\infty)$) in order to ensure that the solution globally exists. One way to overcome this situation is to weight time in a specific space [24, 25]. For this technique, a useful mathematical tool is the generalized form of Ascoli-Arzelà theorem or the Kuratowski measure of noncompactness in the time weighted function space. Based on these methods, several existence results of solutions on an infinite intervals are established in this paper. We also discuss the asymptotic behavior of solutions which are attractive.

The purpose of this paper is to study a fractional evolution equation with Caputo derivative:

$$\begin{cases} {}^C D^{\alpha}_{0+} x(t) = Ax(t) + f(t, x(t)), & t \in [0,\infty), \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where ${}^C D^{\alpha}_{0+}$ is Caputo fractional derivative operator of order $0 < \alpha < 1$, $A$ is the infinitesimal generator of a $C_0$-semigroup of exponential stable linear operators $\{Q(t)\}_{t\geq 0}$ in Banach space $X$, $f : [0,\infty) \times X \to X$ is a given continuous function satisfying certain assumptions and $x_0$ is an element of the Banach space $X$.

The remaining part of the paper is organized as follows. In Section 2, we recall some preliminary concepts of fractional calculus. In Section 3, we establish some lemmas that match with the generalized form of Ascoli-Arzelà theorem. In Section 4, we establish sufficient conditions for the global existence of mild solutions for (1.1) in the cases when semigroups are compact or noncompact. In Section 5, we obtain some results concerning the asymptotic properties of the mild solutions.

## 2 Preliminaries

In this section, we recall some concepts on fractional integrals and derivatives, and state some known lemmas. We denote by $X$ a Banach space with norm $|\cdot|$, while the notation $\mathcal{L}(X)$ stands for the space of all bounded linear operators from $X$ into itself with norm $\|\cdot\|$.

**Definition 2.1.** *The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f : [0,+\infty) \to X$ is defined as*

$$I^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad t > 0,$$

*provided the right side is point-wise defined on* $[0, +\infty)$.

**Definition 2.2.** *The Riemann-Liouville fractional derivative of order* $\alpha \in (0, 1)$ *for a function* $f : [0, +\infty) \to X$ *is defined as*

$$^{RL}D_t^{\alpha} f(t) = \frac{d}{dt}(I_{0+}^{\alpha} f)(t), \quad t > 0.$$

**Definition 2.3.** *The Caputo fractional derivative of order* $\alpha \in (0, 1)$ *for a function* $f : [0, +\infty) \to X$ *is defined as*

$$^{C}D_t^{\alpha} f(t) = {}^{RL}D_t^{\alpha}(f(t) - f'(0)), \quad t > 0.$$

**Definition 2.4.** [23] *Define a Wright function* $M_\alpha(\theta)$ *by*

$$M_\alpha(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-\alpha n)}, \quad 0 < \alpha < 1, \ \theta \in \mathbb{C},$$

*with the following property*

$$M_\alpha(\theta) \geq 0, \quad \text{for } \theta > 0; \quad \int_0^{\infty} \theta^{\delta} M_\alpha(\theta) d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\alpha\delta)}, \quad \text{for } \delta > -1.$$

**Definition 2.5.** [18] *The Mittag-Leffler function* $E_{\alpha,\beta}(\cdot)$ *is defined by*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \ \beta \in \mathbb{R}, \ z \in \mathbb{C}.$$

If $0 < \alpha < 1, \ \beta > 0$, then the asymptotic expansion of $E_{\alpha,\beta}(z)$ as $z \to \infty$ is given by

$$E_{\alpha,\beta}(z) = \begin{cases} \dfrac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) + \varepsilon_{\alpha,\beta}(z), & |\arg z| \leq \dfrac{1}{2}\alpha\pi, \\ \varepsilon_{\alpha,\beta}(z), & |\arg(-z)| < \left(1 - \dfrac{1}{2}\alpha\right)\pi, \end{cases} \tag{2.1}$$

where

$$\varepsilon_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O\left(|z|^{-N}\right), \qquad \text{as } z \to \infty.$$

For short, we set $E_\alpha(\cdot) := E_{\alpha,1}(\cdot)$ and $e_\alpha(\cdot) := E_{\alpha,\alpha}(\cdot)$, from the definition of $E_{\alpha,\beta}$ and [20, Lemma 2], for $t \geq 0$ and some $\omega > 0$, we have

$$0 < E_\alpha(-\omega t^\alpha) \leq 1, \quad 0 < e_\alpha(-\omega t^\alpha) \leq \frac{1}{\Gamma(\alpha)}, \quad \text{and } E_\alpha(0) = 1, \ e_\alpha(0) = \frac{1}{\Gamma(\alpha)}. \tag{2.2}$$

**Remark 2.1.** *In particular, from (2.1), for* $\alpha \in (0, 1)$ *and* $t \in \mathbb{R}$, *we have*

$$\lim_{t \to -\infty} E_\alpha(t) = \lim_{t \to -\infty} e_\alpha(t) = 0.$$

Throughout this paper, we suppose that $A$ is the infinitesimal generator of a $C_0$-semigroup $\{Q(t)\}_{t\geq 0}$ of exponential stable linear operators on Banach space $X$. This means that there exist $M \geq 1$ and some $\omega > 0$ such that

$$\|Q(t)\| \leq Me^{-\omega t}, \quad \text{for } t \geq 0.$$

**Lemma 2.1.** [23] *If the following equation holds:*

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(Ax(s) + f(s, x(s)))ds, \quad \text{for } t \geq 0,$$

*then*

$$x(t) = S_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s, x(s))ds, \quad \text{for } t \geq 0,$$

*where*

$$S_\alpha(t) = \int_0^\infty M_\alpha(\theta)Q(t^\alpha\theta)d\theta, \quad P_\alpha(t) = \int_0^\infty \alpha\theta M_\alpha(\theta)Q(t^\alpha\theta)d\theta.$$

By virtue of Lemma 2.1, we give the following definition of the mild solution of (1.1).

**Definition 2.6.** *By a mild solution of the Cauchy problem (1.1), we mean that the function $x \in C([0, \infty), X)$ satisfies*

$$x(t) = S_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s, x(s))ds, \ t \geq 0.$$

Similar to the proofs of [23], we have the following lemmas.

**Lemma 2.2.** *For any fixed $\omega, t > 0$, $S_\alpha(t)$ and $P_\alpha(t)$ are linear and bounded operators, that is, for any $x \in X$, we have*

$$|S_\alpha(t)x| \leq ME_\alpha(-\omega t^\alpha)|x|, \quad |P_\alpha(t)x| \leq Me_\alpha(-\omega t^\alpha)|x|.$$

**Lemma 2.3.** *$\{S_\alpha(t)\}_{t>0}$ and $\{P_\alpha(t)\}_{t>0}$ are strongly continuous, that is, $\forall\ x \in X$ and $t'' > t' > 0$, we have*

$$|S_\alpha(t'')x - S_\alpha(t')x| \to 0, \quad |P_\alpha(t'')x - P_\alpha(t')x| \to 0, \quad \textit{as } t'' \to t'.$$

**Lemma 2.4.** *Assume that $\{Q(t)\}_{t>0}$ is compact operator. Then $\{S_\alpha(t)\}_{t>0}$ and $\{P_\alpha(t)\}_{t>0}$ are also compact operators.*

Let $D$ be a nonempty subset of $X$. The Kuratowski measure of noncompactness $\beta$ is

$$\beta(D) = \inf\left\{ d > 0 : \ D \ \subset \bigcup_{j=1}^{n} M_j \text{ and } \mathrm{diam}(M_j) \leq d \right\},$$

where the diameter of $M_j$ is given by $\operatorname{diam}(M_j) = \sup\{|x-y| : \ x, y \in M_j\}$, $j = 1, \ldots, n$. Let $J = [0, \infty)$ and $C(J, X)$ be the space consisting of all continuous functions from $J$ into $X$. Moerover, we set

$$C_e(J, X) = \{x \in C(J, X) : \lim_{t\to\infty} e^{-t}|x(t)| = 0\},$$

with the norm $\|x\|_e = \sup_{t\in J} e^{-t}|x(t)| < \infty$. It is obvious that $(C_e(J, X), \|\cdot\|_e)$ is a Banach space.

**Lemma 2.5.** [25] *Let $H \subset C_e(J, X)$ be bounded and equicontinuous on $[0, b]$ with $b > 0$. If, for any $x \in H$, there exists $e^{-t}|x(t)| \to 0$ uniformly as $t \to \infty$, then*

$$\beta_e(H) = \sup_{t\in J} \beta\big(e^{-t}H(t)\big),$$

*where $\beta_e$ is the Kuratowski measure of noncompactness in $C_e(J, X)$.*

For the more details of the definitions, properties and applications of the measure of noncompactness, we refer to the monographs [2] and [23].

**Lemma 2.6.** [7]. *Let $X$ be a Banach space, $D \subset X$ be a bounded closed and convex set, and $\gamma$ be a measure of noncompactness. If the operator $Q : D \to D$ is condensing (that is, $\gamma(Q(D)) < \gamma(D)$), then $Q$ has a fixed point in $D$.*

**Lemma 2.7.** [5, pp.125] *Let $X$ be a Banach space and $W \subset X$ be bounded, then, for each $\varepsilon > 0$, there is a sequence $\{u_n\}_{n=1}^{\infty} \subset W$, such that*

$$\beta(W) \le 2\beta(\{u_n\}_{n=1}^{\infty}) + \varepsilon.$$

**Lemma 2.8.** [16] *Let $X$ be a Banach space and $u_n(t) : [0, \infty) \to X$, $(n = 1, 2, \ldots)$ be the continuous function family. If there exists $\rho \in L^1[0, \infty)$ such that*

$$|u_n(t)| \le \rho(t), \quad t \in [0, \infty), \ n = 1, 2, \ldots,$$

*then $\beta(\{u_n(t)\}_{n=1}^{\infty})$ is integrable on $[0, \infty)$, and*

$$\beta\Big(\Big\{\int_0^{\infty} u_n(t)dt : \ n = 1, 2, \ldots\Big\}\Big) \le 2\int_0^{\infty} \beta(\{u_n(t) : \ n = 1, 2, \ldots\})dt.$$

In the sequel, we need the following generalized form of Ascoli-Arzelà theorem.

**Lemma 2.9.** [25] *The set $H \subset C_e(J, X)$ is relatively compact if and only if the following conditions hold:*

(i) *the set $U = \{y : \ y(t) = e^{-t}x(t), x \in H\}$ is equicontinuous on $[0, b]$ for any $b > 0$;*

(ii) *for any $t \in J$, $U(t) = e^{-t}H(t)$ is relatively compact in $X$;*

(iii) $\lim_{t\to\infty} e^{-t}|x(t)| = 0$ *uniformly for $x \in H$.*

## 3 Some Lemmas

Let us first introduce some assumptions needed for the forthcoming results.

(H0) The operator $\{Q(t)\}_{t>0}$ is equicontinuous, that is, the operator $Q(t)$ is continuous in the uniform operator topology for $t > 0$.

(H1) The function $f(t, x)$ satisfies the Carahéodory type condition, that is, $f(t, \cdot)$ is Lebesgue measurable with respect to all $t$ on $(0, \infty)$, $f(\cdot, x)$ is continuous with respect to each $x$ on $X$.

(H2) There exists a nonnegative Lebesgue measurable function $m : (0, \infty) \to \mathbb{R}^+$ such that

$$|f(t, x)| \leq m(t)|x|^{\sigma}, \quad \text{for all } x \in X, \text{ and a.e. } t > 0,$$

where $\sigma \geq 0$ and $m(\cdot)$ satisfies the following condition:

$$\sup_{t\geq 0} \eta(t) := \sup_{t\geq 0} \frac{M}{\Gamma(\alpha)} e^{-t} \int_0^t (t-s)^{\alpha-1} m(s) e^{\sigma s} ds < 1,$$

with $\lim_{t\to 0} \eta(t) = \lim_{t\to\infty} \eta(t) = 0$.

Let $r_0$ be such that

$$r_0 \geq M|x_0| + \sup_{t\geq 0} \eta(t) r_0^{\sigma}.$$

and $\Omega_{r_0}$ be a subset of $C_e(J, X)$ defined by

$$\Omega_{r_0} = \{x : \ x \in C_e(J, X), \ \|x\|_e \leq r_0\}.$$

It is clear that $\Omega_{r_0}$ is nonempty bounded, closed and convex.

For any $x \in \Omega_{r_0}$, consider the operator $F$ defined by

$$(Fx)(t) = S_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s, x(s)) ds, \text{ for } \ t \in J. \tag{3.1}$$

It is clear that $x$ is a mild solution of (1.1) in $\Omega_{r_0}$ if and only if there exists a fixed point $x^* \in \Omega_{r_0}$, such that $x^* = Fx^*$ holds.

**Lemma 3.1.** *Assume that* (H0)-(H2) *hold. Then* $U = \{y : \ y(t) = e^{-t}(Fx)(t), \ x \in \Omega_{r_0}\}$ *is equicontinuous on* $[0, b]$ *with* $b > 0$ *and* $\lim_{t\to\infty} e^{-t}|(Fx)(t)| = 0$ *uniformly for* $x \in \Omega_{r_0}$.

*Proof.* **Claim I.** $U$ is equicontinuous. By the assumption of $m(\cdot)$ in (H2), there exists a $b > 0$ large enough such that

$$e^{-t} \frac{M}{\Gamma(\alpha)} r_0^{\sigma} \int_0^t (t-s)^{\alpha-1} m(s) e^{\sigma s} ds < \varepsilon/2, \quad t > b. \tag{3.2}$$

Hence, for $t_1, t_2 > b$ and $t_1 < t_2$, by virtue of (H2) and (3.2), for any $x \in \Omega_{r_0}$ we get

$$\begin{aligned}
|y(t_2) - y(t_1)| \leq & \left| e^{-t_2} \int_0^{t_2} (t_2 - s)^{\alpha-1} P_\alpha(t_2 - s) f(s, x(s)) ds \right| \\
& + \left| e^{-t_1} \int_0^{t_1} (t_1 - s)^{\alpha-1} P_\alpha(t_1 - s) f(s, x(s)) ds \right| \\
\leq & M e^{-t_2} \int_0^{t_2} (t_2 - s)^{\alpha-1} e_\alpha(-\omega(t_2 - s)^\alpha) m(s) |x(s)|^\sigma ds \\
& + M e^{-t_1} \int_0^{t_1} (t_1 - s)^{\alpha-1} e_\alpha(-\omega(t_1 - s)^\alpha) m(s) |x(s)|^\sigma ds \\
\leq & \frac{M}{\Gamma(\alpha)} r_0^\sigma e^{-t_2} \int_0^{t_2} (t_2 - s)^{\alpha-1} m(s) e^{\sigma s} ds \\
& + \frac{M}{\Gamma(\alpha)} r_0^\sigma e^{-t_1} \int_0^{t_1} (t - s)^{2q-1} m(s) e^{\sigma s} ds < \varepsilon.
\end{aligned}$$

On the other hand, for any $x \in \Omega_{r_0}$ and $t_1 = 0$, $t_2 \in (0, b]$, it follows by the Hölder's inequality and (2.2) that

$$\begin{aligned}
& |e^{-t_2} (Fx)(t_2) - (Fx)(0)| \\
\leq & |e^{-t_2} S_\alpha(t_2) x_0 - x_0| + \left| e^{-t_2} \int_0^{t_2} (t_2 - s)^{\alpha-1} P_\alpha(t_2 - s) f(s, x(s)) ds \right| \\
\leq & \|e^{-t_2} S_\alpha(t_2) - I\| |x_0| + M e^{-t_2} \int_0^{t_2} (t_2 - s)^{\alpha-1} e_\alpha(-\omega(t_2 - s)^\alpha) m(s) |x(s)|^\sigma ds \\
\leq & \|e^{-t_2} S_\alpha(t_2) - I\| |x_0| + M \eta(t_2) r_0^\sigma.
\end{aligned}$$

From the continuity of power function $e^{-t}$, $t \geq 0$, due to $S_\alpha(0) = I$ and the assumption of $f$, we get $y(t_2) \to y(0)$, as $t_2 \to 0$.

For $0 < t_1 < t_2 \leq b$, we have

$$\begin{aligned}
|y(t_2) - y(t_1)| \leq & |e^{-t_2} S_\alpha(t_2) x_0 - e^{-t_1} S_\alpha(t_1) x_0| \\
& + \left| e^{-t_2} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} P_\alpha(t_2 - s) f(s, x(s)) ds \right| \\
& + \left| e^{-t_2} \int_0^{t_1} \big( (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \big) P_\alpha(t_2 - s) f(s, x(s)) ds \right| \\
& + \left| e^{-t_2} \int_0^{t_1} (t_1 - s)^{\alpha-1} \big( P_\alpha(t_2 - s) - P_\alpha(t_1 - s) \big) f(s, x(s)) ds \right| \\
& + \left| (e^{-t_2} - e^{-t_1}) \int_0^{t_1} (t_1 - s)^{\alpha-1} P_\alpha(t_1 - s) f(s, x(s)) ds \right| \\
=: & I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

By Lemma 2.2, we have

$$I_1 \leq (e^{-t_1} - e^{-t_2}) |S_\alpha(t_2) x_0| + e^{-t_1} |S_\alpha(t_2) x_0 - S_\alpha(t_1) x_0|$$

$$\begin{aligned}&\leq M(e^{-t_1}-e^{-t_2})|x_0|+|S_\alpha(t_2)x_0-S_\alpha(t_1)x_0|\\&\to 0,\quad \text{as } t_2\to t_1.\end{aligned}$$

Now, it can be deduced that $I_2\to 0$ as $t_2\to t_1$ directly. Indeed, since $\eta(t)<1$,

$$\begin{aligned}I_2\leq& Me^{-t_2}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}e_\alpha(-\omega(t_2-s)^\alpha)m(s)|x(s)|^\sigma ds\\ \leq&\frac{M}{\Gamma(\alpha)}e^{-t_2}r_0^\sigma\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}m(s)e^{\sigma s}ds\\ \to& 0,\quad \text{as } t_2\to t_1.\end{aligned}$$

From the assumption of function $m(\cdot)$ in (H2) and noting that

$$((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1})m(s)e^{\sigma s}\leq\frac{2}{\Gamma(\alpha)}(t_1-s)^{\alpha-1}m(s)e^{\sigma s},$$

for a.e. $s\in[0,t_1]$, and the right-hand side of the above inequality is integrable in $s$ from 0 to $t_1$, and $(t_1-s)^{\alpha-1}\to(t_2-s)^{\alpha-1}$ a.e. $[0,t_1]$, then the Lebesgue dominated convergence theorem implies that

$$\int_0^{t_1}\big((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\big)m(s)e^{\sigma s}ds\to 0,\quad \text{as } t_2\to t_1.$$

Therefore, we have

$$\begin{aligned}I_3\leq& Me^{-t_2}\int_0^{t_1}\big((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\big)e_\alpha(-\omega(t_1-s)^\alpha)m(s)|x(s)|^\sigma ds\\ \to& 0,\quad \text{as } t_2\to t_1.\end{aligned}$$

For given $\varepsilon>0$ be small enough, from the assumption (H2), we get

$$\begin{aligned}I_4\leq& e^{-t_2}\int_0^{t_1-\varepsilon}(t_1-s)^{\alpha-1}\big|\big(P_\alpha(t_2-s)-P_\alpha(t_1-s)\big)f(s,x(s))\big|ds\\ &+e^{-t_2}\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\alpha-1}\big|\big(P_\alpha(t_2-s)-P_\alpha(t_1-s)\big)f(s,x(s))\big|ds\\ \leq& e^{-t_2}r_0^\sigma\int_0^{t_1}(t_1-s)^{\alpha-1}m(s)e^{\sigma s}ds\sup_{s\in[0,t_1-\varepsilon]}\|P_\alpha(t_2-s)-P_\alpha(t_1-s)\|\\ &+\frac{2M}{\Gamma(\alpha)}e^{-t_2}\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\alpha-1}m(s)|x(s)|^\sigma ds\\ \leq& I_{41}+I_{42},\end{aligned}$$

where

$$\begin{aligned}I_{41}=&r_0^\sigma\int_0^{t_1}(t_1-s)^{\alpha-1}m(s)e^{\sigma s}ds\sup_{s\in[0,t_1-\varepsilon]}\|P_\alpha(t_2-s)-P_\alpha(t_1-s)\|,\\ I_{42}=&\frac{2M}{\Gamma(\alpha)}e^{-t_1}r_0^\sigma\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\alpha-1}m(s)e^{\sigma s}ds.\end{aligned}$$

One can observe that (H0) implies the continuity of $Q(t)(t > 0)$ in $t$ in the uniform operator topology. Then it follows from Lemma 2.3 that $I_{41} \to 0$ as $t_2 \to t_1$. In a similar manner, $I_2$ tends to zero. Thus we get $I_{42} \to 0$ as $\varepsilon \to 0$. In consequence, $I_4 \to 0$ as $t_2 \to t_1$, and $\varepsilon \to 0$ independently of $x \in \Omega_{r_0}$. Finally, we get

$$\begin{aligned} I_5 \leq & |(e^{-t_2} - e^{-t_1})e^{t_1}|Me^{-t_1} \int_0^{t_1} (t_1 - s)^{\alpha-1} e_\alpha(-\omega(t_1 - s)^\alpha) m(s)|x(s)|^\sigma ds \\ \leq & |(e^{-t_2} - e^{-t_1})e^{t_1}|r_0^\sigma \eta(t_1) \to 0, \quad \text{as } t_2 \to t_1. \end{aligned}$$

Consequently, for any $t_1, t_2 \in [0, b]$ with $t_1 < t_2$, we have $|y(t_2) - y(t_1)| \to 0$ as $t_2 \to t_1$.

To end this proof, for $0 \leq t_1 < b < t_2$, if $t_2 \to t_1$, then $t_2 \to b$ and $t_1 \to b$. Therefore, we obtain

$$|y(t_2) - y(t_1)| \leq |y(t_2) - e^{-b}(Fx)(b)| + |e^{-b}(Fx)(b) - y(t_1)| < \varepsilon, \tag{3.3}$$

as $t_1 \to b$ and $t_2 > b$. Hence, together with the foregoing arguments and (3.3), we have

$$|y(t_2) - y(t_1)| \to 0, \text{ as } t_2 \to t_1$$

independently of $x \in \Omega_{r_0}$. Thus $U = \left\{y : \ y(t) = e^{-t}(Fx)(t), \ x \in \Omega_{r_0}\right\}$ is equicontinuous.

**Claim II.** $\lim_{t\to\infty} e^{-t}|(Fx)(t)| = 0$ uniformly for $x \in \Omega_{r_0}$. For any $x \in \Omega_{r_0}$, by (H2) and Lemma 2.2, we have

$$\begin{aligned} |(Fx)(t)| \leq & |S_\alpha(t)x_0| + \left| \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s, x(s)) ds \right| \\ \leq & ME_\alpha(-\omega t^\alpha)|x_0| + M \int_0^t (t-s)^{\alpha-1} e_\alpha(-\omega(t-s)^\alpha) m(s)|x(s)|^\sigma ds \\ \leq & ME_\alpha(-\omega t^\alpha)|x_0| + e^t \eta(t) r_0^\sigma, \end{aligned} \tag{3.4}$$

which implies that

$$\lim_{t\to\infty} e^{-t}|(Fx)(t)| \leq M \lim_{t\to\infty} e^{-t} E_\alpha(-\omega t^\alpha)|x_0| + \lim_{t\to\infty} \eta(t) = 0.$$

This means that $\lim_{t\to\infty} e^{-t}|(Fx)(t)| = 0$ uniformly for $x \in \Omega_{r_0}$. The proof is completed. □

**Lemma 3.2.** *Assume that* (H0)-(H1) *hold. Then $F$ maps $\Omega_{r_0}$ into itself and $F$ is continuous.*

*Proof.* **Claim I.** $F$ maps $\Omega_{r_0}$ into $\Omega_{r_0}$. In fact, for each $r_0 > 0$ by (2.2) and (3.4), we have

$$\begin{aligned} e^{-t}|(Fx)(t)| \leq & Me^{-t} E_\alpha(-\omega t^\alpha)|x_0| + e^{-t} \int_0^t (t-s)^{\alpha-1} e_\alpha(-\omega(t-s)^\alpha) m(s)|x(s)|^\sigma ds \\ \leq & M|x_0| + \sup_{t\geq 0} \eta(t) r_0^\sigma \leq r_0, \end{aligned}$$

Thus, we get that $F\Omega_{r_0} \subset \Omega_{r_0}$ for $t \in J$.

**Claim II.** $F$ is continuous. Since $\lim_{t\to\infty} \eta(t) = 0$, for $\varepsilon > 0$, there exists a constant $T > 0$ such that

$$\eta(t)r_0^\sigma < \frac{\varepsilon}{2}, \quad \text{for } t > T.$$

Let $\{x_m\}_{m=1}^\infty \subset \Omega_{r_0}$, $x \in \Omega_{r_0}$, and let $\lim_{m\to\infty} x_m = x$. Then, by the continuity of $f$ in (H1) for a.e. $t \in [0, T]$, we have

$$\lim_{m\to\infty} f(t, x_m(t)) = f(t, x(t)).$$

Therefore, for a.e. $t \in [0, T]$,

$$(t-s)^{\alpha-1}e_\alpha(-\omega(t-s)^\alpha)|f(s, x_m(s)) - f(s, x(s))| \le \frac{2}{\Gamma(\alpha)}r_0^\sigma(t-s)^{\alpha-1}m(s)e^{\sigma s}.$$

In addition, the function $(t-s)^{\alpha-1}m(s)e^{\sigma s}$ is integrable for a.e. $s \in [0, t]$ and $t \in [0, T]$. Hence, it follows by the Lebesgue dominated convergence theorem that

$$\begin{aligned}
&e^{-t}|(Fx_m)(t) - (Fx)(t)| \\
\le& e^{-t}\int_0^t (t-s)^{\alpha-1}|P_\alpha(t-s)(f(s, x_m(s)) - f(s, x(s)))|ds \\
\le& M\int_0^t (t-s)^{\alpha-1}e_\alpha(-\omega(t-s)^\alpha)|f(s, x_m(s)) - f(s, x(s))|ds \\
\to& 0, \quad \text{as } m \to \infty.
\end{aligned}$$

On the other hand, for $t > T$, we have

$$\begin{aligned}
&e^{-t}|(Fx_m)(t) - (Fx)(t)| \\
\le& Me^{-t}\int_0^t (t-s)^{\alpha-1}e_\alpha(-\omega(t-s)^\alpha)(|f(s, x_m(s))| + |f(s, x(s))|)ds \\
\le& 2\eta(t)r_0^\sigma < \varepsilon.
\end{aligned} \tag{3.5}$$

Therefore, in view of the above statements, it is obvious that $\|Fx_m - Fx\|_e \to 0$ pointwise on $J$ as $m \to \infty$. By Lemma 3.1, we have that $Fx_m \to Fx$ uniformly on $J$ as $m \to \infty$, that is, $F$ is continuous. The proof is completed. □

## 4 Existence

In this section, we show the main existence results for problem (1.1).

**Theorem 4.1.** *Assume that* (H0)-(H2) *hold. Furthermore, if $f$ satisfies the Lipschitz condition:*

(H3) *for any $x, y \in C_e(J, X)$, there exists a nonnegative constant $L < 1/M$ such that*

$$|f(t, x(t)) - f(t, y(t))| \le L|x(t) - y(t)|, \quad t \in J.$$

*Then the problem* (1.1) *has a unique mild solution.*

*Proof.* By Lemma 3.2, we know that $F$ maps $\Omega_{r_0}$ into itself. It remains to prove that $F$ is contractive by Banach fixed point theorem. Indeed, for any $x, y \in \Omega_{r_0}$, we have

$$\begin{aligned}\|Fx - Fy\|_e \le & M \sup_{t\in J} e^{-t} \int_0^t (t-s)^{\alpha-1} e_\alpha(-\omega(t-s)^\alpha)|f(s,x(s)) - f(s,y(s))|ds \\ \le & \frac{ML}{\Gamma(\alpha)} \sup_{t\in J} e^{-t} \int_0^t (t-s)^{\alpha-1}|x(s) - y(s)|ds \\ \le & \frac{ML}{\Gamma(\alpha)} \sup_{t\in J} \int_0^t (t-s)^{\alpha-1} e^{-(t-s)} ds \|x - y\|_e \\ = & \frac{ML}{\Gamma(\alpha)} \sup_{t\in J} \int_0^t s^{\alpha-1} e^{-s} ds \|x - y\|_e \\ \le & ML\|x - y\|_e,\end{aligned}$$

which shows that the operator $F$ is a contraction. Thus the problem (1.1) has a unique mild solution on $\Omega_{r_0}$. □

## 4.1 The case $Q(t)(t > 0)$ is compact

**Theorem 4.2.** *Assume that $Q(t)(t > 0)$ is compact, and the condition* (H1)-(H2) *hold. Then the Cauchy problem (1.1) admits at least one mild solution.*

*Proof.* Obviously, it is sufficient to show that $x = Fx$ has a fixed point in $\Omega_{r_0}$. Since $Q(t)(t > 0)$ is compact, the condition (H0) holds. By Lemma 3.1, we know that the set $U = \{y : y(t) = e^{-t}(Fx)(t), x \in \Omega_{r_0}\}$ is equicontinuous on $[0, b]$ and $\lim_{t\to\infty} e^{-t}|(Fx)(t)| = 0$ uniformly for $x \in \Omega_{r_0}$. It remains to prove that for each $t \in J$, the set $U(t) = \{y(t) : y(t) = e^{-t}(Fx)(t), x \in \Omega_{r_0}\}$ is relatively compact in $X$ according to Lemma 2.9. Obviously, $U(0)$ is relatively compact in $X$. Let $t \in (0, \infty)$ be fixed. For every $\varepsilon \in (0, t)$ and $\delta > 0$, define an operator $F_{\varepsilon,\delta}$ as follows

$$\begin{aligned}(F_{\varepsilon,\delta}x)(t) = & S_\alpha(t)x_0 \\ & + Q(\varepsilon^\alpha\delta) \int_0^{t-\varepsilon} \int_\delta^\infty \alpha\theta(t-s)^{\alpha-1} M_\alpha(\theta) Q((t-s)^\alpha\theta - \varepsilon^\alpha\delta) f(s, x(s)) d\theta ds.\end{aligned}$$

Since $Q(t)$ is compact for $t > 0$, by Proposition 2.4, we know that $S_\alpha(t)$ is compact. Let $U_{\varepsilon,\delta} = \{y_{\varepsilon,\delta} : y_{\varepsilon,\delta}(t) = e^{-t}(F_{\varepsilon,\delta}x)(t), x \in \Omega_{r_0}\}$. From the compactness of $Q(\varepsilon^\alpha\delta)$, we obtain that the set $U_{\varepsilon,\delta}(t)$ is relatively compact in $X$ for any $\varepsilon \in (0, t)$ and for any $\delta > 0$. Then, for any $x \in \Omega_{r_0}$, $y \in U$ and $y_{\varepsilon,\delta} \in U_{\varepsilon,\delta}$, we have

$$\begin{aligned}|y(t) - y_{\varepsilon,\delta}(t)| = & |e^{-t}(Fx)(t) - e^{-t}(F_{\varepsilon,\delta}x)(t)| \\ \le & \left| e^{-t} \int_0^{t-\varepsilon} \int_0^\delta \alpha\theta(t-s)^{\alpha-1} M_\alpha(\theta) Q((t-s)^\alpha\theta) f(s, x(s)) d\theta ds \right|\end{aligned}$$

$$
\begin{aligned}
&+\left|e^{-t}\int_{t-\varepsilon}^{t}\int_{0}^{\infty}\alpha\theta(t-s)^{\alpha-1}M_\alpha(\theta)Q((t-s)^\alpha\theta)f(s,x(s))d\theta ds\right|\\
\le&\alpha M r_0^\sigma e^{-t}\int_0^{t-\varepsilon}(t-s)^{\alpha-1}m(s)e^{\sigma s}ds\int_0^\delta \theta e^{-\omega(t-s)^\alpha\theta}M_\alpha(\theta)d\theta\\
&+\alpha M r_0^\sigma e^{-t}\int_{t-\varepsilon}^{t}(t-s)^{\alpha-1}m(s)e^{\sigma s}ds\int_0^\infty \theta e^{-\omega(t-s)^\alpha\theta}M_\alpha(\theta)d\theta\\
\le&\alpha\Gamma(\alpha)\eta(t)\int_0^\delta\theta M_\alpha(\theta)d\theta+Mr_0^\sigma e^{-t}\int_{t-\varepsilon}^{t}(t-s)^{\alpha-1}e_\alpha(-\omega(t-s)^\alpha)m(s)e^{\sigma s}ds.
\end{aligned}
$$

In view of $\eta(t)<1$ and $\int_0^\infty\theta M_\alpha(\theta)d\theta=1/\Gamma(\alpha)$, we know that

$$|y(t)-y_{\varepsilon,\delta}(t)|\to 0,\quad \text{as } \varepsilon\to 0,\ \delta\to 0.$$

Therefore, the set $U(t)$ is closed to an arbitrary compact set for $t\in(0,\infty)$. As a result, the set $U(t)=e^{-t}(F\Omega_{r_0})(t)$ is relatively compact set in $X$ for $t\in J$. By Lemma 2.9, we know that $F\Omega_{r_0}$ is a relatively compact set. On the other hand, by Lemma 3.2, we know that $F$ maps $\Omega_{r_0}$ into itself and $F$ is continuous. Hence, $F$ is a completely continuous operator. Therefore, according to Schauder's fixed point theorem, there exists at least one fixed point $x^*\in\Omega_{r_0}$ such that $x^*=Fx^*$ is satisfied. Then $x^*$ is a mild solution of (1.1). The proof is completed. □

### 4.2 The case $Q(t)$ is noncompact

**Theorem 4.3.** *Assume that $Q(t)$ is noncompact, and the conditions* (H0)-(H2) *hold and $f$ satisfies the following condition:*

(H4) *There exists nonnegative constant $l<1/(4M)$ such that for any bounded set $D\subset X$,*

$$\beta(f(t,D))\le l\beta(D),\quad t\in J.$$

*Then the Cauchy problem (1.1) admits at least one mild solution.*

*Proof.* By Lemma 3.2, we know that $F:\Omega_{r_0}\to\Omega_{r_0}$ is bounded and continuous. Let $H=\{y: y(t)=e^{-t}(Fx)(t),\ x\in\Omega_{r_0}\}$. Then $H\subset\Omega_{r_0}\subset C_e(J,X)$ and is bounded. From Lemma 3.1, we have that the set $H(t)$ is equicontinuous on $[0,b]$ for $b>0$. In addition, for any $y\in H$, it follows that $\lim_{t\to\infty}e^{-t}|y(t)|=0$ uniformly. Thus it remains to verify that $F$ is a condensing map according to Lemma 2.6. Let

$$Fx=F_1x+F_2x,\quad \text{for } x\in\Omega_{r_0},$$

where $(F_1x)(t)=S_\alpha(t)x_0$ for $t\in J$ and

$$(F_2x)(t)=\int_0^t(t-s)^{\alpha-1}P_\alpha(t-s)f(s,x(s))ds,\quad \text{for } t\in J.$$

Let $H = \Omega^0_{r_0}$, then $\Omega^0_{r_0} \subset \Omega_{r_0}$, $F\Omega^0_{r_0}$ is bounded, equicontinuous and $e^{-t}|(Fx)(t)| \to 0$ as $t \to \infty$ uniformly for $x \in \Omega^0_{r_0}$ and hence Lemma 2.5 implies that

$$\beta_e(F\Omega^0_{r_0}) = \sup_{t\in J} \beta(e^{-t}(F\Omega^0_{r_0})(t)). \tag{4.1}$$

Moreover, by Lemma 2.5, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset \Omega^0_{r_0}$ such that

$$\begin{aligned}\beta_e\big(F_2\Omega^0_{r_0}\big) =&\sup_{t\in J} \beta\big(e^{-t}(F_2\Omega^0_{r_0})(t)\big)\\ \le& 2\sup_{t\in J} \beta\big(e^{-t}(\{F_2x_n\}_{n=1}^{\infty})(t)\big) + \varepsilon\\ \le& 2\sup_{t\in J} \beta\Big(e^{-t}\int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s,\{x_n(s)\}_{n=1}^{\infty})ds\Big) + \varepsilon.\end{aligned}$$

Therefore, for $t \in J$, according to Lemma 2.8, we have

$$\begin{aligned}\beta_e\big(F_2\Omega^0_{r_0}\big) \le& 4M\sup_{t\in J} e^{-t}\int_0^t (t-s)^{\alpha-1}e_\alpha(-\omega(t-s)^\alpha)\beta\big(f(s,\{x_n(s)\}_{n=1}^{\infty})\big)ds + \varepsilon\\ \le& 4Ml\sup_{t\in J} e^{-t}\int_0^t (t-s)^{\alpha-1}e_\alpha(-\omega(t-s)^\alpha)\beta\big(\{x_n(s)\}_{n=1}^{\infty}\big)ds + \varepsilon\\ =& 4Ml\sup_{t\in J} e^{-t}\int_0^t (t-s)^{\alpha-1}e_\alpha(-\omega(t-s)^\alpha)e^s\beta\big(e^{-s}\{x_n(s)\}_{n=1}^{\infty}\big)ds + \varepsilon\\ \le& \frac{4Ml\beta_e(\Omega^0_{r_0})}{\Gamma(\alpha)}\sup_{t\in J} e^{-t}\int_0^t (t-s)^{\alpha-1}e^s ds + \varepsilon\\ \le& 4Ml\beta_e(\Omega_{r_0}) + \varepsilon.\end{aligned}$$

Since $F_1$ is Lipschitz continuous with constant 0, it follows from the definition of Kuratowski measure of noncompactness that $\beta_e(F_1(\Omega^0_{r_0})) = 0$. By virtue of (4.1) and the arbitrariness of $\varepsilon$, we have

$$\beta_e\big(F\Omega_0\big) \le \beta_e\big(F_1\Omega_0\big) + \beta_e\big(F_2\Omega_0\big) \le 4Ml\beta_e(\Omega_0).$$

Thus, from the above arguments, we find that $F : \Omega_0 \to \Omega_0$ is a condensing operator. Thus, we deduce by Lemma 2.6 that $F$ has a fixed point $x^*$ which is a mild solution of (1.1). The proof is completed. □

**Example 4.1.** *We consider the following fractional diffusion equations*

$$\begin{cases}\partial_t^{1/2}u(t,z) = -\partial_{zz}u(t,z) + e^{-t}\ln(1+|u(t,z)|), \ z\in[0,\pi], \ t>0,\\ u(t,0) = u(t,\pi) = 0, \ t\ge 0,\\ u(0,z) = u_0(z), \quad z\in[0,\pi],\end{cases} \tag{4.2}$$

*where $\partial_t^\alpha$ is the Caputo fractional partial derivative of order $\alpha = 1/2$.*

We define an operator $A$ by $Av = v''$ with the domain

$$D(A) = \{v \in L^2[0,\pi],\ v, v' \text{ absolutely continuous},\ v'' \in L^2[0,\pi],\ v(0) = v(\pi) = 0\}.$$

Then $A$ generates a compact strongly continuous semigroup $\{Q(t)\}_{t>0}$. Clearly, the semigroup is also exponentially stable. In fact, let $\lambda_k = k^2$ and $e_k(z) = \sqrt{2/\pi}\sin(kz)$ for every $k \in \mathbb{N}$. It is clear that $\{-\lambda_k, e_k\}_{k=1}^{\infty}$ is the eigensystem of the operator $A$, where $0 < \lambda_1 \leq \lambda_2 \leq, \cdots, \lambda_k \to \infty$ as $k \to \infty$, and $\{e_k\}_{k=1}^{\infty}$ forms an orthonormal basis of $L^2[0,\pi]$. Then

$$Ax = -\sum_{k=1}^{\infty} \lambda_k (x, e_k) e_k, \quad x \in D(A),$$

where $(\cdot,\cdot)$ is the inner product in $L^2[0,\pi]$. Thus the semigroup is given by

$$Q(t)x = \sum_{k=1}^{\infty} e^{-\lambda_k t}(x, e_k)e_k, \quad x \in L^2[0,\pi].$$

Therefore, we have $|Q(t)x|_{L^2[0,\pi]} \leq e^{-t}|x|_{L^2[0,\pi]}$. This means that the semigroup is also exponentially stable. Consequently, the problem (4.2) can be reformulated as an abstract Cauchy problem (1.1) for $x(t) = u(t,\cdot)$ and $f(t,x) = e^{-t}\ln(1+|x|)$. Clearly, the function $f$ satisfies (H1) and (H2). By Theorem 4.2, the problem (4.2) has at least one mild solution.

## 5 Attractivity

**Definition 5.1.** *The mild solution $x(t)$ of the Cauchy problem* (1.1) *is attractive if $x(t)$ tends to zero as $t \to \infty$.*

In order to obtain the attractivity of mild solution, we need the following lemmas.

**Lemma 5.1.** *[25] If $\mu \in (0,1), \nu \geq 1-\mu, \tau > 0$ and $t > 0$, then*

$$\int_0^t (t-s)^{\mu-1} s^{\nu-1} e^{-\tau s} ds \leq \kappa_1,$$

*where*

$$\kappa_1 = \max\left\{B(\mu,\nu), (\mu+\nu-1)^{\mu+\nu-1} e^{-(\mu+\nu-1)} \tau^{-(\mu+\nu-1)} B(\mu, 1-\mu)\right\} > 0,$$

*and $B(\cdot,\cdot)$ is the Beta function.*

**Lemma 5.2.** [10] *If $\beta, \mu, \omega > 0$, then*

$$t^{1-\beta} \int_0^t (t-s)^{\beta-1} s^{\mu-1} \mathrm{e}^{-\omega s} ds \leq \kappa_2, \ \ t > 0,$$

*where $\kappa_2 = \max\{1, 2^{1-\beta}\}\Gamma(\mu)(1 + \mu(\mu+1)/\beta)\omega^{-\mu} > 0$.*

**Theorem 5.1.** *Assume that* $Q(t)(t > 0)$ *is compact. Let* (H1) *hold and* $f$ *satisfies the following condition*

(H5) *There exist* $K \geq 0$, $\beta \geq 0$, $\delta \geq 0$ *and* $\lambda > 0$ *with* $\lambda > \delta$ *such that* $|f(t,x)| \leq Ke^{-\lambda t}t^{\beta}|x|^{\delta}$ *for* $t \in J$, *and*

$$\frac{MK\kappa_1}{\Gamma(\alpha)} < 1, \quad M|x_0| + \frac{MK\kappa_1 r^{\delta}}{\Gamma(\alpha)} \leq r, \quad \textit{for each} \quad r > 0.$$

*Then the solution of (1.1) is attractive.*

*Proof.* Considering the operator $F$ defined by (3.1), it is obvious that $F\Omega_r \subset \Omega_r$ for each $r > 0$. In fact, by the assumption (H5), for $t > 0$, we have

$$\begin{aligned}
|(Fx)(t)| \leq & ME_{\alpha}(-\omega t^{\alpha})|x_0| + MK\int_0^t (t-s)^{\alpha-1}e_{\alpha}(-\omega(t-s)^{\alpha})s^{\beta}e^{-\lambda s}|x(s)|^{\delta}ds \\
\leq & M|x_0| + \frac{MKr^{\delta}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\beta}e^{-(\lambda-\delta)s}ds \\
\leq & M|x_0| + \frac{MK\kappa_1 r^{\delta}}{\Gamma(\alpha)} \leq r.
\end{aligned}$$

The case of $t = 0$ is trivial. Hence, we get $\|Fx\|_e \leq r$. Let

$$\zeta(t) = \frac{MK}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\beta}e^{-(\lambda-\delta)s}ds.$$

From Lemma 5.2, we know that $\zeta(t) \to 0$ as $t \to \infty$. Since

$$\zeta(t) = \frac{MK}{\Gamma(\alpha)}t^{\alpha+\beta}\int_0^1 (1-s)^{\alpha-1}s^{\beta}e^{-(\lambda-\delta)ts}ds \to 0, \quad \text{as } t \to 0,$$

similar to the proofs of Lemmas 3.1 and 3.2, we get that $\{y :\ y(t) = e^{-t}(Fx)(t),\ x \in \Omega_r\}$ is equicontinuous and $\lim_{t\to\infty} e^{-t}|(Fx)(t)| = 0$ uniformly for $x \in \Omega_r$. Moreover $F$ maps $\Omega_r$ into itself and $F$ is continuous. Then, similar to the arguments used in the proof of Theorem 4.2, there exists at least one mild solution $x$ so that $x = Fx$ of (1.1). By Lemma 5.2, we have

$$\begin{aligned}
|x(t)| \leq & ME_{\alpha}(-\omega t^{\alpha})|x_0| + MK\int_0^t (t-s)^{\alpha-1}e_{\alpha}(-\omega(t-s)^{\alpha})s^{\beta}e^{-\lambda s}|x(s)|^{\delta}ds \\
\leq & ME_{\alpha}(-\omega t^{\alpha})|x_0| + \frac{MKr_1^{\delta}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\beta}e^{-(\lambda-\delta)s}ds \\
\leq & ME_{\alpha}(-\omega t^{\alpha})|x_0| + \frac{MK\kappa_2 r_1^{\delta}}{\Gamma(\alpha)}t^{\alpha-1} \\
\to & 0, \quad \text{as } t \to \infty,
\end{aligned}$$

which shows that the solution $x(t)$ is attractive. $\square$

**Remark 5.1.** *In particular, if $K = 0$ in (H5), then the problem (1.1) is the linear Cauchy problem, and it follows from Theorem 5.1 that (1.1) has at least one mild solution which is attractive.*

Similar to Theorems 4.1 and 5.1, we have the following result.

**Theorem 5.2.** *Assume that* (H0)-(H1) *and* (H4)-(H5) *hold. Then the problem (1.1) admits at least one attractive mild solution. Furthermore, if (H3) holds, then the attractive solution is unique.*

**Example 5.1.** *We can describe the a population of cells, which are distinguished by their individual size, at time $t$ by the number $w(t,s)$ of cells having size $s$ by the following fractional evolution equation (see [9, p.349] and references therein)*

$$\begin{aligned}\partial_t^\alpha w(t,s) =& -\partial_s w(t,s) - \mu(s)w(t,s) - \nu(s)w(t,s)\\ &+ \begin{cases} 4\nu(2s)w(t,2s), & \text{for } \dfrac{\beta}{2} \le s \le \dfrac{1}{2};\\ 0, & \text{for } \dfrac{1}{2} < s \le 1,\end{cases}\end{aligned} \tag{5.1}$$

*with the boundary condition*

$$w\left(t, \frac{\beta}{2}\right) = 0, \quad \text{for } t \ge 0,$$

*and the initial condition*

$$w(0,s) = w_0(s), \quad \text{for } \frac{\beta}{2} \le s \le 1,$$

*where $\partial_t^\alpha$ is Caputo derivative of order $\alpha \in (0,1)$, $\beta > 0$ denotes the minimal cell size, $\mu$ is a positive continuous function on $[\frac{\beta}{2}, 1]$ which is the death rate, and $\nu$ is the division which should be continuous with*

$$\nu(s) > 0, \quad \text{for } s \in (\beta, 1), \quad \text{and } \nu(s) = 0, \text{ otherwise.}$$

*As a natural Banach space we choose $X := L^1[\frac{\beta}{2}, 1]$, in which the norm $f$ of a positive function is the size of the total cell population represented by $f$.*

Next, we define the operators

$$A_0 f := -f - (\mu + \nu)f, \quad \text{with } D(A_0) := \left\{ f \in W^{1,1}\left[\frac{\beta}{2}, 1\right] : \ f\left(\frac{\beta}{2}\right) = 0 \right\},$$

and

$$Bf(s) := \begin{cases} 4\nu(2s)w(t,2s), & \text{for } \dfrac{\beta}{2} \le s \le \dfrac{1}{2};\\ 0, & \text{for } \dfrac{1}{2} < s \le 1.\end{cases}$$

Set $A := A_0 + B$ with $D(A) := D(A_0)$. Based on these notations, the equation (5.1) reduces to an abstract Cauchy problem

$$\begin{cases} {}^C D_t^{\alpha} u(t) = A_0 u(t) + Bu(t), \quad \text{for } t \geq 0, \\ u(0) = w_0, \end{cases} \tag{5.2}$$

for the vector-valued function $u : J \to X$. By [9, Chapter VI, Proposition 1.3], the operator $(A, D(A))$ generates a strongly continuous semigroup $\{Q(t)\}_{t\geq 0}$ on $X$, and by [9, Chapter VI, Proposition 1.4], the semigroup $\{Q(t)\}_{t\geq 0}$ is eventually continuous and even eventually compact for $t > 1 - \frac{\beta}{2}$. Moreover, by [9, Chapter VI, Corollary 1.17], the semigroup $\{Q(t)\}_{t\geq 0}$ is positive on the Banach lattice $X$ and uniformly exponentially stable if and only if

$$\xi(0) := -1 + \int_{\frac{\beta}{2}}^{\frac{1}{2}} 4\nu(2s) e^{-\int_{\sigma}^{2\sigma}(\mu(\tau)+\nu(\tau))d\tau} d\sigma < 0,$$

Then, from Remark 5.1, we know that there exists one attractive mild solution of (5.1).

**Example 5.2.** *Let $X$ be a Banach space. Then we consider the following fractional Cauchy problem*

$$\begin{cases} {}^C D_t^{1/2} x(t) = Ax(t) + Le^{-t}\sin(x(t)), \quad \text{for } t \in [0, \infty), \\ x(0) = x_0, \end{cases} \tag{5.3}$$

*where $A$ generates an exponentially stable semigroup.*

Obviously, let $f(t, x) = Le^{-t}\sin(x)$ for any $x \in X$, then for any bounded set $D \subset X$, (H3)-(H4) hold. In fact, for any $x, y \in \Omega$,

$$\begin{aligned} |f(t, x(t)) - f(t, y(t))| =& |Le^{-t}\sin(x(t)) - Le^{-t}\sin(y(t))| \\ \leq& L|x(t) - y(t)|, \quad t \in J. \end{aligned}$$

According to the definition of Kuratowski measure of noncompactness, we obtain (H4) for each $x \in D$. Then, choosing $L \in [0, 1/4)$, we deduce from Theorem 5.2 that there exists at least one mild solution of (5.1) which is attractive.

**Example 5.3.** *Let we consider the fractional differential equation*

$$\begin{cases} {}^C D_t^{\alpha} x(t) + \lambda x(t) = Ke^{-t}, \quad \text{for } t \in [0, \infty), \\ x(0) = x_0. \end{cases} \tag{5.4}$$

*where $\lambda$ is a real number with $\lambda > 0$, $0 \leq K < \Gamma(\alpha)/\kappa_1$.*

It is clear that $\lambda$ generates an exponentially stable $C_0$-semigroup $Q(t) = e^{-\lambda t}$, for $t \geq 0$. Let $f(t,x) = Ke^{-t}$. Then $f$ satisfies (H1), (H5) and the unique solution of (5.4) is in $C([0,\infty),\mathbb{R})$ which is given by

$$x(t) = E_{\alpha,1}(-\lambda t^{\alpha})x_0 + \int_0^t (t-s)^{\alpha-1} e_{\alpha}(-\lambda(t-s)^{\alpha}) f(s)ds.$$

Obviously, the solution is attractive. In particular, we observe that the solution possesses the same attractive behavior as $\alpha \to 1$ for the case of first order differential equation.

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