Discontinuous Sturm-Liouville Problem with Eigenparameter-Dependent Boundary conditions and Herglotzs transmission

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Abstract. This paper is to study a Sturm-Liouville equation

$$Ly := -p(x)y'' + q(x)y = \lambda y$$

with discontinuities in the case that eigenparameter appears not only in the differential equation but also appears in both the boundary conditions

$$\lambda(\alpha_1' y(-a) - \alpha_2' y'(-a)) - (\alpha_1 y(-a) - \alpha_2 y'(-a)) = 0,$$
$$\lambda(\beta_1' y(b) - \beta_2' y'(b)) + (\beta_1 y(b) - \beta_2 y'(b)) = 0$$

and transmission conditions as

$$-y(0^+)\left(\lambda\eta - \xi - \sum_{i=1}^{N} \frac{b_i^2}{\lambda - c_i}\right) = y'(0^+) - y'(0^-),$$
$$y'(0^-)\left(\lambda\kappa + \zeta - \sum_{j=1}^{M} \frac{a_j^2}{\lambda - d_j}\right) = y(0^+) - y(0^-).$$

In particular, in the space $L^2([-a,b]) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^{N'} \oplus \mathbb{C}^{M'}$, the considered problem can be interpreted as the eigenvalue problem of self-adjoint operator A. Moreover, we construct the Green's function of the considered problem and resolvent operator of A.

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1. Introduction

In the present work, we shall investigate the Sturm-Liouville equation

$$Ly := -p(x)y'' + q(x)y = \lambda y, \ x \in J, \tag{1.1}$$

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where $J = [-a, 0) \cup (0, b]$, subject to the eigenparameter-dependent boundary conditions

$$L_1 y := \lambda(\alpha_1' y(-a) - \alpha_2' y'(-a)) - (\alpha_1 y(-a) - \alpha_2 y'(-a)) = 0, \tag{1.2}$$

$$L_2 y := \lambda(\beta_1' y(b) - \beta_2' y'(b)) + (\beta_1 y(b) - \beta_2 y'(b)) = 0 \tag{1.3}$$

and transmission conditions at the discontinuous point

$$y(0^+)\mu(\lambda) = \Delta' y, \tag{1.4}$$

$$y'(0^-)\nu(\lambda) = \Delta y. \tag{1.5}$$

Here, $p(x) = 1/p_1^2$ for $x \in [-a, 0)$ and $p(x) = 1/p_2^2$ for $x \in (0, b]$; $q(x)$ is real-valued continuous in $[-a, 0) \cup (0, b]$; p_i, α_i, β_i, α_i' and β_i' $(i = 1, 2)$ are nonzero real numbers; a_j, $b_i > 0$ for $i = 1, \cdots, N$ and $j = 1, \cdots, M$, $c_1 < c_2 < \cdots < c_N$, $d_1 < d_2 < \cdots < d_M$. We take η, $\kappa \geq 0$, ξ, $\zeta \in \mathbb{R}$ and N, $M \in \mathbb{N}_0$. $\Delta y = y(0^+) - y(0^-)$, $\Delta' y = y'(0^+) - y'(0^-)$,

$$\mu(\lambda) = -\lambda\eta + \xi + \sum_{i=1}^{N} \frac{b_i^2}{\lambda - c_i}, \ \nu(\lambda) = \lambda\kappa + \zeta - \sum_{j=1}^{M} \frac{a_j^2}{\lambda - d_j}.$$

Furthermore, we reconsider

$$\frac{1}{\mu(\lambda)} = \sigma - \sum_{i=1}^{N'} \frac{\varepsilon_i^2}{\lambda - \gamma_i}, \ \frac{1}{\nu(\lambda)} = \tau + \sum_{j=1}^{M'} \frac{\epsilon_j^2}{\lambda - \delta_j}$$

with σ, $\tau \in \mathbb{R}$ and ε_i, $\epsilon_j > 0$, for $i = 1, \cdots, N'$, $j = 1, \cdots, M'$; $\gamma_1 < \gamma_2 < \cdots < \gamma_{N'}$, $\delta_1 < \delta_2 < \cdots < \delta_{M'}$.

The investigation of the eigenvalue and eigenfunction of Sturm-Liouville problem is required in many mathematical physics problems. When the eigenparameter appears not only in the differential equation but also in the boundary conditions, highly excellent results have been obtained (see [3,4,11–14,24, 26]), such as, the operator-theoretic formulations (see [7, 8]), the expansion theorems and dependence of the eigenvalues branches in Hilbert space. The substantial investigation of the spectral theory for Sturm-Liouville problems of ordinary differential equations with λ-dependent boundary conditions

$$(a_k\lambda + b_k)y(0) = (c_k\lambda + d_k)(py)'(0), \quad (-1)^k(a_k d_k - b_k c_k) \leq 0, k = 0, 1$$

was undertaken by Binding in [4]. It can be seen that Pruüfer transformation is an effective method together with simple geometrical arguments in this article and yields a comprehensive Sturm theory for variable end condition problems with $(-1)^k(a_k d_k - b_k c_k) \leq 0$. Meanwhile, Since the Pruüfer transformation does not work well for the version of discrete, Gao et al. overcame the lack of the Pruüfer transformation by introducing two new functions and obtained the existence of the eigenvalues, the sign-changing times of the eigenfunctions and the interlacing results of the eigenvalues in [12, 13]. It is noteworthy that these works mainly focus on some properties of eigenvalues and eigenfunctions. Besides, no matter the continuous cases or discrete cases, Sturm-Liouville eigenvalue problem is not equivalent to a linear operator spectral problem. Here, our goal is to make some discontinuity restrictions for the equation and boundary conditions (Herglotzs transmission) in the continuous cases. To deal with the discontinuities, some conditions are necessary, such as point interactions, impulsive conditions, transmission conditions, jump conditions or interface conditions. The other is by reducing the eigenvalue problem to a spectral problem of linear operator in the Hilbert space to establish some properties of linear operators.

While direct and inverse problem for Sturm-Liouville equation with the eigenparameter-dependent boundary conditions have been extensively studied, very little is known about operator spectral theory associated with transmission conditions dependent on eigenparmeter, which arise in many physical problems, such as, heat and mass transfer problem (see [16]), in vibrating problem (see [21]) and diffraction problem (see [2]). However, the presence of discontinuities produces essential qualitative modifications in the investigation of the operators.

Recall the related problems mentioned in the previous literature, which have obtained general spectral theory and methods of boundary value problems with the eigenparameter in the boundary conditions. But the considered problems in these works only dealt with the continuous coefficient ($p(x)$) or the discontinuity of boundary conditions at a certain point. Recently, similar problems for differential equation with constant coefficients ($p(x) \equiv 1$) were investigated in [5] and [6]. Meanwhile, in [3], the author considered the Sturm-Liouville equation, for which the coefficient of the highest derivative may exists discontinuity and two transmission conditions are given in this discontinuous point.

As a result of this, we are interested in two types of generalizations of classical Sturm-Liouville problems. Firstly, we considered more general Sturm-Liouville equation in which the coefficient of the highest derivative may have the discontinuity at one point of the considered interval. Moreover, we allow boundary conditions and transmission conditions (Nevanlinna-Herglotz functions) dependent on the eigenparameter. In section 2, the operator formulation is established and it is possible to interpret the problem (1.1)-(1.5) as the eigenvalue problem for a self-adjoint operator. The fundamental solutions and characteristic determinant are given in Section 3. Based on the operator formulation in the Hilbert space, the resolvent operator and self-adjointness of A is constructed in last section.

Remark 1.1. (i) if λ is a pole of $\mu(\lambda)$, then (1.4) becomes $y(0^+) = 0$ and (1.5) becomes $y(0^-)\nu(\lambda) = -y(0^-)$, resulting in two separate eigenvalues problem on the intervals $(-a, 0)$ and $(0, b)$;

(ii) if λ is a zero of $\mu(\lambda)$, then (1.4) becomes $\Delta' y = 0$;

(iii) if λ is a pole of $\nu(\lambda)$, then (1.5) becomes $y'(0^-) = 0$ and (1.4) becomes $y(0^+)\mu(\lambda) = y'(0^+)$, again resulting in two separate eigenvalues problem on the intervals $(-a, 0)$ and $(0, b)$;

(iv) if λ is a zero of $\nu(\lambda)$, then (1.4) becomes $\Delta y = 0$.

2. An Operator Formulation in the Hilbert Space

In this section, we introduce a special inner product in the Hilbert space

$$\mathcal{H} = L^2(-a,b) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^{N'} \oplus \mathbb{C}^{M'}$$

and a symmetric linear operator A defined on $\mathcal{H}$. Moreover, the problem (1.1)-(1.5) can be considered as the eigenvalue problem of the operator A.

Define

$$\rho_1 := \begin{vmatrix} \alpha_1' & \alpha_1 \\ \alpha_2' & \alpha_2 \end{vmatrix} > 0, \ \rho_2 := \begin{vmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{vmatrix} > 0.$$

For a short exposition, we use the following notations:

$$\begin{gathered} \mathbf{f}^1 := f_i^1, \ \mathbf{f}^2 := f_j^2; \\ f_1 := \alpha_1' f(-a) - \alpha_2' f'(-a), \ f_2 := \beta_1' f(b) - \beta_2' f'(b), \\ f^1 := \alpha_1 f(-a) - \alpha_2 f'(-a), \ f^2 := \beta_1 f(b) - \beta_2 f'(b). \end{gathered}$$

For η, $\kappa > 0$, we introduce a new inner product in $\mathcal{H}$ by

$$\begin{aligned} \langle F, G \rangle := & p_1^2 \int_{-a}^{0} f(x)\bar{g}(x)dx + p_2^2 \int_0^b f(x)\bar{g}(x)dx \\ & + \frac{1}{\rho_1} f_1 \bar{g}_1 + \frac{1}{\rho_2} f_2 \bar{g}_2 + \langle \mathbf{f}^1, \mathbf{g}^1 \rangle_1 + \langle \mathbf{f}^2, \mathbf{g}^2 \rangle_1 \end{aligned} \tag{2.1}$$

for

$$F := (f, f_1, f_2, \mathbf{f}^1, \mathbf{f}^2)^T, \ G := (g, g_1, g_2, \mathbf{g}^1, \mathbf{g}^2)^T \in \mathcal{H},$$

where $\langle \cdot, \cdot \rangle_1$ denotes Euclidean inner product.

In the Hilbert space $\mathcal{H}$, we consider the operator A which is defined by

$$A \begin{pmatrix} f \\ f_1 \\ f_2 \\ \mathbf{f}^1 \\ \mathbf{f}^2 \end{pmatrix} = \begin{pmatrix} Lf \\ f^1 \\ -f^2 \\ \boldsymbol{\varepsilon}\Delta' f + [\gamma_i]\mathbf{f}^1 \\ \boldsymbol{\epsilon}\Delta f + [\delta_j]\mathbf{f}^2 \end{pmatrix} = \begin{pmatrix} Lf \\ \alpha_1 f(-a) - \alpha_2 f'(-a) \\ -(\beta_1 f(b) - \beta_2 f'(b)) \\ \boldsymbol{\varepsilon}\Delta' f + [\gamma_i]\mathbf{f}^1 \\ \boldsymbol{\epsilon}\Delta f + [\delta_j]\mathbf{f}^2 \end{pmatrix}$$

with the domain

$$\begin{aligned} D(A) = \big\{ F = & (f, f_1, f_2, \mathbf{f}^1, \mathbf{f}^2)^T : f \in AC[-a,b], f' \in AC[-a,0) \cup (0,b], \\ Lf \in L^2(-a,b), & - f(0^+) + \sigma\Delta' f - \langle \mathbf{f}^1, \boldsymbol{\varepsilon} \rangle_1 = 0, \ f'(0^-) - \tau\Delta f - \langle \mathbf{f}^2, \boldsymbol{\epsilon} \rangle_1 = 0 \big\}, \end{aligned}$$

where $[\gamma_i] := \operatorname{diag}(\gamma_1, \cdots, \gamma_{N'})$, $[\delta_j] := \operatorname{diag}(\delta_1, \cdots, \gamma_{M'})$, $\boldsymbol{\varepsilon} := (\varepsilon_i)$ and $\boldsymbol{\epsilon} := (\epsilon_j)$.

In the above results, the case of $\eta = 0$ can be obtained by replacing $\boldsymbol{\varepsilon}$, $\boldsymbol{\gamma}$, σ, $f(0^+)$, $\Delta' f$ and N' with $\boldsymbol{b}$, $\boldsymbol{c}$, $-\xi$, $-\Delta' f$, $f(0^+)$ and N, while replacement of $\boldsymbol{\epsilon}$, $\boldsymbol{\delta}$, τ, $f(0^-)$, Δf and M' by $\boldsymbol{a}$, $\boldsymbol{d}$, $-\zeta$, $-\Delta f$, $f'(0^-)$ and N yields the case of $\kappa = 0$.

Lemma 2.1. *The domain $D(A)$ is dense in $\mathcal{H}$.*

Proof. We only give the proof of the case of $\eta, \kappa > 0$, the other cases are similar.

Let $W = (w, f_1, f_2, \mathbf{f}^1, \mathbf{f}^2)^T \in \mathcal{H}$, where $w \in C^\infty[-a, 0) \cup (0, b]$ with $w(-a) = w'(-a) = w(b) = w'(b) = 0$ and satisfies the condition

$$w(0^-) = \sigma\langle \mathbf{f}^1, \boldsymbol{\varepsilon}\rangle_1 + (1-\sigma)\langle \mathbf{f}^2, \boldsymbol{\epsilon}\rangle_1,\ w(0^+) = (\sigma - 1)\langle \mathbf{f}^1, \boldsymbol{\varepsilon}\rangle_1 - \sigma\langle \mathbf{f}^2, \boldsymbol{\epsilon}\rangle_1,$$
$$w'(0^-) = -\tau\langle \mathbf{f}^1, \boldsymbol{\varepsilon}\rangle_1 + (\tau+1)\langle \mathbf{f}^2, \boldsymbol{\epsilon}\rangle_1,\ w'(0^+) = (1-\tau)\langle \mathbf{f}^1, \boldsymbol{\varepsilon}\rangle_1 + \tau\langle \mathbf{f}^2, \boldsymbol{\epsilon}\rangle_1.$$

Meanwhile,

$$\Delta w = \langle \mathbf{f}^2, \boldsymbol{\epsilon}\rangle_1 - \langle \mathbf{f}^1, \boldsymbol{\varepsilon}\rangle_1,\ \Delta' w = \langle \mathbf{f}^1, \boldsymbol{\varepsilon}\rangle_{1-}\langle \mathbf{f}^2, \boldsymbol{\epsilon}\rangle_1.$$

Then, it is easy to verify that $W \in D(A)$. Next, as long as it is proved that the elements in $\mathcal{H}$ can be approximated by the elements in $D(A)$, the desired result can be obtained.

Since

$$(C_0^\infty(-a,0) \oplus C_0^\infty(0,b)) \oplus \{0\} \oplus \{0\} \oplus \{\mathbf{0}\} \oplus \{\mathbf{0}\} \subseteq D(A)$$

and

$$\overline{(C_0^\infty(-a,0) \oplus C_0^\infty(0,b))} \supset L^2(-a,b),$$

then there exists a sequence $\{m_n\} \in C_0^\infty(-a,0) \oplus C_0^\infty(0,b)$ with $m_n \to f - w$ as $n \to \infty$, where $M_n := (m_n, 0, 0, \mathbf{0}, \mathbf{0})^T \in D(A)$. Therefore, $W + M_n \to F$ as $n \to \infty$ giving that $\overline{D(A)} \supset \mathcal{H}$. The proof the Lemma is complete. □

Theorem 2.2. *The operator eigenvalue problem $AF = \lambda F$ and the considered problem (1.1)-(1.5) is equivalent in the sense of that λ is an eigenvalue of $AF = \lambda F$ and the eigenfunction is the first components of the corresponding eigenelements of the operator A. Moreover, for η, $\kappa > 0$, we have following results:*

(i) if $\lambda \neq \gamma_i$ for all $i = 1, 2, \cdots, N'$, then $\mathbf{f}^1 = (\lambda I - [\gamma_i])^{-1}\boldsymbol{\varepsilon}\Delta' f$, while if $\lambda = \gamma_I$ for some $I \in \{1, \cdots, N'\}$, then $\mathbf{f}^1 = \frac{-f(0^+)}{\varepsilon_I} e^I$;
(ii) if $\lambda \neq \delta_j$ for all $j = 1, 2, \cdots, M'$, then $\mathbf{f}^2 = (\lambda I - [\delta_i])^{-1}\boldsymbol{\epsilon}\Delta f$, while if $\lambda = \gamma_J$ for some $J \in \{1, \cdots, M'\}$, then $\mathbf{f}^2 = \frac{f(0^-)}{\epsilon_J} e^J$, where $\mathbf{e}^n$ is the vector in $\mathbb{R}^n$ with all entries 0 except the n-th which is 1.

Proof. We just need to show that eigenelement y of the operator A obeys the boundary conditions (1.2)-(1.3) and transfer conditions (1.4)-(1.5). It is clear that y satisfies (1.2)-(1.3). The definition of A implies $\gamma_i f_i^1 + \varepsilon_i \Delta' f = \lambda f_i^1$ for all i. Meanwhile, the domain of A gives $-f(0^+) + \sigma\Delta' f - \langle \mathbf{f}^1, \boldsymbol{\varepsilon}\rangle_1 = 0$. Thus, if $\lambda \neq \gamma_i$ for all i, then

$$f(0^+) = \left(\sigma - \sum_{i=1}^{N'} \frac{\varepsilon_i^2}{\lambda - \gamma_i}\right)\Delta' f.$$

If $\lambda = \gamma_I$ for some $I \in \{1, \cdots, N'\}$, then $-f(0^+) - \langle f_I^1, \varepsilon_I\rangle_1 = 0$. That is, $f_I^1 = \frac{-f(0^+)}{\varepsilon_I}$. Hence, y satisfies (1.4).

Similarly, if $\lambda \neq \delta_j$ for all j, then $f_j^2 = \frac{\epsilon_j}{\lambda-\delta_j}\Delta f$ and

$$f'(0^-) = \left(\tau + \sum_{j=1}^{M'} \frac{\epsilon_j^2}{\lambda - \delta_j}\right)\Delta f.$$

while $\lambda = \delta_J$ for some $j \in \{1, \cdots, M'\}$ the domain condition forces $f_J^2 = \frac{f'(0^-)}{\epsilon_J}$ from which (1.5) follows. The proof of the Theorem is complete. □

Theorem 2.3. *The linear operator A is symmetric.*

Proof. Let F, $G \in D(A)$. Then it follows from the equation (1.1) and the relation (2.1) that

$$\begin{aligned}
&\langle AF, G\rangle - \langle F, AG\rangle \\
=&(f\bar{g}')(0^-) - (f'\bar{g})(0^-) + (f'\bar{g})(0^+) - (f\bar{g}')(0^+) \\
+&(f'\bar{g})(-a) - (f\bar{g}')(-a) + (f\bar{g}')(b) - (f'\bar{g})(b) \\
+&\frac{1}{\rho_1} f^1\bar{g}_1 - \frac{1}{\rho_2} f^2\bar{g}_2 + \langle \boldsymbol{\varepsilon}\Delta' f + [\gamma_i]\mathbf{f}^1, \mathbf{g}^1\rangle_1 + \langle \boldsymbol{\epsilon}\Delta f + [\delta_j]\mathbf{f}^2, \mathbf{g}^2\rangle_1 \\
-&\frac{1}{\rho_1} f_1\bar{g}^1 + \frac{1}{\rho_2} f_2\bar{g}^2 - \langle \mathbf{f}^1, \boldsymbol{\varepsilon}\Delta' g + [\gamma_i]\mathbf{g}^1\rangle_1 - \langle \mathbf{f}^2, \boldsymbol{\epsilon}\Delta g + [\delta_j]\mathbf{g}^2\rangle_1.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\rho_1^{-1}(f^1\bar{g}_1 - f_1\bar{g}^1) &= (f\bar{g}')(-a) - (\bar{g}f')(-a), \\
\rho_2^{-1}(f^2\bar{g}_2 - f_2\bar{g}^2) &= -((f\bar{g}')(b) - (\bar{g}f')(b)).
\end{aligned}$$

Meanwhile, the vector components satisfy

$$\begin{aligned}
\langle \boldsymbol{\varepsilon}\Delta' f + [\gamma_i]\mathbf{f}^1, \mathbf{g}^1\rangle_1 - \langle \mathbf{f}^1, \boldsymbol{\varepsilon}\Delta' g + [\gamma_i]\mathbf{g}^1\rangle_1 &= \langle \boldsymbol{\varepsilon}\Delta' f, \mathbf{g}^1\rangle_1 - \langle \mathbf{f}^1, \boldsymbol{\varepsilon}\Delta' g\rangle_1, \\
\langle \boldsymbol{\epsilon}\Delta f + [\delta_j]\mathbf{f}^2, \mathbf{g}^2\rangle_1 - \langle \mathbf{f}^2, \boldsymbol{\epsilon}\Delta g + [\delta_j]\mathbf{g}^2\rangle_1 &= \langle \boldsymbol{\epsilon}\Delta f, \mathbf{g}^2\rangle_1 - \langle \mathbf{f}^2, \boldsymbol{\epsilon}\Delta g\rangle_1,
\end{aligned}$$

and the domain conditions $D(A)$ implies

$$\begin{aligned}
\langle \boldsymbol{\varepsilon}\Delta' f, \mathbf{g}^1\rangle_1 - \langle \mathbf{f}^1, \boldsymbol{\varepsilon}\Delta' g\rangle_1 &= \Delta' f[-\bar{g}(0^+) + \sigma\Delta'\bar{g}] - \Delta'\bar{g}[-f(0^+) + \sigma\Delta' f], \\
\langle \boldsymbol{\epsilon}\Delta f, \mathbf{g}^2\rangle_1 - \langle \mathbf{f}^2, \boldsymbol{\epsilon}\Delta g\rangle_1 &= \Delta f[\bar{g}'(0^-) - \tau\Delta\bar{g}] - \Delta\bar{g}[f'(0^-) - \tau\Delta f].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle \boldsymbol{\varepsilon}\Delta' f + [\gamma_i]\mathbf{f}^1, \mathbf{g}^1\rangle_1 - \langle \mathbf{f}^1, \boldsymbol{\varepsilon}\Delta' g + [\gamma_i]\mathbf{g}^1\rangle_1 &= \Delta'\bar{g}f(0^+) - \Delta' f\bar{g}(0^+), \\
\langle \boldsymbol{\epsilon}\Delta f + [\delta_j]\mathbf{f}^2, \mathbf{g}^2\rangle_1 - \langle \mathbf{f}^2, \boldsymbol{\epsilon}\Delta g + [\delta_j]\mathbf{g}^2\rangle_1 &= \Delta f\bar{g}'(0^-) - \Delta\bar{g}f'(0^-).
\end{aligned}$$

Direct computation yields

$$\begin{aligned}
&-[(f'\bar{g} - f\bar{g}')(0^-) - (f'\bar{g} - f\bar{g}')(0^+)] \\
=&-f(0^+)\Delta'\bar{g} + \bar{g}(0^+)\Delta' f - \bar{g}'(0^-)\Delta f + f'(0^-)\Delta\bar{g}.
\end{aligned}$$

Thus, $\langle AF, G\rangle - \langle F, AG\rangle = 0$ and so A is symmetric. The proof the Theorem is complete. □

Corollary 2.4. *All eigenvalues of the Sturm-Liouville problem (1.1)-(1.5) are real.*

Corollary 2.5. *Let λ_1 and λ_2 be two different eigenvalues of the problem (1.1)-(1.5). Then the corresponding the eigenfunctions $u_1(x)$ and $u_2(x)$ are orthogonal in the sense of*

$$p_1^2\int_{-a}^{0} u_1(x)u_2(x)dx + p_2^2\int_{0}^{b} f_1(x)f_2(x)dx + \frac{1}{\rho_1}f_1(u_1)f_1(u_2)$$
$$+\frac{1}{\rho_2}f_2(u_1)f_2(u_2) + \langle \mathbf{f}^1(u_1), \mathbf{g}^1(u_2)\rangle_1 + \langle \mathbf{f}^2(u_1), \mathbf{g}^2(u_2)\rangle_1 = 0.$$

3. Fundamental Solutions and Characteristic Determinant

Lemma 3.1. ([5]) *All eigenvalues of (1.1)-(1.5) not at poles of $\mu(\lambda)$ or $\nu(\lambda)$ are geometrically simple. In the case, the transmission conditions (1.4)-(1.5) can be expressed as*

$$\begin{pmatrix} y(0^+) \\ y'(0^+) \end{pmatrix} = \begin{pmatrix} 1 & \nu(\lambda) \\ \mu(\lambda) & 1+\mu(\lambda)\nu(\lambda) \end{pmatrix} \begin{pmatrix} y(0^-) \\ y'(0^-) \end{pmatrix}.$$

Lemma 3.2. ([20]) *Let the real valued $q(x)$ be continuous in $[-a,b]$, and $f(\lambda)$ and $g(\lambda)$ are given entire functions. Then for any $\lambda\in\mathbb{C}$, the equation*

$$-p(x)u'' + q(x)u = \lambda u, \quad x\in[-a,b]$$

has a unique solution $u = u(x,\lambda)$ satisfying the initial conditions

$$u(a) = f(\lambda),\ u'(a) = g(\lambda)\ (or\ u(b) = f(\lambda),\ u'(b) = g(\lambda)).$$

For each fixed $x\in[-a,b]$, $u = u(x,\lambda)$ is an entire function of λ.

Lemma 3.3. *Let $u_-(x,\lambda)$ be the solution of the equation (1.1) in the interval $[-a,0)$ satisfying the initial conditions*

$$u_-(-a) = -\alpha_2 + \lambda\alpha_2', \quad u_-'(-a) = -\alpha_1 + \lambda\alpha_1' \tag{3.1}$$

and $v_(x,\lambda)$ denote the solution of the equation (1.1) in the interval $(0,b]$ satisfying the terminal conditions

$$v_+(b) = -\beta_2 + \lambda\beta_2', \quad v_+'(b) = -\beta_1 + \lambda\beta_1'. \tag{3.2}$$

Then $W[u_-,v_+]$ is independent of x. Moreover, it is a function of λ.

Proof. Direct computation, we have

$$\frac{\partial}{\partial x}W[u_-(x,\lambda),v_+(x,\lambda)] = u_-(x,\lambda)\frac{\partial^2}{\partial x^2}v_+(x,\lambda) - v_+(x,\lambda)\frac{\partial^2}{\partial x^2}u_-(x,\lambda)$$
$$= \frac{qv_+ - \lambda v_+}{p}u_- - \frac{qu_- - \lambda u_-}{p}v_+ = 0.$$

It follows that $W[u_-(x,\lambda),v_+(x,\lambda)]$ is constant on $[-a,0)\cup(0,b]$ and by virtue of Lemma 3.2, it is a function of λ. The proof of the Lemma is complete. □

Theorem 3.4. *The Sturm-Liouville equation (1.1) exists two fundamental solutions on whole $[-a,0)\cup(0,b]$ satisfying the boundary conditions (1.2)-(1.3) and transfer conditions (1.4)-(1.5). Moreover, the eigenvalue λ of the problem (1.1)-(1.5) is consist of the zero of the characteristic determinant.*

Proof. First, we extend $u_-(x,\lambda)$ and $v_+(x,\lambda)$ by the zero function to $[-a,0)\cup(0,b]$, i.e. we define

$$\tilde{u}_-(x,\lambda)=\begin{cases}u_-(x,\lambda), & x\in[-a,0),\\ 0, & x\in(0,b]\end{cases}$$

and

$$\tilde{v}_+(x,\lambda)=\begin{cases}0, & x\in[-a,0),\\ v_+(x,\lambda), & x\in(0,b].\end{cases}$$

Furthermore, by Lemma 3.1, we know that the eigenvalue not coinciding with a pole of $\mu(\lambda)$ or $\nu(\lambda)$. It's possible to extend $u_-(x,\lambda)$, $x\in[-a,0)$ and $v_+(x,\lambda)$, $x\in(0,b]$ by nontrivial solution $u_+(x,\lambda)$, $x\in(0,b]$ and $v_-(x,\lambda)$, $x\in[-a,0)$ of the equation (1.1) obeying the conditions

$$\begin{pmatrix}u_+(0^+)\\ u'_+(0^+)\end{pmatrix}=\begin{pmatrix}1 & \nu(\lambda)\\ \mu(\lambda) & 1+\mu(\lambda)\nu(\lambda)\end{pmatrix}\begin{pmatrix}u_-(0^-)\\ u'_-(0^-)\end{pmatrix}$$

and

$$\begin{pmatrix}v_-(0^-)\\ v'_-(0^-)\end{pmatrix}=\begin{pmatrix}1+\mu(\lambda)\nu(\lambda) & -\nu(\lambda)\\ -\mu(\lambda) & 1\end{pmatrix}\begin{pmatrix}v_+(0^+)\\ v'_+(0^+)\end{pmatrix}.$$

Moreover, let us define two linearly independent fundamental solutions of the equation (1.1) on the whole $[-a,0)\cup(0,b]$ as

$$u(x,\lambda)=\begin{cases}u_-(x,\lambda), & x\in[-a,0),\\ u_+(x,\lambda), & x\in(0,b],\end{cases}\tag{3.3}$$

$$v(x,\lambda)=\begin{cases}v_-(x,\lambda), & x\in[-a,0),\\ v_+(x,\lambda), & x\in(0,b].\end{cases}\tag{3.4}$$

In fact, the initial value condition implies

$$\begin{cases}u_+(0^+)=u_-(0^-)+\nu(\lambda)u'_-(0^-),\\ u'_+(0^+)=\mu(\lambda)u_-(0^-)+(1+\mu(\lambda)\nu(\lambda))u'_-(0^-)\end{cases}$$

and

$$\begin{cases}v_-(0^-)=(1+\mu(\lambda)\nu(\lambda))v_+(0^+)-\nu(\lambda)v'_+(0^+),\\ v'_-(0^-)=-\mu(\lambda)v_+(0^+)+v'_+(0^+).\end{cases}$$

Then easy to verify $W[u_+,v_-]|_0\neq 0$.

It must note that $u(x,\lambda)$ and $v(x,\lambda)$ satisfy boundary conditions (1.2)-(1.3) and transmission conditions (1.4)-(1.5).

Let

$$y(x,\lambda)=\varphi(\lambda)u(x,\lambda)+\psi(\lambda)v(x,\lambda).\tag{3.5}$$

If $y(x,\lambda)$ satisfy transfer conditions (1.4)-(1.5), then

$$\begin{cases} U_1(y,\lambda) = -(y(0^+)(\lambda\eta-\xi)+\Delta' y)\prod_{i=1}^{N}(\lambda-c_i) \\ \qquad\qquad +y(0^+)\sum_{i=1}^{N} b_i^2 \prod_{k\neq i}(\lambda-c_k), \\ U_2(y,\lambda) = (y'(0^-)(\lambda\kappa+\zeta)+\Delta y)\prod_{j=1}^{M}(\lambda-d_i) \\ \qquad\qquad -y'(0^-)\sum_{j=1}^{M} a_j^2 \prod_{k\neq j}(\lambda-d_k) \end{cases} \tag{3.6}$$

and any solution to equation (1.1) on $[-a,0)\cup(0,b]$ satisfying the boundary conditions (1.2)-(1.3) must be of the form (3.5).

Next, we prove that the second part. The relation (3.6) implies

$$U_k(y,\lambda) = \varphi(\lambda)U_k(u,\lambda)+\psi(\lambda)U_k(v,\lambda), \quad k=1,2. \tag{3.7}$$

That is, λ is an eigenvalues of (1.1)-(1.5) with the eigenfunction y as defined in (3.5), if and only if $U_1(y,\lambda)=0$ and $U_2(y,\lambda)=0$. Moreover, these two equations exist nontrivial solution $\varphi(\lambda)$ and $\psi(\lambda)$ if and only if

$$\omega(\lambda) = \det\begin{pmatrix} U_1(u,\lambda) & U_1(v,\lambda) \\ U_2(u,\lambda) & U_2(v,\lambda) \end{pmatrix} = 0.$$

Therefore, it is shown that each eigenvalue is zero of the function $\omega(\lambda)$. The proof of the Theorem is complete. □

Let $W[u(x,\lambda),v(x,\lambda)] =: \varpi(\lambda)$. In view of Theorem 3.4, we have

$$\Phi(x,\lambda) := \frac{v(x,\lambda)}{\varpi(\lambda)}\int_{-a}^{x} u(t,\lambda)h(t)dt + \frac{u(x,\lambda)}{\varpi(\lambda)}\int_{x}^{b} v(t,\lambda)h(t)dt, \;\; h\in L^2(-a,b).$$

and the Green's function of the problem (1.1)-(1.5) is given by

$$G(x,t;\lambda) = \begin{cases} \dfrac{u(t,\lambda)v(x,\lambda)}{\varpi(\lambda)}, & t<x, \;\; t\in[-a,0)\cup(0,b], \\ \dfrac{v(t,\lambda)u(x,\lambda)}{\varpi(\lambda)}, & x<t, \;\; t\in[-a,0)\cup(0,b]. \end{cases}$$

Theorem 3.5. *Let*

$$g(x,\lambda) = \int_{-a}^{b} G(x,t;\lambda)h(t)dt := T_\lambda h. \tag{3.8}$$

Then $g(x,\lambda)$ is a solution of the equation $(\lambda-L)g = ph$ on $[-a,0)\cup(0,b]$. Moreover, g obeys the boundary conditions (1.2)-(1.3) and transfer conditions (1.4)-(1.5).

Proof. The relation (3.8) implies

$$g\varpi(\lambda) = v(x,\lambda)\int_{-a}^{x} u(t,\lambda)h(t)dt + u(x,\lambda)\int_{x}^{b} v(t,\lambda)h(t)dt. \tag{3.9}$$

Furthermore, we have

$$\frac{\partial}{\partial x}g\varpi(\lambda) = \frac{\partial}{\partial x}v(x,\lambda)\int_{-a}^{x} u(t,\lambda)h(t)dt + \frac{\partial}{\partial x}u(x,\lambda)\int_{x}^{b} v(t,\lambda)h(t)dt \tag{3.10}$$

and

$$\begin{aligned}
&p\frac{\partial^2}{\partial x^2}g\varpi(\lambda)\\
=&\frac{\partial^2}{\partial x^2}v(x,\lambda)\int_{-a}^{x}u(t,\lambda)h(t)dt+\frac{\partial^2}{\partial x^2}u(x,\lambda)\int_{x}^{b}v(t,\lambda)h(t)dt+ph(x)\varpi\\
=&(q-\lambda)g\varpi(\lambda)+ph\varpi(\lambda).
\end{aligned} \tag{3.11}$$

Therefore, $(\lambda-L)g=ph$ holds.

It remains only to show that g satisfies the (1.2)-(1.3) and (1.4)-(1.5). In view of (3.9) and (3.10), we have

$$g(-a)=\frac{u(-a,\lambda)}{\varpi(\lambda)}\int_{-a}^{b}v(t,\lambda)h(t)dt,\quad g'(-a)=\frac{u'(-a,\lambda)}{\varpi(\lambda)}\int_{-a}^{b}v(t,\lambda)h(t)dt,$$

By the Theorem 3.4, u obeys (1.2). Then, g satisfies (1.2). Similarly, g satisfies (1.3). Moreover,

$$\begin{pmatrix}g(0^{\pm})\\ g'(0^{\pm})\end{pmatrix}=\frac{1}{\varpi(\lambda)}\begin{pmatrix}v(0^{\pm})\\ v'(0^{\pm})\end{pmatrix}\int_{-a}^{0}u(t)h(t)dt+\frac{1}{\varpi(\lambda)}\begin{pmatrix}u(0^{\pm})\\ u'(0^{\pm})\end{pmatrix}\int_{0}^{b}v(t)h(t)dt.$$

Obviously, (1.4) and (1.5) are obeyed. The proof of the Theorem is complete. □

4. The Resolvent Operator of A

In this section, we apply the properties of operator A to study the resolvent operator in the Hilbert space $\mathcal{H}$. We first consider nonhomogeneous boundary conditions

$$-\boldsymbol{\varepsilon}\Delta' f+(\lambda I-[\gamma_i])\mathbf{f}^1=p\mathbf{h}^1, \tag{4.1}$$

$$-\boldsymbol{\epsilon}\Delta f+(\lambda I-[\delta_j])\mathbf{f}^2=p\mathbf{h}^2. \tag{4.2}$$

Meanwhile, the domain of the operator A implies

$$-f(0^+)+\sigma\Delta' f-\langle\mathbf{f}^1,\boldsymbol{\varepsilon}\rangle_1=0, \tag{4.3}$$

$$f'(0^-)-\tau\Delta f-\langle\mathbf{f}^2,\boldsymbol{\epsilon}\rangle_1=0. \tag{4.4}$$

If $\lambda\neq\gamma_i$ for all i, then from (4.1) we have

$$-f(0^+)+\sigma\Delta' f-\langle(\lambda I-[\gamma_i])^{-1}(p\mathbf{h}^1+\boldsymbol{\varepsilon}\Delta' f),\boldsymbol{\varepsilon}\rangle_1.$$

Using (1.4), we get

$$-f(0^+)+\frac{1}{\mu(\lambda)}\Delta' f=\langle p\mathbf{h}^1,(\lambda I-[\gamma_i])^{-1}\boldsymbol{\varepsilon}\rangle_1.$$

If $\lambda = \gamma_I$ for some $I \in \{1, \cdots, N'\}$, then from (4.1) we have $\Delta' f = -\frac{ph^1_I}{\varepsilon_I}$. For $i \in \{1, \cdots, N'\} \setminus I$, $f^1_i = \frac{h^1_i + \varepsilon_i \Delta' f}{\gamma_I - \gamma_i}$. Thus, from (4.3) we get

$$-f(0^+) - \sigma \frac{ph^1_I}{\varepsilon_I} - \sum_{i \neq I} \frac{\varepsilon_i}{\varepsilon_I} \frac{\varepsilon_I ph^1_i - \varepsilon_i ph^1_I}{\gamma_I - \gamma_i} = \varepsilon_I f^1_I.$$

Similarly, if $\lambda \neq \delta_j$ for all j, then

$$f'(0^-) - \frac{1}{\nu(\lambda)} \Delta f = \langle p\mathbf{h}^2, (\lambda I - [\delta_j])^{-1} \boldsymbol{\epsilon} \rangle_1.$$

If $\lambda = \delta_J$ for some $J \in \{1, \cdots, M'\}$, then the relation

$$f'(0^-) + \tau \frac{ph^2_J}{\epsilon_J} - \sum_{j \neq J} \frac{\epsilon_j}{\epsilon_J} \frac{\epsilon_J ph^2_j - \epsilon_j ph^2_J}{\delta_J - \delta_j} = \epsilon_J f^2_J.$$

holds.

Therefore, the operator equation $(\lambda I - A)Y = H$,

$$H = (ph, ph_1, ph_2, p\mathbf{h}^1, p\mathbf{h}^2)^T \in L((-a, b)) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^{N'} \oplus \mathbb{C}^{M'}$$

is equivalent to the discontinuous boundary value problem consisting of the differential equation

$$-p(x)y'' + q(x)y = \lambda y(x) - ph(x), \ x \in J,$$

together with eigenparameter-dependent boundary conditions

$$\lambda(\alpha'_1 y(-a) - \alpha'_2 y'(-a)) - (\alpha_1 y(-a) - \alpha_2 y'(-a)) = ph_1,$$
$$\lambda(\beta'_1 y(b) - \beta'_2 y'(b)) + (\beta_1 y(b) - \beta_2 y'(b)) = ph_2$$

and transfer conditions (the case of $\lambda \neq \gamma_i$ and $\lambda \neq \delta_j$)

$$y(0^+)\mu(\lambda) - \Delta' y = \langle p\mathbf{h}^1, (\lambda I - [\gamma_i])^{-1} \boldsymbol{\varepsilon} \rangle,$$
$$y'(0^+)\nu(\lambda) - \Delta y = \langle p\mathbf{h}^2, (\lambda I - [\delta_j])^{-1} \boldsymbol{\epsilon} \rangle.$$

We consider the resolvent set $\rho(A) = \{\lambda \in \mathbb{C} | (\lambda I - A)^{-1} \in D(A)\}$. Then, we need to show $(\lambda I - A)^{-1}$ is the resolvent operator, just prove $(\lambda I - A)^{-1} \in D(A)$.

Theorem 4.1. *Let λ be not an eigenvalue of operator A. Then*

$$(\lambda I - A)^{-1} H = \begin{pmatrix} T_\lambda h \\ (T_\lambda h)_1 \\ (T_\lambda h)_2 \\ (\lambda I - [\gamma_i])^{-1} \boldsymbol{\varepsilon} \Delta' T_\lambda h \\ (\lambda I - [\delta_j])^{-1} \boldsymbol{\epsilon} \Delta' T_\lambda h \end{pmatrix} =: \tilde{G}h.$$

Proof. We know from Corollary 2.4 that each $\lambda \in \mathbb{C}$ is not an eigenvalue of A. Thus, the resolvent operator $(\lambda I - A)^{-1}$ exists. It remains only to show $\tilde{G}h \in$

$D(A)$. The definition of $T_\lambda h$ implies that $g \in AC[-a,b]$, $g' \in AC[-a,0)\cup(0,b]$ and obey the boundary condition (1.2)-(1.3). Moreover, the equalities

$$g(0^+) = (T_\lambda h)(0^+) = \left(\sigma - \sum_{i=1}^{N'} \frac{\varepsilon_i^2}{\lambda - \gamma_i}\right)\Delta'(T_\lambda h) = \sigma\Delta' g - \langle \mathbf{g}^1, \boldsymbol{\varepsilon}\rangle_1, \quad \lambda \neq \gamma_i.$$

and

$$g'(0^-) = (T_\lambda h)'(0^-) = \left(\tau + \sum_{j=1}^{M'} \frac{\epsilon_j^2}{\lambda - \delta_j}\right)\Delta(T_\lambda h) = \tau\Delta' g - \langle \mathbf{g}^2, \boldsymbol{\epsilon}\rangle_1 \quad \lambda \neq \delta_j.$$

hold. Meanwhile, Remark 2.1 implies that if $\lambda = \gamma_I$ for some $I \in \{1, \cdots, N'\}$, then $(\Delta' T_\lambda h) = 0$ and $\mathbf{g}^1 = \frac{-y(0^+)}{\varepsilon_I} e^I$; Therefore,

$$g(0^+) = -\langle \mathbf{g}^1, \boldsymbol{\varepsilon}\rangle_1 = \sigma\Delta' g - \langle \mathbf{g}^1, \boldsymbol{\varepsilon}\rangle_1.$$

Similarly, we have

$$g'(0^-) = \langle \mathbf{g}^2, \boldsymbol{\epsilon}\rangle_1 = \tau\Delta g + \langle \mathbf{g}^1, \boldsymbol{\epsilon}\rangle_1.$$

Thus, $\tilde{G}h \in D(A)$ and the desired result holds. The proof of the Theorem is complete. □

Theorem 4.2. *Let $R(\lambda, A) = (\lambda I - A)^{-1}$. Then*

$$\|R(\lambda, A)H\| \leq |\mathrm{Im}\lambda|^{-1}\|H\|, \quad H \in \mathcal{H}$$

holds for all $\lambda \in \mathbb{C}$ such that $\mathrm{Im}\lambda \neq 0$.

Proof. Let $H = (ph, ph_1, ph_2, p\mathbf{h}_3, p\mathbf{h}_4)^T$ be any element of $\mathcal{H}$ and $Y = R(\lambda, A)H$. Since $(\lambda I - A)Y = H$, we have

$$\langle AY, Y\rangle = \langle \lambda Y - H, Y\rangle = \lambda\langle Y, Y\rangle - \langle H, Y\rangle$$

and

$$\langle Y, AY\rangle = \langle Y, \lambda Y - H\rangle = \bar{\lambda}\langle Y, Y\rangle - \overline{\langle H, Y\rangle},$$

which imply $|\mathrm{Im}\lambda|\|Y\|^2 = |\mathrm{Im}(H, Y)|$. On the other side, in view of Cauchy-Schwartz inequality, we have

$$|\mathrm{Im}(H, Y)| \leq |(H, Y)| \leq \|H\|\|Y\|.$$

Therefore, the inequality

$$\|R(\lambda, A)H\| = \|Y\| \leq |\mathrm{Im}\lambda|^{-1}\|H\|, \quad H \in \mathcal{H}$$

holds. The proof of the Theorem is complete. □

It should be mentioned that for $\lambda \in \mathbb{C}$ such that $\mathrm{Im} \neq 0$, the resolvent operator exists and for all $\lambda \in \rho(A)$ is regular point of A.

Theorem 4.3. *The operator A is self-adjoint in $\mathcal{H}$.*

Proof. We know from Lemma 2,1 and Theorem 2.2 that A is a densely symmetric operator in $\mathcal{H}$. To show that A is self-adjoint , it remains only to verify that $D(A^*) = D(A)$, where A^* is the adjoint of A.

Since $D(A) \subseteq D(A^*)$, we only show that $D(A^*) \subseteq D(A)$. Let $H \in D(A^*)$. Then

$$\langle AH, G\rangle = \langle H, A^*G\rangle \text{ for all } G \in D(A). \tag{4.5}$$

It follows from (4.5) that

$$\langle (iI - A)G, H\rangle = \langle G, (-iI - A^*)H\rangle. \tag{4.6}$$

Note that $\lambda = -i$ is a regular point. Then, we let

$$(iI - A)Y = -iH - A^*H, \quad Y \in D(A). \tag{4.7}$$

Substituting (4.7) in (4.6) and taking into account that A is symmetric, we have

$$\langle (iI - A)G, H\rangle = \langle (iI - A)G, Y\rangle. \tag{4.8}$$

Similarly, from $\lambda = i$ ia a regular point, we let $G = R(i, A)(H - Y)$. Then by (4.8), we have $H = Y$ and thus $H \in D(A)$. The proof of this Theorem is complete. □

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