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RESEARCH ARTICLE

Relatively exact controllability of fractional stochastic delay system driven by Lévy noise

Jizhao Huang[1] | Danfeng Luo[1]

[1]Department of Mathematics, Guizhou University, Guiyang, 550025, Guizhou, China

Correspondence
Danfeng Luo, Department of Mathematics, Guizhou University, Guiyang, 550025, Guizhou, China.
Email: dfluo@gzu.edu.cn

In this article, we consider the relatively exact controllability of fractional stochastic delay system (FSDS) driven by Lévy noise. Firstly, we derive the solution of linear FSDS via delayed matrix functions of Mittag-Leffler (M-L). Subsequently, by virtue of the controllability Grammian matrix, we explore the relatively exact controllability of linear FSDS. In addition, with the aid of Jensen's inequality, Hölder's inequality and Itô's isometry, the existence and uniqueness of the considered nonlinear FSDS are investigated by employing Banach contraction principle. Thereafter, the relatively exact controllability of nonlinear FSDS is discussed. Finally, the theoretical results are supported through an example.

KEYWORDS:
fractional calculus, relatively exact controllability, stochastic delay system, delayed Mittag-Leffler matrices

MSC CLASSIFICATION
34A08; 93B05; 34K50; 60J65

1 | INTRODUCTION

1.1 | Previous works

Fractional calculus has been of great interest in the past few decades since it has been confirmed to be a powerful tool with more accurate results in the mathematical modeling of many phenomena occurring in physics and engineering. This is due to its wide application in various areas, particularly in control engineering,[1,2] physics,[3,4] economics[5] and image processing;[6] for more detail, we can pay attention to previous studies.[7,8,9]

The time-delay system refers to a kind of dynamic system whose current state is affected by the past state. In addition, time delay not only makes the system more complex, instability and oscillation, but also makes it more difficult to derive the explicit solution of the system. In recent years, the representation of solutions of the linear fractional delay differential equations (FDDEs) has attracted many researchers since it is much more useful than the implicit solution. In the work of Li et al,[13] the researchers obtained an explicit formula for solutions of the Caputo-type linear FDDEs by adopting two parameters delayed matrix function of M-L. Xiao et al[14] derived the solution of linear delay system by means of conformable delayed exponential matrix. In the work of Mahmudov,[15] the author found construction of solutions of the linear Riemann-Liouville FDDEs of order $1 < 2\alpha \leq 2$ by applying delayed matrix functions of sine and cosine. For more results about the representation of solutions to the linear FDDEs, one can refer to other works.[16,17,18]

As we all know, the concept of controllability was first proposed by Kalman in 1960, and it soon became a basic concept in modern control theory, which has been crucial to the development of control theory and engineering , and we are able to discover lots of applications in robotics, optimal control of linear system, and flight control system. Up to now, controllability of fractional system is still one of the heated research topics. Due to controllability as a qualitative property of the dynamic system, it is vital to discover the representation of its solution. In the work of Joice et al,[21] the authors gave the representation of solution by virtue of Laplace transform and considered controllability criteria for FDDEs. Recently, in the work of Li et al,[22] by means of delayed matrix functions of M-L, the researchers sought the representation of solution and presented controllability of the linear FDDEs. For more detail about controllability, we can pay attention to previous works.[23,24,25,26]

In the real world, many practical systems affected by random disturbances from the external environment are usually modeled as stochastic differential equations (SDEs). In many literatures, we can find that the noise process many researchers assume in stochastic questions is the Wiener process. For instance, the relatively exact controllability of FSDS with Wiener noise was discussed by adopting delayed matrix functions of sine and cosine in the work of wang et al.[27] In addition, for many other researched results about stochastic systems with Wiener noise, one can see the papers.[28,29] As a matter of fact, noise is not only continuous like Wiener noise, but also has discontinuous fluctuations, such as sudden changes in some natural environments (earthquakes, hurricanes, epidemics, etc.), which can be described as jumping process or more general Lévy noise. It is worth mentioning that the authors considered the controllability of the fractional neutral stochastic system with Lévy noise, in the work of Rajendran et al.[30] In the work of Su et al,[31] approximate controllability of second order SDEs with Lévy process was presented. However, there are few papers considering the controllability of FSDS with Lévy noise.

1.2 | Major contributions

Motivated by the articles mentioned above, we are concerned with relatively exact controllability of FSDS in this paper:

$$\begin{cases} {}^C D^{\alpha}_{0^+} y(t) = & Ay(t-\tau) + Bu(t) + Cu(t-\delta(t)) + G(t, y(t), y(t-\tau)) + \Delta(t, y(t), y(t-\tau))\frac{\mathrm{d}w(t)}{\mathrm{d}t} \\ & + \int_Z g(t, y(t), y(t-\tau), z)\frac{\mathrm{d}\tilde{N}(t,z)}{\mathrm{d}t}, t \in [0,b], \tau > 0, \\ y(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{1}$$

where ${}^C D^{\alpha}_{0^+} y(\cdot)$ is the Caputo derivative, $1 < 2\alpha \leq 2$, $y(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^n$ show state and control vectors, respectively. $A, B, C \in \mathbb{R}^{n\times n}$ are any matrices, $\delta(t) \geq 0$ is the time-varying delay, and $u(t) = 0$, for all $t \in [-\delta(0), 0]$. The functions $G : [0, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\Delta : [0, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ and $g : [0, b] \times \mathbb{R}^n \times \mathbb{R}^n \times Z \to \mathbb{R}^n$ are continuous. $\mathrm{d}\tilde{N}(t, z) = \widetilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - \upsilon(\mathrm{d}z)\mathrm{d}t$ is a compensated Poisson random measure, where $\upsilon(\mathrm{d}z)$ and $N(\mathrm{d}t, \mathrm{d}z)$ show a σ-finite Lévy measure on $(Z, \mathcal{B}(Z))$ and the Poisson random measure related to Poisson point process on $Z \in \mathcal{B}(\mathbb{R}^n)$, respectively. The initial function $\varphi \in C^1([-\tau, 0], \mathbb{R}^n)$, and $\{w(s), s \in [0, b]\}$ denotes a standard d-dimensional Wiener process. The delay function $\delta : [0, b] \to \mathbb{R}$ is continuously differentiable.

Compared with some recent works,[27,29,30,31,32,33,34,35,36,37] the major contributions and difficulties of this article are mainly reflected in the following three aspects:

- (C_1) In the literatures,[27,29,31,33,34] the literatures studied the controllability of dynamical systems without delay term in control, but the system we study has the control delay which is a continuous function regarding the variable t. It is worth remark that time varying term make it harder to construct controllability Grammain matrix. However we deal with the time varying term using the time lead function in this paper.

- (C_2) To the best of our information, there are few papers considering the controllability of FSDS with Lévy noise in literature. Compared with these works,[30,32,33,35,37] the system we study not only has the state delay, but also has the stochastic term with Lévy noise, so it is more generalized. The delay term and stochastic term make the proof of controllability more complicated.

- (C_3) In the study of various controllability of the nonlinear dynamical systems, many papers[30,31,36] have adopted a stronger Lipschitz condition. However, we adopt the weaker condition to study the controllability of the nonlinear dynamical system in this article.

Arrangements for the rest of this article are outlined below. In Section 2, we establish some necessary results required for the subsequent sections. In Section 3, by using the controllability Grammain matrix, the relatively exact controllability of linear FSDS is obtained. In addition, by virtue of Banach contraction principle, the existence and uniqueness of nonlinear FSDS are considered. Then, we demonstrate the relatively exact controllability result of the nonlinear FSDS. Finally, we give an example to verify the correctness of our conclusion in Section 4.

2 | PRELIMINARY

For the sake of the smooth follow-up work, we give briefly the preparatory work in this section. Let $\{\Omega, \mathscr{F}, \mathbf{P}\}$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathscr{F}_0$ containing all $\mathbf{P}$-null sets); $\mathbf{E}(\cdot)$ denotes the expectation regarding the measure $\mathbf{P}$; $L_2\left(\Omega, \mathscr{F}_b, \mathbb{R}^n\right)$ denotes the Hilbert space of all $\mathscr{F}_b$-measurable square integrable random variables with values in $\mathbb{R}^n$; $L_2^{\mathscr{F}}([0,b],\mathbb{R}^n)$ is the Hilbert space of all square integrable and $\mathscr{F}_t$-measurable processes with values in $\mathbb{R}^n$. $\mathbb{L}(\mathbb{R}^n,\mathbb{R}^n)$ denotes the space of all linear transformation. Furthermore, $C\left([0,b], L_2(\Omega,\mathscr{F},\mathbf{P},\mathbb{R}^n)\right)$ shows the Banach space of all square integrable and $\mathscr{F}$-adapted processes $y(t)$ with norm $\|\cdot\|_c$, where $\|y\|_C^2 = \sup\limits_{t\in[0,b]} \mathbf{E}\|y(t)\|^2$. We consider the matrix norm

$$\|A\| = \max\left\{\sum_{i=1}^n |a_{i1}|, \sum_{i=1}^n |a_{i2}|, \ldots, \sum_{i=1}^n |a_{in}|\right\}.$$

For the initial value $\|\varphi\|_C^2 = \sup\limits_{t\in[-\tau,0]} \mathbf{E}\|\varphi(t)\|^2$, and $\mathcal{U}_{ad} = L_2^{\mathscr{F}}([0,b],\mathbb{R}^n)$ denotes set of all admissible controls. Moreover, we let $k_3 = \max\{\|\varphi\|_C^2, \|\varphi'\|_C^2\}$, where $\|\varphi\|_C^2 = \sup\limits_{t\in[-\tau,0]} \mathbf{E}\|\varphi(t)\|^2$, $\|\varphi'\|_C^2 = \sup\limits_{t\in[-\tau,0]} \mathbf{E}\|\varphi'(t)\|^2$. Now, we introduce the time lead function $r(t) : [0, b-\delta(b)] \to [0,b]$ such that $r[t-\delta(t)] = t$ for $t \in [0,b]$.

Definition 1. (see Zhou et al[8]) The Riemann-Liouville fractional integral for f is described by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \quad 0<\alpha\leq 1,$$

and the Caputo derivative for f is ${}^C D^\alpha f = I^{1-\alpha} f'$, that is,

$${}^C D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} f'(s)\mathrm{d}s.$$

Definition 2. (see Gorenflo et al[10]) The M-L function is described by

$$\mathbf{M}_{\alpha,\beta}(\varpi) = \sum_{k=0}^\infty \frac{\varpi^k}{\Gamma(k\alpha+\beta)}.$$

In particular, for $\beta = 1$,

$$\mathbf{M}_{\alpha,1}(\varpi) = \mathbf{M}_\alpha(\varpi) = \sum_{k=0}^\infty \frac{\varpi^k}{\Gamma(k\alpha+1)}.$$

Definition 3. (see Li et al[13]) Delayed matrix function of M-L $\mathbb{Z}_\tau^{At^\alpha} : \mathbb{R}\to\mathbb{R}^{n\times n}$ is described by

$$\mathbb{Z}_\tau^{At^\alpha} = \begin{cases} \Theta, & -\infty < t \leq -\tau, \\ I, & -\tau < t \leq 0, \\ I + A\frac{t^\alpha}{\Gamma(\alpha+1)} + A^2\frac{(t-\tau)^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + A^p\frac{(t-(p-1)\tau)^{p\alpha}}{\Gamma(p\alpha+1)}, & (p-1)\tau < t \leq p\tau, p\in\mathbb{N}, \end{cases}$$

where I and Θ show the identity and zero matrices, respectively.

Definition 4. (see Li et al[13]) Delayed matrix function of M-L $\mathbb{Z}_{\tau,\alpha}^{At^\alpha} : \mathbb{R}\to\mathbb{R}^{n\times n}$ is described by

$$\mathbb{Z}_{\tau,\alpha}^{At^\alpha} = \begin{cases} \Theta, & -\infty < t \leq -\tau, \\ I\frac{(\tau+t)^{\alpha-1}}{\Gamma(\alpha)}, & -\tau < t \leq 0, \\ I\frac{(\tau+t)^{\alpha-1}}{\Gamma(\alpha)} + A\frac{t^{2\alpha-1}}{\Gamma(\alpha+\alpha)} + A^2\frac{(t-\tau)^{3\alpha-1}}{\Gamma(2\alpha+\alpha)} + \cdots + A^p\frac{(t-(p-1)\tau)^{(p+1)\alpha-1}}{\Gamma(p\alpha+\alpha)}, & (p-1)\tau < t \leq p\tau, p\in\mathbb{N}. \end{cases}$$

Lemma 1. (see Li et al[13]) For $(p-1)\tau < t \leq p\tau$ and $p\in\mathbb{N}$, we obtain

$$\left\|\mathbb{Z}_\tau^{At^\alpha}\right\| \leq \mathbf{M}_\alpha\left(\|A\|t^\alpha\right).$$

Lemma 2. For $(p-1)\tau < t \leq p\tau$ and $p\in\mathbb{N}$, we have

$$\left\|\mathbb{Z}_{\tau,\alpha}^{At^\alpha}\right\| \leq (t+\tau)^{\alpha-1}\mathbf{M}_{\alpha,\alpha}\left(\|A\|(t+\tau)^\alpha\right).$$

Proof. For $\forall t \in ((p-1)\tau, p\tau]$, we can get the following

$$\begin{aligned}
\left\|\mathbb{Z}_{\tau,\alpha}^{At^\alpha}\right\| \leq& \frac{(t+\tau)^{\alpha-1}}{\Gamma(\alpha)} + \|A\|\frac{t^{2\alpha-1}}{\Gamma(\alpha+\alpha)} + \|A\|^2\frac{(t-\tau)^{3\alpha-1}}{\Gamma(2\alpha+\alpha)} + \cdots + \|A\|^p\frac{(t-(p-1)\tau)^{(p+1)\alpha-1}}{\Gamma(p\alpha+\alpha)} \\
\leq& \frac{(t+\tau)^{\alpha-1}}{\Gamma(\alpha)} + \|A\|\frac{(t+\tau)^{2\alpha-1}}{\Gamma(\alpha+\alpha)} + \|A\|^2\frac{(t+\tau)^{3\alpha-1}}{\Gamma(2\alpha+\alpha)} + \cdots + \|A\|^p\frac{(t+\tau)^{(p+1)\alpha-1}}{\Gamma(p\alpha+\alpha)} \\
=&(t+\tau)^{\alpha-1}\left(\frac{1}{\Gamma(\alpha)} + \|A\|\frac{(t+\tau)^{\alpha}}{\Gamma(\alpha+\alpha)} + \|A\|^2\frac{(t+\tau)^{2\alpha}}{\Gamma(2\alpha+\alpha)} + \cdots + \|A\|^p\frac{(t+\tau)^{p\alpha}}{\Gamma(p\alpha+\alpha)}\right) \\
\leq&(t+\tau)^{\alpha-1}\sum_{k=0}^{\infty}\frac{(\|A\|(t+\tau)^\alpha)^k}{\Gamma(k\alpha+\alpha)} \\
=&(t+\tau)^{\alpha-1}\mathbf{M}_{\alpha,\alpha}\left(\|A\|(t+\tau)^\alpha\right).
\end{aligned}$$

□

Lemma 3. (see Li et al[13]) Let $f \in C([0,b],\mathbb{R}^n)$. A solution $y(t) \in C([0,b],\mathbb{R}^n)$ of the following system:

$$\begin{cases} {}^C D_{0^+}^{\alpha} y(t) = Ay(t-\tau) + f(t), t \in [0,b], \tau > 0, & f \in C([0,b],\mathbb{R}^n), \\ y(t) = \varphi(t), & -\tau \leq t \leq 0, \end{cases}$$

can be represented as

$$y(t) = \mathbb{Z}_{\tau}^{At^\alpha}\varphi(-\tau) + \int_{-\tau}^{0} \mathbb{Z}_{\tau}^{A(t-\tau-s)^\alpha}\varphi'(t)\mathrm{d}t + \int_{0}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} f(t)\mathrm{d}t.$$

3 | MAIN RESULTS

3.1 | Linear case

In this section, we investigate the following the linear stochastic delay control system:

$$\begin{cases} {}^C D_{0^+}^{\alpha} y(t) = & Ay(t-\tau) + Bu(t) + Cu(t-\delta(t)) + \widetilde{\Delta}(t)\frac{\mathrm{d}w(t)}{\mathrm{d}t} \\ & +\int_Z \widetilde{g}(t,z)\frac{\mathrm{d}\tilde{N}(t,z)}{\mathrm{d}t}, \quad t \in [0,b], \tau > 0, \\ y(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{2}$$

where $\widetilde{\Delta} : [0,b] \to \mathbb{R}^{n\times d}$ and $\widetilde{g} : [0,b]\times Z \to \mathbb{R}^n$ are continuous. The corresponding deterministic system can be given by

$$\begin{cases} {}^C D_{0^+}^{\alpha} y(t) = & Ay(t-\tau) + Bu(t) + Cu(t-\delta(t)) + f(t), \quad t \in [0,b], \tau > 0, \\ y(t) = \varphi(t), & t \in [-\tau, 0]. \end{cases} \tag{3}$$

By the Lemma 3, the solution of the delay system (3) can be represented as

$$\begin{aligned} y(t) =&\mathbb{Z}_{\tau}^{At^\alpha}\varphi(-\tau) + \int_{-\tau}^{0} \mathbb{Z}_{\tau}^{A(t-\tau-s)^\alpha}\varphi'(s)\mathrm{d}s + \int_{0}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} Bu(s)\mathrm{d}s \\ &+ \int_{0}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} Cu(s-\delta(s))\mathrm{d}s + \int_{0}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} f(s)\mathrm{d}s. \end{aligned} \tag{4}$$

Taking $s-\delta(s)=\upsilon$ and using the function $r(t)$, we obtain $s=r(s-\delta(s))=r(\upsilon)$. Then (4) can be represented as

$$
\begin{aligned}
y(t) =&\mathbb{Z}_{\tau}^{At^{\alpha}}\varphi(-\tau)+\int_{-\tau}^{0}\mathbb{Z}_{\tau}^{A(t-\tau-s)^{\alpha}}\varphi'(s)\mathrm{d}s+\int_{0}^{t}\mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}}Bu(s)\mathrm{d}s\\
&+\int_{-\delta(0)}^{t-\delta(t)}\mathbb{Z}_{\tau,\alpha}^{A(t-\tau-r(s))^{\alpha}}Cu(s)r'(s)\mathrm{d}s+\int_{0}^{t}\mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}}f(s)\mathrm{d}s\\
=&\mathbb{Z}_{\tau}^{At^{\alpha}}\varphi(-\tau)+\int_{-\tau}^{0}\mathbb{Z}_{\tau}^{A(t-\tau-s)^{\alpha}}\varphi'(s)\mathrm{d}s+\int_{0}^{t-\delta(t)}\left(\mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}}B+\mathbb{Z}_{\tau,\alpha}^{A(t-\tau-r(s))^{\alpha}}Cr'(s)\right)u(s)\mathrm{d}s\\
&+\int_{t-\delta(t)}^{t}\mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}}Bu(s)\mathrm{d}s+\int_{0}^{t}\mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}}f(s)\mathrm{d}s,
\end{aligned}
\tag{5}
$$

and substituting $t=b$ in (5) , we have

$$
\begin{aligned}
y(b) =&\mathbb{Z}_{\tau}^{Ab^{\alpha}}\varphi(-\tau)+\int_{-\tau}^{0}\mathbb{Z}_{\tau}^{A(b-\tau-s)^{\alpha}}\varphi'(s)\mathrm{d}s+\int_{0}^{b-\delta(b)}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^{\alpha}}Cr'(s)\right)u(s)\mathrm{d}s\\
&+\int_{b-\delta(b)}^{b}\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}Bu(s)\mathrm{d}s+\int_{0}^{b}\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}f(s)\mathrm{d}s.
\end{aligned}
$$

The definition of the control operator $\mathcal{L}_b\in\mathbb{L}\left(L_2^{\mathscr{F}}([0,b],\mathbb{R}^n),L_2\left(\Omega,\mathscr{F}_t,\mathbb{R}^n\right)\right)$ can be given by

$$
\mathcal{L}_b u=\int_{0}^{b-\delta(b)}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^{\alpha}}Cr'(s)\right)u(s)\mathrm{d}s+\int_{b-\delta(b)}^{b}\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}Bu(s)\mathrm{d}s,
$$

and its adjoint $\mathcal{L}_b^*:L_2\left(\Omega,\mathscr{F}_t,\mathbb{R}^n\right)\to L_2^{\mathscr{F}}([0,b],\mathbb{R}^n)$ is defined as

$$
\mathcal{L}_b^* z=\begin{cases}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-t)^{\alpha}}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(t))^{\alpha}}Cr'(s)\right)^*\mathbf{E}\left\{z\mid\mathscr{F}_t\right\}, & t\in[0,b-\delta(b)],\\ B^*\mathbb{Z}_{\tau,\alpha}^{A^*(b-\tau-t)^{\alpha}}\mathbf{E}\left\{z\mid\mathscr{F}_t\right\}, & t\in[b-\delta(b),b],\end{cases}
$$

where B^* shows the transpose of B.

The definition of the linear controllability operator $\Pi_{\tau}^{b}\in\mathbb{L}\left(L_2\left(\Omega,\mathscr{F}_t,\mathbb{R}^n\right),L_2\left(\Omega,\mathscr{F}_t,\mathbb{R}^n\right)\right)$ can be given by

$$
\begin{aligned}
\Pi_{\tau}^{b} =&\mathcal{L}_b\mathcal{L}_b^*\{\cdot\}\\
=&\int_{0}^{b-\delta(b)}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^{\alpha}}Cr'(s)\right)\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^{\alpha}}Cr'(s)\right)^*\mathbf{E}\left\{\cdot\mid\mathscr{F}_s\right\}\mathrm{d}s\\
&+\int_{b-\delta(b)}^{b}\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}BB^*\mathbb{Z}_{\tau,\alpha}^{A^*(b-\tau-s)^{\alpha}}\mathbf{E}\left\{\cdot\mid\mathscr{F}_s\right\}\mathrm{d}s,
\end{aligned}
\tag{6}
$$

and the deterministic controllability Grammian matrix $\Theta_{\tau}^{b}\in\mathbb{L}(\mathbb{R}^n,\mathbb{R}^n)$:

$$
\begin{aligned}
\Theta_{\tau}^{b}=&\int_{0}^{b-\delta(b)}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^{\alpha}}Cr'(s)\right)\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^{\alpha}}Cr'(s)\right)^*\mathrm{d}s\\
&+\int_{b-\delta(b)}^{b}\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}BB^*\mathbb{Z}_{\tau,\alpha}^{A^*(b-\tau-s)^{\alpha}}\mathrm{d}s.
\end{aligned}
\tag{7}
$$

Definition 5. (see Wang et al[27]) The set $h(t) = \left\{ y(t), u_t \right\}$ is called the complete state of the system (1) at time t.

Definition 6. (see Wang et al[27]) For an arbitrary complete state $h(t)$ and any $y_1 \in \mathbb{R}^n$, the stochastic system (1) is called relatively controllable on [0,b] if there exists a control vector $u(t) \in \mathbb{R}^n$ such that the stochastic system (1) has a solution y that satisfies the initial condition $y(t) = \varphi(t), t \in [-\tau, 0]$ and $y(b) = y_1$.

Definition 7. (see Klamka et al[20]) For an arbitrary initial state y_0, the stochastic system (1) is called relatively exactly controllable on $[0, b]$ if any $y \in L_2\left(\Omega, \mathscr{F}_b, \mathbb{R}^n\right)$ can be exactly reached from y_0 at time b that is, if

$$\mathcal{R}_b\left(\mathcal{U}_{ad}\right) = L_2\left(\Omega, \mathscr{F}_b, \mathbb{R}^n\right),$$

where $\mathcal{R}_b\left(\mathcal{U}_{ad}\right) = \{y(b,u) \in L_2\left(\Omega, \mathscr{F}_b, \mathbb{R}^n\right) : u(\cdot) \in \mathcal{U}_{ad}\}$.

Lemma 4. (see Mahmudov et al[19]) For $\forall y \in L_2\left(\Omega, \mathscr{F}_b, \mathbb{R}^n\right)$, $\exists q(\cdot) \in L_2^{\mathscr{F}}\left([0,b], \mathbb{R}^{n\times d}\right)$ such that

$$y = \mathbf{E}y + \int_0^b q(\xi)\mathrm{d}w(\xi),$$

$$\Pi_\tau^b y = \Theta_\tau^b \mathbf{E}y + \int_0^b \Theta_\tau^b(\xi) q(\xi)\mathrm{d}w(\xi).$$

Lemma 5. The controllability matrix (7) is nonsingular if and only if the system (3) is relatively controllable on [0, b].

Proof. **Necessity:** Suppose Θ_τ^b is nonsingular, its inverse $[\Theta_\tau^b]^{-1}$ is well defined. The control function $u(t) \in \mathbb{R}^n$ is defined as

$$u(t) = \begin{cases} \left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-t)^\alpha} B + \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(t))^\alpha} C r'(t)\right)^* [\Theta_\tau^b]^{-1}\beta, & t \in [0, b-\delta(b)], \\ B^* \mathbb{Z}_{\tau,\alpha}^{A^*(b-\tau-t)^\alpha} [\Theta_\tau^b]^{-1}\beta, & t \in [b-\delta(b), b], \end{cases} \tag{8}$$

where $\beta = y_1 - \mathbb{Z}_\tau^{Ab^\alpha}\varphi(-\tau) - \int_{-\tau}^0 \mathbb{Z}_\tau^{A(b-\tau-s)^\alpha}\varphi'(s)ds - \int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} f(s)ds$, and the vector $y_1 \in \mathbb{R}^n$ is chosen arbitrarily.

Letting $t = b$ in (5) and inserting (8) in (5), one can derive

$$\begin{aligned} y(b) =& \mathbb{Z}_\tau^{Ab^\alpha}\varphi(-\tau) + \int_{-\tau}^0 \mathbb{Z}_\tau^{A(b-\tau-s)^\alpha}\varphi'(s)\mathrm{d}s \\ &+ \int_0^{b-\delta(b)} \left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} B + \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^\alpha} C r'(s)\right)\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} B + \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^\alpha} C r'(s)\right)^* [\Theta_\tau^b]^{-1}\beta\mathrm{d}s \\ &+ \int_{b-\delta(b)}^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} B B^* \mathbb{Z}_{\tau,\alpha}^{A^*(b-\tau-s)^\alpha}[\Theta_\tau^b]^{-1}\beta\mathrm{d}s + \int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} f(s)\mathrm{d}s \\ =& \mathbb{Z}_\tau^{Ab^\alpha}\varphi(-\tau) + \int_{-\tau}^0 \mathbb{Z}_\tau^{A(b-\tau-s)^\alpha}\varphi'(s)\mathrm{d}s + \Theta_\tau^b[\Theta_\tau^b]^{-1}\beta + \int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} f(s)\mathrm{d}s \\ =& y_1. \end{aligned}$$

Further, by the Lemma 3, the initial condition $y(t) = \varphi(t)$, $t \in [-\tau, 0]$ holds. Hence, by the Definition 6, the system (3) is relatively controllable.

Sufficiency: We prove our result by means of contradiction. Suppose that the Grammian matrix Θ_τ^b is singular, and there exists a nonzero state vector $\tilde{a} \in \mathbb{R}^n$ such that

$$\begin{aligned} 0 =& \tilde{a}^* \Theta_\tau^b \tilde{a} \\ =& \int_0^{b-\delta(b)} \tilde{a}^* \left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} B + \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^\alpha} C r'(s)\right)\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} B + \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^\alpha} C r'(s)\right)^* \tilde{a}\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+\int_{b-\delta(b)}^{b} \tilde{a}^{*}\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}} BB^{*}\mathbb{Z}_{\tau,\alpha}^{A^{*}(b-\tau-s)^{\alpha}}\tilde{a}\mathrm{d}s\\
&=\int_{0}^{b-\delta(b)} \left[\tilde{a}^{*}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^{\alpha}}Cr'(s)\right)\right]\left[\tilde{a}^{*}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^{\alpha}}Cr'(s)\right)\right]^{*}\mathrm{d}s\\
&+\int_{b-\delta(b)}^{b}\left[\tilde{a}^{*}\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B\right]\left[\tilde{a}^{*}\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B\right]^{*}\mathrm{d}s\\
&=\int_{0}^{b-\delta(b)}\left\|\tilde{a}^{*}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^{\alpha}}Cr'(s)\right)\right\|^{2}\mathrm{d}s+\int_{b-\delta(b)}^{b}\left\|\tilde{a}^{*}\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B\right\|^{2}\mathrm{d}s,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\tilde{a}^{*}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^{\alpha}}Cr'(s)\right)&=\underbrace{(0\cdots0)}_{n-times},\quad \forall s\in[0,b-\delta(b)],\\
\tilde{a}^{*}\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B&=\underbrace{(0\cdots0)}_{n-times},\quad \forall s\in[b-\delta(b),b].
\end{aligned}
\tag{9}
$$

Since system (3) is relatively controllable on [0, b], in light of the Definition 6, there exists a control function $u_0(t)$ that steers the complete state to zero at time b, i.e.,

$$
\begin{aligned}
y(b)=&\mathbb{Z}_{\tau}^{Ab^{\alpha}}\varphi(-\tau)+\int_{-\tau}^{0}\mathbb{Z}_{\tau}^{A(b-\tau-s)^{\alpha}}\varphi'(s)\mathrm{d}s+\int_{0}^{b-\delta(b)}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^{\alpha}}Cr'(s)\right)u_0(s)\mathrm{d}s\\
&+\int_{b-\delta(b)}^{b}\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}Bu_0(s)\mathrm{d}s+\int_{0}^{b}\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}f(s)\mathrm{d}s\\
=&\mathbf{0},
\end{aligned}
\tag{10}
$$

where $\mathbf{0}$ shows the n-dimensional zero vector. Moreover, in light of the Definition 6, there exists a control function $u_1(t)$ that drives the complete state to $\tilde{a}$ at time b, then

$$
\begin{aligned}
y(b)=&\mathbb{Z}_{\tau}^{Ab^{\alpha}}\varphi(-\tau)+\int_{-\tau}^{0}\mathbb{Z}_{\tau}^{A(b-\tau-s)^{\alpha}}\varphi'(s)\mathrm{d}s+\int_{0}^{b-\delta(b)}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^{\alpha}}Cr'(s)\right)u_1(s)\mathrm{d}s\\
&+\int_{b-\delta(b)}^{b}\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}Bu_1(s)\mathrm{d}s+\int_{0}^{b}\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}f(s)\mathrm{d}s\\
=&\tilde{a}.
\end{aligned}
\tag{11}
$$

Combining the formula (10) and (11), one can get

$$
\begin{aligned}
\tilde{a}=&\int_{0}^{b-\delta(b)}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^{\alpha}}Cr'(s)\right)(u_1(s)-u_0(t))\mathrm{d}s\\
&+\int_{b-\delta(b)}^{b}\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B(u_1(s)-u_0(t))\mathrm{d}s,
\end{aligned}
$$

and multiply by $\tilde{a}^*$ on both sides of the above equation. We get

$$\tilde{a}^*\tilde{a} = \int_0^{b-\delta(b)} \tilde{a}^* \left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} B + \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^\alpha} C r'(s) \right) (u_1(s) - u_0(t)) \mathrm{d}s + \int_{b-\delta(b)}^{b} \tilde{a}^* \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} B (u_1(s) - u_0(t)) \mathrm{d}s.$$

We acquire $\tilde{a}^*\tilde{a} = 0$, from the fact (9), i.e., $\tilde{a} = 0$. This is a contradiction to nonzero vector $\tilde{a}$. Thus, Θ_τ^b is nonsingular. □

Theorem 1. The deterministic system (3) is relatively controllable on $[0, b]$ if and only if the stochastic system (2) is relatively exactly controllable.

Proof. **Necessity:** From the Lemma 5, the system (3) is relatively controllable, and then one can obtain the controllability Grammian matrix $\Theta_\tau^b(\xi)$ is strictly positive definite and nonsingular for $\forall \xi \in [0, b]$. Thus, for $y \in \mathbb{R}^n$, there exist some $\lambda > 0$ such that

$$\langle \Theta_\tau^b(\xi) y, y \rangle \geq \lambda \|y\|^2, \quad \xi \in [0, b].$$

By the Lemma 4, we establish the following formula

$$y = \mathbf{E}y + \int_0^b q(\xi) \mathrm{d}w(\xi),$$

$$\Pi_\tau^b y = \Theta_\tau^b \mathbf{E}y + \int_0^b \Theta_\tau^b(\xi) q(\xi) \mathrm{d}w(\xi).$$

To express $\mathbf{E}\langle \Pi_\tau^b y, y \rangle$ in terms of $\langle \Theta_\tau^b \mathbf{E}y, \mathbf{E}y \rangle$, one can get

$$\begin{aligned}
\mathbf{E}\langle \Pi_\tau^b y, y \rangle =& \mathbf{E} \left\langle \Theta_\tau^b \mathbf{E}y + \int_0^b \Theta_\tau^b(\xi) q(\xi) \mathrm{d}w(\xi), \mathbf{E}y + \int_0^b q(\xi) \mathrm{d}w(\xi) \right\rangle \\
=& \langle \Theta_\tau^b \mathbf{E}y, \mathbf{E}y \rangle + \mathbf{E} \int_0^b \langle \Theta_\tau^b(\xi) q(\xi), q(\xi) \rangle \mathrm{d}\xi \\
\geq& \lambda \mathbf{E}\|y\|^2 + \lambda \mathbf{E} \int_0^b \|q(\xi)\|^2 \mathrm{d}\xi \\
\geq& \lambda \mathbf{E}\|y\|^2,
\end{aligned}$$

which implies that the operator Π_τ^b is strictly positive definite and $[\Pi_\tau^b]^{-1}$ is bounded. Further, a control function $u(t) \in \mathcal{U}_{ad} = L_2^{\mathscr{F}}([0, b], \mathbb{R}^n)$ is defined as

$$u(t) = \begin{cases} \left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-t)^\alpha} B + \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(t))^\alpha} C r'(t) \right)^* \mathbf{E} \left\{ [\Pi_\tau^b]^{-1} \beta \mid \mathscr{F}_t \right\}, & t \in [0, b - \delta(b)], \\ B^* \mathbb{Z}_{\tau,\alpha}^{A^*(b-\tau-t)^\alpha} \mathbf{E} \left\{ [\Pi_\tau^b]^{-1} \beta \mid \mathscr{F}_t \right\}, & t \in [b - \delta(b), b], \end{cases} \tag{12}$$

where

$$\begin{aligned}
\beta =& y_1 - \mathbb{Z}_\tau^{Ab^\alpha} \varphi(-\tau) - \int_{-\tau}^0 \mathbb{Z}_\tau^{A(b-\tau-s)^\alpha} \varphi'(s) \mathrm{d}s \\
&- \int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} \widetilde{\Delta}(s) \mathrm{d}w(s) - \int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} \int_Z \widetilde{g}(s, z) \tilde{N}(\mathrm{d}s, \mathrm{d}z),
\end{aligned}$$

and vector $y_1 \in \mathbb{R}^n$ is chosen arbitrarily.

By the Lemma 3, the solution of (2) can be given by

$$
\begin{aligned}
y(t) =& \mathbb{Z}_{\tau}^{At^{\alpha}} \varphi(-\tau) + \int_{-\tau}^{0} \mathbb{Z}_{\tau}^{A(t-\tau-s)^{\alpha}} \varphi'(s)\mathrm{d}s \\
&+ \int_{0}^{t-\delta(t)} \left(\mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}} B + \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-r(s))^{\alpha}} Cr'(s) \right) u(s)\mathrm{d}s + \int_{t-\delta(t)}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}} Bu(s)\mathrm{d}s \\
&+ \int_{0}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}} \widetilde{\Delta}(s)\mathrm{d}w(s) + \int_{0}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}} \int_{Z} \widetilde{g}(s,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z).
\end{aligned} \tag{13}
$$

Substituting $t = b$ in (13) and inserting (12) in (13), one can obtain

$$
\begin{aligned}
y(b) =& \mathbb{Z}_{\tau}^{Ab^{\alpha}} \varphi(-\tau) + \int_{-\tau}^{0} \mathbb{Z}_{\tau}^{A(b-\tau-s)^{\alpha}} \varphi'(s)\mathrm{d}s \\
&+ \int_{0}^{b-\delta(b)} \left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}} B + \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^{\alpha}} Cr'(s) \right) \left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}} B + \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^{\alpha}} Cr'(s) \right)^{*} \mathbf{E}\left\{ [\Pi_{\tau}^{b}]^{-1}\beta \mid \mathscr{F}_{s} \right\} \mathrm{d}s \\
&+ \int_{b-\delta(b)}^{b} \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}} BB^{*} \mathbb{Z}_{\tau,\alpha}^{A^{*}(b-\tau-s)^{\alpha}} \mathbf{E}\left\{ [\Pi_{\tau}^{b}]^{-1}\beta \mid \mathscr{F}_{s} \right\} \mathrm{d}s \\
&+ \int_{0}^{b} \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}} \widetilde{\Delta}(s)\mathrm{d}w(s) + \int_{0}^{b} \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}} \int_{Z} \widetilde{g}(s,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z) \\
=& \mathbb{Z}_{\tau}^{Ab^{\alpha}} \varphi(-\tau) + \int_{-\tau}^{0} \mathbb{Z}_{\tau}^{A(b-\tau-s)^{\alpha}} \varphi'(s)\mathrm{d}s + \Pi_{\tau}^{b}[\Pi_{\tau}^{b}]^{-1}\beta \\
&+ \int_{0}^{b} \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}} \widetilde{\Delta}(s)\mathrm{d}w(s) + \int_{0}^{b} \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}} \int_{Z} \widetilde{g}(s,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z) \\
=& y_1.
\end{aligned}
$$

Further, by the Lemma 3, the initial condition $y(t) = \varphi(t)$, $t \in [-\tau, 0]$ holds. Thus (2) is relatively exactly controllable on $[0, b]$ by the Definition 7.

Sufficiency: By Lemma 5, in order to prove that the system (3) is relatively controllable, one just prove Θ_{τ}^{b} is positive definite. The proof is similar to the proof of Lemma 5, so it is omitted. □

Next, we will give the result about the minimum energy control.

Lemma 6. Suppose that the linear stochastic system (2) is relatively exactly controllable on $[0, b]$. Hence, for any target $y_1 \in L_2\left(\Omega, \mathscr{F}_b, \mathbb{R}^n\right)$, any $\tilde{\Delta}, \tilde{g}$, the control

$$
u_0(t) = \begin{cases} \left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-t)^{\alpha}} B + \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(t))^{\alpha}} Cr'(t) \right)^{*} \mathbf{E}\left\{ [\Pi_{\tau}^{b}]^{-1}\beta \mid \mathscr{F}_{t} \right\}, & t \in [0, b-\delta(b)], \\ B^{*} \mathbb{Z}_{\tau,\alpha}^{A^{*}(b-\tau-t)^{\alpha}} \mathbf{E}\left\{ [\Pi_{\tau}^{b}]^{-1}\beta \mid \mathscr{F}_{t} \right\}, & t \in [b-\delta(b), b], \end{cases} \tag{14}
$$

where

$$
\beta = y_1 - \mathbb{Z}_{\tau}^{Ab^{\alpha}} \varphi(-\tau) - \int_{-\tau}^{0} \mathbb{Z}_{\tau}^{A(b-\tau-s)^{\alpha}} \varphi'(s)\mathrm{d}s
$$

$$-\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\widetilde{\Delta}(s)\mathrm{d}w(s)-\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\int_Z \widetilde{g}(s,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z),$$

transfers the system (2) from y_0 to y_1 at time b. In addition, among all the admissible controls $u_m(t)$ driving y_0 to y_1 at time b, the control function $u_0(t)$ minimizes the integral performance index $J(u)=\mathbf{E}\int_0^b \|u(t)\|^2\mathrm{d}t$.

Proof. Since the stochastic linear system (2) is relatively exactly controllable on $[0,b]$, the linear controllability operator Π_τ^b is invertible and its inverse $[\Pi_\tau^b]^{-1}\in\mathbb{L}\left(L_2\left(\Omega,\mathscr{F}_b,\mathbb{R}^n\right),L_2\left(\Omega,\mathscr{F}_b,\mathbb{R}^n\right)\right)$. Substituting the control $u_0(t)$ into (13) at time b and using Lemma 3, we can obtain

$$\begin{aligned}
y(b)=&\mathbb{Z}_\tau^{Ab^\alpha}\varphi(-\tau)+\int_{-\tau}^0 \mathbb{Z}_\tau^{A(b-\tau-s)^\alpha}\varphi'(s)\mathrm{d}s\\
&+\int_0^{b-\delta(b)}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^\alpha}Cr'(s)\right)u_0(s)\mathrm{d}s+\int_{b-\delta(b)}^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}Bu_0(s)\mathrm{d}s\\
&+\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\widetilde{\Delta}(s)\mathrm{d}w(s)+\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\int_Z \widetilde{g}(s,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\\
=&\mathbb{Z}_\tau^{Ab^\alpha}\varphi(-\tau)+\int_{-\tau}^0 \mathbb{Z}_\tau^{A(b-\tau-s)^\alpha}\varphi'(s)\mathrm{d}s\\
&+\int_0^{b-\delta(b)}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^\alpha}Cr'(s)\right)\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^\alpha}Cr'(s)\right)^*\mathbf{E}\left\{[\Pi_\tau^b]^{-1}\beta\mid\mathscr{F}_s\right\}\mathrm{d}s\\
&+\int_{b-\delta(b)}^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}BB^*\mathbb{Z}_{\tau,\alpha}^{A^*(b-\tau-s)^\alpha}\mathbf{E}\left\{[\Pi_\tau^b]^{-1}\beta\mid\mathscr{F}_s\right\}\mathrm{d}s\\
&+\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\widetilde{\Delta}(s)\mathrm{d}w(s)+\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\int_Z \widetilde{g}(s,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\\
=&\mathbb{Z}_\tau^{Ab^\alpha}\varphi(-\tau)+\int_{-\tau}^0 \mathbb{Z}_\tau^{A(b-\tau-s)^\alpha}\varphi'(s)\mathrm{d}s+\Pi_\tau^b[\Pi_\tau^b]^{-1}\beta\\
&+\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\widetilde{\Delta}(s)\mathrm{d}w(s)+\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\int_Z \widetilde{g}(s,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\\
=&y_1,
\end{aligned}\tag{15}$$

and $y(t)=\varphi(t)$, $t\in[-\tau,0]$.

To show that the control function $u_0(t)$ is optimal for J, we assume the control $u_1(t)\in\mathcal{U}_{ad}$, $t\in[0,b]$ drives y_0 to y_1 at time b and $y(t)=\varphi(t)$, $t\in[-\tau,0]$. Hence, one can get

$$\begin{aligned}
y(b)=&\mathbb{Z}_\tau^{Ab^\alpha}\varphi(-\tau)+\int_{-\tau}^0 \mathbb{Z}_\tau^{A(b-\tau-s)^\alpha}\varphi'(s)\mathrm{d}s+\int_0^{b-\delta(b)}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^\alpha}Cr'(s)\right)u_1(s)\mathrm{d}s\\
&+\int_{b-\delta(b)}^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}Bu_1(s)\mathrm{d}s+\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\widetilde{\Delta}(s)\mathrm{d}w(s)+\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\int_Z \widetilde{g}(s,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\\
=&y_1.
\end{aligned}\tag{16}$$

Combining the formula (15) and (16), one can obtain

$$\int_0^{b-\delta(b)} \left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} B + \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^\alpha} C r'(s)\right)(u_1(s)-u_0(s))\mathrm{d}s + \int_{b-\delta(b)}^{b} \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} B(u_1(s)-u_0(s))\mathrm{d}s$$
$$= 0.$$

Therefore

$$\begin{aligned}
0 =& \mathbf{E}\Bigg\langle \int_0^{b-\delta(b)} \left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} B + \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^\alpha} C r'(s)\right)(u_1(s)-u_0(s))\mathrm{d}s \\
&+ \int_{b-\delta(b)}^{b} \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} B(u_1(s)-u_0(s))\mathrm{d}s, \mathbf{E}\left\{[\Pi_\tau^b]^{-1}\beta \mid \mathscr{F}_b\right\}\Bigg\rangle \\
=& \mathbf{E}\Bigg\langle \int_0^{b-\delta(b)} \left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} B + \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^\alpha} C r'(s)\right)(u_1(s)-u_0(s))\mathrm{d}s, \mathbf{E}\left\{[\Pi_\tau^b]^{-1}\beta \mid \mathscr{F}_b\right\}\Bigg\rangle \\
&+ \mathbf{E}\Bigg\langle \int_{b-\delta(b)}^{b} \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} B(u_1(s)-u_0(s))\mathrm{d}s, \mathbf{E}\left\{[\Pi_\tau^b]^{-1}\beta \mid \mathscr{F}_b\right\}\Bigg\rangle \\
=& \mathbf{E}\int_0^{b-\delta(b)} \left\langle (u_1(s)-u_0(s)), \left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} B + \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^\alpha} C r'(s)\right)^* \mathbf{E}\left\{[\Pi_\tau^b]^{-1}\beta \mid \mathscr{F}_s\right\}\right\rangle \mathrm{d}s \\
&+ \mathbf{E}\int_{b-\delta(b)}^{b} \left\langle (u_1(s)-u_0(s)), B^*\mathbb{Z}_{\tau,\alpha}^{A^*(b-\tau-s)^\alpha} \mathbf{E}\left\{[\Pi_\tau^b]^{-1}\beta \mid \mathscr{F}_s\right\}\right\rangle \mathrm{d}s \\
=& \mathbf{E}\int_0^{b-\delta(b)} \langle (u_1(s)-u_0(s)), u_0(s)\rangle\, \mathrm{d}s + \mathbf{E}\int_{b-\delta(b)}^{b} \langle (u_1(s)-u_0(s)), u_0(s)\rangle\, \mathrm{d}s \\
=& \mathbf{E}\int_0^{b} \langle (u_1(s)-u_0(s)), u_0(s)\rangle\, \mathrm{d}s.
\end{aligned}$$

Thus,

$$\mathbf{E}\int_0^{b} \langle (u_1(s)-u_0(s)), u_0(s)\rangle\, \mathrm{d}s = 0.$$

Further, one can obtain

$$\begin{aligned}
\mathbf{E}\int_0^b \|u_1(t)\|^2\mathrm{d}t =& \mathbf{E}\int_0^b \langle u_1(t)-u_0(t)+u_0(t), u_1(t)-u_0(t)+u_0(t)\rangle\,\mathrm{d}t \\
=& \mathbf{E}\int_0^b \|u_1(t)-u_0(t)\|^2\mathrm{d}t + \mathbf{E}\int_0^b \|u_0(t)\|^2\mathrm{d}t + 2\mathbf{E}\int_0^b \langle u_1(t)-u_0(t), u_0(t)\rangle\,\mathrm{d}t \\
=& \mathbf{E}\int_0^b \|u_1(t)-u_0(t)\|^2\mathrm{d}t + \mathbf{E}\int_0^b \|u_0(t)\|^2\mathrm{d}t \\
\geq& \mathbf{E}\int_0^b \|u_0(t)\|^2\mathrm{d}t.
\end{aligned}$$

Thus, the control $u_0(t) \in \mathcal{U}_{ad}$ is optimal for J. □

3.2 | Nonlinear case

Before stating the main result, we first give the following hypotheses:

- (H_1) The functions $G \in C([0,b] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $\Delta \in C([0,b] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^{n\times d})$ and $g \in C([0,b] \times \mathbb{R}^n \times \mathbb{R}^n \times Z, \mathbb{R}^n)$, and there exist $L_G(t), L_\Delta(t), L_g(t) \in L^\gamma\left([0,b], \mathbb{R}^+\right)$ and $\gamma > 1$ such that
  - (i)
$$\|G(t,\omega_1,v_1) - G(t,\omega_2,v_2)\|^2 \leq L_G(t)(\|\omega_1-\omega_2\|^2 + \|v_1 - v_2\|^2), t \in [0,b], \omega_1, \omega_2, v_1, v_2 \in \mathbb{R}^n;$$
  - (ii)
$$\|\Delta(t,\omega_1,v_1) - \Delta(t,\omega_2,v_2)\|^2 \leq L_\Delta(t)(\|\omega_1-\omega_2\|^2 + \|v_1 - v_2\|^2), t \in [0,b], \omega_1, \omega_2, v_1, v_2 \in \mathbb{R}^n;$$
  - (iii)
$$\int_Z \|g(t,\omega_1,v_1,z) - g(t,\omega_2,v_2,z)\|^2 v(\mathrm{d}z) \leq L_g(t)(\|\omega_1-\omega_2\|^2 + \|v_1 - v_2\|^2), t \in [0,b], \omega_1, \omega_2, v_1, v_2 \in \mathbb{R}^n.$$
- (H_2) Set $k_1 = \|\Theta_\tau^b\|^2, k_2 = \|[\Pi_\tau^b]^{-1}\|^2$ and
$$\begin{aligned} K =& 8b^{\frac{1}{\rho}} \left(\mathbf{M}_{\alpha,\alpha}(\|A\|(b+\tau))^\alpha\right)^2 (b+\tau)^{2\alpha-2} \\ &\times (b\|L_G\|_{L^\gamma([0,b],\mathbb{R}^+)} + \|L_\Delta\|_{L^\gamma([0,b],\mathbb{R}^+)} + \|L_g\|_{L^\gamma([0,b],\mathbb{R}^+)})(1+3k_1k_2) < 1. \end{aligned}$$
- (H_3) Set $M_1 = \sup_{t\in[0,b]} \|G(t,0,0)\|^2, M_2 = \sup_{t\in[0,b]} \|\Delta(t,0,0)\|^2, M_3 = \sup_{t\in[0,b]} \int_Z \|g(t,0,0,z)\|^2 v(\mathrm{d}z)$.

Now we give the solution of (1) with the form:

$$\begin{aligned} y(t) =& \mathbb{Z}_\tau^{At^\alpha} \varphi(-\tau) + \int_{-\tau}^{0} \mathbb{Z}_\tau^{A(t-\tau-s)^\alpha} \varphi'(s)\mathrm{d}s \\ &+ \int_0^{t-\delta(t)} \left(\mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} B + \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-r(s))^\alpha} C r'(s)\right) u_y(s)\mathrm{d}s + \int_{t-\delta(t)}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} B u_y(s)\mathrm{d}s \\ &+ \int_0^t \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} G(s,y(s),y(s-\tau))\mathrm{d}s + \int_0^t \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} \Delta(s,y(s),y(s-\tau))\mathrm{d}w(s) \\ &+ \int_0^t \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} \int_Z g(s,y(s),y(s-\tau),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z). \end{aligned} \tag{17}$$

Further, a control function $u_y(t)$ can be formulated by

$$u_y(t) = \begin{cases} \left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-t)^\alpha} B + \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(t))^\alpha} C r'(t)\right)^* \mathbf{E}\left\{[\Pi_\tau^b]^{-1}\beta \mid \mathscr{F}_t\right\}, & t \in [0, b-\delta(b)], \\ B^* \mathbb{Z}_{\tau,\alpha}^{A^*(b-\tau-t)^\alpha} \mathbf{E}\left\{[\Pi_\tau^b]^{-1}\beta \mid \mathscr{F}_t\right\}, & t \in [b-\delta(b), b], \end{cases} \tag{18}$$

where

$$\begin{aligned} \beta =& y_1 - \mathbb{Z}_\tau^{Ab^\alpha} \varphi(-\tau) - \int_{-\tau}^0 \mathbb{Z}_\tau^{A(b-\tau-s)^\alpha} \varphi'(s)\mathrm{d}s \\ &- \int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} G(s,y(s),y(s-\tau))\mathrm{d}s - \int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} \Delta(s,y(s),y(s-\tau))\mathrm{d}w(s) \\ &- \int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} \int_Z g(s,y(s),y(s-\tau),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z), \end{aligned}$$

and the vector $y_1 \in \mathbb{R}^n$ is chosen arbitrarily.

Inserting (18) in (17), one can get that the control function $u_y(t)$ drives y_0 to the desired vector y_1 at time b. Furthermore, we introduce the Banach space $\mathcal{A} := \mathcal{C}\left([-\tau, b], L_2(\Omega, \mathcal{F}, \mathbf{P}, \mathbb{R}^n)\right)$ with norm $\|y\|_{\mathcal{C}}^2 = \sup_{t\in[-\tau,b]} \mathbf{E}\|y(t)\|^2 < \infty$. Using the control function, the operator $\Phi : \mathcal{A} \to \mathcal{A}$ is described by

$$
\begin{aligned}
(\Phi y)(t) =& \mathbb{Z}_{\tau}^{At^{\alpha}} \varphi(-\tau) + \int_{-\tau}^{0} \mathbb{Z}_{\tau}^{A(t-\tau-s)^{\alpha}} \varphi'(s)\mathrm{d}s \\
&+ \int_{0}^{t-\delta(t)} \left(\mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}} B + \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-r(s))^{\alpha}} Cr'(s)\right) u_y(s)\mathrm{d}s + \int_{t-\delta(t)}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}} Bu_y(s)\mathrm{d}s \\
&+ \int_{0}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}} G(s, y(s), y(s-\tau))\mathrm{d}s + \int_{0}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}} \Delta(s, y(s), y(s-\tau))\mathrm{d}w(s) \\
&+ \int_{0}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}} \int_{Z} g(s, y(s), y(s-\tau), z)\tilde{N}(\mathrm{d}s, \mathrm{d}z),
\end{aligned}
$$

and it follows from the Lemma 6, that if the operator Φ has a fixed point, thus the stochastic system (1) has a solution $y(t)$ regarding $u_y(\cdot) \in \mathcal{U}_{ad}$, and $(\Phi y)(b) = y(b) = y_1$, $y(t) = \varphi(t)$, $t \in [-\tau, 0]$, namely, the stochastic system (1) is relatively exactly controllable.

In what follows, we derive the relatively exact controllability result for system (1) by virtue of fixed point theorem.

Theorem 2. If the hypotheses $(H1)$-$(H3)$ hold and the stochastic system (2) is relatively exactly controllable. Thus, the stochastic system (1) is relatively exactly controllable.

Proof. In order to prove that the system (1) is relatively exactly controllable, we divide our proof into the following two steps.

Step 1. We show that Φ maps $\mathcal{A}$ into itself.

For $\forall y \in \mathcal{A}$ and $t \in [0, b]$. According to Jensen's inequality, one can get the following:

$$
\begin{aligned}
\mathbf{E}\|(\Phi y)(t)\|^2 =& \mathbf{E}\Bigg\| \mathbb{Z}_{\tau}^{At^{\alpha}} \varphi(-\tau) + \int_{-\tau}^{0} \mathbb{Z}_{\tau}^{A(t-\tau-s)^{\alpha}} \varphi'(s)\mathrm{d}s \\
&+ \int_{0}^{t-\delta(t)} \left(\mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}} B + \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-r(s))^{\alpha}} Cr'(s)\right) u_y(s)\mathrm{d}s + \int_{t-\delta(t)}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}} Bu_y(s)\mathrm{d}s \\
&+ \int_{0}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}} G(s, y(s), y(s-\tau))\mathrm{d}s + \int_{0}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}} \Delta(s, y(s), y(s-\tau))\mathrm{d}w(s) \\
&+ \int_{0}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}} \int_{Z} g(s, y(s), y(s-\tau), z)\tilde{N}(\mathrm{d}s, \mathrm{d}z) \Bigg\|^2 \\
\leq& 6\mathbf{E}\left\| \mathbb{Z}_{\tau}^{At^{\alpha}} \varphi(-\tau) \right\|^2 + 6\mathbf{E}\left\| \int_{-\tau}^{0} \mathbb{Z}_{\tau}^{A(t-\tau-s)^{\alpha}} \varphi'(s)\mathrm{d}s \right\|^2 \\
&+ 6\mathbf{E}\left\| \int_{0}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}} G(s, y(s), y(s-\tau))\mathrm{d}s \right\|^2 + 6\mathbf{E}\left\| \int_{0}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}} \Delta(s, y(s), y(s-\tau))\mathrm{d}w(s) \right\|^2 \\
&+ 6\mathbf{E}\left\| \int_{0}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^{\alpha}} \int_{Z} g(s, y(s), y(s-\tau), z)\tilde{N}(\mathrm{d}s, \mathrm{d}z) \right\|^2
\end{aligned}
$$

$$+ 6\mathbf{E}\left\| \int_{0}^{t-\delta(t)} \left(\mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} B + \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-r(s))^\alpha} Cr'(s) \right) u_y(s)\mathrm{d}s + \int_{t-\delta(t)}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} Bu_y(s)\mathrm{d}s \right\|^2$$

$$:= J_1 + J_2 + J_3 + J_4 + J_5 + J_6.$$

With the aid of the Lemma 1, we have

$$\begin{aligned} J_1 =& 6\mathbf{E}\left\| \mathbb{Z}_{\tau}^{At^\alpha} \varphi(-\tau) \right\|^2 \\ \leq& 6\left(\mathbf{M}_\alpha\left(\|A\| b^\alpha \right) \right)^2 \mathbf{E}\, \|\varphi(-\tau)\|^2 . \end{aligned}$$

By the Lemma 1, one can obtain

$$\begin{aligned} J_2 =& 6\mathbf{E}\left\| \int_{-\tau}^{0} \mathbb{Z}_{\tau}^{A(t-\tau-s)^\alpha} \varphi'(s)\mathrm{d}s \right\|^2 \\ \leq& 6\tau^2 \left(\mathbf{M}_\alpha\left(\|A\| b^\alpha \right) \right)^2 \mathbf{E}\, \left\|\varphi'(\eta)\right\|^2 , \end{aligned}$$

where $\varphi'(\eta) = \max\limits_{s\in[-\tau,0]} \varphi'(s)$.

As for the third term, by employing Hölder's inequality, (H1), (H3) and the Lemma 2, we have the following:

$$\begin{aligned} J_3 =& 6\mathbf{E}\left\| \int_{0}^{t} \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} G(s, y(s), y(s-\tau))\mathrm{d}s \right\|^2 \\ \leq& 6b \int_{0}^{t} \left\| \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} \right\|^2 \mathbf{E}\, \|G(s, y(s), y(s-\tau))\|^2 \,\mathrm{d}s \\ \leq& 12b \int_{0}^{t} \left\| \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} \right\|^2 \mathbf{E}\, \|G(s, y(s), y(s-\tau)) - G(s,0,0)\|^2 \,\mathrm{d}s + 12b \int_{0}^{t} \left\| \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} \right\|^2 \mathbf{E}\, \|G(s,0,0)\|^2 \,\mathrm{d}s \\ \leq& 12b \int_{0}^{t} \left\| \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} \right\|^2 L_G(s)\mathbf{E}(\|y(s)\|^2 + \|y(s-\tau)\|^2)\mathrm{d}s + 12b \int_{0}^{t} \left\| \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} \right\|^2 M_1 \mathrm{d}s \\ \leq& 24b \left(\int_{0}^{t} \left\| \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} \right\|^{2\varrho} \mathrm{d}s \right)^{1/\varrho} \left(\int_{0}^{t} L_G^{\gamma}(s)\mathrm{d}s \right)^{1/\gamma} \|y\|_C^2 \\ &+ 12b^2 M_1 (b+\tau)^{2\alpha-2} \left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau) \right)^\alpha \right)^2 \\ \leq& 24b^{\frac{\varrho+1}{\varrho}} (b+\tau)^{2\alpha-2} \left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau) \right)^\alpha \right)^2 \|L_G\|_{L^\gamma([0,b],\mathbb{R}^+)} \|y\|_C^2 \\ &+ 12b^2 M_1 (b+\tau)^{2\alpha-2} \left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau) \right)^\alpha \right)^2 , \end{aligned}$$

where $1/\varrho + 1/\gamma = 1, \varrho, \gamma > 1$.

By virtue of the Hölder's inequality, (H1), (H3), Lemma 2 and Itô's isometry, we have

$$\begin{aligned} J_4 \leq& 6\mathbf{E}\left\| \int_{0}^{b} \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} \Delta(s, y(s), y(s-\tau))\mathrm{d}w(s) \right\|^2 \\ =& 6 \int_{0}^{b} \left\| \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} \right\|^2 \mathbf{E}\, \|\Delta(s, y(s), y(s-\tau))\|^2 \,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\leq 12\int_0^b \left\|\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\right\|^2 \mathbf{E}\left\|\Delta(s,y(s),y(s-\tau))-\Delta(s,0,0)\right\|^2 \mathrm{d}s + 12\int_0^b \left\|\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\right\|^2 \left\|\Delta(s,0,0)\right\|^2 \mathrm{d}s \\
&\leq 12\int_0^b \left\|\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\right\|^2 L_\Delta(s)\mathbf{E}(\|y(s)\|^2+\|y(s-\tau)\|^2)\mathrm{d}s + 12\int_0^b \left\|\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\right\|^2 M_2\mathrm{d}s \\
&\leq 24\left(\int_0^b \left\|\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\right\|^{2\varrho}\mathrm{d}s\right)^{1/\varrho}\left(\int_0^b L_\Delta^\gamma(s)\mathrm{d}s\right)^{1/\gamma}\|y\|_C^2 \\
&\quad + 12bM_2(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}(\|A\|(b+\tau))^\alpha\right)^2 \\
&\leq 24b^{\frac{1}{\varrho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}(\|A\|(b+\tau))^\alpha\right)^2\|L_\Delta\|_{L^\gamma([0,b],\mathbb{R}^+)}\|y\|_C^2 \\
&\quad + 12bM_2(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}(\|A\|(b+\tau))^\alpha\right)^2.
\end{aligned}
$$

By means of the Hölder's inequality, (H1), (H3), the Lemma 2 and Itô's isometry, one can obtain

$$
\begin{aligned}
J_5 &\leq 6\mathbf{E}\left\|\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\int_Z g(s,y(s),y(s-\tau),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right\|^2 \\
&=6\mathbf{E}\int_0^b\int_Z \left\|\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} g(s,y(s),y(s-\tau),z)\right\|^2 \upsilon(\mathrm{d}z)\mathrm{d}s \\
&\leq 12\mathbf{E}\int_0^b \left\|\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\right\|^2\int_Z \|g(s,y(s),y(s-\tau),z)-g(s,0,0,z)\|^2\upsilon(\mathrm{d}z)\mathrm{d}s \\
&\quad + 12\mathbf{E}\int_0^b \left\|\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\right\|^2\int_Z \|g(s,0,0,z)\|^2\upsilon(\mathrm{d}z)\mathrm{d}s \\
&\leq 12\int_0^b \left\|\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\right\|^2 L_g(s)\mathbf{E}(\|y(s)\|^2+\|y(s-\tau)\|^2)\mathrm{d}s + 12\int_0^b \left\|\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\right\|^2 M_3\mathrm{d}s \\
&\leq 24\left(\int_0^b \left\|\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\right\|^{2\varrho}\mathrm{d}s\right)^{1/\varrho}\left(\int_0^b L_g^\gamma(s)\mathrm{d}s\right)^{1/\gamma}\|y\|_C^2 \\
&\quad + 12bM_3(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}(\|A\|(b+\tau))^\alpha\right)^2 \\
&\leq 24b^{\frac{1}{\varrho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}(\|A\|(b+\tau))^\alpha\right)^2\|L_g\|_{L^\gamma([0,b],\mathbb{R}^+)}\|y\|_C^2 \\
&\quad + 12bM_3(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}(\|A\|(b+\tau))^\alpha\right)^2.
\end{aligned}
$$

Using the Jensen's inequality, we get

$$
\begin{aligned}
J_6 &\leq 6\mathbf{E}\left\|\int_0^{b-\delta(b)}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^\alpha}Cr'(s)\right)u_y(s)\mathrm{d}s + \int_{b-\delta(b)}^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}Bu_y(s)\mathrm{d}s\right\|^2 \\
&=6\mathbf{E}\left\|\int_0^{b-\delta(b)}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^\alpha}Cr'(s)\right)\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^\alpha}Cr'(s)\right)^*\mathbf{E}\left\{[\Pi_\tau^b]^{-1}\beta \mid \mathscr{F}_s\right\}\mathrm{d}s\right. \\
&\quad \left. + \int_{b-\delta(b)}^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}BB^*\mathbb{Z}_{\tau,\alpha}^{A^*(b-\tau-s)^\alpha}\mathbf{E}\left\{[\Pi_\tau^b]^{-1}\beta \mid \mathscr{F}_s\right\}\mathrm{d}s\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\le 36\|\Theta_\tau^b\|^2\|[\Pi_\tau^b]^{-1}\|^2 \Bigg[\mathbf{E}\left\|y_1\right\|^2 + \mathbf{E}\left\|\mathbb{Z}_\tau^{Ab^\alpha}\varphi(-\tau)\right\|^2 + \mathbf{E}\left\|\int_{-\tau}^{0}\mathbb{Z}_\tau^{A(b-\tau-s)^\alpha}\varphi'(s)\mathrm{d}s\right\|^2 \\
&\quad + \mathbf{E}\left\|\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} G(s,y(s),y(s-\tau))\mathrm{d}s\right\|^2 + \mathbf{E}\left\|\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\Delta(s,y(s),y(s-\tau))\mathrm{d}w(s)\right\|^2 \\
&\quad + \mathbf{E}\left\|\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\int_Z g(s,y(s),y(s-\tau),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right\|^2\Bigg] \\
&\le 36k_1k_2\Big[\mathbf{E}\left\|y_1\right\|^2 + \left(\mathbf{M}_\alpha\left(\|A\|b^\alpha\right)\right)^2\mathbf{E}\left\|\varphi(-\tau)\right\|^2 + \tau^2\left(\mathbf{M}_\alpha\left(\|A\|b^\alpha\right)\right)^2\mathbf{E}\left\|\varphi'(\eta)\right\|^2 \\
&\quad + 4b^{\frac{\rho+1}{\rho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2\|L_G\|_{L^\gamma([0,b],\mathbb{R}^+)}\|y\|_C^2 + 2b^2M_1(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2 \\
&\quad + 4b^{\frac{1}{\rho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2\|L_\Delta\|_{L^\gamma([0,b],\mathbb{R}^+)}\|y\|_C^2 + 2bM_2(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2 \\
&\quad + 4b^{\frac{1}{\rho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2\|L_g\|_{L^\gamma([0,b],\mathbb{R}^+)}\|y\|_C^2 + 2bM_3(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2\Big].
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\mathbf{E}\|(\Phi y)(t)\|^2 \le\; & J_1+J_2+J_3+J_4+J_5+J_6 \\
\le\; & 6\left(\mathbf{M}_\alpha\left(\|A\|b^\alpha\right)\right)^2\mathbf{E}\left\|\varphi(-\tau)\right\|^2 + 6\tau^2\left(\mathbf{M}_\alpha\left(\|A\|b^\alpha\right)\right)^2\mathbf{E}\left\|\varphi'(\eta)\right\|^2 \\
& + 24b^{\frac{\rho+1}{\rho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2\|L_G\|_{L^\gamma([0,b],\mathbb{R}^+)}\|y\|_C^2 \\
& + 12b^2M_1(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2 \\
& + 24b^{\frac{1}{\rho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2\|L_\Delta\|_{L^\gamma([0,b],\mathbb{R}^+)}\|y\|_C^2 \\
& + 12bM_2(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2 \\
& + 24b^{\frac{1}{\rho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2\|L_g\|_{L^\gamma([0,b],\mathbb{R}^+)}\|y\|_C^2 \\
& + 12bM_3(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2 \\
& + 36k_1k_2\Big[\mathbf{E}\left\|y_1\right\|^2 + \left(\mathbf{M}_\alpha\left(\|A\|b^\alpha\right)\right)^2\mathbf{E}\left\|\varphi(-\tau)\right\|^2 + \tau^2\left(\mathbf{M}_\alpha\left(\|A\|b^\alpha\right)\right)^2\mathbf{E}\left\|\varphi'(\eta)\right\|^2 \\
& + 4b^{\frac{\rho+1}{\rho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2\|L_G\|_{L^\gamma([0,b],\mathbb{R}^+)}\|y\|_C^2 + 2b^2M_1(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2 \\
& + 4b^{\frac{1}{\rho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2\|L_\Delta\|_{L^\gamma([0,b],\mathbb{R}^+)}\|y\|_C^2 + 2bM_2(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2 \\
& + 4b^{\frac{1}{\rho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2\|L_g\|_{L^\gamma([0,b],\mathbb{R}^+)}\|y\|_C^2 + 2bM_3(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2\Big] \\
\le\; & 36k_1k_2\mathbf{E}\left\|y_1\right\|^2 + 6\left(\mathbf{M}_\alpha\left(\|A\|b^\alpha\right)\right)^2k_3(1+6k_1k_2) + 6\tau^2\left(\mathbf{M}_\alpha\left(\|A\|b^\alpha\right)\right)^2k_3(1+6k_1k_2) \\
& + 12b(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2(bM_1+M_2+M_3)(1+6k_1k_2) \\
& + 24b^{\frac{1}{\rho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau))^\alpha\right)^2\left(b\|L_G\|_{L^\gamma([0,b],\mathbb{R}^+)} + \|L_\Delta\|_{L^\gamma([0,b],\mathbb{R}^+)} + \|L_g\|_{L^\gamma([0,b],\mathbb{R}^+)}\right) \\
& \times(1+6k_1k_2)\,\|y\|_C^2.
\end{aligned}
$$

From above it follows that there exists a constant $C>0$ such that

$$
\mathbf{E}\|(\Phi y)(t)\|^2 \le C(1+\|y\|_C^2).
$$

Therefore Φ maps $\mathcal{A}$ into $\mathcal{A}$.

Step 2. We show that Φ is a contraction mapping. For $\forall x, y \in \mathcal{A}$ and $t \in [0, b]$, by adopting the Jensen's inequality, one can get

$$\begin{aligned}
\mathbf{E}\|(\Phi x)(t) - (\Phi y)(t)\|^2 \leq & 4\mathbf{E}\left\| \int_0^t \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} \left[G(s, x(s), x(s-\tau)) - G(s, y(s), y(s-\tau))\right] \mathrm{d}s \right\|^2 \\
& + 4\mathbf{E}\left\| \int_0^t \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} \left[\Delta(s, x(s), x(s-\tau)) - \Delta(s, y(s), y(s-\tau))\right] \mathrm{d}w(s) \right\|^2 \\
& + 4\mathbf{E}\left\| \int_0^t \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} \int_Z \left[g(s, x(s), x(s-\tau), z) - g(s, y(s), y(s-\tau), z)\right] \tilde{N}(\mathrm{d}s, \mathrm{d}z) \right\|^2 \\
& + 4\mathbf{E}\left\| \int_0^{t-\delta(t)} \left(\mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} B + \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-r(s))^\alpha} C r'(s)\right) \left[u_x(s) - u_y(s)\right] \mathrm{d}s \right. \\
& \left. + \int_{t-\delta(t)}^t \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} B \left[u_x(s) - u_y(s)\right] \mathrm{d}s \right\|^2 \\
:= & I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

By virtue of Hölder's inequality, the Lemma 2 and (H1), it is easy to obtain

$$\begin{aligned}
I_1 = & 4\mathbf{E}\left\| \int_0^t \mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha} \left[G(s, x(s), x(s-\tau)) - G(s, y(s), y(s-\tau))\right] \mathrm{d}s \right\|^2 \\
\leq & 4b \int_0^t \left\|\mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha}\right\|^2 \mathbf{E}\left\|G(s, x(s), x(s-\tau)) - G(s, y(s), y(s-\tau))\right\|^2 \mathrm{d}s \\
\leq & 4b \int_0^t \left\|\mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha}\right\|^2 L_G(s) \mathbf{E}(\|x(s) - y(s)\|^2 + \|x(s-\tau) - y(s-\tau)\|^2) \mathrm{d}s \\
\leq & 8b \left(\int_0^t \left\|\mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha}\right\|^{2\varrho} \mathrm{d}s\right)^{1/\varrho} \left(\int_0^t L_G^\gamma(s) \mathrm{d}s\right)^{1/\gamma} \sup_{t\in[-\tau,b]} \mathbf{E}\|x(t) - y(t)\|^2 \\
\leq & 8b^{\frac{\varrho+1}{\varrho}} (b+\tau)^{2\alpha-2} \left(\mathbf{M}_{\alpha,\alpha}(\|A\|(b+\tau))^\alpha\right)^2 \|L_G\|_{L^\gamma([0,b],\mathbb{R}^+)} \|x - y\|_C^2 .
\end{aligned}$$

By means of the Hölder's inequality, (H1), Lemma 2 and Itô's isometry, one can obtain

$$\begin{aligned}
I_2 \leq & 4\mathbf{E}\left\| \int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha} \left[\Delta(s, x(s), x(s-\tau)) - \Delta(s, y(s), y(s-\tau))\right] \mathrm{d}w(s) \right\|^2 \\
= & 4 \int_0^b \left\|\mathbb{Z}_{\tau,\alpha}^{A(t-\tau-s)^\alpha}\right\|^2 \mathbf{E}\left\|\Delta(s, x(s), x(s-\tau)) - \Delta(s, y(s), y(s-\tau))\right\|^2 \mathrm{d}s \\
\leq & 4 \int_0^b \left\|\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\right\|^2 L_\Delta(s) \mathbf{E}(\|x(s) - y(s)\|^2 + \|x(s-\tau) - y(s-\tau)\|^2) \mathrm{d}s \\
\leq & 8 \left(\int_0^b \left\|\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^\alpha}\right\|^{2\varrho} \mathrm{d}s\right)^{1/\varrho} \left(\int_0^b L_\Delta^\gamma(s) \mathrm{d}s\right)^{1/\gamma} \sup_{t\in[-\tau,b]} \mathbf{E}\|x(t) - y(t)\|^2
\end{aligned}$$

$$\leq 8b^{\frac{1}{\varrho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau)\right)^{\alpha}\right)^{2}\|L_{\Delta}\|_{L^{\gamma}([0,b],\mathbb{R}^{+})}\|x-y\|_{C}^{2}.$$

By applying Hölder's inequality, (H1), Lemma 2 and Itô's isometry, one can get

$$\begin{aligned}
I_3 \leq & 4\mathbf{E}\left\|\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}\int_Z [g(s,x(s),x(s-\tau),z)-g(s,y(s),y(s-\tau),z)]\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right\|^2 \\
= & 4\int_0^b\int_Z \left\|\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}\right\|^2 \mathbf{E}\,\|g(s,x(s),x(s-\tau),z)-g(s,y(s),y(s-\tau),z)\|^2\,\upsilon(\mathrm{d}z)\mathrm{d}s \\
= & 4\int_0^b \left\|\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}\right\|^2 \int_Z \mathbf{E}\,\|g(s,x(s),x(s-\tau),z)-g(s,y(s),y(s-\tau),z)\|^2\,\upsilon(\mathrm{d}z)\mathrm{d}s \\
\leq & 4\int_0^b \left\|\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}\right\|^2 L_g(s)\mathbf{E}(\|x(s)-y(s)\|^2+\|x(s-\tau)-y(s-\tau)\|^2)\mathrm{d}s \\
\leq & 8\left(\int_0^b \left\|\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}\right\|^{2\varrho}\mathrm{d}s\right)^{1/\varrho}\left(\int_0^b L_g^{\gamma}(s)\mathrm{d}s\right)^{1/\gamma}\sup_{t\in[-\tau,b]}\mathbf{E}\,\|x(t)-y(t)\|^2 \\
\leq & 8b^{\frac{1}{\varrho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau)\right)^{\alpha}\right)^{2}\|L_{g}\|_{L^{\gamma}([0,b],\mathbb{R}^{+})}\|x-y\|_{C}^{2}.
\end{aligned}$$

One can apply the Jensen's inequality to derive that

$$\begin{aligned}
I_4 \leq & 4\mathbf{E}\left\|\int_0^{b-\delta(b)}\left(\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B+\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-r(s))^{\alpha}}Cr'(s)\right)\left[u_x(s)-u_y(s)\right]\mathrm{d}s\right. \\
& \left.+\int_{b-\delta(b)}^{b}\mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}B\left[u_x(s)-u_y(s)\right]\mathrm{d}s\right\|^2 \\
\leq & 12\|\Theta_{\tau}^{b}\|^2\|[\Pi_{\tau}^{b}]^{-1}\|^2\left[\mathbf{E}\left\|\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}[G(s,x(s),x(s-\tau))-G(s,y(s),y(s-\tau))]\,\mathrm{d}s\right\|^2\right. \\
& +\mathbf{E}\left\|\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}[\Delta(s,x(s),x(s-\tau))-\Delta(s,y(s),y(s-\tau))]\,\mathrm{d}w(s)\right\|^2 \\
& \left.+\mathbf{E}\left\|\int_0^b \mathbb{Z}_{\tau,\alpha}^{A(b-\tau-s)^{\alpha}}\int_Z [g(s,x(s),x(s-\tau),z)-g(s,y(s),y(s-\tau),z)]\,\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right\|^2\right] \\
\leq & 12k_1k_2\left[2b^{\frac{\varrho+1}{\varrho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau)\right)^{\alpha}\right)^{2}\|L_{G}\|_{L^{\gamma}([0,b],\mathbb{R}^{+})}\|x-y\|_{C}^{2}\right. \\
& +2b^{\frac{1}{\varrho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau)\right)^{\alpha}\right)^{2}\|L_{\Delta}\|_{L^{\gamma}([0,b],\mathbb{R}^{+})}\|x-y\|_{C}^{2} \\
& \left.+2b^{\frac{1}{\varrho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau)\right)^{\alpha}\right)^{2}\|L_{g}\|_{L^{\gamma}([0,b],\mathbb{R}^{+})}\|x-y\|_{C}^{2}\right].
\end{aligned}$$

From the results of $I_1 - I_4$, we get following:

$$\begin{aligned}
\mathbf{E}\|(\Phi x)(t)-(\Phi y)(t)\|^2 \leq & I_1+I_2+I_3+I_4 \\
\leq & 8b^{\frac{\varrho+1}{\varrho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau)\right)^{\alpha}\right)^{2}\|L_{G}\|_{L^{\gamma}([0,b],\mathbb{R}^{+})}\|x-y\|_{C}^{2} \\
& +8b^{\frac{1}{\varrho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau)\right)^{\alpha}\right)^{2}\|L_{\Delta}\|_{L^{\gamma}([0,b],\mathbb{R}^{+})}\|x-y\|_{C}^{2}
\end{aligned}$$

$$
\begin{aligned}
&+ 8b^{\frac{1}{\varrho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau)\right)^{\alpha}\right)^{2}\|L_g\|_{L^{\gamma}([0,b],\mathbb{R}^+)}\|x-y\|_C^2\\
&+ 12k_1k_2\Big[2b^{\frac{\varrho+1}{\varrho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau)\right)^{\alpha}\right)^{2}\|L_G\|_{L^{\gamma}([0,b],\mathbb{R}^+)}\|x-y\|_C^2\\
&+ 2b^{\frac{1}{\varrho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau)\right)^{\alpha}\right)^{2}\|L_\Delta\|_{L^{\gamma}([0,b],\mathbb{R}^+)}\|x-y\|_C^2\\
&+ 2b^{\frac{1}{\varrho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau)\right)^{\alpha}\right)^{2}\|L_g\|_{L^{\gamma}([0,b],\mathbb{R}^+)}\|x-y\|_C^2\Big]\\
=&8b^{\frac{1}{\varrho}}(b+\tau)^{2\alpha-2}\left(\mathbf{M}_{\alpha,\alpha}\left(\|A\|(b+\tau)\right)^{\alpha}\right)^{2}\\
&\times(b\|L_G\|_{L^{\gamma}([0,b],\mathbb{R}^+)}+\|L_\Delta\|_{L^{\gamma}([0,b],\mathbb{R}^+)}+\|L_g\|_{L^{\gamma}([0,b],\mathbb{R}^+)})(1+3k_1k_2)\|x-y\|_C^2\\
=&K\|x-y\|_C^2.
\end{aligned}
$$

Since $K<1$ by (H2), Φ is a contraction mapping. Therefore, it has an unique fixed point $y\in\mathcal{A}$ with $u_y(\cdot)\in\mathcal{U}_{ad}$, which is the solution of (1) and it satisfies initial function $y(t)=\varphi(t)$, $t\in[-\tau,0]$. Therefore, the stochastic system (1) is relatively exactly controllable. □

4 | EXAMPLE

Let us consider the following specified nonlinear FSDS:

$$
\begin{cases}
{}^C D_{0^+}^{0.6}y(t) = & Ay(t-0.5)+Bu(t)+Cu(t-t/2)+G(t,y(t),y(t-0.5))+\Delta(t,y(t),y(t-0.5))\frac{\mathrm{d}w(t)}{\mathrm{d}t}\\
 & +\int_Z g(t,y(t),y(t-0.5),z)\frac{\mathrm{d}\tilde{N}(t,z)}{\mathrm{d}t},\ t\in[0,1],\\
y(t)=\varphi(t), & t\in[-0.5,0],
\end{cases}
\tag{19}
$$

where $y(t)=(y_1(t),y_2(t))^T$.

In the matrix form, we have $A=\begin{pmatrix}0.2 & 0\\ 0.4 & 0.6\end{pmatrix}$, $B=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$, $C=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$,
$G(t,y(t),y(t-0.5)=\begin{pmatrix}e^{0.8t-13}(y_1(t)+y_1(t-0.5))\\ e^{0.8t-13}(y_2(t)+y_2(t-0.5))\end{pmatrix}$, $g(t,y(t),y(t-0.5))=e^{-6}\begin{pmatrix}z_1t\\ z_2\cos y_2\end{pmatrix}$,
$\Delta(t,y(t),y(t-0.5))=\begin{pmatrix}(e^{0.5t-7}-0.3te^{-6})(y_1(t)+y_1(t-0.5))\\ (e^{0.5t-7}-0.3te^{-6})(y_2(t)+y_2(t-0.5))\end{pmatrix}$, $\varphi(t)=\begin{pmatrix}t\\ 2t\end{pmatrix}$.
The controllability Grammian matrix of linear system corresponding of system (19) is

$$
\begin{aligned}
\Theta_{0.5}^{1}=&\int_0^{0.5}\left(\mathbb{Z}_{0.5,0.6}^{A(0.5-s)^{0.6}}B+2\mathbb{Z}_{0.5,0.6}^{A(0.5-2s)^{\alpha}}C\right)\left(\mathbb{Z}_{0.5,0.6}^{A(0.5-s)^{0.6}}B+2\mathbb{Z}_{0.5,0.6}^{A(0.5-2s)^{0.6}}C\right)^{*}\mathrm{d}s\\
&+\int_{0.5}^{1}\mathbb{Z}_{0.5,0.6}^{A(0.5-s)^{0.6}}BB^{*}\mathbb{Z}_{0.5,0.6}^{A^{*}(0.5-s)^{0.6}}\mathrm{d}s\\
=&\int_0^{0.25}\left(\mathbb{Z}_{0.5,0.6}^{A(0.5-s)^{0.6}}B+2\mathbb{Z}_{0.5,0.6}^{A(0.5-2s)^{\alpha}}C\right)\left(\mathbb{Z}_{0.5,0.6}^{A(0.5-s)^{0.6}}B+2\mathbb{Z}_{0.5,0.6}^{A(0.5-2s)^{0.6}}C\right)^{*}\mathrm{d}s\\
&+\int_{0.25}^{0.5}\left(\mathbb{Z}_{0.5,0.6}^{A(0.5-s)^{0.6}}B+2\mathbb{Z}_{0.5,0.6}^{A(0.5-2s)^{\alpha}}C\right)\left(\mathbb{Z}_{0.5,0.6}^{A(0.5-s)^{0.6}}B+2\mathbb{Z}_{0.5,0.6}^{A(0.5-2s)^{0.6}}C\right)^{*}\mathrm{d}s\\
&+\int_{0.5}^{1}\mathbb{Z}_{0.5,0.6}^{A(0.5-s)^{0.6}}BB^{*}\mathbb{Z}_{0.5,0.6}^{A^{*}(0.5-s)^{0.6}}\mathrm{d}s.
\end{aligned}
$$

The delayed matrix function of Mittag-Leffler is

$$\mathbb{Z}_{0.5,0.6}^{At^{0.6}} = \begin{cases} \Theta, & -\infty < t \leq -0.5, \\ I\frac{(0.5+t)^{-0.4}}{\Gamma(0.6)}, & -0.5 < t \leq 0, \\ I\frac{(0.5+t)^{-0.4}}{\Gamma(0.6)} + A\frac{t^{0.2}}{\Gamma(1.2)}, & 0 < t \leq 0.5. \end{cases}$$

By calculation, one can derive the controllability Grammian matrix

$$\Theta_{0.5}^{1} = \begin{pmatrix} 22.1072 & 1.8401 \\ 1.8401 & 23.5164 \end{pmatrix},$$

and its inverse

$$[\Theta_{0.5}^{1}]^{-1} = \begin{pmatrix} 0.0455 & -0.0036 \\ -0.0036 & 0.0428 \end{pmatrix}.$$

We can see that $\Theta_{0.5}^{1}$ is positive definite, so the corresponding linear system of stochastic system (19) is relatively exactly controllable on [0,1]. Therefore, we get

$$\langle \Theta_{0.5}^{1} y, y \rangle = \begin{pmatrix} 22.1072y_1^2 + 1.8401y_2^2 \\ 1.8401y_1^2 + 23.5164y_2^2 \end{pmatrix} \geq \lambda \|y\|^2,$$

where $0 < \lambda \leq 1.8401$, and one can obtain $k_2 = 0.5434$. Letting $\varrho = \gamma = 2$. It is easy to see that G, Δ and g satisfy the hypotheses of Theorem 2. Therefore, the nonlinear FSDS (19) is relatively exactly controllable on [0,1].

5 | CONCLUSIONS

In this paper, the main purpose is to investigate the relatively exact controllability of FSDS driven by Lévy noise via delayed matrix functions of M-L. By applying the controllability Grammian matrix, we obtain the relatively exact controllability of linear FSDS. In addition, by adopting the fixed point theorem, the existence and uniqueness of nonlinear FSDS are discussed. Various inequality scaling techniques, such as Hölder's inequality, Jensen's inequality and Itô's isometry, are used in the derivation. Further, the relatively exact controllability of nonlinear FSDS is established. Finally, the theoretical results are supported through an example.

Our future research topic will focus on fractional fuzzy impulsive stochastic system and will explore to derive the controllability of the addressed system. Compared with fractional impulsive stochastic system without fuzzy environment, the controllability of fractional fuzzy impulsive stochastic system is relatively new. At the same time, we will also face many difficulties, such as how to investigate the controllability of fractional system in the fuzzy space, how to give the control function in the fuzzy space, how to study fractional system with impulses in the fuzzy space, and how to deal with stochastic term driven by Liu process in the fuzzy space.

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CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

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□