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RESEARCH ARTICLE

Decomposition of the displacements of thin-walled beams with rectangular cross-section

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Abstract

The aim of this paper is to decompose the displacements of thin-walled beams with rectangular cross-section. The decomposition is accompanied by estimates of all its terms with respect to the norm of the strain tensor. Korn's inequality is also given.

KEYWORDS:
linear elasticity, elementary displacement, Kirchhoff-Love displacement, Bernoulli-Navier displacement, residual displacement.
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1 | INTRODUCTION

The first work on thin elastic structures dates back to the 19th century. It was carried out by Euler, Bernoulli, Navier and Kirchhoff (among others). This work was continued and completed in the 20th century by physicists such as Timoshenko and Love (among others). All these authors started from the displacements of a beam or a plate and gave approximations: the Bernoulli-Navier displacements (below BN displacements) or Kirchhoff-Love displacements (below KL displacements). Then, to solve elasticity problems, they neglected certain components of the stress tensors.
For several decades, mathematicians have been interested in the elasticity problems of thin structures. They began by transforming the structure (beam or plate) by expanding in the direction(s) of the small dimension(s) in order to work in a fixed domain. They then treated elasticity problems as minimisation problems or they used PDE techniques for singular variational problems. They have shown that the asymptotic behavior of the solutions of elasticity problems are BN or KL displacements, and they have also shown that certain components of the stress tensors vanish.
Both approaches have their limitations.
The mathematical approach cannot easily be extended to structures formed by a large number of beams or plates. The approach of the early pioneers (mechanicians and physicists) is the most natural. But restricting the displacements of beams or plates to BN or KL displacements is not enough, so they have added some assumptions about the stress tensors. In their decompositions, shearing and warping are missing. It should be noted that it is not easy to deal with these last small parts of the displacements. To deal with them, we need accurate estimates of all the terms of the decomposition. However, we can establish a simple rule for using the residual displacements (shearing+warping): in the strain and stress tensors, it is sufficient to neglect the partial derivative(s) of these terms in the direction(s) of the larger dimension(s) of the structures; i.e. we keep only the partial derivative(s) of these terms in the smallest dimension or dimensions (if there are several of the same order) (see Theorem 2 and [15,16]).

It's a truism that a thin-walled beam with a rectangular cross-section is neither a beam nor a plate. But on closer inspection, this structure looks much more like a plate than a beam. It has thickness 2δ, width 2ε and length L ($0 < 2\delta < 2\varepsilon < L$), each of its pieces of length 2ε is a small plate. This is why we start by treating this structure as a plate.
We therefore decompose any displacement of the thin-walled beam as the sum of a KL displacement and a residual displacement (we use the simplified version of the displacement decomposition of a plate obtained in [16]).

Any displacement $u \in W^{1,p}(\Omega_{\varepsilon,\delta})$ is written as

$$u(x) = U^\diamond_{KL}(x) + \widetilde{u}^{pl}(x) = \underbrace{\begin{pmatrix} \mathcal{U}^\diamond_1(x') - x_3 \dfrac{\partial \mathcal{U}^\diamond_3}{\partial x_1}(x') \\ \mathcal{U}^\diamond_2(x') - x_3 \dfrac{\partial \mathcal{U}^\diamond_3}{\partial x_2}(x') \\ \mathcal{U}^\diamond_3(x') \end{pmatrix}}_{\text{Kirchhoff-Love displacement}} + \underbrace{\widetilde{u}^{pl}(x)}_{\text{residual displacement}} \qquad \text{for a.e. } x \text{ in } \Omega_{\varepsilon,\delta}.$$

where $\Omega_{\varepsilon,\delta} \doteq P_\varepsilon \times (-\delta, \delta)$, $P_\varepsilon \doteq (0, L) \times (-\varepsilon, \varepsilon)$ and $\mathcal{U}^\diamond_1$, $\mathcal{U}^\diamond_1 \in W^{1,p}(P_\varepsilon)$, $\mathcal{U}^\diamond_3 \in W^{2,p}(P_\varepsilon)$, $\widetilde{u}^{pl} = \widetilde{u}^{pl}_1\mathbf{e}_1 + \widetilde{u}^{pl}_2\mathbf{e}_2 + \widetilde{u}^{pl}_3\mathbf{e}_3 \in W^{1,p}(\Omega_{\varepsilon,\delta})^3$.

The KL displacement $U^\diamond_{KL}$ can now be considered as a displacement of the $3D$ beam $B_\varepsilon = (0, L) \times (-\varepsilon, \varepsilon)^2$. As a displacement of this beam, it could be decompose as the sum of a BN displacement and a residual displacement. Unfortunately, this does not work. A straightforward calculation shows that the contributions of membrane displacement $U^\diamond_m = \mathcal{U}^\diamond_1\mathbf{e}_1 + \mathcal{U}^\diamond_2\mathbf{e}_2$ and bending $\mathcal{U}^\diamond_3$ to the strain tensor are not of the same order. That is why we take a different approach. First, we consider $U^\diamond_m$ as a displacement of the $2D$ thin beam P_ε and we decompose it as the sum of a BN displacement and a residual displacement (see[15]). This gives us $\mathcal{U}_1 \in W^{1,p}(0, L)$, $\mathcal{U}_2 \in W^{2,p}(0, L)$ and $\widetilde{u}_m = \widetilde{u}_1\mathbf{e}_1 + \widetilde{u}_2\mathbf{e}_2 \in W^{1,p}(P_\varepsilon)^2$ such that

$$\mathcal{U}^\diamond_m(x') = \begin{pmatrix} \mathcal{U}_1(x_1) - x_2 \dfrac{d\mathcal{U}_2}{dx_1}(x_1) \\ \mathcal{U}_2(x_1) \end{pmatrix} + \widetilde{u}_m(x') \quad \text{for a.e. } x' \text{ in } P_\varepsilon.$$

We continue by dealing with bending $\mathcal{U}^\diamond_3$. As x_2 is close to 0 ($|x_2| < \varepsilon$), we develop it as follows:

$$\mathcal{U}^\diamond_3(x') = \mathcal{U}^\diamond_3(x_1, 0) + x_2 \frac{\partial \mathcal{U}^\diamond_3}{\partial x_2}(x_1, 0) + \widetilde{\mathcal{U}}^\diamond_3(x').$$

Unfortunately, the functions $\mathcal{U}^\diamond_3(\cdot, 0)$, $\dfrac{\partial \mathcal{U}^\diamond_3}{\partial x_2}(\cdot, 0)$ and the last one above are not smooth enough to be used in a PDE equation. That is why we are replacing them with functions that are much better suited to PDE equations. We show that there exist $\mathcal{U}_3 \in W^{2,p}(0, L)$, $\Theta \in W^{2,p}(0, L)$, $\widetilde{u}_3 \in W^{2,p}(P_\varepsilon)$ such that

$$\mathcal{U}^\diamond_3(x') = \mathcal{U}_3(x_1) + x_2\Theta(x_1) + \widetilde{u}_3(x') \qquad \text{for a.e. } x' \in P_\varepsilon.$$

We therefore arrive at the following decomposition of u:

$$u(x) = U_{BN}(x) + \widetilde{u}^{tw}(x) = U_{BN}(x) - x_2x_3 \frac{d\Theta}{dx_1}(x_1)\mathbf{e}_1 + \widetilde{u}^{kl}(x) + \widetilde{u}^{pl}(x),$$

$$U_{BN}(x) = \begin{pmatrix} \mathcal{U}_1(x_1) - x_2 \dfrac{d\mathcal{U}_2}{dx_1}(x_1) - x_3 \dfrac{d\mathcal{U}_3}{dx_1}(x_1) \\ \mathcal{U}_2(x_1) - x_3\Theta(x_1) \\ \mathcal{U}_3(x_1) + x_2\Theta(x_1) \end{pmatrix}, \quad \widetilde{u}^{kl}(x) = \begin{pmatrix} \widetilde{u}_1(x') - x_3 \dfrac{\partial \widetilde{u}_3}{\partial x_1}(x') \\ \widetilde{u}_2(x') - x_3 \dfrac{\partial \widetilde{u}_3}{\partial x_2}(x') \\ \widetilde{u}_3(x') \end{pmatrix}. \tag{1}$$

for a.e. x in $\Omega_{\varepsilon,\delta}$.

The first and main part of the above decomposition is a BN displacement, the second term: the displacement $\widetilde{u}^{tw}$ is the residual part of the decomposition of u. Displacement $\widetilde{u}^{tw}$ is the sum of 3 terms. First $-x_2x_3 \dfrac{d\Theta}{dx_1}(x_1)$, where Θ is the torsion angle, and the KL displacement $\widetilde{u}^{kl}$, these two terms give information on shearing and warping of the cross-sections $\{x_1\} \times \omega_{\varepsilon,\delta}$, $x_1 \in (0, L)$. The last term $\widetilde{u}^{pl}$ represents shearing and warping of the fibers $\{x'\} \times (-\delta, \delta)$, $x' \in P_\varepsilon$. These terms are smaller than those in the main part but we cannot neglect them as they play an important role in the strain and stress tensors. In the end, we can see that the decomposition of the displacements of a thin-walled beam resembles that of a beam (at least in its main part: the BN displacement).

Such a decomposition is only of interest if we can give an order of magnitude for the various terms that make it up, which is done in Theorem 1).

As a general reference on elasticity, we refer the reader to[1,3,5]. For mathematical modeling of plates we refer to[2] and[4] for rods. There is an abundance of literature written by mechanicians on the study of thin-walled beams (see e.g.[6,7,8]). A mathematical

study of the thin-walled beams with rectangular cross-sections using Γ-convergence is given in [9]. The decomposition of displacements is presented in [10,12] for curved beams, in [11,15] for straight beams, in [16] for plates, the decomposition of the deformations is presented in [13] for beams and [14] for shells. In these papers we also find references to the decomposition of displacements or deformations of structures made up of a large number of rods, plates, or plate and rod(s).
The paper is organized as follows:

- In Section 2 we introduce the main notations.
- In Section 3 we decompose any displacement of the thin-walled beam as the sum of a Kirchhoff-Love displacement and a residual displacement.
- In Section 4 we detail the (1) writing of a displacement and we give all the estimates (see Theorem 1).
- In Section 5, we choose a sequence of displacements of the thin-walled beam $\Omega_{\varepsilon,\delta}$ whose strain tensor has a L^p norm of order $(\varepsilon\delta)^{1+1/p}$. In Theorem 2, besides the limits of the terms of the decomposition, we give the asymptotic behavior of the strain tensor using the limits of the terms of the decomposition.
- In Subsection 6.1, we give an application of our decomposition. We choose a classical loading of the structure and derive the limit elasticity problem (see Theorem 3) posed in the rescaled domain $\Omega = (0, L) \times (-1, 1)^2$ and then the variational problems satisfied by the limit terms in the Bernoulli-Navier displacement. In Subsection 6.3, the thin-walled beam is made of a homogeneous and isotropic material, in this case we rewrite the results of the previous subsection.
- Appendix (Section 7) is devoted to some technical results.

In this work, the constants appearing in the estimates will always be independent from ε, δ and L. As a rule the Latin indices i, j, k and l take values in $\{1, 2, 3\}$ while the Greek indices α and β in $\{1, 2\}$. We also use the Einstein convention of summation over repeated indices.

2 | NOTATIONS

We denote by $|\cdot|$ the euclidian norm of $\mathbb{R}^3$ and by $\cdot$ the associated scalar product. The euclidian space $\mathbb{R}^3$ is referred to the orthonormal frame $\big(O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\big)$.

In this paper L is a fixed parameter while ε and δ are two small parameters satisfying $0 < 2\delta < 2\varepsilon < L$, they will simultaneously tend to 0 as well as $\dfrac{\delta}{\varepsilon}$.
Denote

- $P_\varepsilon \doteq (0, L) \times (-\varepsilon, \varepsilon)$, $\quad \Omega_{\varepsilon,\delta} \doteq P_\varepsilon \times (-\delta, \delta)$ the mid-surface and the thin-walled beam,
- $\omega_{\varepsilon,\delta} \doteq (-\varepsilon, \varepsilon) \times (-\delta, \delta)$ the reference cross-section,
- $\Gamma_{\varepsilon,\delta} \doteq \{0\} \times \omega_{\varepsilon,\delta}$ the clamped part,
- $\gamma_\varepsilon \doteq \{0\} \times (-\varepsilon, \varepsilon)$ the clamped part of the mid-surface,
- $\Omega \doteq (0, L) \times (-1, 1)^2$ the re-scaled thin-walled beam,
- $P \doteq (0, L) \times (-1, 1)$ the re-scaled mid surface,
- $\omega \doteq (-1, 1)^2$ the re-scaled reference cross-section, $\Gamma \doteq \{0\} \times \omega$,
- for every $v \in W^{1,p}(\Omega_{\varepsilon,\delta})^3$, $1 \le p \le \infty$, the strain tensor of v is
$$e(v) = \frac{1}{2}\Big((\nabla v)^T + \nabla v\Big), \qquad e_{ij}(v) = \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big).$$
  $e(v)$ is the 3×3 symmetric matrix whose entries are the $e_{ij}(v)$'s,

3 | DECOMPOSITION OF A THIN-WALLED BEAM DISPLACEMENT VIA A KIRCHHOFF-LOVE DISPLACEMENT

In this section we decompose every displacement as the sum of a Kirchhoff-Love displacement plus a residual displacement. Below, we use the function $\rho_\delta \in W^{1,\infty}(\mathbb{R})$ defined by

$$\rho_\delta(x_1) = \begin{cases} 0 & \text{if } 0 \le x_1 \le \delta, \\ \dfrac{1}{\delta}(x_1 - \delta) & \text{if } \delta \le x_1 \le 2\delta, \\ 1 & \text{if } x_1 \ge 2\delta. \end{cases}$$

Note that

$$\forall x_1 \in \mathbb{R}, \qquad 0 \le \frac{d\rho_\delta}{dx_1}(x_1) \le \frac{1}{\delta}.$$

Proposition 1. For every displacement u belonging to $W^{1,p}(\Omega_{\varepsilon,\delta})^3$ there exist a Kirchhoff-Love displacement and a residual displacement such that

$$u(x) = U^\diamond_{KL}(x) + \widetilde{u}^{pl}(x) = \underbrace{\begin{pmatrix} \mathcal{U}^\diamond_1(x') - x_3 \dfrac{\partial \mathcal{U}^\diamond_3}{\partial x_1}(x') \\ \mathcal{U}^\diamond_2(x') - x_3 \dfrac{\partial \mathcal{U}^\diamond_3}{\partial x_2}(x') \\ \mathcal{U}^\diamond_3(x') \end{pmatrix}}_{\text{Kirchhoff-Love displacement}} + \underbrace{\widetilde{u}^{pl}(x)}_{\text{residual displacement}} \tag{2}$$

for a.e. x in $\Omega_{\varepsilon,\delta}$.
$\mathcal{U}^\diamond_m = \mathcal{U}^\diamond_1 \mathbf{e}_1 + \mathcal{U}^\diamond_2 \mathbf{e}_2$ is the membrane displacement, $\mathcal{U}^\diamond_3$ is the bending and $\widetilde{u}^{pl}$ satisfies

$$\int_{-\delta}^{\delta} \widetilde{u}^{pl}_1(x', x_3)dx_3 = \int_{-\delta}^{\delta} \widetilde{u}^{pl}_2(x', x_3)dx_3 = 0 \qquad \text{for a.e. } x' \in P_\varepsilon. \tag{3}$$

We have

$$\mathcal{U}^\diamond_m \in W^{1,p}(P_\varepsilon)^2, \quad \mathcal{U}^\diamond_3 \in W^{2,p}(P_\varepsilon), \quad \widetilde{u}^{pl} \in W^{1,p}(\Omega_{\varepsilon,\delta})^3$$

and the following estimates:

$$\begin{aligned} &\|e_{\alpha\beta}(\mathcal{U}^\diamond_m)\|_{L^p(P_\varepsilon)} \le \frac{C}{\delta^{1/p}} \|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}, \\ &\Big\| \frac{\partial^2 \mathcal{U}^\diamond_3}{\partial x_\alpha \partial x_\beta} \Big\|_{L^p(P_\varepsilon)} \le \frac{C}{\delta^{1+1/p}} \|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}, \\ &\|\widetilde{u}^{pl}\|_{L^p(\Omega_{\varepsilon,\delta})} + \delta \|\nabla \widetilde{u}^{pl}\|_{L^p(\Omega_{\varepsilon,\delta})} \le C\delta \|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}. \end{aligned} \tag{4}$$

The constants do not depend on ε, δ and L.
Moreover, if $u = 0$ a.e. on $\Gamma_{\varepsilon,\delta}$ then

$$\mathcal{U}^\diamond = 0, \qquad \nabla \mathcal{U}^\diamond_3 = 0 \quad \text{a.e. on } \gamma_\varepsilon, \qquad \widetilde{u}^{pl} = 0 \quad \text{a.e. on } \Gamma_{\varepsilon,\delta}.$$

Proof. First, we decompose u as the sum of an elementary displacement and a warping (see Theorem 5 in Subsection 7.1). Then, we extend u to $\Omega'_{\varepsilon,\delta}$ (see Proposition 3 in Subsection 7.2). For simplicity, we still write u the extension of u to the thin-walled beam $\Omega'_{\varepsilon,\delta}$.
This gives

$$u(x) = \mathcal{U}^{**}(x') + x_3 \mathcal{R}^{**}(x') + \overline{u}^{**}(x) \qquad \text{for a.e. } x = (x', x_3) \in \Omega'_{\varepsilon,\delta} \tag{5}$$

where

$$\mathcal{U}^{**} \in W^{1,p}(P'_\varepsilon)^3, \quad \mathcal{R}^{**} \in W^{1,p}(P'_\varepsilon)^2, \quad \overline{u}^{**} \in W^{1,p}(\Omega'_{\varepsilon,\delta})^3.$$

These terms satisfy the estimates (52).

Case 1: The thin-walled beam $\Omega_{\varepsilon,\delta}$ is not clamped.

Set $Y = (0,1)^2$ and

$$\Xi_\varepsilon \doteq \{\xi \in \mathbb{Z}^2 \mid \delta(\xi + Y) \subset P'_\varepsilon\}, \qquad \widehat{P}'_\varepsilon = \text{interior}\Big(\bigcup_{\xi\in\Xi_{\varepsilon,\delta}} \delta(\xi + \widetilde{Y})\Big).$$

We have

$$P_\varepsilon \subset \widehat{P}'_\varepsilon \subset P'_\varepsilon.$$

Now, we are in position to construct the Kirchhoff-Love displacement associated to u. To do this, we follow the lines of the proof of Theorem 5.2 and its Corollary 1 in [16] (remember that all we have to do is change $\mathcal{U}^{**}$ and $\mathcal{R}^{**}$). This gives the estimates (4) with constants independent of ε, δ and L since these estimates are based on those of (52).

Case 2: The thin-walled beam $\Omega_{\varepsilon,\delta}$ is clamped on $\Gamma_{\varepsilon,\delta}$.

In this casee we replace the above decomposition (5) by the following one:

$$u(x) = \mathcal{U}^{***}(x') + x_3\mathcal{R}^{***}(x') + \overline{u}^{***}(x) \qquad \text{for a.e. } x = (x', x_3) \in \Omega'_{\varepsilon,\delta} \tag{6}$$

where

$$\begin{aligned}&\mathcal{U}^{***} = \mathcal{U}_1^{**}\mathbf{e}_1 + \mathcal{U}_2^{**}\mathbf{e}_2 + \rho_\delta\mathcal{U}_3^{**}\mathbf{e}_3, \qquad \mathcal{R}^{***} = \rho_\delta\mathcal{R}^{**} \qquad \text{a.e. in } P'_\varepsilon\\ &\overline{u}^{***}(x) = (\mathcal{U}^{**}(x') - \mathcal{U}^{***}(x')) + x_3(1-\rho_\delta(x'))\mathcal{R}^{**}(x') + \overline{u}^{**}(x) \quad \text{for a.e. } x \text{ in } \Omega'_{\varepsilon,\delta}.\end{aligned}$$

We have only modified $\mathcal{U}_3^{**}$ and $\mathcal{R}^{**}$.

Since in this case $\mathcal{U}_3^{**}$ and $\mathcal{R}^{**}$ vanish on $\{0\}\times(-3\varepsilon,3\varepsilon)$. Estimate $(52)_3$ and the Poincaré inequality yield

$$\|\mathcal{R}^{**}\|_{L^p(C_{\varepsilon,\delta})} \le C\delta\|\nabla\mathcal{R}^{**}\|_{L^2(P'_\varepsilon)} \le \frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})} \text{ where } C_{\varepsilon,\delta} = (0,2\delta)\times(-3\varepsilon,3\varepsilon). \tag{7}$$

Then, the above together with $(52)_5$ lead to

$$\|\nabla\mathcal{U}_3^{**}\|_{L^p(C_{\varepsilon,\delta})} \le \frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})} \tag{8}$$

and then, using the Poincaré inequality

$$\|\mathcal{U}_3^{**}\|_{L^p(C_{\varepsilon,\delta})} \le C\delta^{1-1/p}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}. \tag{9}$$

A straightforward calculation leads to

$$\begin{aligned}&\|\overline{u}^{***}\|_{L^p(\Omega'_{\varepsilon,\delta})} \le C\delta\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}, \qquad \|\nabla\overline{u}^{***}\|_{L^p(\Omega'_{\varepsilon,\delta})} \le C\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})},\\ &\delta\|\nabla\mathcal{R}^{***}\|_{L^p(P'_\varepsilon)} + \|e_{\alpha\beta}(\mathcal{U}^{***})\|_{L^p(P'_\varepsilon)} + \Big\|\frac{\partial\mathcal{U}_3^{***}}{\partial x_\alpha} + \mathcal{R}_\alpha^{**}\Big\|_{L^p(P'_\varepsilon)} \le \frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}.\end{aligned}$$

The constants do not depend on ε, δ and L.

We are now in a position to construct a Kirchhoff-Love displacement vanishing on $\Gamma_{\varepsilon,\delta}$. To do this, we proceed as in Step 1.

For the conditions (3), we refer to [16] Section 6. □

4 | FROM A KIRCHHOFF-LOVE DISPLACEMENT TO A BERNOULLI-NAVIER DISPLACEMENT OF THE THIN-WALLED BEAM

Theorem 1. Any displacement $u \in W^{1,p}(\Omega_{\varepsilon,\delta})^3$ is the sum of a Bernoulli-Navier displacement U_{BN} and a residual displacement $\widetilde{u}^{tw}$

$$u(x) = U_{BN}(x) + \widetilde{u}^{tw}(x) = U_{BN}(x) - x_2x_3\frac{d\Theta}{dx_1}(x_1)\mathbf{e}_1 + \widetilde{u}^{kl}(x) + \widetilde{u}^{pl}(x),$$

$$U_{BN}(x) = \begin{pmatrix}\mathcal{U}_1(x_1) - x_2\dfrac{d\mathcal{U}_2}{dx_1}(x_1) - x_3\dfrac{d\mathcal{U}_3}{dx_1}(x_1)\\ \mathcal{U}_2(x_1) - x_3\Theta(x_1)\\ \mathcal{U}_3(x_1) + x_2\Theta(x_1)\end{pmatrix}, \quad \widetilde{u}^{kl}(x) = \begin{pmatrix}\widetilde{u}_1(x') - x_3\dfrac{\partial\widetilde{u}_3}{\partial x_1}(x')\\ \widetilde{u}_2(x') - x_3\dfrac{\partial\widetilde{u}_3}{\partial x_2}(x')\\ \widetilde{u}_3(x')\end{pmatrix} \tag{10}$$

for a.e. x in $\Omega_{\varepsilon,\delta}$, where $\mathcal{U}_1 \in W^{1,p}(0,L)$, $\mathcal{U}_2$, $\mathcal{U}_3$, $\Theta \in W^{2,p}(0,L)$ and $\widetilde{u}_m = \widetilde{u}_1\mathbf{e}_1 + \widetilde{u}_2\mathbf{e}_2 \in W^{1,p}(P_\varepsilon)^2$, $\widetilde{u}_3 \in W^{2,p}(P_\varepsilon)$, $\widetilde{u}^{pl} \in W^{1,p}(\Omega_{\varepsilon,\delta})^3$.

We have the following estimates:

$$
\begin{aligned}
&\Big\|\frac{d\mathcal{U}_1}{dx_1}\Big\|_{L^p(0,L)}\le C\frac{\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}}{(\varepsilon\delta)^{1/p}},\\
&\Big\|\frac{d\Theta}{dx_1}\Big\|_{L^p(0,L)}+\varepsilon\Big\|\frac{d^2\Theta}{dx_1^2}\Big\|_{L^p(0,L)}\le\frac{C}{\delta}\frac{\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}}{(\varepsilon\delta)^{1/p}},\\
&\Big\|\frac{d^2\mathcal{U}_2}{dx_1^2}\Big\|_{L^p(0,L)}\le\frac{C}{\varepsilon}\frac{\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}}{(\varepsilon\delta)^{1/p}},\quad\Big\|\frac{d^2\mathcal{U}_3}{dx_1^2}\Big\|_{L^p(0,L)}\le\frac{C}{\delta}\frac{\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}}{(\varepsilon\delta)^{1/p}},\\
&\|\widetilde{u}_3\|_{L^p(P_\varepsilon)}+\varepsilon\|\nabla\widetilde{u}_3\|_{L^p(P_\varepsilon)}+\varepsilon^2\|D^2\widetilde{u}_3\|_{L^p(P_\varepsilon)}\le\frac{C\varepsilon^2}{\delta^{1+1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})},\\
&\|\widetilde{u}_m\|_{L^p(P_\varepsilon)}+\varepsilon\|\nabla\widetilde{u}_m\|_{L^p(P_\varepsilon)}\le C\frac{\varepsilon}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})},\\
&\|\widetilde{u}^{pl}\|_{L^p(\Omega_{\varepsilon,\delta})}+\delta\|\nabla\widetilde{u}^{pl}\|_{L^p(\Omega_{\varepsilon,\delta})}\le C\delta\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}.
\end{aligned}
\tag{11}
$$

The constants do not depend on ε, δ and L.
Moreover if $u=0$ a.e. on $\Gamma_{\varepsilon,\delta}$ then

$$
\begin{aligned}
&\mathcal{U}_1(0)=\Theta(0)=\mathcal{U}_2(0)=\mathcal{U}_3(0)=\frac{d\mathcal{U}_2}{dx_1}(0)=\frac{d\mathcal{U}_3}{dx_1}(0)=\frac{d\Theta}{dx_1}(0)=0,\\
&\text{and}\quad\widetilde{u}^{kl}=0,\quad\widetilde{u}^{pl}=0\quad\text{a.e. on }\Gamma_{\varepsilon,\delta}.
\end{aligned}
\tag{12}
$$

Proof. We decompose $u\in W^{1,p}(\Omega_{\varepsilon,\delta})^3$ as (2).
Step 1. We transform the membrane displacement associated to $U^{\diamond}_{KL}$.
The membrane part of the Kirchhoff-Love displacement $U^{\diamond}_{KL}$ is

$$\mathcal{U}^{\diamond}_m(x')=\mathcal{U}^{\diamond}_1(x')\mathbf{e}_1+\mathcal{U}^{\diamond}_2(x')\mathbf{e}_2\qquad\text{for a.e. }x'=(x_1,x_2)\in P_\varepsilon.$$

This is a displacement of the $2D$ beam P_ε. From (4) we have

$$\|e(\mathcal{U}^{\diamond}_m)\|_{L^p(P_\varepsilon)}\le\frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}.\tag{13}$$

Now, we want to decompose $\mathcal{U}^{\diamond}_m$ as the sum of a $2D$ Bernoulli-Navier displacement and a residual displacement.
In [15] we have dealed with $3D$ displacements of thin rods. Here, we can consider $\mathcal{U}^{\diamond}_m$ as a displacement of the $3D$ rod $B_\varepsilon=(0,L)\times(-\varepsilon,\varepsilon)^2$. This displacement does not depend on the third variable x_3 and its third component is equal to 0. Before obtaining a Bernoulli-Navier displacement, in [15] we have decomposed any displacement as the sum of an elementary displacement and a warping (see [12,15]). Here, this gives

$$\mathcal{U}^{\diamond}_m=\mathcal{U}^*+\mathcal{R}^*\wedge(x_2\mathbf{e}_2+x_3\mathbf{e}_3)+\overline{u}^*\quad\text{a.e. in }B_\varepsilon$$

where $\mathcal{U}^*,\ \mathcal{R}^*\in W^{1,p}(0,L)^3$ and $\overline{u}^*\in W^{1,p}(B_\varepsilon)^3$. Component $\mathcal{U}^*$ is the mean value of $\mathcal{U}^{\diamond}_m$ on the cross-sections, so $\mathcal{U}^*_3=0$. Component $\mathcal{R}^*$ is the mean value of certain moments of u on the cross-sections (see [12,15]), since the third component of $\mathcal{U}^{\diamond}_m$ is equal to 0 we obtain $\mathcal{R}^*_1=\mathcal{R}^*_2=0$. After this first decomposition, we have

$$\mathcal{U}^{\diamond}_m(x')=\big(\mathcal{U}^*_1(x_1)-\mathcal{R}^*_3(x_1)\big)\mathbf{e}_1+\mathcal{U}^*_2(x_1)\mathbf{e}_2+\overline{u}^*(x')\qquad\text{for a.e. }x'=(x_1,x_2)\in P_\varepsilon.$$

Then, in [15] we have constructed the Bernoulli-Navier displacement by setting $\mathcal{U}_1=\mathcal{U}^*_1$, $\mathcal{U}_2$ is constructed using $\mathcal{U}^*_2$ and $\mathcal{R}^*_3$. This gives $\mathcal{U}_1\in W^{1,p}(0,L)$, $\mathcal{U}_2\in W^{2,p}(0,L)$ and $\widetilde{u}_m=\widetilde{u}_1\mathbf{e}_1+\widetilde{u}_2\mathbf{e}_2\in W^{1,p}(P_\varepsilon)^2$ such that

$$\mathcal{U}^{\diamond}_m(x')=\begin{pmatrix}\mathcal{U}_1(x_1)-x_2\dfrac{d\mathcal{U}_2}{dx_1}(x_1)\\ \mathcal{U}_2(x_1)\end{pmatrix}+\widetilde{u}_m(x')\quad\text{for a.e. }x'\text{ in }P_\varepsilon.\tag{14}$$

We have the following estimates (see [15]):

$$
\begin{aligned}
&\Big\|\frac{d\mathcal{U}_1}{dx_1}\Big\|_{L^p(0,L)}+\varepsilon\Big\|\frac{d^2\mathcal{U}_2}{dx_1^2}\Big\|_{L^p(0,L)}\le\frac{C}{\varepsilon^{1/p}}\|e(\mathcal{U}^{\diamond}_m)\|_{L^p(P_\varepsilon)}\le\frac{C}{(\varepsilon\delta)^{1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})},\\
&\|\widetilde{u}_m\|_{L^p(P_\varepsilon)}+\varepsilon\|\nabla\widetilde{u}_m\|_{L^p(P_\varepsilon)}\le C\varepsilon\|e(\mathcal{U}^{\diamond}_m)\|_{L^p(P_\varepsilon)}\le C\frac{\varepsilon}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}.
\end{aligned}
$$

The constants do not depend on ε, δ and L. The residual displacement $\widetilde{u}_m$ satisfies (see [15])

$$\int_{-\varepsilon}^{\varepsilon} \widetilde{u}_1(\cdot, x_2) dx_2 = 0 \quad \text{for a.e. } x_1 \in (0, L). \tag{15}$$

Step 3. We transform the bending $\mathcal{U}_3^\diamond$.

Now, we treat the remaining terms of the Kirchhoff-Love displacement $U_{KL}^\diamond$.

Proposition 5 in Appendix gives $\mathcal{U}_3 \in W^{2,p}(0,L)$, $\Theta \in W^{2,p}(0,L)$ and $\widetilde{u}_3 \in W^{2,p}(P_\varepsilon)$ such that

$$\mathcal{U}_3^\diamond = \mathcal{U}_3 + x_2\Theta + \widetilde{u}_3$$

and the estimates

$$\begin{aligned}
&\Big\| \frac{d^2\mathcal{U}_3}{dx_1^2} \Big\|_{L^p(0,L)} \le \frac{C}{\varepsilon^{1/p}} \Big\| \frac{\partial^2 \mathcal{U}_3^\diamond}{\partial x_1^2} \Big\|_{L^p(P_\varepsilon)} \le \frac{C}{\delta(\varepsilon\delta)^{1/p}} \|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})},\\
&\Big\| \frac{d\Theta}{dx_1} \Big\|_{L^p(0,L)} + \varepsilon \Big\| \frac{d^2\Theta}{dx_1^2} \Big\|_{L^p(0,L)} \le \frac{C}{\varepsilon^{1/p}} \Big\| D^2\mathcal{U}_3^\diamond \Big\|_{L^p(P_\varepsilon)} \le \frac{C}{\delta(\varepsilon\delta)^{1/p}} \|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})},\\
&\|\widetilde{u}_3\|_{L^p(P_\varepsilon)} + \varepsilon \|\nabla \widetilde{u}_3\|_{L^p(P_\varepsilon)} + \varepsilon^2 \|D^2 \widetilde{u}_3\|_{L^p(P_\varepsilon)} \le C\varepsilon^2 \|D^2\mathcal{U}_3^\diamond\|_{L^p(P_\varepsilon)} \le \frac{C\varepsilon^2}{\delta^{1+1/p}} \|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}.
\end{aligned}$$

Component $\widetilde{u}_3$ satisfies (see Proposition 5)

$$\int_{-\varepsilon}^{\varepsilon} \widetilde{u}_3(\cdot, x_2) dx_2 = \int_{-\varepsilon}^{\varepsilon} \widetilde{u}_3(\cdot, x_2) x_2 dx_2 = 0 \quad \text{for a.e. } x_1 \in (0, L). \tag{16}$$

The Kirchhoff-Love displacement $U_{KL}^\diamond$ is then written as follows:

$$U_{KL}^\diamond(x) = \begin{pmatrix} \mathcal{U}_1(x_1) - x_2 \dfrac{d\mathcal{U}_2}{dx_1}(x_1) - x_3 \dfrac{d\mathcal{U}_3}{dx_1}(x_1) \\ \mathcal{U}_2(x_1) - x_3\Theta(x_1) \\ \mathcal{U}_3(x_1) + x_2\Theta(x_1) \end{pmatrix} - x_2x_3 \frac{d\Theta}{dx_1}(x_1)\mathbf{e}_1 + \widetilde{u}^{kl}(x)$$

$$\widetilde{u}^{kl}(x) = \begin{pmatrix} \widetilde{u}_1(x') - x_3 \dfrac{\partial \widetilde{u}_3}{\partial x_1}(x') \\ \widetilde{u}_2(x') - x_3 \dfrac{\partial \widetilde{u}_3}{\partial x_2}(x') \\ \widetilde{u}_3(x') \end{pmatrix} \qquad \text{for a.e. } x \in \Omega_{\varepsilon,\delta}.$$

If $u = 0$ on $\Gamma_{\varepsilon,\delta}$ then, by Proposition 1, the Kirchhoff-Love displacement $U_{KL}^\diamond$ and the residual displacement $\widetilde{u}^{pl}$ (given by the decomposition (2)) vanish on $\Gamma_{\varepsilon,\delta}$. By construction of the fields $\mathcal{U}_1,\ \mathcal{U}_2,\ \mathcal{U}_3,\ \dfrac{d\mathcal{U}_3}{dx_1},\ \Theta,\ \dfrac{d\Theta}{dx_1}$ and $\widetilde{u}_m$ these functions also vanish on $\Gamma_{\varepsilon,\delta}$. □

Proposition 2 (Korn type inequalities)**.** Let u be a displacement in $W^{1,p}(\Omega_{\varepsilon,\delta})$, $p \in (1,\infty)$. We assume the thin-walled beam clamped on $\Gamma_{\varepsilon,\delta}$. Then, we have

$$\begin{aligned}
&\|u_1\|_{L^p(\Omega_{\varepsilon,\delta})} \le CL\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})},\\
&\|u_2\|_{L^p(\Omega_{\varepsilon,\delta})} \le \frac{CL^2}{\varepsilon}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}, \qquad \|u_3\|_{L^p(\Omega_{\varepsilon,\delta})} \le \frac{CL^2}{\delta}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})},\\
&\sum_{i=1}^{3} \Big\| \frac{\partial u_i}{\partial x_i} \Big\|_{L^p(\Omega_{\varepsilon,\delta})} \le C\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}, \quad \Big\| \frac{\partial u_2}{\partial x_1} \Big\|_{L^p(\Omega_{\varepsilon,\delta})} + \Big\| \frac{\partial u_1}{\partial x_2} \Big\|_{L^p(\Omega_{\varepsilon,\delta})} \le \frac{CL}{\varepsilon}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})},\\
&\Big\| \frac{\partial u_3}{\partial x_1} \Big\|_{L^p(\Omega_{\varepsilon,\delta})} + \Big\| \frac{\partial u_1}{\partial x_3} \Big\|_{L^p(\Omega_{\varepsilon,\delta})} \le \frac{CL}{\delta}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})},\\
&\Big\| \frac{\partial u_3}{\partial x_2} \Big\|_{L^p(\Omega_{\varepsilon,\delta})} + \Big\| \frac{\partial u_2}{\partial x_3} \Big\|_{L^p(\Omega_{\varepsilon,\delta})} \le \frac{CL}{\varepsilon}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}.
\end{aligned}$$

The constants do not depend on ε, δ and L.

Proof. We decompose u as (10). The estimates of this proposition are the consequences of those in (11). Indeed, the Poincaré inequality and $(11)_{1,2,4,5}$ give

$$\begin{aligned}
&\|\mathcal{U}_1\|_{L^p(0,L)} \le \frac{CL}{(\varepsilon\delta)^{1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}, \quad \|\Theta\|_{L^p(0,L)} \le \frac{CL}{\delta(\varepsilon\delta)^{1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})},\\
&\Big\|\frac{d\mathcal{U}_2}{dx_1}\Big\|_{L^p(0,L)} \le \frac{CL}{\varepsilon(\varepsilon\delta)^{1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}, \qquad \Big\|\frac{d\mathcal{U}_3}{dx_1}\Big\|_{L^p(0,L)} \le \frac{CL}{\delta(\varepsilon\delta)^{1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}.
\end{aligned} \tag{17}$$

The last two inequalities and again the Poincaré inequality imply that

$$\|\mathcal{U}_2\|_{L^p(0,L)} \le \frac{CL^2}{\varepsilon(\varepsilon\delta)^{1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}, \qquad \|\mathcal{U}_3\|_{L^p(0,L)} \le \frac{CL^2}{\delta(\varepsilon\delta)^{1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}. \tag{18}$$

The constants do not depend on ε, δ and L. The inequalities above and the estimates (11) lead to those in the proposition. $\square$

5 | ASYMPTOTIC BEHAVIOR OF A SEQUENCE OF DISPLACEMENTS

First, we recall the definition of the dimension reduction operator.

Definition 1. For ϕ measurable function on $\Omega_{\varepsilon,\delta}$, the dimension reduction operator $\Pi_{\varepsilon,\delta}$ is defined as follows:

$$\Pi_{\varepsilon,\delta}(\phi)(x_1, X_2, X_3) = \phi(x_1, \varepsilon X_2, \delta X_3) \qquad \text{for a.e. } (x_1, X_2, X_3) \in \Omega.$$

$\Pi_{\varepsilon,\delta}(\phi)$ is a measurable function on Ω.

We easily check that

1. for any $\phi \in L^p(\Omega_{\varepsilon,\delta})$, $1 \le p \le \infty$
$$\|\Pi_{\varepsilon,\delta}(\phi)\|_{L^p(\Omega)} = \frac{1}{(\varepsilon\delta)^{1/p}}\|\phi\|_{L^p(\Omega_{\varepsilon,\delta})}, \tag{19}$$
2. for any $\phi \in W^{1,p}(\Omega_{\varepsilon,\delta})$, $1 \le p \le \infty$
$$\frac{\partial\Pi_{\varepsilon,\delta}(\phi)}{\partial x_1} = \Pi_{\varepsilon,\delta}\Big(\frac{\partial\phi}{\partial x_1}\Big), \quad \frac{\partial\Pi_{\varepsilon,\delta}(\phi)}{\partial X_2} = \varepsilon\Pi_{\varepsilon,\delta}\Big(\frac{\partial\phi}{\partial x_2}\Big), \quad \frac{\partial\Pi_{\varepsilon,\delta}(\phi)}{\partial X_3} = \delta\Pi_{\varepsilon,\delta}\Big(\frac{\partial\phi}{\partial x_3}\Big). \tag{20}$$

Let u be a displacement belonging to $W^{1,p}(\Omega_{\varepsilon,\delta})^3$, decomposed as (10).

The strain tensor of u is given by the sum of 3×3 symmetric matrices defined a.e. in $\Omega_{\varepsilon,\delta}$ by

$$e(u) = \begin{pmatrix} \dfrac{d\mathcal{U}_1}{dx_1} - x_2\dfrac{d^2\mathcal{U}_2}{dx_1^2} - x_3\dfrac{d^2\mathcal{U}_3}{dx_1^2} & * & * \\ -x_3\dfrac{d\Theta}{dx_1} & 0 & * \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -x_2x_3\dfrac{d^2\Theta}{dx_1^2} + \dfrac{\partial\widetilde{u}_1}{\partial x_1} - x_3\dfrac{\partial^2\widetilde{u}_3}{\partial x_1^2} & * & * \\ \dfrac{1}{2}\Big(\dfrac{\partial\widetilde{u}_1}{\partial x_2} + \dfrac{\partial\widetilde{u}_2}{\partial x_1}\Big) - x_3\dfrac{\partial^2\widetilde{u}_3}{\partial x_1\partial x_2} & \dfrac{\partial\widetilde{u}_2}{\partial x_2} - x_3\dfrac{\partial^2\widetilde{u}_3}{\partial x_2^2} & * \\ 0 & 0 & 0 \end{pmatrix} + e(\widetilde{u}^{pl}). \tag{21}$$

Denote for $p \in (1, \infty)$

$$\begin{aligned}
&\mathbb{D} \doteq W^{1,p}(0,L) \times W^{2,p}(0,L)^2 \times W^{1,p}(0,L),\\
&\mathbb{D}^{(p)}_{W^{kl}} \doteq \Big\{\overline{\phi} \in W^{1,p}(-1,1)^2 \times W^{2,p}(-1,1) \ \mid \ \int_{-1}^{1}\overline{\phi}(t)dt = 0, \quad \int_{-1}^{1}\overline{\phi}_3(t)tdt = 0\Big\},\\
&\mathbb{D}^{(p)}_{W^{pl}} \doteq \Big\{\overline{\phi}^{pl} \in W^{1,p}(-1,1)^3 \ \mid \ \int_{-1}^{1}\overline{\phi}^{pl}(t)dt = 0\Big\}.
\end{aligned} \tag{22}$$

We equip $\mathbb{D}^{(p)}_{W^{kl}}$ and $\mathbb{D}^{(p)}_{W^{pl}}$ with the semi-norms

$$\begin{aligned}
&\|\overline{\phi}\|_{kl,p} = \Big\|\frac{d\overline{\phi}_1}{dt}\Big\|_{L^p(-1,1)} + \Big\|\frac{d\overline{\phi}_2}{dt}\Big\|_{L^p(-1,1)} + \Big\|\frac{d^2\overline{\phi}}{dt^2}\Big\|_{L^p(-1,1)}, \qquad \forall\overline{\phi} \in \mathbb{D}^{(p)}_{W^{kl}},\\
&\|\overline{\phi}^{pl}\|_{pl,p} = \Big\|\frac{d\overline{\phi}^{pl}}{dt}\Big\|_{L^p(-1,1)}, \qquad \forall\overline{\phi}^{pl} \in \mathbb{D}^{(p)}_{W^{pl}}.
\end{aligned}$$

We easily check that these semi-norms are norms equivalent to the usual norms of these spaces.

For every

$$\big(\Phi,\Psi,\overline{\Phi},\overline{\Phi}^{pl}\big)\in\mathbb{D}\times L^p(0,L;\mathbb{D}^{(p)}_{Wkl})\times L^p(P;\mathbb{D}^{(p)}_{Wpl})$$

where $\Phi=(\Phi_1,\Phi_2,\Phi_3)$ we define the 3×3 symmetric tensor $E\big(\Phi,\Psi,\overline{\Phi},\overline{\Phi}^{pl}\big)$ by

$$E\big(\Phi,\Psi,\overline{\Phi},\overline{\Phi}^{pl}\big)\doteq\begin{pmatrix}\dfrac{d\Phi_1}{dx_1}-X_2\dfrac{d^2\Phi_2}{dx_1^2}-X_3\dfrac{d^2\Phi_3}{dx_1^2} & * & *\\ -X_3\dfrac{d\Theta}{dx_1}+\dfrac{1}{2}\dfrac{\partial\overline{\Phi}_1}{\partial X_2} & \dfrac{\partial\overline{\Phi}_2}{\partial X_2}-X_3\dfrac{\partial^2\overline{\Phi}_3}{\partial X_2^2} & *\\ \dfrac{1}{2}\dfrac{\partial\overline{\Phi}^{pl}_1}{\partial X_3} & \dfrac{1}{2}\dfrac{\partial\overline{\Phi}^{pl}_2}{\partial X_3} & \dfrac{\partial\overline{\Phi}^{pl}_3}{\partial X_3}\end{pmatrix}\tag{23}$$

From now on, we assume that $\{(\varepsilon,\delta)\}$ is a sequence of strictly positive real numbers such that

$$\varepsilon\to 0,\qquad \delta\to 0,\qquad \frac{\delta}{\varepsilon}\to 0.$$

Denote for $p\in(1,\infty)$

$$\begin{aligned}W^{1,p}_\gamma(0,L)&\doteq\big\{\phi\in W^{1,p}(0,L)\mid \phi(0)=0\big\},\\ W^{2,p}_\gamma(0,L)&\doteq\Big\{\phi\in W^{2,p}(0,L)\mid \phi(0)=\frac{d\phi}{dx_1}(0)=0\Big\},\\ \mathbb{D}_{\gamma,p}&\doteq W^{1,p}_\gamma(0,L)\times W^{2,p}_\gamma(0,L)^2\times W^{1,p}_\gamma(0,L).\end{aligned}\tag{24}$$

Theorem 2. Let $\{u_{\varepsilon,\delta}\}_{\varepsilon,\delta}$ be a sequence of displacements belonging to $W^{1,p}(\Omega_{\varepsilon,\delta})^3$, $p\in(1,\infty)$, decomposed as (10). Suppose the thin-walled beam clamped on $\Gamma_{\varepsilon,\delta}$ and

$$\|e(u_{\varepsilon,\delta})\|_{L^p(\Omega_{\varepsilon,\delta})}\le C(\varepsilon\delta)^{1+1/p}\tag{25}$$

where the constant does not depend on ε and δ.

There exist a subsequence of $\{(\varepsilon,\delta)\}$, still denoted $\{(\varepsilon,\delta)\}$, $(\mathcal{U},\ \Theta)\in\mathbb{D}_{\gamma,p}$ and $\overline{U}\in L^p(0,L;\mathbb{D}^{(p)}_{Wkl})$, $\overline{U}^{pl}\in L^p(P;\mathbb{D}^{(p)}_{Wpl})$ such that

$$\begin{aligned}&\frac{1}{\varepsilon\delta}\mathcal{U}_{\varepsilon,\delta,1}\rightharpoonup\mathcal{U}_1\quad\text{weakly in }\ W^{1,p}_\gamma(0,L),\\ &\frac{1}{\delta}\mathcal{U}_{\varepsilon,\delta,2}\rightharpoonup\mathcal{U}_2\quad\text{weakly in }\ W^{2,p}_\gamma(0,L),\\ &\frac{1}{\varepsilon}\mathcal{U}_{\varepsilon,\delta,3}\rightharpoonup\mathcal{U}_3\quad\text{weakly in }\ W^{2,p}_\gamma(0,L),\\ &\frac{1}{\varepsilon}\Theta_{\varepsilon,\delta}\rightharpoonup\Theta\quad\text{weakly in }\ W^{1,p}_\gamma(0,L),\\ &\frac{d^2\Theta_{\varepsilon,\delta}}{dx_1^2}\rightharpoonup 0\quad\text{weakly in }\ L^p(0,L)\end{aligned}\tag{26}$$

and

$$\frac{1}{\varepsilon\delta}\Pi_{\varepsilon,\delta}\big(e(u_{\varepsilon,\delta})\big)\rightharpoonup E\big(\mathcal{U},\Theta,\overline{U},\overline{U}^{pl}\big)\quad\text{weakly in }\ L^p(\Omega)^{3\times 3}.\tag{27}$$

Moreover we have

$$\begin{aligned}&\frac{1}{\varepsilon\delta}\Pi_{\varepsilon,\delta}(u_{\varepsilon,\delta,1})\longrightarrow\mathcal{U}_1-X_2\frac{d\mathcal{U}_2}{dx_1}-X_3\frac{d\mathcal{U}_3}{dx_1}\quad\text{strongly in }\ L^p(\Omega),\\ &\frac{1}{\delta}\Pi_{\varepsilon,\delta}(u_{\varepsilon,\delta,2})\longrightarrow\mathcal{U}_2\quad\text{strongly in }\ L^p(\Omega),\\ &\frac{1}{\varepsilon}\Pi_{\varepsilon,\delta}(u_{\varepsilon,\delta,3})\longrightarrow\mathcal{U}_3\quad\text{strongly in }\ L^p(\Omega)\end{aligned}\tag{28}$$

Proof. Convergences (26) are the consequences of the estimates (17)-(18)-(25) and the properties (19)-(20) of the operator Π_δ.

Now, from $(11)_{6,7,8,9,10}$-(25) and the properties (19)-(20) of $\Pi_{\varepsilon,\delta}$ we deduce that

$$
\begin{aligned}
&\Big\|\Pi_{\varepsilon,\delta}\big(\widetilde{u}_{\varepsilon,\delta,\alpha}\big)\Big\|_{L^p(\Omega)}+\Big\|\frac{\partial\Pi_{\varepsilon,\delta}\big(\widetilde{u}_{\varepsilon,\delta,\alpha}\big)}{\partial X_2}\Big\|_{L^p(\Omega)}\le C\varepsilon^2\delta,\qquad \Big\|\frac{\partial\Pi_{\varepsilon,\delta}\big(\widetilde{u}_{\varepsilon,\delta,\alpha}\big)}{\partial x_1}\Big\|_{L^p(\Omega)}\le C\varepsilon\delta,\\
&\Big\|\Pi_{\varepsilon,\delta}\big(\widetilde{u}_{\varepsilon,\delta,3}\big)\Big\|_{L^p(\Omega)}+\Big\|\frac{\partial\Pi_{\varepsilon,\delta}\big(\widetilde{u}_{\varepsilon,\delta,3}\big)}{\partial X_2}\Big\|_{L^p(\Omega)}+\Big\|\frac{\partial^2\Pi_{\varepsilon,\delta}\big(\widetilde{u}_{\varepsilon,\delta,3}\big)}{\partial X_2^2}\Big\|_{L^p(\Omega)}\le C\varepsilon^3,\\
&\Big\|\frac{\partial\Pi_{\varepsilon,\delta}\big(\widetilde{u}_{\varepsilon,\delta,3}\big)}{\partial x_1}\Big\|_{L^p(\Omega)}\le C\varepsilon^2,\quad \Big\|\frac{\partial^2\Pi_{\varepsilon,\delta}\big(\widetilde{u}_{\varepsilon,\delta,3}}{\partial x_1^2}\Big)\Big\|_{L^p(\Omega)}\le C\varepsilon,\quad \Big\|\frac{\partial^2\Pi_{\varepsilon,\delta}\big(\widetilde{u}_{\varepsilon,\delta,3}\big)}{\partial x_1\partial X_2}\Big\|_{L^p(\Omega)}\le C\varepsilon^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\|\Pi_{\varepsilon,\delta}\big(\widetilde{u}^{pl}_{\varepsilon,\delta}\big)\Big\|_{L^p(\Omega)}+\Big\|\frac{\partial\Pi_{\varepsilon,\delta}\big(\widetilde{u}^{pl}_{\varepsilon,\delta}\big)}{\partial X_3}\Big\|_{L^p(\Omega)}\le C\varepsilon\delta^2,\\
&\Big\|\frac{\partial\Pi_{\varepsilon,\delta}\big(\widetilde{u}^{pl}_{\varepsilon,\delta}\big)}{\partial x_1}\Big\|_{L^p(\Omega)}\le C\varepsilon\delta,\qquad \Big\|\frac{\partial\Pi_{\varepsilon,\delta}\big(\widetilde{u}^{pl}_{\varepsilon,\delta}\big)}{\partial X_2}\Big\|_{L^p(\Omega)}\le C\varepsilon^2\delta.
\end{aligned}
$$

Then, there exist a subsequence of $\{(\varepsilon,\delta)\}$, still denoted $\{(\varepsilon,\delta)\}$, $\widetilde{U}\in L^p(0,L;W^{1,p}(-1,1))^2\oplus L^p(0,L;W^{2,p}(-1,1))$ such that

$$
\begin{aligned}
&\frac{1}{\varepsilon^2\delta}\Pi_{\varepsilon,\delta}(\widetilde{u}_{\varepsilon,\delta,\alpha})\rightharpoonup\widetilde{U}_\alpha\quad\text{weakly in }\ L^p(0,L;W^{1,p}(-1,1)),\\
&\frac{1}{\varepsilon\delta}\Pi_{\varepsilon,\delta}\Big(\frac{\partial\widetilde{u}_{\varepsilon,\delta,\alpha}}{\partial x_1}\Big)=\frac{1}{\varepsilon\delta}\frac{\partial\Pi_{\varepsilon,\delta}(\widetilde{u}_{\varepsilon,\delta,\alpha})}{\partial x_1}\rightharpoonup 0\quad\text{weakly in }\ L^p(\Omega),\\
&\frac{1}{\varepsilon^3}\Pi_{\varepsilon,\delta}(\widetilde{u}_{\varepsilon,\delta,3})\rightharpoonup\widetilde{U}_3\quad\text{weakly in }\ L^p(0,L;W^{2,p}(-1,1)),\\
&\frac{1}{\varepsilon^2}\Pi_{\varepsilon,\delta}\Big(\frac{\partial\widetilde{u}_{\varepsilon,\delta,3}}{\partial x_1}\Big),\ \frac{1}{\varepsilon}\Pi_{\varepsilon,\delta}\Big(\frac{\partial^2\widetilde{u}_{\varepsilon,\delta,3}}{\partial x_1^2}\Big),\ \frac{1}{\varepsilon}\Pi_{\varepsilon,\delta}\Big(\frac{\partial^2\widetilde{u}_{\varepsilon,\delta,3}}{\partial x_1\partial x_2}\Big)\rightharpoonup 0\quad\text{weakly in }\ L^p(\Omega)
\end{aligned}
\tag{29}
$$

and $\widetilde{U}^{pl}\in L^p(P;W^{1,p}(-1,1))^3$ such that

$$
\begin{aligned}
&\frac{1}{\varepsilon\delta^2}\Pi_{\varepsilon,\delta}(\widetilde{u}^{pl}_{\varepsilon,\delta})\rightharpoonup\widetilde{U}^{pl}\quad\text{weakly in }\ L^p(P;W^{1,p}(-1,1))^3,\\
&\frac{1}{\varepsilon\delta}\Pi_{\varepsilon,\delta}\Big(\frac{\partial\widetilde{u}^{pl}_{\varepsilon,\delta}}{\partial x_1}\Big)=\frac{1}{\varepsilon\delta}\frac{\partial\Pi_{\varepsilon,\delta}(\widetilde{u}^{pl}_{\varepsilon,\delta})}{\partial x_1}\rightharpoonup 0\quad\text{weakly in }\ L^p(\Omega)^3,\\
&\frac{1}{\varepsilon\delta}\Pi_{\varepsilon,\delta}\Big(\frac{\partial\widetilde{u}^{pl}_{\varepsilon,\delta}}{\partial x_2}\Big)=\frac{1}{\varepsilon^2\delta}\frac{\partial\Pi_{\varepsilon,\delta}(\widetilde{u}^{pl}_{\varepsilon,\delta})}{\partial X_2}\rightharpoonup 0\quad\text{weakly in }\ L^p(\Omega)^3.
\end{aligned}
\tag{30}
$$

The strong convergences (28) are the consequences of the fact that the sequences $\Big\{\dfrac{1}{\varepsilon\delta}\Pi_{\varepsilon,\delta}(u_{\varepsilon,\delta,1})\Big\}_{\varepsilon,\delta}$, $\Big\{\dfrac{1}{\delta}\Pi_{\varepsilon,\delta}(u_{\varepsilon,\delta,2})\Big\}_{\varepsilon,\delta}$, $\Big\{\dfrac{1}{\varepsilon}\Pi_{\varepsilon,\delta}(u_{\varepsilon,\delta,3})\Big\}_{\varepsilon,\delta}$ are uniformly bounded in $W^{1,p}(\Omega)$ and the compact embedding of $W^{1,p}(\Omega)$ in $L^p(\Omega)$.

Then, convergence (27) follows from convergences (26)-(29)-(30).

Equalities (3)-(15)-(16) yield

$$
\begin{aligned}
&\int_{-1}^{1}\widetilde{U}^{pl}_1(\cdot,X_3)dX_3=\int_{-1}^{1}\widetilde{U}^{pl}_2(\cdot,X_3)dX_3=0\qquad\text{a.e. in }\ P,\\
&\int_{-1}^{1}\widetilde{U}_1(\cdot,X_2)dX_2=0\qquad\text{a.e. in }\ (0,L),\\
&\int_{-1}^{1}\widetilde{U}_3(\cdot,X_2)dX_2=\int_{-1}^{1}\widetilde{U}_3(\cdot,X_2)X_2dX_2=0\qquad\text{a.e. in }\ (0,L).
\end{aligned}
$$

The conditions $\displaystyle\int_{-1}^{1}\widetilde{U}^{pl}_3(\cdot,X_3)dX_3=0$ a.e. in P and $\displaystyle\int_{-1}^{1}\widetilde{U}_2(\cdot,X_2)dX_2=0$ a.e. in $(0,L)$ are missing to get $\widetilde{U}^{pl}\in L^p(P;\mathbb{D}^{(p)}_{W^{pl}})$ and $\widetilde{U}\in L^p(0,L;\mathbb{D}^{(p)}_{W^{kl}})^1$.

Despite this absence of conditions, it should be noted that these functions are only involved in the strain tensor via their partial derivative with respect to their last variable. We can note that $\widetilde{U}_3^{pl}$ and $\overline{U}_3^{pl} = \widetilde{U}_3^{pl} - \frac{1}{2}\int_{-1}^{1} \widetilde{U}_3^{pl}(\cdot, X_3)dX_3$ have the same partial derivative with respect to X_3. That is why in the strain tensor limit we replace $\widetilde{U}^{pl}$ by $\overline{U}^{pl}$ with $\overline{U}_\alpha^{pl} = \widetilde{U}_\alpha^{pl}$. In the strain tensor limit we also replace $\widetilde{U}$ by $\overline{U}$ with $\overline{U}_i = \widetilde{U}_i$, $i \in \{1,3\}$ and $\overline{U}_2 = \widetilde{U}_2 - \frac{1}{2}\int_{-1}^{1} \widetilde{U}_2(\cdot, X_2)dX_2$. Of course we have $\overline{U} \in L^p(0, L; \mathbb{D}^{(p)}_{W^{kl}})$ and $\overline{U}^{pl} \in L^p(P; \mathbb{D}^{(p)}_{W^{pl}})$. □

As a consequence of the above theorem

Corollary 1. We have

$$\frac{1}{\varepsilon^2\delta}\Pi_{\varepsilon,\delta}\big(u_{\varepsilon,\delta} - U_{BN,\varepsilon,\delta}\big) \rightharpoonup -X_2X_3\frac{d\Theta}{dx_1}\mathbf{e}_1 + \begin{pmatrix} \widetilde{U}_1 \\ \widetilde{U}_2 - X_3\dfrac{\partial \widetilde{U}_3}{\partial X_2} \\ 0 \end{pmatrix} \quad \text{weakly in } L^p(\Omega)^3.$$

We equip the space $L^p(0, L; \mathbb{D}^{(p)}_{W^{kl}})$ (resp. $L^p(P; \mathbb{D}^{(p)}_{W^{pl}})$) with the norm

$$\forall \overline{\Phi} \in L^p(0, L; \mathbb{D}^{(p)}_{W}), \qquad \|\overline{\Phi}\|_{W^{kl,p}} = \Big\|\frac{\partial \overline{\Phi}_1}{\partial X_2}\Big\|_{L^p(\Omega)} + \Big\|\frac{\partial \overline{\Phi}_2}{\partial X_2}\Big\|_{L^p(\Omega)} + \Big\|\frac{\partial^2 \overline{\Phi}_3}{\partial X_2^2}\Big\|_{L^p(\Omega)},$$

$$(\text{resp.} \quad \forall \overline{\Phi}^{pl} \in L^p(P; \mathbb{D}^{(p)}_{W^{pl}}), \qquad \|\overline{\Phi}^{pl}\|_{W^{pl,p}} = \Big\|\frac{\partial \overline{\Phi}^{pl}}{\partial X_3}\Big\|_{L^p(\Omega)}).$$

These norms are equivalent to the usual norms of these spaces.

Lemma 1. For every

$$\big(\Phi, \Psi, \overline{\Phi}, \overline{\Phi}^{pl}\big) \in \mathbb{D} \times L^p(0, L; \mathbb{D}^{(p)}_{W^{kl}}) \times L^p(P; \mathbb{D}^{(p)}_{W^{pl}})$$

we have

$$\begin{aligned} &\Big\|\frac{d\Phi_1}{dx_1}\Big\|_{L^p(0,L)} + \Big\|\frac{d^2\Phi_2}{dx_1^2}\Big\|_{L^p(0,L)} + \Big\|\frac{d^2\Phi_3}{dx_1}\Big\|_{L^p(0,L)} + \Big\|\frac{d\Psi}{dx_1}\Big\|_{L^p(0,L)} \\ &+\Big\|\overline{\Phi}\Big\|_{W^{kl,p}} + \Big\|\overline{\Phi}^{pl}\Big\|_{W^{pl,p}} \le C\Big\|E\big(\Phi, \Psi, \overline{\Phi}, \overline{\Phi}^{pl}\big)\Big\|_{L^p(\Omega)}. \end{aligned} \tag{31}$$

Proof. From the expression (23) of $E\big(\Phi, \Psi, \overline{\Phi}, \overline{\Phi}^{pl}\big)$ we first obtain

$$\Big\|\frac{d\Phi_1}{dx_1}\Big\|_{L^p(0,L)} + \Big\|\frac{d^2\Phi_2}{dx_1^2}\Big\|_{L^p(0,L)} + \Big\|\frac{d^2\Phi_3}{dx_1}\Big\|_{L^p(0,L)} + \Big\|\overline{\Phi}^{pl}\Big\|_{W^{pl}} \le C\Big\|E\big(\Phi, \Psi, \overline{\Phi}, \overline{\Phi}^{pl}\big)\Big\|_{L^p(\Omega)}.$$

Remind that if ϕ, ψ are functions in $L^p(P)$ then

$$\|\phi\|_{L^p(P)} + \|\psi\|_{L^p(P)} \le C\Big\|\phi + X_3\psi\|_{L^p(\Omega)}.$$

The constant only depends on p.
Hence, we get

$$\begin{aligned} &\Big\|\frac{d\Psi}{dx_1}\Big\|_{L^p(0,L)} + \Big\|\frac{\partial \overline{\Phi}_1}{\partial X_2}\Big\|_{L^p(P)} \le C\Big\|E\big(\Phi, \Psi, \overline{\Phi}, \overline{\Phi}^{pl}\big)\Big\|_{L^p(\Omega)}, \\ &\Big\|\frac{\partial \overline{\Phi}_2}{\partial X_2}\Big\|_{L^p(P)} + \Big\|\frac{\partial^2 \overline{\Phi}_3}{\partial X_2^2}\Big\|_{L^p(\Omega)} \le C\Big\|E\big(\Phi, \Psi, \overline{\Phi}, \overline{\Phi}^{pl}\big)\Big\|_{L^p(\Omega)}. \end{aligned}$$

This completes the proof of (31). □

[1] A more complete decomposition of the displacements of the plates and beams would show that these quantities are in fact equal to 0.

6 ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF A LINEAR ELASTICITY PROBLEM

6.1 The linear elasticity problem

Denote

$$
\begin{aligned}
&H^1_{\Gamma_{\varepsilon,\delta}}(\Omega_{\varepsilon,\delta}) \doteq \Big\{ v \in H^1(\Omega_{\varepsilon,\delta}) \mid v = 0 \quad \text{a.e. on } \Gamma_{\varepsilon,\delta} \Big\},\\
&H^1_{\Gamma}(\Omega) \doteq \Big\{ v \in H^1(\Omega) \mid v = 0 \quad \text{a.e. on } \Gamma \Big\},\\
&\mathbb{D}_\gamma \doteq H^1_\gamma(0,L) \times \big(H^2_\gamma(0,L)\big)^2 \times H^1_\gamma(0,L),\\
&\mathbb{D}_W \doteq L^2(0,L;\mathbb{D}^{(2)}_{W^{kl}}) \times L^2(P;\mathbb{D}^{(2)}_{W^{pl}}).
\end{aligned}
$$

For $1 \le i,\ j,\ k,\ l \le 3$, let a_{ijkl} be in $L^\infty(\omega)$ and satisfy the symmetry conditions

$$
a_{ijkl}(X_2,X_3) = a_{jikl}(X_2,X_3) = a_{klij}(X_2,X_3) \quad \text{for a.e. } (X_2,X_3) \in \omega
$$

as well as the coercivity condition

$$
a_{ijkl}(X_2,X_3)\xi_{ij}\xi_{kl} \ge c_0 \xi_{ij}\xi_{ij} \qquad \text{for a.e. } (X_2,X_3) \in \Omega \tag{32}
$$

for every 3×3 symmetric matrix $\xi = (\xi_{ij})$ (c_0 is a given strictly positive number).
The coefficients $a_{ijkl,\varepsilon,\delta}$ of the Hooke tensor are given by

$$
a_{ijkl,\varepsilon,\delta}(x) = a_{ijkl}\Big(\frac{x_2}{\varepsilon},\frac{x_3}{\delta}\Big) \quad \text{for a.e. } \ x \in \Omega_{\varepsilon,\delta}.
$$

The constitutive law of the materials is the relation between the strain tensor and the stress tensor,

$$
\sigma_{ij,\varepsilon,\delta}(v) = a_{ijkl,\varepsilon,\delta}e_{kl}(v), \qquad \forall v \in H^1_{\Gamma_{\varepsilon,\delta}}(\Omega_{\varepsilon,\delta})^3.
$$

For simplify we consider only applied body forces.
The displacement $u_{\varepsilon,\delta} \in H^1_{\Gamma_{\varepsilon,\delta}}(\Omega_{\varepsilon,\delta})^3$ of the thin-walled beam is the solution of the following elasticity problem:

$$
\begin{cases}
\displaystyle\int_{\Omega_{\varepsilon,\delta}} \sigma_{ij,\varepsilon,\delta}(u_{\varepsilon,\delta})e_{ij}(v)\,dx = \int_{\Omega_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x)\cdot v(x)\,dx, \quad f_{\varepsilon,\delta} \in L^2(\Omega_{\varepsilon,\delta})^3\\
\forall v \in H^1_{\Gamma_{\varepsilon,\delta}}(\Omega_{\varepsilon,\delta})^3.
\end{cases} \tag{33}
$$

Due to the above assumptions on the $a_{ijkl,\varepsilon,\delta}$'s, the Lax-Milgram theorem applied to problem (33) implies that this problem has a unique solution.
We make the assumption that the applied body forces $f_{\varepsilon,\delta}$ are of the form

$$
f_{\varepsilon,\delta}(x) = \varepsilon\delta\Big[\Big(f_1(x_1) + \frac{x_2}{\varepsilon}g_2(x_1) + \frac{x_3}{\delta}g_3(x_1)\Big)\mathbf{e}_1 + \Big(\varepsilon f_2(x_1) - \frac{x_3}{\delta}g_1(x_1)\Big)\mathbf{e}_2 + \Big(\delta f_3(x_1) + \frac{x_2\delta}{\varepsilon^2}g_1(x_1)\Big)\mathbf{e}_3\Big],
$$

where $f = (f_1,f_2,f_3)$ and $g = (g_1,g_2,g_3)$ belong to $L^2(0,L)^3$.
This allows us to obtain an a priori estimate of $u_{\varepsilon,\delta}$. Using the decomposition (10) for a $u \in H^1_{\Gamma_{\varepsilon,\delta}}(\Omega_{\varepsilon,\delta})^3$ and estimates $(11)_{6,7,9,11}$, we first have

$$
\begin{aligned}
\Big|\int_{\Omega_{\varepsilon,\delta}} f_{\varepsilon,\delta}\cdot\big(u - U_{BN}\big)\,dx\Big| &\le C\varepsilon\|f_{\varepsilon,\delta}\|_{L^2(\Omega_{\varepsilon,\delta})}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}\\
&\le C\varepsilon^2(\varepsilon\delta)^{3/2}\big(\|f\|_{L^2(0,L)} + \|g\|_{L^2(0,L)}\big)\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}
\end{aligned} \tag{34}
$$

and then

$$
\begin{aligned}
\int_{\Omega_{\varepsilon,\delta}} f_{\varepsilon,\delta}\cdot U_{BN}\,dx &= 4(\varepsilon\delta)^2\Big(\int_0^L f_1\,\mathcal{U}_1\,dx_1 + \int_0^L \varepsilon f_2\,\mathcal{U}_2\,dx_1 + \int_0^L \delta f_3\,\mathcal{U}_3\,dx_1\Big)\\
&\quad + 4(\varepsilon\delta)^2\Big(-\frac{\varepsilon}{3}\int_0^L g_2\frac{d\mathcal{U}_2}{dx_1}\,dx_1 - \frac{\delta}{3}\int_0^L g_3\frac{d\mathcal{U}_3}{dx_1}\,dx_1 + \frac{2\varepsilon}{3}\int_0^L g_1\Theta\,dx_1\Big).
\end{aligned}
$$

Hence, from $(17)_{1,2,3,4}$ and $(18)_{1,2}$ we deduce that

$$\Big|\int_{\Omega_{\varepsilon,\delta}} f_{\varepsilon,\delta}\cdot U_{BN}\,dx\Big| \le C(\varepsilon\delta)^{3/2}\big(\|f\|_{L^2(0,L)}+\|g\|_{L^2(0,L)}\big)\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}. \tag{35}$$

The constant does not depend on ε and δ.
Applying the estimates (34)-(35) for $u_{\varepsilon,\delta}$ taken as test function in (33), give the estimate

$$\|e(u_{\varepsilon,\delta})\|_{L^p(\Omega_{\varepsilon,\delta})} \le C(\varepsilon\delta)^{3/2}\big(\|f\|_{L^2(0,L)}+\|g\|_{L^2(0,L)}\big). \tag{36}$$

6.2 | The rescaled limit problem

Theorem 3. Let $u_{\varepsilon,\delta}$ be the solution of the elasticity problem (33). Then, there exists $\big(\mathcal{U},\Theta,\overline{U},\overline{U}^{pl}\big)\in\mathbb{D}_\gamma\times\mathbb{D}_W$ such that for the whole sequence $\{(\varepsilon,\delta)\}$ the convergences (26) and the following hold:

$$\frac{1}{\varepsilon\delta}\Pi_{\varepsilon,\delta}\big(e(u_{\varepsilon,\delta})\big)\longrightarrow E\big(\mathcal{U},\Theta,\overline{U},\overline{U}^{pl}\big)\quad\text{stronglyly in}\quad L^2(\Omega)^{3\times3}. \tag{37}$$

The quadruplet $\big(\mathcal{U},\Theta,\overline{U},\overline{U}^{pl}\big)$ belonging to $\mathbb{D}_\gamma\times\mathbb{D}_W$ is the solution of the variational problem

$$\begin{aligned}&\int_\Omega a_{ijkl}E_{ij}\big(\mathcal{U},\Theta,\overline{U},\overline{U}^{pl}\big)E_{kl}\big(\Phi,\Psi,\overline{\Phi},\overline{\Phi}^{pl}\big)\,dx_1dX_2dX_3\\ &=4\Big(\int_0^L f\cdot\Phi\,dx_1-\frac{1}{3}\int_0^L g_\alpha\frac{d\Phi_\alpha}{dx_1}dx_1+\frac{2}{3}\int_0^L g_1\Psi\,dx_1\Big),\quad\forall\big(\Phi,\Psi,\overline{\Phi},\overline{\Phi}^{pl}\big)\in\mathbb{D}_\gamma\times\mathbb{D}_W.\end{aligned} \tag{38}$$

Proof. The solution to problem (33) satisfies (36). So, there exists a subsequence of $\{(\varepsilon,\delta)\}$, still denoted $\{(\varepsilon,\delta)\}$ and $\big(\mathcal{U},\Theta,\overline{U},\overline{U}^{pl}\big)\in\mathbb{D}_\gamma\times\mathbb{D}_W$ such that convergences (26)-(27) hold.
Let $\big(\Phi,\Psi\big)$ be in $\mathbb{D}_\gamma$, such that $\Psi\in W^{2,p}_\gamma(0,L)$, and $\big(\overline{\Phi},\overline{\Phi}^{pl}\big)\in\mathbb{D}_W\cap\big(H^1_\Gamma(\Omega)^3\times H^1_\Gamma(\Omega)^3\big)$.
Now, consider the test displacement

$$\begin{aligned}\phi_{\varepsilon,\delta}(x)=&\begin{pmatrix}\varepsilon\delta\Phi_1(x_1)-x_2\delta\dfrac{d\Phi_2}{dx_1}(x_1)-x_3\varepsilon\dfrac{d\Phi_3}{dx_1}(x_1)\\ \delta\Phi_2(x_1)-x_3\varepsilon\Psi(x_1)\\ \varepsilon\Phi_3(x_1)+x_2\varepsilon\Psi(x_1)\end{pmatrix}-x_2x_3\varepsilon\frac{d\Psi}{dx_1}(x_1)\mathbf{e}_1\\ &+\varepsilon^2\delta\begin{pmatrix}\overline{\Phi}_1\Big(x_1,\dfrac{x_2}{\varepsilon}\Big)-x_3\dfrac{\varepsilon}{\delta}\dfrac{\partial\overline{\Phi}_3}{\partial x_1}\Big(x_1,\dfrac{x_2}{\varepsilon}\Big)\\ \overline{\Phi}_2\Big(x_1,\dfrac{x_2}{\varepsilon}\Big)-\dfrac{x_3}{\delta}\dfrac{\partial\overline{\Phi}_3}{\partial X_2}\Big(x_1,\dfrac{x_2}{\varepsilon}\Big)\\ \dfrac{\varepsilon}{\delta}\overline{\Phi}_3\Big(x_1,\dfrac{x_2}{\varepsilon}\Big)\end{pmatrix}+\varepsilon\delta^2\overline{\Phi}^{pl}\Big(x_1,\frac{x_2}{\varepsilon},\frac{x_3}{\delta}\Big)\quad\text{for a.e. } x \text{ in } \Omega_{\varepsilon,\delta}.\end{aligned}$$

A straightforward calculation gives

$$\frac{1}{\varepsilon\delta}\Pi_{\varepsilon,\delta}\big(e(\phi_{\varepsilon,\delta})\big)\longrightarrow E\big(\Phi,\Psi,\overline{\Phi},\overline{\Phi}^{pl}\big)\quad\text{strongly in } L^2(\Omega)^{3\times3}.$$

In (33), we take $\phi_{\varepsilon,\delta}$ as test function, we transform the RHS and LHS of this equality thanks to $\Pi_{\varepsilon,\delta}$, we divide by $(\varepsilon\delta)^3$ and finally we pass to the limit. We obtain (38) with $\big(\Phi,\Psi,\overline{\Phi},\overline{\Phi}^{pl}\big)$. Then, a density argument gives (38) for all $\big(\Phi,\Psi,\overline{\Phi},\overline{\Phi}^{pl}\big)\in\mathbb{D}_\gamma\times\mathbb{D}_W$. Due to (31) and the Lax-Milgram theorem, problem (38) has a unique solution. As a consequence, the whole sequences converge to their limits. Proceeding as usual we show the strong convergence (37). ☐

6.3 | The system satisfied by $(\mathcal{U}, \Theta)$

Now, we express the displacements $\overline{U}$ and $\overline{U}^{pl}$ in terms of $\mathcal{U}$ and Θ.
Set

$$\mathbf{M}^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{M}^2 = \begin{pmatrix} -X_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{M}^3 = \begin{pmatrix} -X_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{M}^4 = \begin{pmatrix} 0 & -X_3 & 0 \\ -X_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The 4 pairs of correctors are the solutions to ($m \in \{1, 2, 3, 4\}$)

$$\begin{cases} \big(\overline{\chi}^{(m)}, \overline{\chi}^{pl,(m)}\big) \in \mathbb{D}^{(2)}_{W^{kl}} \times \mathbb{D}^{(2)}_{W^{pl}}, \\ \displaystyle\int_P a_{ijkl}\Big(\mathbf{M}^m_{ij} + E_{ij}\big(0, 0, \overline{\chi}^{(m)}, \overline{\chi}^{pl,(m)}\big)\Big) E_{kl}\big(0, 0, \overline{\Phi}, \overline{\Phi}^{pl}\big)\, dX_2 dX_3 = 0 \\ \forall \big(\overline{\Phi}, \overline{\Phi}^{pl}\big) \in \mathbb{D}^{(2)}_{W^{kl}} \times \mathbb{D}^{(2)}_{W^{pl}}. \end{cases} \tag{39}$$

So, we get

$$\big(\overline{U}, \overline{U}^{pl}\big) = \frac{dU_1}{dx_1}\big(\overline{\chi}^{(1)}, \overline{\chi}^{pl,(1)}\big) + \frac{d^2U_2}{dx_1^2}\big(\overline{\chi}^{(2)}, \overline{\chi}^{pl,(2)}\big) + \frac{d^2U_3}{dx_1^2}\big(\overline{\chi}^{(3)}, \overline{\chi}^{pl,(3)}\big) + \frac{d\Theta}{dx_1}\big(\overline{\chi}^{(4)}, \overline{\chi}^{pl,(4)}\big).$$

Theorem 4. The pair $\big(\mathcal{U}, \Theta) \in \mathbb{D}_\gamma$ is the unique solution to the variational problem

$$\begin{aligned} &\int_0^L A \frac{d}{dx_1}\begin{pmatrix} \mathcal{U}_1 \\ \mathcal{U}_2 \\ \mathcal{U}_3 \\ \Theta \end{pmatrix} \cdot \frac{d}{dx_1}\begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Psi \end{pmatrix} = 4\Big(\int_0^L f \cdot \Phi\, dx_1 - \frac{1}{3}\int_0^L g_\alpha \frac{d\Phi_\alpha}{dx_1} dx_1 + \frac{2}{3}\int_0^L g_1 \Psi\, dx_1 \Big), \\ &\forall \big(\Phi, \Psi, \overline{\Phi}, \overline{\Phi}^{pl}\big) \in \mathbb{D}_\gamma \end{aligned} \tag{40}$$

where the entries of the 4×4 symmetric matrix A are given by ($(m, n) \in \{1, 2, 3, 4\}^2$)

$$A_{mn} = \int_P a_{ijkl}\big(\mathbf{M}^m_{ij} + E_{ij}\big(0, 0, \overline{\chi}^{(m)}, \overline{\chi}^{pl,(m)}\big)\big)\big(\mathbf{M}^n_{kl} + E_{kl}\big(0, 0, \overline{\chi}^{(n)}, \overline{\chi}^{pl,(n)}\big)\big)\, dX_2 dX_3.$$

This matrix is definite positive.

Proof. Let ξ be a vector in $\mathbb{R}^4$. We have

$$A\xi \cdot \xi = \sum_{m,n=1}^4 \int_P a_{ijkl}\xi_m\xi_n\big(\mathbf{M}^m_{ij} + E_{ij}\big(0, 0, \overline{\chi}^{(m)}, \overline{\chi}^{pl,(m)}\big)\big)\big(\mathbf{M}^n_{kl} + E_{kl}\big(0, 0, \overline{\chi}^{(n)}, \overline{\chi}^{pl,(n)}\big)\big)\, dX_2 dX_3.$$

Set

$$M(\xi) = \begin{pmatrix} \xi_1 - X_2\xi_2 - X_3\xi_3 & -X_3\xi_4 & 0 \\ -X_3\xi_4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \big(\overline{\chi}(\xi), \overline{\chi}^{pl}(\xi)\big) = \sum_{m=1}^4 \xi_m\big(\overline{\chi}^{(m)}, \overline{\chi}^{pl,(m)}\big).$$

This allows us to rewrite $A\xi \cdot \xi$ as

$$A\xi \cdot \xi = \int_P a_{ijkl}\xi_m\xi_n\big(\mathbf{M}_{ij}(\xi) + E_{ij}\big(0, 0, \overline{\chi}(\xi), \overline{\chi}^{pl}(\xi)\big)\big)\big(\mathbf{M}_{kl}(\xi) + E_{kl}\big(0, 0, \overline{\chi}(\xi), \overline{\chi}^{pl}(\xi)\big)\big)\, dX_2 dX_3.$$

Thanks to (32), we deduce that

$$A\xi \cdot \xi \geq c_0 \int_P \Big|\mathbf{M}_{ij}(\xi) + E_{ij}\big(0, 0, \overline{\chi}(\xi), \overline{\chi}^{pl}(\xi)\big)\Big|^2 dX_2 dX_3.$$

Now, proceeding as in Lemma 1 leads to

$$A\xi \cdot \xi \geq C\big(|\xi|^2 + \|\overline{\chi}(\xi)\|^2_{W^{kl,2}} + \|\overline{\chi}^{pl}(\xi)\|^2_{W^{pl,2}}\big)$$

where C is a constant strictly positive. ◻

6.4 | The case of a homogeneous and isotropic material

In this subsection, we consider a thin-walled beam made of a homogeneous and isotropic material. So, we have

$$a_{ijkl} \doteq \lambda\boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{kl} + \mu\big(\boldsymbol{\delta}_{ik}\boldsymbol{\delta}_{jl} + \boldsymbol{\delta}_{il}\boldsymbol{\delta}_{jk}\big), \qquad \{i,j,k,l\} \in \{1,2,3\}^4$$

where $\boldsymbol{\delta}_{ij}$ is the Kronecker symbol and λ, μ the Lamé's constants.
For all $\xi \in \mathbb{R}^4$ we consider the problem satisfies by $\big(\overline{\chi}(\xi), \overline{\chi}^{pl}(\xi)\big) \in \mathbb{D}^{(2)}_{W^{kl}} \times \mathbb{D}^{(2)}_{W^{pl}}$. We have

$$\begin{aligned}
&\int_\omega \Big\{\Big[\lambda\big(\xi_1 - X_2\xi_2 - X_3\xi_3\big) + (\lambda+2\mu)\Big(\frac{\partial\overline{\chi}_2(\xi)}{\partial X_2} - X_3\frac{\partial^2\overline{\chi}_3(\xi)}{\partial X_2^2}\Big) + \lambda\frac{\partial\overline{\chi}^{pl}_3(\xi)}{\partial X_3}\Big]\Big(\frac{\partial\overline{\Phi}_2}{\partial X_2} - X_3\frac{\partial^2\overline{\Phi}_3}{\partial X_2^2}\Big)\\
&+\Big[\lambda\big(\xi_1 - X_2\xi_2 - X_3\xi_3\big) + \lambda\Big(\frac{\partial\overline{\chi}_2(\xi)}{\partial X_2} - X_3\frac{\partial^2\overline{\chi}_3(\xi)}{\partial X_2^2}\Big) + (\lambda+2\mu)\frac{\partial\overline{\chi}^{pl}_3(\xi)}{\partial X_3}\Big]\frac{\partial\overline{\Phi}^{pl}_3}{\partial X_3}\Big\}dX_2dX_3 = 0,\\
&\forall\big(\overline{\Phi}, \overline{\Phi}^{pl}\big) \in \mathbb{D}^{(2)}_{W^{kl}} \times \mathbb{D}^{(2)}_{W^{pl}}
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
&\int_\omega \frac{\partial\overline{\chi}^{pl}_1(\xi)}{\partial X_3}\frac{\partial\overline{\Phi}^{pl}_1}{\partial X_3}dX_2dX_3 = 0, \qquad \int_\omega \frac{\partial\overline{\chi}^{pl}_2(\xi)}{\partial X_3}\frac{\partial\overline{\Phi}^{pl}_2}{\partial X_3}dX_2dX_3 = 0,\\
&\int_\omega \Big(-X_3\xi_4 + \frac{\partial\overline{\chi}_1(\xi)}{\partial X_2}\Big)\frac{\partial\overline{\Phi}_1}{\partial X_2}dX_2dX_3 = 0, \qquad \forall\big(\overline{\Phi}, \overline{\Phi}^{pl}\big) \in \mathbb{D}^{(2)}_{W^{kl}} \times \mathbb{D}^{(2)}_{W^{pl}}.
\end{aligned} \tag{42}$$

A straightforward calculation leads to

$$\begin{aligned}
&\overline{\chi}_1(\xi)(X_2) = 0, \quad \overline{\chi}_2(\xi)(X_2) = -\nu\Big(X_2\xi_1 - \Big(\frac{X_2^2}{2} - \frac{1}{6}\Big)\xi_2\Big), \quad \overline{\chi}_3(\xi)(X_2) = -\nu\Big(\frac{X_2^2}{2} - \frac{1}{6}\Big)\xi_3,\\
&\overline{\chi}^{pl}_1(\xi)(X_2,X_3) = \overline{\chi}^{pl}_2(\xi)(X_2,X_3) = 0, \qquad \overline{\chi}^{pl}_3(\xi)(X_2,X_3) = -\nu\Big(\xi_1X_3 - X_2X_3\xi_2 - \Big(\frac{X_3^2}{2} - \frac{1}{6}\Big)\xi_3\Big)
\end{aligned}$$

where $\nu = \dfrac{\lambda}{2(\lambda+\mu)}$ is the Poisson coefficient.
So, we get

$$\begin{aligned}
&\overline{U}_1 = 0, \quad \overline{U}_2 = -\nu\Big(X_2\frac{dU_1}{dx_1} - \Big(\frac{X_2^2}{2} - \frac{1}{6}\Big)\frac{d^2U_2}{dx_1^2}\Big), \quad \overline{U}_3 = -\nu\Big(\frac{X_2^2}{2} - \frac{1}{6}\Big)\frac{d^2U_3}{dx_1^2},\\
&\overline{U}^{pl}_1 = \overline{U}^{pl}_2 = 0, \qquad \overline{U}^{pl}_3 = -\nu\Big(X_3\frac{dU_1}{dx_1} - X_2X_3\frac{d^2U_2}{dx_1^2} - \Big(\frac{X_3^2}{2} - \frac{1}{6}\Big)\frac{d^2U_3}{dx_1^2}\Big).
\end{aligned}$$

Problem (43) becomes

$$\begin{aligned}
&E\int_0^L \frac{d\mathcal{U}_1}{dx_1}\frac{d\Phi_1}{dx_1}dx_1 = \int_0^L f_1\,\Phi_1\,dx_1, \qquad \mu\int_0^L \frac{d\Theta}{dx_1}\frac{d\Psi}{dx_1}dx_1 = 2\int_0^L g_1\Psi\,dx_1,\\
&\frac{E}{3}\int_0^L \frac{d^2\mathcal{U}_\alpha}{dx_1^2}\frac{d^2\Phi_\alpha}{dx_1^2}dx_1 = \int_0^L f_\alpha\Phi_\alpha\,dx_1 - \frac{1}{3}\int_0^L g_\alpha\frac{d\Phi_\alpha}{dx_1}dx_1, \qquad \forall\big(\Phi,\Psi\big) \in \mathbb{D}_\gamma
\end{aligned} \tag{43}$$

where $E = \dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ is the Young modulus.

Now, we can reconstruct the solution to problem (33). We obtain

$$u_{\varepsilon,\delta}(x) \approx \begin{pmatrix} \varepsilon\delta\mathcal{U}_1(x_1) - x_2\delta\dfrac{d\mathcal{U}_2}{dx_1}(x_1) - x_3\varepsilon\dfrac{d\mathcal{U}_3}{dx_1}(x_1) \\ \delta\mathcal{U}_2(x_1) - x_3\varepsilon\Theta(x_1) \\ \varepsilon\mathcal{U}_3(x_1) + x_2\varepsilon\Theta(x_1) \end{pmatrix} - x_2x_3\varepsilon\frac{d\Theta}{dx_1}(x_1)\mathbf{e}_1$$
$$+\varepsilon^2\begin{pmatrix} -x_3\varepsilon\dfrac{\partial\overline{U}_3}{\partial x_1}\Big(x_1,\dfrac{x_2}{\varepsilon}\Big) \\ \delta\overline{U}_2\Big(x_1,\dfrac{x_2}{\varepsilon}\Big) - x_3\dfrac{\partial\overline{U}_3}{\partial X_2}\Big(x_1,\dfrac{x_2}{\varepsilon}\Big) \\ \varepsilon\overline{U}_3\Big(x_1,\dfrac{x_2}{\varepsilon}\Big) \end{pmatrix} + \varepsilon\delta^2\overline{U}_3^{pl}\Big(x_1,\frac{x_2}{\varepsilon},\frac{x_3}{\delta}\Big)\mathbf{e}_3$$

and for the stress tensor we have

$$\sigma(u_{\varepsilon,\delta})(x) \approx \begin{pmatrix} E\Big(\dfrac{d\mathcal{U}_1}{dx_1} - X_2\dfrac{d^2U_2}{dx_1^2} - X_3\dfrac{d^2U_3}{dx_1^2}\Big) & -2\mu X_3\dfrac{d\Theta}{dx_1} & 0 \\ -2\mu X_3\dfrac{d\Theta}{dx_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

7 | APPENDIX

7.1 | Elementary plate displacement of the thin-walled beam

Definition 2. An elementary displacement of the thin-walled beam $\Omega_{\varepsilon,\delta}$ (considered as a plate of thickness 2δ) is a displacement $v \in L^1(\Omega_{\varepsilon,\delta})^3$ written in the form

$$v(x', x_3) = \mathcal{V}(x') + x_3\mathcal{A}(x') \quad \text{for a.e. } x = (x', x_3) \in \Omega_{\varepsilon,\delta}.$$

The component $\mathcal{V}$ belongs to $L^1(P_\varepsilon)^3$ while $\mathcal{A} = \mathcal{A}_1\mathbf{e}_1 + \mathcal{A}_2\mathbf{e}_2$ is in $L^1(P_\varepsilon)^2$.
Here, $\mathcal{V}$ gives the mid-surface displacement and $x_3\mathcal{A}(x')$ represents a "small rotation" of the fiber $\{x'\} \times (-\delta, \delta)$, whose axis is directed by $-\mathcal{A}_2(x')\mathbf{e}_1 + \mathcal{A}_1(x')\mathbf{e}_2$ and whose angle is approximately $|\mathcal{A}(x')|$.

To any displacement $u \in L^1(\Omega_{\varepsilon,\delta})^3$ we associate an elementary displacement $U^*_{e\ell} \in L^1(\Omega_{\varepsilon,\delta})^3$ and a warping $\overline{u}^* \in L^1(\Omega_{\varepsilon,\delta})^3$

$$\begin{aligned} u(x) &= U^*_{e\ell}(x) + \overline{u}^*(x) \\ U^*_{e\ell}(x) &= \mathcal{U}^*(x') + x_3\mathcal{R}^*(x') \end{aligned} \qquad \text{for a.e. } x = (x', x_3) \in \Omega_{\varepsilon,\delta} \tag{44}$$

so that

$$\int_{-\delta}^{\delta} \overline{u}^*(\cdot, x_3)dx_3 = 0, \quad \int_{-\delta}^{\delta} \overline{u}_1^*(\cdot, x_3)x_3dx_3 = \int_{-\delta}^{\delta} \overline{u}_2^*(\cdot, x_3)x_3dx_3 = 0 \quad \text{a.e. in } P_\varepsilon. \tag{45}$$

The above equalities determine $\mathcal{U}^*(x')$ and $\mathcal{R}^*(x')$ in terms of u and integrals on the fiber $\{x'\} \times (-\delta, \delta)$ (see [12]). We have

$$\begin{aligned} \mathcal{U}^*(x') &= \frac{1}{2\delta}\int_{-\delta}^{\delta} u(x', x_3)dx_3, \\ \mathcal{R}^*(x') &= \frac{3}{2\delta^3}\int_{-\delta}^{\delta} x_3\big(u_1(x', x_3)\mathbf{e}_1 + u_2(x', x_3)\mathbf{e}_2\big)dx_3, \end{aligned} \qquad \text{for a.e. } x' \in P_\varepsilon.$$

Theorem 5 (Theorem 4.1 in[12]). Let u be a displacement in $W^{1,p}(\Omega_{\varepsilon,\delta})^3$, $p \in (1,\infty)$, decomposed as (44). The terms $\mathcal{U}^*$, $\mathcal{R}^*$ and $\overline{u}^*$ of this decomposition satisfy

$$\begin{aligned}
&\mathcal{U}^* \in W^{1,p}(P_\varepsilon)^3, \quad \mathcal{R}^* \in W^{1,p}(P_\varepsilon)^2, \quad \overline{u}^* \in W^{1,p}(\Omega_{\varepsilon,\delta})^3,\\
&\|\overline{u}^*\|_{L^p(\Omega_{\varepsilon,\delta})} \le C\delta\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}, \qquad \|\nabla\overline{u}^*\|_{L^p(\Omega_{\varepsilon,\delta})} \le C\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})},\\
&\delta\|\nabla\mathcal{R}^*\|_{L^p(P_\varepsilon)} + \|e_{\alpha\beta}(\mathcal{U}^*)\|_{L^p(P_\varepsilon)} + \Big\|\frac{\partial \mathcal{U}^*_3}{\partial x_\alpha} + \mathcal{R}^*_\alpha\Big\|_{L^p(P_\varepsilon)} \le \frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}.
\end{aligned} \tag{46}$$

The constants do not depend on ε, δ and L.

Proof. In[12] Theorem 4.1 we have considered a plate whose mid-surface is a bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary. We have proved that the constants in the estimates given in[12] Theorem 4.1 are independent of δ. In fact, these constants depend only on the boundary of the mid surface and on p.
Now, if we revisit the proof of[12] Theorem 4.1 bearing in mind that the mid-surface of the thin-walled beam is P_ε, we realize that what is important is to fill $\Omega_{\varepsilon,\delta}$ with parallelotopes whose dimensions we control.
Set

$$N_{\varepsilon,\delta} = \Big[\frac{\varepsilon}{\delta}\Big], \qquad N_\delta = \Big[\frac{L}{\delta}\Big], \qquad l_{\varepsilon,\delta} = \frac{\varepsilon}{N_{\varepsilon,\delta}}, \qquad l_\delta = \frac{L}{N_\delta}$$

where $[t]$ is the integer part of $t \in \mathbb{R}$. We have

$$\delta \le l_{\varepsilon,\delta} \le 2\delta, \qquad \delta \le l_\delta \le 2\delta.$$

Denote $Y_{\varepsilon,\delta} \doteq (0, l_\delta)\times(0, l_{\varepsilon,\delta})\times(-\delta,\delta)$. Note that $Y_{\varepsilon,\delta}$ has a diameter less than $R_\delta = 4\delta$ and it contains a ball of radius $r_\delta = \delta/2$. This is important because the estimates in[12] Theorem 4.1 are controlled by the ratio $R_\delta/r_\delta \le 8$.
Observe that $\Omega_{\varepsilon,\delta}$ can be entirely filled with parallelotopes isometric to $Y_{\varepsilon,\delta}$, two by two with empty intersections.
It now remains to follow the lines of the proof of[12] Theorem 4.1 to obtain the estimates (46) with constants independent of ε, δ and L. ☐

7.2 | Extension of a thin-walled beam displacement

Denote

$$\begin{aligned}
&P^{(1)}_\varepsilon \doteq (-L,L)\times(-\varepsilon,\varepsilon), && P^{(2)}_\varepsilon \doteq (-L,2L)\times(-\varepsilon,\varepsilon), && P^{(3)}_\varepsilon \doteq (-L,2L)\times(-\varepsilon,3\varepsilon),\\
&\Omega^{(1)} \doteq P^{(1)}_\varepsilon\times(-\delta,\delta), && \Omega^{(2)} \doteq P^{(2)}_\varepsilon\times(-\delta,\delta), && \Omega^{(3)} \doteq P^{(3)}_\varepsilon\times(-\delta,\delta),\\
&P'_\varepsilon \doteq (-L,2L)\times(-3\varepsilon,3\varepsilon), && \omega'_{\varepsilon,\delta} \doteq (-3\varepsilon,3\varepsilon)\times(-\delta,\delta), && \Omega'_{\varepsilon,\delta} \doteq P'_\varepsilon\times(-\delta,\delta).
\end{aligned}$$

Proposition 3. There exists an extension operator $\mathcal{P}_\varepsilon$ from $W^{1,p}(\Omega_{\varepsilon,\delta})^3$ into $W^{1,p}(\Omega'_{\varepsilon,\delta})^3$, $p \in (1,\infty)$, satisfying

$$\forall u \in W^{1,p}(\Omega_{\varepsilon,\delta})^3, \quad \mathcal{P}_\varepsilon(u) \in W^{1,p}(\Omega'_{\varepsilon,\delta})^3, \quad \mathcal{P}_\varepsilon(u)_{|\Omega_{\varepsilon,\delta}} = u, \quad \Big\|e(\mathcal{P}_\varepsilon(u))\Big\|_{L^p(\Omega'_{\varepsilon,\delta})} \le C\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}.$$

The constant does not depend on ε, δ and L.

Moreover, if $u = 0$ a.e. on $\Gamma_{\varepsilon,\delta}$ then $\mathcal{P}_\varepsilon(u) = 0$ a.e. in $(-L,0)\times\omega'_{\varepsilon,\delta}$.

Proof. Construction of $\mathcal{P}_\varepsilon(u)$.

We decompose u as (44).

Step 1. Extension of u to the thin-walled beam $\Omega^{(2)}_{\varepsilon,\delta}$.

First, if $u = 0$ a.e. on $\{0\}\times\omega_{\varepsilon,\delta}$ then we extend u by 0 in $(-L,0)\times\omega_{\varepsilon,\delta}$. Obviously the terms of the decomposition of u (see (44)) are also extended by 0 in $(-L,0)\times\omega_{\varepsilon,\delta}$.

Otherwise, we set

$$\begin{aligned}
&\mathcal{U}^{*1}(x') = \mathcal{U}^{*}(x') && \text{for a.e. } x' \in P_\varepsilon,\\
&\mathcal{U}^{*1}(x') = 4\mathcal{U}^{*}\Big(-\frac{x_1}{2}, x_2\Big) - 3\mathcal{U}^{*}(-x_1, x_2) && \text{for a.e. } x' \in (-L,0)\times(-\varepsilon,\varepsilon),\\
&\mathcal{R}^{*1}(x') = \mathcal{R}^{*}(x') && \text{for a.e. } x' \in P_\varepsilon,\\
&\mathcal{R}^{*1}(x') = -2\mathcal{R}^{*}\Big(-\frac{x_1}{2}, x_2\Big) + 3\mathcal{R}^{*}(-x_1, x_2) && \text{for a.e. } x' \in (-L,0)\times(-\varepsilon,\varepsilon),\\
&\overline{u}^{*1}(x) = \overline{u}^{*}(x) && \text{for a.e. } x \in \Omega_{\varepsilon,\delta},\\
&\overline{u}^{*1}(x) = \overline{u}^{*}(-x_1, x_2, x_3) && \text{for a.e. } x \in (-L,0)\times\omega_{\varepsilon,\delta}.
\end{aligned}$$

We have

$$\mathcal{U}^{*1} \in W^{1,p}(P_\varepsilon^{(1)})^3, \quad \mathcal{R}^{*1} \in W^{1,p}(P_\varepsilon^{(1)})^2, \quad \overline{u}^{*1} \in W^{1,p}(\Omega_{\varepsilon,\delta}^{(1)})^3.$$

Using the estimates (46), we easily check that

$$\begin{aligned}
&\|\overline{u}^{*1}\|_{L^p(\Omega^{(1)}_{\varepsilon,\delta})} \le C\delta\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}, \qquad \|\nabla\overline{u}^{*1}\|_{L^p(\Omega^{(1)}_{\varepsilon,\delta})} \le C\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})},\\
&\delta\|\nabla\mathcal{R}^{*1}\|_{L^p(P^{(1)}_\varepsilon)} + \|e_{\alpha\beta}(\mathcal{U}^{*1})\|_{L^p(P^{(1)}_\varepsilon)} + \Big\|\frac{\partial\mathcal{U}^{*1}_3}{\partial x_\alpha} + \mathcal{R}^{*1}_\alpha\Big\|_{L^p(P^{(1)}_\varepsilon)} \le \frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}.
\end{aligned} \tag{47}$$

We set

$$u^{*1}(x) = \mathcal{U}^{*1}(x') + x_3\mathcal{R}^{*1}(x') + \overline{u}^{*1}(x) \qquad \text{for a.e. } x \in \Omega^{(1)}_{\varepsilon,\delta}.$$

Thus, we have $u^{*1} \in W^{1,p}(\Omega^{(1)}_{\varepsilon,\delta})^3$. A straightforward calculation yields

$$\|e(u^{*1})\|_{L^p(\Omega^{(1)}_{\varepsilon,\delta})} \le C\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}. \tag{48}$$

The constants do not depend on ε, δ and L.

We proceed in a similar way to extend u and the terms of its decomposition in $(L,2L)\times\omega_{\varepsilon,\delta}$. We denote u^{*2} the extension of u to the domain $(-L,2L)\times\omega_{\varepsilon,\delta}$ and $\mathcal{U}^{*2}$, $\mathcal{R}^{*2}$, $\overline{u}^{*2}$ the terms of its decomposition. The estimates (47) and (48) are still valid replacing $\Omega^{(1)}_{\varepsilon,\delta}$ by $\Omega^{(2)}_{\varepsilon,\delta}$, of course the constants are always independent of ε, δ and L.

Hence, we have

$$\begin{aligned}
&\|\overline{u}^{*2}\|_{L^p(\Omega^{(2)}_{\varepsilon,\delta})} \le C\delta\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}, \qquad \|\nabla\overline{u}^{*2}\|_{L^p(\Omega^{(2)}_{\varepsilon,\delta})} \le C\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})},\\
&\delta\|\nabla\mathcal{R}^{*2}\|_{L^p(P^{(2)}_\varepsilon)} + \|e_{\alpha\beta}(\mathcal{U}^{*2})\|_{L^p(P^{(2)}_\varepsilon)} + \Big\|\frac{\partial\mathcal{U}^{*2}_3}{\partial x_\alpha} + \mathcal{R}^{*2}_\alpha\Big\|_{L^p(P^{(2)}_\varepsilon)} \le \frac{C}{\delta^{2/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}.
\end{aligned} \tag{49}$$

So, $u^{*2} \in W^{1,p}(\Omega^{(2)}_{\varepsilon,\delta})^3$ and is decomposed as

$$u^{*2}(x) = \mathcal{U}^{*2}(x') + x_3\mathcal{R}^{*2}(x') + \overline{u}^{*2}(x) \qquad \text{for a.e. } x \in \Omega^{(2)}_{\varepsilon,\delta}.$$

It satisfies

$$\|e(u^{*2})\|_{L^p(\Omega^{(2)}_{\varepsilon,\delta})} \le C\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}. \tag{50}$$

Step 2. Extension to the thin-walled beam $\Omega^{(3)}_{\varepsilon,\delta}$.

We set

$$\begin{aligned}
&\mathcal{U}^{*3}(x') = \mathcal{U}^{*2}(x') && \text{for a.e. } x' \in P^{(2)}_\varepsilon,\\
&\mathcal{U}^{*3}(x') = 4\mathcal{U}^{*2}\Big(x_1, \frac{3\varepsilon - x_2}{2}\Big) - 3\mathcal{U}^{*2}(x_1, 2\varepsilon - x_2) && \text{for a.e. } x' \in (-L,2L)\times(\varepsilon,3\varepsilon),\\
&\mathcal{R}^{*3}(x') = \mathcal{R}^{*2}(x') && \text{for a.e. } x' \in P^{(2)}_\varepsilon,\\
&\mathcal{R}^{*3}(x') = -2\mathcal{R}^{*2}\Big(x_1, \frac{3\varepsilon - x_2}{2}\Big) + 3\mathcal{R}^{*2}(x_1, 2\varepsilon - x_2) && \text{for a.e. } x' \in (-L,2L)\times(\varepsilon,3\varepsilon),\\
&\overline{u}^{*3}(x) = \overline{u}^{*2}(x) && \text{for a.e. } x \in \Omega^{(2)}_{\varepsilon,\delta},\\
&\overline{u}^{*3}(x) = \overline{u}^{*2}(x_1, 2\varepsilon - x_2, x_3) && \text{for a.e. } x \in (-L,2L)\times(\varepsilon,3\varepsilon)\times(-\delta,\delta).
\end{aligned}$$

Here, using the estimates (47), we obtain

$$
\begin{aligned}
&\mathcal{U}^{*3} \in W^{1,p}(P_\varepsilon^{(3)})^3, \quad \mathcal{R}^{*3} \in W^{1,p}(P_\varepsilon^{(3)})^2, \quad \overline{u}^{*3} \in W^{1,p}(\Omega_{\varepsilon,\delta}^{(3)})^3,\\
&\|\overline{u}^{*3}\|_{L^p(\Omega^{(3)}_{\varepsilon,\delta})} \le C\delta\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}, \qquad \|\nabla\overline{u}^{*3}\|_{L^p(\Omega^{(3)}_{\varepsilon,\delta})} \le C\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})},\\
&\delta\|\nabla\mathcal{R}^{*3}\|_{L^p(P_\varepsilon^{(3)})} + \|e_{\alpha\beta}(\mathcal{U}^{*3})\|_{L^p(P_\varepsilon^{(3)})} + \Big\|\frac{\partial\mathcal{U}_3^{*3}}{\partial x_\alpha} + \mathcal{R}_\alpha^{*3}\Big\|_{L^p(P_\varepsilon^{(3)})} \le \frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}.
\end{aligned}
\tag{51}
$$

We set

$$
u^{*3}(x) = \mathcal{U}^{*3}(x') + x_3\mathcal{R}^{*3}(x') + \widetilde{u}^{*3}(x) \qquad \text{for a.e. } x \in \Omega^{(3)}_{\varepsilon,\delta}.
$$

Thus, we have $u^{*3} \in W^{1,p}(\Omega^{(3)}_{\varepsilon,\delta})^3$ and

$$
\|e(u^{*3})\|_{L^p(\Omega^{(3)}_{\varepsilon,\delta})} \le C\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}.
$$

The constants do not depend on ε, δ and L.

Step 3. Extension to the thin-walled beam $\Omega'_{\varepsilon,\delta}$.

We proceed as in Step 2 to extend u^{*3} and the terms of its decomposition in $\Omega'_{\varepsilon,\delta}$. We denote u^{**} the extension of u to the domain $\Omega'_{\varepsilon,\delta}$ and $\mathcal{U}^{**}$, $\mathcal{R}^{**}$, $\widetilde{u}^{**}$ the terms of its decomposition. The estimates (47) and (48) are still valid replacing $\Omega^{(1)}_{\varepsilon,\delta}$ by $\Omega^{(2)}_{\varepsilon,\delta}$, of course the constants are always independent of ε, δ and L.

We finally obtain

$$
\begin{aligned}
&\mathcal{U}^{**} \in W^{1,p}(P'_\varepsilon)^3, \quad \mathcal{R}^{**} \in W^{1,p}(P'_\varepsilon)^2, \quad \overline{u}^{**} \in W^{1,p}(\Omega'_{\varepsilon,\delta})^3,\\
&\|\overline{u}^{**}\|_{L^p(\Omega'_{\varepsilon,\delta})} \le C\delta\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}, \qquad \|\nabla\overline{u}^{**}\|_{L^p(\Omega'_{\varepsilon,\delta})} \le C\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})},\\
&\delta\|\nabla\mathcal{R}^{**}\|_{L^p(P'_\varepsilon)} + \|e_{\alpha\beta}(\mathcal{U}^{**})\|_{L^p(P'_\varepsilon)} + \Big\|\frac{\partial\mathcal{U}_3^{**}}{\partial x_\alpha} + \mathcal{R}_\alpha^{**}\Big\|_{L^p(P'_\varepsilon)} \le \frac{C}{\delta^{1/p}}\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}.
\end{aligned}
\tag{52}
$$

We set

$$
\mathcal{P}_\varepsilon(u)(x) = \mathcal{U}^{**}(x') + x_3\mathcal{R}^{**}(x') + \overline{u}^{**}(x) \qquad \text{for a.e. } x \in \Omega'_{\varepsilon,\delta}.
$$

We have $\mathcal{P}_\varepsilon(u) \in W^{1,p}(\Omega'_{\varepsilon,\delta})^3$ and

$$
\|e\big(\mathcal{P}_\varepsilon(u)\big)\|_{L^p(\Omega'_{\varepsilon,\delta})} \le C\|e(u)\|_{L^p(\Omega_{\varepsilon,\delta})}.
$$

The constants do not depend on ε, δ and L. □

7.3 | Decomposition of functions defined on P_ε

Proposition 4. Let ϕ be in $W^{1,p}(P_\varepsilon)$, $p \in (1,\infty)$. There exist $\Phi \in W^{1,p}(0,L)$ and $\overline{\phi} \in W^{1,p}(P_\varepsilon)$ such that

$$
\phi = \Phi + \overline{\phi} \quad \text{a.e. in } \ P_\varepsilon
$$

with the following estimates

$$
\begin{aligned}
&\big\|\Phi\big\|_{L^p(0,L)} \le \frac{C}{\varepsilon^{1/p}}\big\|\phi\big\|_{L^p(P_\varepsilon)}, \qquad \Big\|\frac{d\Phi}{dx_1}\Big\|_{L^p(0,L)} \le \frac{C}{\varepsilon^{1/p}}\Big\|\frac{\partial\phi}{\partial x_1}\Big\|_{L^p(P_\varepsilon)},\\
&\big\|\overline{\phi}\big\|_{L^p(P_\varepsilon)} \le C\varepsilon\Big\|\frac{\partial\phi}{\partial x_2}\Big\|_{L^p(P_\varepsilon)},\\
&\Big\|\frac{\partial\overline{\phi}}{\partial x_1}\Big\|_{L^p(P_\varepsilon)} \le C\Big\|\frac{\partial\phi}{\partial x_1}\Big\|_{L^p(P_\varepsilon)}, \qquad \Big\|\frac{\partial\overline{\phi}}{\partial x_2}\Big\|_{L^p(P_\varepsilon)} \le \Big\|\frac{\partial\phi}{\partial x_2}\Big\|_{L^p(P_\varepsilon)}.
\end{aligned}
\tag{53}
$$

Furthermore, if $\dfrac{\partial^2\phi}{\partial x_1\partial x_2}$ belongs to $L^p(P_\varepsilon)$ then

$$
\Big\|\frac{\partial\overline{\phi}}{\partial x_1}\Big\|_{L^p(P_\varepsilon)} \le C\varepsilon\Big\|\frac{\partial^2\phi}{\partial x_1\partial x_2}\Big\|_{L^p(P_\varepsilon)}.
\tag{54}
$$

The constants only depend on p.

Proof. We set

$$
\Phi(x_1) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\phi(x_1,x_2)dx_2 \ \text{ for a.e. } x_1 \text{ in } (0,L) \quad \text{and} \quad \overline{\phi} = \phi - \Phi.
$$

We have $\Phi \in W^{1,p}(0,L)$ and $\overline{\phi} \in W^{1,p}(P_\varepsilon)$. The derivative of Φ is

$$\frac{d\Phi}{dx_1}(x_1) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \frac{\partial \phi}{\partial x_1}(x_1,x_2)dx_2 \quad \text{for a.e. in } (0,L).$$

Then, the Hölder inequality yields $(53)_{1,2}$, from which we obtain $(53)_4$. Since we have $\dfrac{\partial \phi}{\partial x_2} = \dfrac{\partial \overline{\phi}}{\partial x_2}$ estimate $(53)_5$ follows.

Observe that $\displaystyle\int_{-\varepsilon}^{\varepsilon} \overline{\phi}(x_1,x_2)dx_2 = 0$ for a.e. x_1 in $(0,L)$. Thus, the Poincaré-Wirtinger inequality leads to $(53)_3$.

We have also $\displaystyle\int_{-\varepsilon}^{\varepsilon} \frac{\partial \overline{\phi}}{\partial x_1}(x_1,x_2)dx_2 = 0$ for a.e. x_1 in $(0,L)$. Hence, if $\dfrac{\partial^2 \phi}{\partial x_1 \partial x_2}$ belongs to $L^p(P_\varepsilon)$ then the Poincaré-Wirtinger inequality leads to (54).

□

Proposition 5. Let ϕ be in $W^{2,p}(P_\varepsilon)$, $p \in (1,\infty)$. There exist $\Phi,\ \Psi \in W^{2,p}(0,L)$ and $\widetilde{\phi} \in W^{2,p}(P_\varepsilon)$ such that

$$\phi = \Phi + x_2\Psi + \widetilde{\phi} \quad \text{a.e. in } P_\varepsilon$$

with the following estimates:

$$\begin{aligned}
&\|\Phi\|_{L^p(0,L)} \le \frac{C}{\varepsilon^{1/p}}\|\phi\|_{L^p(P_\varepsilon)}, && \Big\|\frac{d\Phi}{dx_1}\Big\|_{L^p(0,L)} \le \frac{C}{\varepsilon^{1/p}}\Big\|\frac{\partial \phi}{\partial x_1}\Big\|_{L^p(P_\varepsilon)},\\
&\Big\|\frac{d^2\Phi}{dx_1^2}\Big\|_{L^p(0,L)} \le \frac{C}{\varepsilon^{1/p}}\Big\|\frac{\partial^2 \phi}{\partial x_1^2}\Big\|_{L^p(P_\varepsilon)}, &&\\
&\|\Psi\|_{L^p(0,L)} \le \frac{C}{\varepsilon^{1/p}}\Big\|\frac{\partial \phi}{\partial x_2}\Big\|_{L^p(P_\varepsilon)}, && \Big\|\frac{d\Psi}{dx_1}\Big\|_{L^p(0,L)} \le \frac{C}{\varepsilon^{1/p}}\Big\|\frac{\partial^2 \phi}{\partial x_1\partial x_2}\Big\|_{L^p(P_\varepsilon)},\\
&\Big\|\frac{d^2\Psi}{dx_1^2}\Big\|_{L^p(0,L)} \le \frac{C}{\varepsilon^{1+1/p}}\Big\|\frac{\partial^2 \phi}{\partial x_1^2}\Big\|_{L^p(P_\varepsilon)}, &&\\
&\|\widetilde{\phi}\|_{L^p(P_\varepsilon)} \le C\varepsilon^2\Big\|\frac{\partial^2 \phi}{\partial x_2^2}\Big\|_{L^p(P_\varepsilon)}, && \Big\|\frac{\partial\widetilde{\phi}}{\partial x_2}\Big\|_{L^p(P_\varepsilon)} \le C\varepsilon\Big\|\frac{\partial^2 \phi}{\partial x_2^2}\Big\|_{L^p(P_\varepsilon)},\\
&\Big\|\frac{\partial\widetilde{\phi}}{\partial x_1}\Big\|_{L^p(P_\varepsilon)} \le C\varepsilon\Big\|\frac{\partial^2 \phi}{\partial x_1\partial x_2}\Big\|_{L^p(P_\varepsilon)}, && \Big\|\frac{\partial^2\widetilde{\phi}}{\partial x_2^2}\Big\|_{L^p(P_\varepsilon)} \le \Big\|\frac{\partial^2 \phi}{\partial x_2^2}\Big\|_{L^p(P_\varepsilon)},\\
&\Big\|\frac{\partial^2\widetilde{\phi}}{\partial x_1^2}\Big\|_{L^p(P_\varepsilon)} \le C\Big\|\frac{\partial^2 \phi}{\partial x_1^2}\Big\|_{L^p(P_\varepsilon)}, && \Big\|\frac{\partial^2\widetilde{\phi}}{\partial x_1\partial x_2}\Big\|_{L^p(P_\varepsilon)} \le C\Big\|\frac{\partial^2 \phi}{\partial x_1\partial x_2}\Big\|_{L^p(P_\varepsilon)}.
\end{aligned} \tag{55}$$

The constants only depend on p.

Proof. Step 1. We define Φ, Ψ and $\widetilde{\phi}$.

We set

$$\Phi(x_1) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \phi(x_1,x_2)dx_2 \ \text{ for a.e. } x_1 \text{ in } (0,L),$$

$$\Psi(x_1) = \frac{3}{2\varepsilon^3}\int_{-\varepsilon}^{\varepsilon} \phi(x_1,x_2)x_2dx_2 \ \text{ for a.e. } x_1 \text{ in } (0,L),$$

$$\text{and} \quad \widetilde{\phi}(x_1,x_2) = \phi(x_1,x_2) - \Phi(x_1) - x_2\Psi(x_1) \quad \text{for a.e. } (x_1,x_2) \in P_\varepsilon.$$

We have $\Phi,\ \Psi \in W^{2,p}(0,L)$ and $\widetilde{\phi} \in W^{2,p}(P_\varepsilon)$.

Step 2. We prove the estimates $(55)_{1,2,3,4,5,6}$.

First, as in Proposition 4, we prove $(55)_{1,2,3}$.

Now, observe that

$$\Psi(x_1)=\frac{3}{2\varepsilon^3}\int_{-\varepsilon}^{\varepsilon}\phi(x_1,x_2)x_2dx_2=\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}3\frac{\overline{\phi}(x_1,x_2)}{\varepsilon}\frac{x_2}{\varepsilon}dx_2\quad\text{for a.e. } x_1 \text{ in } (0,L)$$

where $\overline{\phi}=\phi-\Phi$.
Set

$$\psi(x_1,x_2)=3\frac{\overline{\phi}(x_1,x_2)}{\varepsilon}\frac{x_2}{\varepsilon}\quad\text{for a.e. } (x_1,x_2)\in P_\varepsilon.$$

Function ψ belongs to $W^{2,p}(P_\varepsilon)$. From the estimates in Proposition 4 and a straightforward calculation we deduce that

$$\begin{aligned}&\|\psi\|_{L^p(P_\varepsilon)}\le C\Big\|\frac{\partial\phi}{\partial x_2}\Big\|_{L^p(P_\varepsilon)},\quad \Big\|\frac{\partial\psi}{\partial x_1}\Big\|_{L^p(P_\varepsilon)}\le C\Big\|\frac{\partial^2\phi}{\partial x_1\partial x_2}\Big\|_{L^p(P_\varepsilon)},\\ &\Big\|\frac{\partial^2\psi}{\partial x_1\partial x_2}\Big\|_{L^p(P_\varepsilon)}\le\frac{C}{\varepsilon}\Big\|\frac{\partial^2\phi}{\partial x_1\partial x_2}\Big\|_{L^p(P_\varepsilon)}.\end{aligned}$$

Then, again from the estimates in Proposition 4 we obtain $(55)_{4,5,6}$.

Step 3. We prove the estimates $(55)_{7,8,9}$.

Observe that

$$\begin{aligned}&\int_{-\varepsilon}^{\varepsilon}\widetilde{\phi}(x_1,x_2)dx_2=0\quad\text{for a.e. } x_1\in(0,L),\\ &\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{\partial\widetilde{\phi}}{\partial x_2}(x_1,x_2)dx_2=\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{\partial\phi}{\partial x_2}(x_1,x_2)dx_2-\Psi(x_1)\quad\text{for a.e. } x_1\in(0,L).\end{aligned}$$

We have

$$\Psi(x_1)=\frac{3}{2\varepsilon^3}\int_{-\varepsilon}^{\varepsilon}\phi(x_1,x_2)x_2dx_2=-\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{\partial\phi}{\partial x_2}(x_1,x_2)\frac{3(x_2^2-\varepsilon^2)}{2\varepsilon^2}dx_2.$$

So, since $\displaystyle\int_{-\varepsilon}^{\varepsilon}\frac{3x_2^2-\varepsilon^2}{2\varepsilon^2}dx_2=0$

$$\begin{aligned}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{\partial\widetilde{\phi}}{\partial x_2}(x_1,x_2)dx_2&=\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{\partial\phi}{\partial x_2}(x_1,x_2)\frac{3x_2^2-\varepsilon^2}{2\varepsilon^2}dx_2\\ &=\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\overline{\frac{\partial\phi}{\partial x_2}}(x_1,x_2)\frac{3x_2^2-\varepsilon^2}{2\varepsilon^2}dx_2\end{aligned}\qquad\text{for a.e. } x_1\in(0,L).\tag{56}$$

Estimate $(53)_3$ applied with ϕ replaced by $\dfrac{\partial\phi}{\partial x_2}$ gives

$$\Big\|\overline{\frac{\partial\phi}{\partial x_2}}\Big\|_{L^p(P_\varepsilon)}\le C\varepsilon\Big\|\frac{\partial^2\phi}{\partial x_2^2}\Big\|_{L^p(P_\varepsilon)}.$$

As a consequence of the above estimate and equality (56) we obtain

$$\Big\|\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{\partial\widetilde{\phi}}{\partial x_2}(\cdot,x_2)dx_2\Big\|_{L^p(0,L)}\le C\varepsilon^{1-1/p}\Big\|\frac{\partial^2\phi}{\partial x_2^2}\Big\|_{L^p(P_\varepsilon)}.\tag{57}$$

We can now use the Poincaré-Wirtinger inequality with the function $\dfrac{\partial\phi}{\partial x_2}$. Estimate $(53)_3$ yields

$$\Big\|\frac{\partial\phi}{\partial x_2}-\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{\partial\phi}{\partial x_2}(\cdot,x_2)dx_2\Big\|_{L^p(P_\varepsilon)}\le C\varepsilon\Big\|\frac{\partial^2\phi}{\partial x_2^2}\Big\|_{L^p(P_\varepsilon)}.$$

The above together with (57) lead to

$$\left\|\frac{\partial\phi}{\partial x_2}-\Psi\right\|_{L^p(P_\varepsilon)}\le C\varepsilon\left\|\frac{\partial^2\phi}{\partial x_2^2}\right\|_{L^p(P_\varepsilon)}.$$

Again the Poincaré-Wirtinger inequality

$$\left\|\phi-\Phi-x_2\Psi\right\|_{L^p(P_\varepsilon)}\le C\varepsilon^2\left\|\frac{\partial^2\phi}{\partial x_2^2}\right\|_{L^p(P_\varepsilon)}.$$

This proves $(55)_{7,8}$.
We have $\dfrac{\partial^2\widetilde{\phi}}{\partial x_2^2}=\dfrac{\partial^2\phi}{\partial x_2^2}$. This gives $(55)_{10}$. Estimate $(55)_9$ is a consequence of (54) and $(55)_5$. Estimate $(55)_{11}$ comes from $(55)_3$-$(55)_6$. Estimate $(55)_{12}$ is a consequence of $(55)_5$. □

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