k-sparse signal recovery via unrestricted ℓ_{1-2}-minimization

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Abstract

In the field of compressed sensing, ℓ_{1-2}-minimization model can recover the sparse signal well. In dealing with the ℓ_{1-2}-minimization problem, most of the existing literatures use the DCA algorithm to solve the unrestricted ℓ_{1-2}-minimization model, i.e. model (2). Although experiments have proved that the unrestricted ℓ_{1-2}-minimization model can recover the original sparse signal, the theoretical proof has not been established yet. This paper mainly proves theoretically that the unrestricted ℓ_{1-2}-minimization model can recover the sparse signal well, and makes an experimental study on the parameter λ in the unrestricted minimization model. The experimental results show that increasing the size of parameter λ in model (2) appropriately can improve the recovery success rate. However, when λ is sufficiently large, increasing λ will not increase the recovery success rate.
Keywords: compressed sensing, ℓ_{1-2}-minimization, DCA algorithm, k-sparse signal

1 Introduction

Compressed sensing is an effective data recovery technology. It mainly recovers high-dimensional unknown signals from low-dimensional measurement by finding the sparse solution. Its mathematical model can be expressed as

$$\min_{x\in R^n} \|x\|_0 \qquad s.t. \qquad Ax = y,$$

where $A \in R^{m\times n}$ is the measurement matrix, y is the measurement, $\|x\|_0$ represents the number of non-zero components in x, and $m \ll n$. We call the above mathematical model ℓ_0-minimization model.

The ℓ_0-minimization problem is NP-hard and thus computationally infeasible in high dimensional sets [1]. In order to solve the ℓ_0-minimization problem, a popular method is to replace it with ℓ_1-minimization model. The mathematical expression of ℓ_1-minimization model is

$$\min_{x\in R^n} \|x\|_1 \qquad s.t. \qquad Ax = y,$$

where $\|x\|_1 = \sum_{i=1}^n |x|$. The existing literature has shown that when the measurement matrix meets certain properties, such as null space property [1, 2], coherence [8], cumulative coherence [9], restricted orthogonality constant [7] and restricted isometry property [3–6], ℓ_1-minimization model can well solve the ℓ_0-minimization problem.

Although ℓ_1-minimization problem has considerable results, it is not exactly equivalent to ℓ_0-minimization problem [10, 11]. Hence, ℓ_{1-2}-minimization problem [12–15] has been put forward to replace ℓ_1-minimization problem in whice case ℓ_1-minimization problem does not execute well.

The mathematical expression of ℓ_{1-2}-minimizatiol model is as follows:

$$\operatorname*{argmin}_{x\in R^n} \|x\|_1 - \|x\|_2 \qquad \text{subject to} \qquad Ax = y, \tag{1}$$

where $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$. Its unrestricted model is as follows:

$$\min_{x\in R^n} \|x\|_1 - \|x\|_2 + \frac{\lambda}{2}\|Ax - y\|_2^2 \tag{2}$$

Existing literature has shown that ℓ_{1-2}-minimizatiol model has stronger ability to recover the original data than ℓ_1-minimizatiol model [14, 15]. However, because the ℓ_{1-2}-minimizatiol model is a nonconvex optimization problem, it is not so easy to solve this model. At present, paper [13] uses the DCA algorithm to solve the unrestricted ℓ_{1-2}-minimizatiol model. Although the experimental results show that their algorithm is very effective, the theoretical proof that the unrestricted ℓ_{1-2}-minimizatiol model can recover the original data has not been established yet. Therefore, it is very meaningful to establish a theory to prove that the unrestricted ℓ_{1-2}-minimizatiol model can recover the original data. The main content of this paper is to establish this result.

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The main contribution of this paper includes two aspects. (i) We theoretically prove that the unrestricted ℓ_{1-2}-minimizatiol model can effectively restore the original sparse data; (ii) We use DCA algorithm to study the influence of the size of parameter λ on the experimental results. It is found that increasing the size of parameter λ in model (2) appropriately can improve the recovery success rate. However, when λ is sufficiently large, increasing λ will not increase the recovery success rate.

2 Preliminary

In this paper, we denote $[n]=\{1,2,\cdots n\}$, supp(x) $=\{i|x_i\neq 0\}$. $S\subset[n]$ is a subscript set. $\overline{S}$ is the complement of S. $|S|$ is the cardinal of S. x_S is a vector related to x, meaning $(x_S)_i=x_i$ for $i\in S$, and otherwise $(x_S)_i=0$. A^T is transpose of A.

Definition 1 *For $S\subset[n]$ and each number s, s-restricted isometry constant of A is the smallest $\delta_s\in(0,1)$ such that*

$$(1-\delta_s)\|x\|_2^2\leq\|Ax\|_2^2\leq(1+\delta_s)\|x\|_2^2$$

for all subsets S with $|S|\leq s$ and all $\|x\|_0\leq s$. The matrix A is said to satisfy the s-RIP with δ_s.

3 main

In this section, we give the theoretical results.

Theorem 1 *Suppose x_0 is s-sparse vector with $S=supp(x_0)$, x^* is a solution of (2) with $y=Ax_0+e$, where $\|e\|_2=\epsilon$, if matrix A satisfies some $s+s_1$-RIP with δ_{s+s_1} such that*

$$(1-\delta_{s+s_1})\frac{\sqrt{s_1}-1}{\sqrt{s_1}+\sqrt{s}}-(1+\delta_{s_1})\frac{\sqrt{s}+1}{\sqrt{s_1}+\sqrt{s}}>0$$

then we have

$$\|x^*-x_0\|_2\leq C\epsilon, \tag{3}$$

where C is a constant.

Proof: Since x^* is a solution of (2), then we have

$$\|x^*\|_1-\|x^*\|_2+\frac{\lambda}{2}\|Ax^*-y\|_2^2\leq\|x_0\|_1-\|x_0\|_2+\frac{\lambda}{2}\|Ax_0-y\|_2^2. \tag{4}$$

Setting $v=x^*-x_0$, (4) yields that

$$\begin{aligned}&\|Ax^*-y\|_2^2-\|Ax_0-y\|_2^2\leq\frac{2}{\lambda}(\|x_0\|_1-\|x^*\|_1-\|x_0\|_2+\|x^*\|_2)\\&\leq\frac{2}{\lambda}(\|x_0\|_1-\|x_0+v_S+v_{\overline{S}}\|_1+\|v\|_2)=\frac{2}{\lambda}(\|x_0\|_1-\|x_0+v_S\|_1-\|v_{\overline{S}}\|_1+\|v\|_2)\\&\leq\frac{2}{\lambda}(\|v_S\|_1-\|v_{\overline{S}}\|_1+\|v\|_2).\end{aligned}\tag{5}$$

On the other hand, for any $\alpha\in R^n$, $\beta\in R^n$, $t>0$, it holds that

$$-\|\alpha-\beta\|_2^2\leq\frac{1}{t}(\|\alpha\|_2^2-\|\beta\|_2^2)+\frac{1}{t(t+1)}\|\beta\|_2^2. \tag{6}$$

Taking $\alpha=Ax^*-y$, $\beta=Ax_0-y$, and by the fact $\|Ax_0-y\|_2=\epsilon$ and (5), we obtain

$$\begin{aligned}-\|Av\|_2^2&\leq\frac{1}{t}(\|Ax^*-y\|_2^2-\|Ax_0-y\|_2^2)+\frac{1}{t(t+1)}\|Ax_0-y\|_2^2\\&\leq\frac{2}{t\lambda}(\|v_S\|_1-\|v_{\overline{S}}\|_1+\|v\|_2)+\frac{1}{t(t+1)}\epsilon^2.\end{aligned}\tag{7}$$

Since $|S|\leq s$, Cauchy-Schwarz inequality yields $\|v_S\|_1\leq\sqrt{s}\|v_S\|_2$. So (7) implies that

$$\|v_{\overline{S}}\|_1\leq\sqrt{s}\|v_S\|_2+\|v\|_2+\frac{t\lambda}{2}\|Av\|_2^2+\frac{\lambda}{2(t+1)}\epsilon^2. \tag{8}$$

Since (8) holds for all $t>0$, hence by taking $t=0$, we can get

$$\|v_{\overline{S}}\|_1 \le \sqrt{s}\|v_S\|_2 + \|v\|_2 + \frac{\lambda}{2}\epsilon^2. \tag{9}$$

Now, we estimate $\|Av\|_2^2$. Note that for any $r>0$, it holds that

$$\|\alpha-\beta\|_2^2 \le (1+r)\|\alpha\|_2^2 + (1+\frac{1}{r})\|\beta\|_2^2.$$

We apply above result to $Av = Ax^* - y - (Ax_0 - y)$ and combine with (5) ant the fact $\|Ax_0-y\|_2 = \epsilon$ to obtain

$$\begin{aligned}
\|Av\|_2^2 &\le (1+r)\|Ax^*-y\|_2^2 + (1+\frac{1}{r})\|Ax_0-y\|_2^2 \\
&= (1+r)(\|Ax^*-y\|_2^2 - \|Ax_0-y\|_2^2) + (2+r+\frac{1}{r})\|Ax_0-y\|_2^2 \\
&\le (1+r)\frac{2}{\lambda}(\|v_S\|_1 - \|v_{\overline{S}}\|_1 + \|v\|_2) + (2+r+\frac{1}{r})\epsilon^2 \\
&\le (1+r)\frac{2}{\lambda}(\sqrt{s}\|v_S\|_2 + \|v\|_2) + (2+r+\frac{1}{r})\epsilon^2 \\
&\le \frac{2(1+r)(\sqrt{s}+1)}{\lambda}\|v\|_2 + (2+r+\frac{1}{r})\epsilon^2.
\end{aligned} \tag{10}$$

Now we divide $\overline{S}$ into subsets of size s_1. Suppose $\overline{S} = \{k_1, k_2 \cdots k_{n-|S|}\}$ with $|v_{k_i}| \ge |v_{k_j}|$ for all $1 \le i < j \le n-|S|$. Let $S_j = \{k_l : (j-1)s_1 + 1 \le l \le js_1\}$, $j = 1, 2, \cdots$. Then we have $\|v_{S_{j+1}}\|_\infty \le \frac{\|v_{S_j}\|_1}{s_1}$, which yields $\|v_{S_{j+1}}\|_2^2 \le \frac{\|v_{S_j}\|_1^2}{s_1} = (\frac{\|v_{S_j}\|_1}{\sqrt{s_1}})^2$. Therefore

$$\sum_{j\ge2}\|v_{S_j}\|_2 \le \frac{1}{\sqrt{s_1}}\sum_{j\ge1}\|v_{S_j}\|_1 = \frac{1}{\sqrt{s_1}}\|v_{\overline{S}}\|_1. \tag{11}$$

Thus, it follows from (9) that

$$\sum_{j\ge2}\|v_{S_j}\|_2 \le \frac{1}{\sqrt{s_1}}(\sqrt{s}\|v_S\|_2 + \|v\|_2 + \frac{\lambda}{2}\epsilon^2). \tag{12}$$

Detoting $S\cup S_1$ by S_{01}, then we obtain

$$\begin{aligned}
\|v\|_2 &\le \|v_{S_{01}}\|_2 + \sum_{j\ge2}\|v_{S_j}\|_2 \\
&\le \|v_{S_{01}}\|_2 + \frac{1}{\sqrt{s_1}}(\sqrt{s}\|v_S\|_2 + \|v\|_2 + \frac{\lambda}{2}\epsilon^2) \\
&\le (1+\sqrt{\frac{s}{s_1}})\|v_{S_{01}}\|_2 + \frac{1}{\sqrt{s_1}}\|v\|_2 + \frac{\lambda\epsilon^2}{2\sqrt{s_1}}.
\end{aligned} \tag{13}$$

Thus, we can get

$$\|v_{S_{01}}\|_2 \ge \frac{\sqrt{s_1}-1}{\sqrt{s_1}+\sqrt{s}}\|v\|_2 - \frac{\lambda\epsilon^2}{2\sqrt{s_1}+2\sqrt{s}}. \tag{14}$$

$$\begin{aligned}
&\|Av\|_2 \ge \|Av_{01}\|_2 - \sum_{j\ge2}\|Av_{S_j}\|_2 \\
&\ge \sqrt{1-\delta_{s+s_1}}\|v_{01}\|_2 - \sqrt{1+\delta_{s_1}}\sum_{j\ge2}\|v_{S_j}\|_2 \\
&\ge (1-\delta_{s+s_1})\|v_{01}\|_2 - (1+\delta_{s_1})\sum_{j\ge2}\|v_{S_j}\|_2 \\
&\ge (1-\delta_{s+s_1})\|v_{01}\|_2 - (1+\delta_{s_1})(\frac{1}{\sqrt{s_1}}(\sqrt{s}\|v_S\|_2 + \|v\|_2 + \frac{\lambda}{2}\epsilon^2)) \\
&\ge (1-\delta_{s+s_1} - (1+\delta_{s_1})\frac{\sqrt{s}}{\sqrt{s_1}})\|v_{01}\|_2 - (1+\delta_{s_1})(\frac{1}{\sqrt{s_1}}(\|v\|_2 + \frac{\lambda}{2}\epsilon^2)) \\
&\ge (1-\delta_{s+s_1} - (1+\delta_{s_1})\frac{\sqrt{s}}{\sqrt{s_1}})(\frac{\sqrt{s_1}-1}{\sqrt{s_1}+\sqrt{s}}\|v\|_2 - \frac{\lambda\epsilon^2}{2\sqrt{s_1}+2\sqrt{s}}) - (1+\delta_{s_1})(\frac{1}{\sqrt{s_1}}(\|v\|_2 + \frac{\lambda}{2}\epsilon^2)) \\
&= ((1-\delta_{s+s_1})\frac{\sqrt{s_1}-1}{\sqrt{s_1}+\sqrt{s}} - (1+\delta_{s_1})\frac{\sqrt{s}+1}{\sqrt{s_1}+\sqrt{s}})\|v\|_2 - (\frac{1-\delta_{s_1+s}}{\sqrt{s_1}+\sqrt{s}} + \frac{(1+\delta_{s_1})\sqrt{s_1}}{s_1+\sqrt{s_1 s}})\frac{\lambda\epsilon^2}{2}.
\end{aligned} \tag{15}$$

Setting $a = (1-\delta_{s+s_1})\frac{\sqrt{s_1}-1}{\sqrt{s_1}+\sqrt{s}} - (1+\delta_{s_1})\frac{\sqrt{s}+1}{\sqrt{s_1}+\sqrt{s}}$, $b = (\frac{1-\delta_{s_1+s}}{\sqrt{s_1}+\sqrt{s}} + \frac{(1+\delta_{s_1})\sqrt{s_1}}{s_1+\sqrt{s_1 s}})\frac{\lambda\epsilon}{2}$, $c = \frac{2(1+r)(\sqrt{s}+1)}{\lambda\epsilon}$, $d = (2+r+\frac{1}{r})$, we know that b, c, d are all positive numbers. In addition, the conditions in the theorem show that $a > 0$. Combine (10) and (15), we have

$$a\|v\|_2 \leq \sqrt{c\epsilon\|v\|_2 + d\epsilon^2} + b\epsilon. \tag{16}$$

Setting $x = \frac{\|v\|_2}{\epsilon}$, then (16) is equivalent to

$$ax - b \leq \sqrt{cx + d}. \tag{17}$$

Figure 1 shows the images of function $ax - b$ and function $\sqrt{cx+d}$ when a, b, c, d are positive numbers. Figure 1 shows that there is a unique constant x_1 such that $ax_1 - b = \sqrt{cx_1 + d}$ and shows that when $0 < x < x_1$, it holds that $ax - b \leq \sqrt{cx+d}$. Therefore we have $\|x^* - x_0\|_2 = \|v\|_2 \leq x_1\epsilon$. $\square$

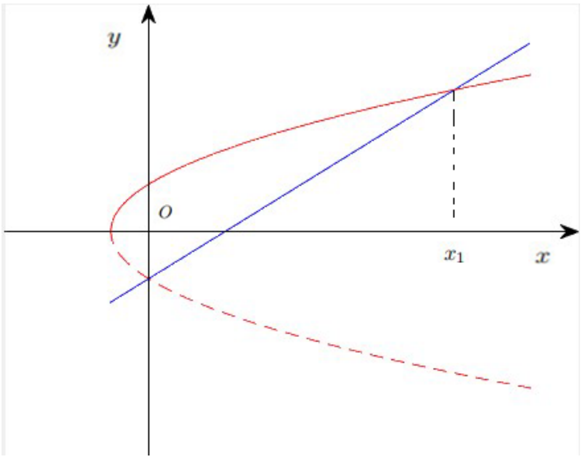

Figure 1: Two function images

Remark 1 *From the proof of Theorem 1, it is noted that, s_1 has not been fixed yet. So we can use this freedom to pick s_1 so that $a > 0$.*

4 Selection of parameter λ

In this section, we will use DCA algorithm to study the influence of the size of parameter λ on the ability of model (2) to recover the original signal. We first give the specific DCA algorithm for model (2).

4.1 DCA algorithm

Since $0 \in \partial\|0\|_2$, and $\frac{x}{\|x\|_2} \in \partial\|x\|_2$ when $x \neq 0$, we give the following algorithm to solve model (2).

Algorithm 1:

Input: $k = 0$, A, y, $x^0 = 0$, ϵ, Outmaxtimes

WHILE($k < Outmaxtimes$)

If $x^k = 0$

$v = 0$;

ELSE

$v = \frac{x^k}{\|x^k\|_2}$

ENDIF

$x^{k+1} = \text{argmin}\left\{\|x\|_1 - \langle x, v\rangle + \frac{\lambda}{2}\|Ax - y\|_2^2\right\}$;

$x^* = x^{k+1}$;

IF $\frac{\|x^{k+1} - x^k\|_2}{\max\{1, \|x^k\|_2\}} > \epsilon$

$k = k + 1$;

CONTINUE;

ENDIF;

BREAK;

ENDWILE;

Output: x^*

There is no analytical solution in the seventh step of Algorithm 1. We use the idea of ADMM algorithm to design a sub algorithm to approximate its solution. The seventh step in algorithm 1 is equivalent to solving the following problems

$$\min_{x,z\in R^n} \|z\|_1 - \langle v, x\rangle + \frac{\lambda}{2}\|Ax-y\|_2^2 \qquad s.t. \qquad z = x \tag{18}$$

The extended Lagrange function of (18) is

$$L(x,z,\alpha) = \|z\|_1 - \langle v, x\rangle + \frac{\lambda}{2}\|Ax-y\|_2^2 + \langle \alpha, z-x\rangle + \frac{\delta}{2}\|z-x\|_2^2. \tag{19}$$

According to ADMM algorithm, we get the following iterative formula

$$x^{k+1} = \operatorname*{argmin}_{x\in R^n}\{-\langle v, x\rangle + \frac{\lambda}{2}\|Ax-y\|_2^2 - \langle \alpha^k, x\rangle + \frac{\delta}{2}\|z^k - x\|_2^2\}. \tag{20}$$

$$z^{k+1} = \operatorname*{argmin}_{z\in R^n}\{\|z\|_1 + \langle \alpha^k, z\rangle + \frac{\delta}{2}\|z - x^{k+1}\|_2^2\}. \tag{21}$$

$$\alpha^{k+1} = \alpha^k + \delta(z^{k+1} - x^{k+1}). \tag{22}$$

Next, we give the specific sub algorithm.

Algorithm 2: sub algorithm

Input: $k = 0$, A, y, z^0, α^0, λ, δ, v, ϵ^{rel}, ϵ^{abs}, $inmaxtime > 0$

WHILE($k < inmaxtime$)

  $x^{k+1} = (\lambda A^TA + \delta I)^{-1}(v + \lambda A^Ty + \alpha^k + \delta z^k)$

  $z^{k+1} = soft(x^{k+1} - \frac{\alpha^k}{\delta}, \frac{1}{\delta})$

  $\alpha^{k+1} = \alpha^k + \delta(z^{k+1} - x^{k+1})$

  $x^* = x^{k+1}$;

  Set $r = x^{k+1} - z^{k+1}$, $s = \delta(z^{k+1} - z^k)$.

  IF $\|r\|_2 \leqslant \sqrt{n}\epsilon^{abs} + \epsilon^{rel} max\{\|x^{k+1}\|_2, \|z^{k+1}\|_2\}$&& $\|s\|_2 \leqslant \sqrt{n}\epsilon^{abs} + \epsilon^{res}\|\alpha^{k+1}\|_2$

    BREAK;

  ENDIF;

  $k = k+1$;

ENDWHILE;

Output: x*

4.2 Test of the size of parameter λ

In this section, we will test the impact of the size of parameter λ on the ability of model (2) to recover the original signal. We set other parameters of Algorithm 1 and Algorithm 2 first.

In this paper, two experiments are carried out. In the first experiment, we choose Gaussian matrix $A \in R^{64\times 256}$ as the measurement matrix, and in the second experiment, we choose Gaussian matrix $A \in R^{128\times 512}$ as the measurement matrix. The measurement matrices are row linear independent. The other parameters are the same in the two experiments. In Algorithm 1, we choose $\epsilon = 10^{-4}$, $Outmaxtimes = 31$, $x^0 = 0$. We take random sparse vector $x \in R^n$ as analog signals respectively. The position of non-zero elements on the x is random. We take $y = Ax + e$, where e is a Gaussian noise with $\|e\|_2 = 10^{-4}$. In Algorithm 2, we take $\alpha^0 = 0$, $z^0 = 0$, $\delta = 1$, $\epsilon^{rel} = 10^{-5}$, $\epsilon^{abs} = 10^{-2}\epsilon^{rel}$, $inmaxtime = 6000$ and the values of v and λ are the same as those in Algorithm 1. Assuming that x^* is the result of the algorithm and x is the analog signal, if $\frac{\|x^*-x\|_2}{\|x\|_2} < 10^{-3}$, then the algorithm is considered to have successfully restored the original signal.

Figure 2 is the first experiment. Its analog signal sparsity s is 12, 14, 16, 18, 20. The measurement matrix A is a 64×256 order Gaussian matrix, and the test parameters are $\lambda = 10$, 20, 30, 40 and 50 respectively. As can be seen from Figure 2, with the increase of sparsity s, the recovery success rate of the experiment decreases, and the larger the parameter λ, the higher the recovery success rate.

Figure 3 is the second experiment. Its analog signal sparsity s is 24, 28, 32, 36, 40. The measurement matrix A is a 128×512 order Gaussian matrix, and the test parameters are $\lambda = 20$, 40, 60, 80 and 100 respectively. It can be seen from Figure 3 that the recovery success rate of the experiment decreases with the increase of sparsity s. However, different from the first experiment, in this experiment, when the parameters $\lambda = 60$, $\lambda = 80$ and $\lambda = 100$, their recovery success rate is the same, and all three case are higher than when $\lambda = 20$ and

$\lambda = 40$. Combined with Figure 2 and Figure 3, we know that increasing the size of parameter λ appropriately can improve the recovery success rate. However, when λ is sufficiently large, increasing λ will not increase the recovery success rate.

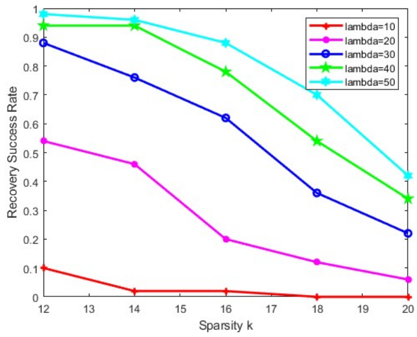

Figure 2: Measurement matrix $A \in R^{64\times 256}$

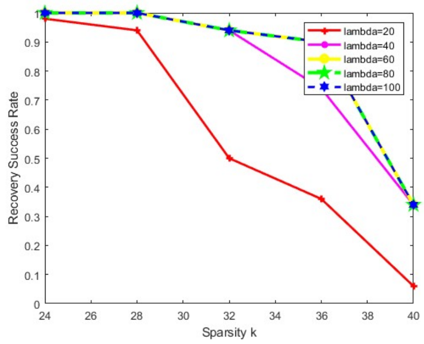

Figure 3: Measurement matrix $A \in R^{128\times 512}$

5 Conclusion

Using RIP condition, this paper proves that the unrestricted ℓ_{1-2}-minimization model can recover the original sparse signal. Data experiments show that the unrestricted ℓ_{1-2}-minimization model of the size of the parameters λ in the model has a great impact on the ability of the model to recover data.

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