Travelling waves for generalized Fisher–Kolmogorov equation with discontinuous density dependent diffusion

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Dedicated to Jean Mawhin in celebration of his 80th birthday.

Abstract. We are concerned with the existence and qualitative properties of travelling wave solutions for a quasilinear reaction-diffusion equation on the real line. We consider a non-Lipschitz reaction term of Fisher–KPP type and a discontinuous diffusion coefficient that allows for degenerations and singularities at equilibrium points. We investigate the joint influence of the reaction and diffusion terms on the existence and nonexistence of travelling waves and, assuming these terms are of power-type near equilibria, we provide classification of solutions based on their asymptotic properties. Our approach provides a broad theoretical background for the mathematical treatment of rather general models not only in population dynamics but also in other applied sciences and engineering.

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1 Introduction

This paper is concerned with the quasilinear reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(d(u)\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) + g(u), \quad (x,t) \in \mathbb{R} \times [0,+\infty), \quad p > 1, \tag{1.1}$$

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and its travelling wave solutions, i.e., bounded non-constant solutions of the form $u(x,t) = U(x-ct)$, where U is the profile of the travelling wave and $c \in \mathbb{R}$ denotes the (unknown) speed of propagation. We consider the reaction term $g \in C[0,1]$ to be of the so-called Fisher-KPP type, i.e.,

$$g(0) = g(1) = 0, \quad g > 0 \ \text{ in } \ (0,1). \tag{1.2}$$

The diffusion coefficient $d = d(s)$ is a rather general function in the sense that it need not be continuous in $[0,1]$ or even in $(0,1)$. In particular, d may vanish or be singular at one or both endpoints and it may also have discontinuities of the first kind at a finite number of points in $(0,1)$. Its properties will be specified in the next section.

For $p = 2$ and sufficiently smooth reaction and diffusion terms, equation (1.1) has been widely studied, starting with the classical papers [12] by Fisher and [13] by Kolmogorov, Petrovsky and Piskunov, both from 1937. We now mention some basic results and applications concerning special cases of (1.1). In [13], the authors investigated travelling wave solutions of the semilinear equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(u) \tag{1.3}$$

with $g \in C^1[0,1]$ satisfying (1.2) together with $g'(0) > 0$ and $g'(u) \le g'(0)$ for all $u \in (0,1)$. A particular case of (1.3) was also considered by Fisher to model the spread of advantageous gene in a population uniformly distributed in a one-dimensional habitat. In [12] he suggested that if an advantageous mutation occurs, a wave of increase in the mutant gene frequency can be expected at the expense of its parent. Assuming that the gene only occurs in two forms and that the rate of diffusion per generation is governed by the law

$$-d\,\frac{\partial u}{\partial x}, \quad d > 0,$$

the frequency u of the mutant gene then satisfies the equation

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + mu(1-u)$$

where $m > 0$ is the intensity of selection in favour of the mutant gene and $(1-u)$ is the frequency of the non–mutant gene. Note that the constants d and m in the equation above can be absorbed by a suitable rescaling. Fisher's genetical context was also explored in detail in [2, 3] considering a population of diploid individuals carrying a pair of alleles that occur in two forms, denoted by a and A. The population is then divided into three classes – homozygotes (aa, AA) and heterozygotes (aA) – whose linear densities as well as death rates generally differ. The general assumptions on $g = g(s)$ in this model are $g \in C^1[0,1]$, $g(0) = g(1) = 0$. Other relevant properties of the function g are specified by distinguishing what genotype is more viable than the two others. Condition (1.2) with $g'(0) > 0$ corresponds to the *heterozygote intermediate case*, in which the viability of the heterozygotes is between the viabilities of the homozygotes. Other possible interpretations can be found in [16, 19]. The basic result states that there exists a number $c^* > 0$ such that for each $c \ge c^*$ equation (1.3) possesses a monotone decreasing travelling wave solution $u(x,t) = U(x-ct)$ satisfying boundary conditions $U(-\infty) = 1$, $U(+\infty) = 0$. Moreover,

$$2\sqrt{g'(0)} \le c^* \le 2\sqrt{\sup_{u \in (0,1)} \frac{g(u)}{u}},$$

see [3].

In many populations, density dependent dispersal has been observed, such as migration to less inhabited areas as the population gets more crowded. This phenomenon can be incorporated by considering non-constant diffusion coefficient $d = d(s)$. Certain models also suggest that besides $d(s) > 0$ it is also reasonable to assume that d degenerates at 0, i.e., $d(0) = 0$, cf. [16, 18]. Particular cases for $p = 2$ were investigated in, e.g., [1, 6, 16, 17]. A systematic treatment of (1.1) with $p = 2$ and degenerate diffusion was given in [18]. The authors prove the existence of monotone travelling wave solutions for $c \geq c^* > 0$ and distinguish *front-type* and *sharp-type* profiles. The latter appears for $c = c^*$ and, in contrast with front-type solutions, the leading edge of the wave reaches 0 in a finite time with a negative slope. The assumptions on d and g were rather strong, such as that both function are of class $C^2[0,1]$ and the second derivative of d does not vanish in $[0,1]$. The results were extended in [14] for $g \in C[0,1]$ and $d \in C^1[0,1]$, $d > 0$ in $(0,1)$, $d(0) = 0$, using upper and lower solutions method. Sharp-type profiles appear if $d'(0) > 0$ whereas $d'(0) = +\infty$ leads to front-type solutions only. The authors also discuss the case of doubly degenerate diffusion when $d(0) = d(1) = 0$. Even weaker assumptions on $d = d(s)$ were considered in [9] allowing singularities at both equilibrium points 0 and 1. Existence of travelling waves is shown, however, information regarding possible "sharpness" is obtained only by assuming power-type behaviour of the reaction and diffusion terms.

Recently, models with $p > 1$ also appeared in literature, see for instance [4, 8, 11] and their references. In our previous work [10] we derived existence and uniqueness results for a continuous bistable reaction term

$$g(0) = g(s_*) = g(1) = 0, \; g < 0 \text{ in } (0, s_*), \; g > 0 \text{ in } (s_*, 1)$$

and a possibly discontinuous diffusion coefficient $d = d(s)$ with the properties listed in the next section. To our best knowledge, discontinuous diffusion has not been considered in models with Fisher-KPP type reaction term. In the bistable case, piecewise constant diffusion coefficient has appeared in [20], physically motivated by phenomena in which the diffusion constant drops abruptly.

This paper is organized as follows. In Section 2 we introduce a new definition of solution that accounts for discontinuities as well as possible degenerations and singularities of the diffusion coefficient. The sign of wave speed c and basic properties of the travelling wave profile are discussed at the beginning of Section 3. We then prove the equivalence with a first order problem, resulting in Proposition 3.3. We study this problem in detail in Section 4 and subsequently interpret the existence and nonexistence results in terms of the travelling wave profile in Section 5, Theorems 5.1, 5.3. In contrast with the results obtained in [14] for $p = 2$ and [11] for $p > 1$, our existence theorem does not provide information about whether the solution attains values 0 and/or 1. We investigate this phenomenon in the case of power-type behaviour of $d = d(s)$ and $g = g(s)$ near 0 and 1 in Section 6.

2 Preliminaries

Let $g : [0,1] \to \mathbb{R}$, $g \in C[0,1]$ satisfy (1.2). The diffusion coefficient $d : [0,1] \to \mathbb{R}$ is supposed to be nonnegative lower semicontinuous and $d > 0$ in $(0,1)$. There exist $0 = s_0 < s_1 < s_2 < \cdots <$

$s_n < s_{n+1} = 1$ such that $d|_{(s_i,s_{i+1})} \in C(s_i, s_{i+1})$, $i = 0, \ldots, n$, and d has discontinuity of the first kind (finite jump) at s_i, $i = 1, \ldots, n$.

We use the moving coordinate $z = x - ct$ and write $u(x,t) = U(x - ct) = U(z)$. For the sake of simplicity we write $(\,\cdot\,)'$ instead of $\frac{\mathrm{d}}{\mathrm{d}z}(\,\cdot\,)$. Then (1.1) transforms into

$$\left(d(U(z))\left|U'(z)\right|^{p-2}U'(z)\right)' + cU'(z) + g(U(z)) = 0. \tag{2.1}$$

Discontinuities of $d = d(s)$ imply that we cannot expect (2.1) to be satisfied in a classical sense. Therefore, below we introduce the definition of more general solution of (2.1).

A function $U \in C(\mathbb{R})$ is piecewise C^1 (denoted $U \in \widehat{C}^1(\mathbb{R})$) provided that there is a discrete set $D_U \subset \mathbb{R}$, i.e, a set which consists of isolated points, such that $U \in C^1(\mathbb{R} \setminus D_U)$.

Definition 2.1. Let $U : \mathbb{R} \to [0,1]$, $U \in \widehat{C}^1(\mathbb{R})$. We denote

$$M_U := \{z \in \mathbb{R} : U(z) = s_i,\ i = 1, 2, \ldots, n\}, \quad N_U := \{z \in \mathbb{R} : U(z) = 0 \text{ or } U(z) = 1\}.$$

Then U is called a solution of (2.1) if

(a) $\partial M_U \cup \partial N_U = D_U$.

(b) For any $z \in \partial M_U$ there exist finite one sided derivatives $U'(z-)$, $U'(z+)$ and

$$L(z) := \left|U'(z-)\right|^{p-2}U'(z-)\lim_{\xi\to z-} d(U(\xi)) = \left|U'(z+)\right|^{p-2}U'(z+)\lim_{\xi\to z+} d(U(\xi)).$$

(c) Function $v : \mathbb{R} \to \mathbb{R}$ defined by

$$v(z) := \begin{cases} d(U(z))\,|U'(z)|^{p-2}\,U'(z), & z \notin \partial M_U \cup \partial N_U, \\ 0, & z \in \partial N_U, \\ L(z), & z \in \partial M_U \end{cases}$$

is continuous and for any $z, \hat{z} \in \mathbb{R}$

$$v(\hat{z}) - v(z) + c\,(U(\hat{z}) - U(z)) + \int_z^{\hat{z}} g(U(\xi))\,\mathrm{d}\xi = 0. \tag{2.2}$$

Moreover, $\lim\limits_{z\to\pm\infty} v(z) = 0$ if either $\lim\limits_{z\to-\infty} U(z) = 1$ and $\lim\limits_{z\to+\infty} U(z) = 0$ or else $\lim\limits_{z\to-\infty} U(z) = 0$ and $\lim\limits_{z\to+\infty} U(z) = 1$.

Remark 2.2. It follows from Definition 2.1 (a) that for any solution U of (2.1) both sets ∂M_U and ∂N_U are discrete. The existence of the derivative $U'(z)$ (or $U'(z-)$, $U'(z+)$) for $z \in \partial N_U$ depends on asymptotic behaviour of d near 0 and 1.

Remark 2.3. It follows from $g(s) > 0$, $s \in (0,1)$, that $\operatorname{int} M_U = \emptyset$. Indeed, let $\tilde{\xi} \in \operatorname{int} M_U$. Then there is an open neighbourhood $\mathcal{U}(\tilde{\xi})$ of $\tilde{\xi}$ such that $\mathcal{U}(\tilde{\xi}) \subset \operatorname{int} M_U$, i.e., there exists $i \in \{1, 2, \ldots, n\}$ such that $U(z) = s_i$ for all $z \in \mathcal{U}(\tilde{\xi})$. In particular, $U'(z) = 0$ for all $z \in \mathcal{U}(\tilde{\xi})$. Choosing $z, \hat{z} \in \mathcal{U}(\tilde{\xi})$, $z \neq \hat{z}$ in (2.2), we arrive at $\int_z^{\hat{z}} g(s_i)\,\mathrm{d}\xi = 0$, a contradiction. Therefore, $M_U = \partial M_U$.

Remark 2.4. Constant functions

$$U_0(z) = 0, \quad U_1(z) = 1, \quad z \in \mathbb{R},$$

are solutions of (2.1). Here, $M_{U_0} = M_{U_1} = \emptyset$, $N_{U_0} = N_{U_1} = \mathbb{R}$. It follows from the properties of d and g that those are the only constant solutions of (2.1) and they are called equilibria.

Remark 2.5. Let $p = 2$, $d \equiv 1$ and $g \in C^1[0,1]$. Let $U = U(z)$ be a solution of (2.1) in the sense of Definition 2.1. Then $M_U = \emptyset$, $N_U = \emptyset$ and the equation (2.1) holds pointwise. Therefore, $U \in C^2(\mathbb{R})$ is a classical solution. For more general d we have to employ the first integral (2.2) due to the lack of differentiability of the solution U.

Remark 2.6. Let $z \notin M_U \cup N_U$, $\hat{z} = z + h$, $h \neq 0$. Since M_U and N_U are closed sets we can choose $|h|$ so small that $\hat{z} \notin M_U \cup N_U$. Divide (2.2) by h and let $h \to 0$. Then, by Definition 2.1 (a), the derivative $U'(z)$ exists and

$$v'(z) + cU'(z) + g(U(z)) = 0. \tag{2.3}$$

In particular, v is differentiable in $z \notin M_U \cup N_U$.

Remark 2.7. Let $z \in M_U$, $\hat{z} = z + h$, $h < 0$. Divide (2.2) by h and let $h \to 0$. Then by Definition 2.1 we get

$$v'(z-) + cU'(z-) + g(U(z)) = 0.$$

In particular, $v'(z-)$ exists and it is finite. Similarly, we derive

$$v'(z+) + cU'(z+) + g(U(z)) = 0$$

and $v'(z+)$ exists and it is finite.

3 Equivalent first order ODE

Let U be a solution of (2.1) satisfying boundary conditions

$$\lim_{z \to -\infty} U(z) = 1 \qquad \text{and} \qquad \lim_{z \to +\infty} U(z) = 0. \tag{3.1}$$

Passing to the limit for $z \to -\infty$ in (2.2) and writing z in place of $\hat{z}$, we obtain that

$$v(z) + c\,(U(z) - 1) + \int_{-\infty}^{z} g(U(\sigma))\,\mathrm{d}\sigma = 0 \tag{3.2}$$

holds for any $z \in \mathbb{R}$. On the other hand, passing to the limit for $\hat{z} \to +\infty$ in (2.2), we obtain that

$$v(z) + cU(z) - \int_{z}^{+\infty} g(U(\sigma))\,\mathrm{d}\sigma = 0 \tag{3.3}$$

holds for any $z \in \mathbb{R}$. Passing to the limit for $\hat{z} \to +\infty$ and $z \to -\infty$ in (2.2) yields

$$-c + \int_{-\infty}^{+\infty} g(U(\sigma))\,\mathrm{d}\sigma = 0.$$

Since $g > 0$ in $(0,1)$ and U satisfies (3.1), we get $c > 0$.

Lemma 3.1. *Let $U = U(z)$, $z \in \mathbb{R}$, be a solution of* (2.1), (3.1) *and assume $\xi \in N_U$. Then the following two alternatives occur:*

(i) *if $U(\xi) = 0$ then $U(z) = 0$ for every $z \geq \xi$;*

(ii) *if $U(\xi) = 1$ then $U(z) = 1$ for every $z \leq \xi$.*

Proof. (i) Let $U(\xi) = 0$ and there exists $\xi_* > \xi$ such that $U(\xi_*) > 0$. Taking ξ_* closer to ξ if necessary, we may assume that also $U(\xi_*) < 1$. Then $g(U(\xi_*)) > 0$ and therefore $\int_\xi^{+\infty} g(U(\sigma))\,\mathrm{d}\sigma > 0$. From the definition of v we get $v(\xi) = 0$ and from (3.3) with $z = \xi$ we deduce $\int_\xi^{+\infty} g(U(\sigma))\,\mathrm{d}\sigma = 0$, a contradiction.
(ii) Assume $U(\xi) = 1$ and there is some $\xi_* < \xi$ such that $U(\xi_*) < 1$. Taking ξ_* closer to ξ if necessary, we can guarantee also $U(\xi_*) > 0$. Hence $g(U(\xi_*)) > 0$ and so $\int_{-\infty}^{\xi} g(U(\sigma))\,\mathrm{d}\sigma > 0$. From the definition of v we have $v(\xi) = 0$ and from (3.2) with $z = \xi$ we deduce $\int_{-\infty}^{\xi} g(U(\sigma))\,\mathrm{d}\sigma = 0$, a contradiction. $\square$

Lemma 3.2. *Let $U = U(z)$, $z \in \mathbb{R}$, be a solution of* (2.1), (3.1). *Then U is nonincreasing in $\mathbb{R}$. Moreover, for $z \notin N_U$ we have $U'(z) < 0$ if $z \notin M_U$ and $U'(z-) < 0$, $U'(z+) < 0$ if $z \in M_U$. In other words, U is (strictly) decreasing at any point $z \in \mathbb{R}$ such that $0 < U(z) < 1$.*

Proof. Let $\xi \notin N_U$ be such that $U'(\xi-) = 0$. Then it follows from Remarks 2.6 and 2.7 depending on whether $\xi \notin M_U \cup N_U$ or $\xi \in M_U$, respectively, that

$$v'(\xi-) = -g(U(\xi)) < 0.$$

Since $v(\xi) = 0$, there exists left neighbourhood $\mathcal{U}_-(\xi)$ of the point ξ such that for all $z \in \mathcal{U}_-(\xi)$ we have $v(z) > 0$. Taking $\mathcal{U}_-(\xi)$ smaller if necessary, we may assume that $N_U \cap \mathcal{U}_-(\xi) = \emptyset$. Since $d(U(z)) > 0$, $z \in \mathcal{U}_-(\xi)$, from $v(z) > 0$ we deduce that for any $z \in \mathcal{U}_-(\xi)$ we have also $U'(z-) > 0$, $U'(z+) > 0$. However, this implies that $U(z) < U(\xi)$, $z \in \mathcal{U}_-(\xi)$. Since, by Definition 2.1, $U'(\xi+) = 0$, we deduce similarly as above that there is also a right neighbourhood $\mathcal{U}_+(\xi)$ of ξ such that $U(z) < U(\xi)$, $z \in \mathcal{U}_+(\xi)$. Therefore, ξ is the point of strict local maximum for U. Since $U(z) \to 1$ as $z \to -\infty$ and $U(\xi) < 1$, there is $\xi_* \in (-\infty, \xi)$ such that $U(\xi) \leq U(\xi_*) < 1$. Let $\xi^* \in [\xi_*, \xi]$ be a global minimizer for U over the compact interval $[\xi_*, \xi]$. Then $U(\xi^*) < U(\xi) \leq U(\xi_*) < 1$ and therefore $\xi^* \in (\xi_*, \xi)$. In particular, ξ^* is also a local minimizer for U. If $U(\xi^*) = 0$, i.e., $\xi^* \in N_U$, then Lemma 3.1 forces $U(\xi) = 0$, contradicting $\xi \notin N_U$. Therefore, we have $\xi^* \notin N_U$. If $\xi^* \notin M_U$ then $U'(\xi^*)$ exists and hence $U'(\xi^*) = 0$ (ξ^* is a local minimizer for U). We can prove as above that ξ^* is a strict local maximizer for U, a contradiction. Finally, if $\xi^* \in M_U$ then from Definition 2.1 (b) and $d(U(\xi^*)) > 0$ we conclude $\operatorname{sgn} U'(\xi^*_-) = \operatorname{sgn} U'(\xi^*_+)$. But ξ^* being local minimizer for U implies that $U'(\xi^*_-) \leq 0$ and $U'(\xi^*_+) \geq 0$. Hence, $U'(\xi^*_-) = U'(\xi^*_+) = 0$, i.e., $U'(\xi^*) = 0$ and we proceed as above. This concludes the proof. $\square$

It follows from Lemmas 3.1 and 3.2 that function $U = U(z)$, $z \in \mathbb{R}$, which solves (2.1), (3.1) is a nonincreasing function in $\mathbb{R}$ and there is an open interval $(z_0, z_1) \subset \mathbb{R}$, $-\infty \leq z_0 < z_1 \leq +\infty$, such that U is strictly decreasing in (z_0, z_1),

$$\lim_{z\to z_0+} U(z) = 1 \quad \text{and} \quad U(z) = 1 \qquad \text{if} \quad -\infty < z \le z_0,$$
$$\lim_{z\to z_1-} U(z) = 0 \quad \text{and} \quad U(z) = 0 \qquad \text{if} \quad z_1 \le z < +\infty.$$

Moreover, $M_U = \{\xi_1, \xi_2, \ldots, \xi_n\}$ where $U(\xi_i) = s_i$, $i = 1, 2, \ldots, n$. In particular, $\operatorname{int} M_U = \emptyset$ and $M_U = \partial M_U$. For all $z \notin M_U \cup N_U$ we have $U'(z) < 0$ and for all $z \in M_U$ we have $U'(z-) < 0$ and $U'(z+) < 0$. The function U is continuous and piecewise C^1 in the sense that $U|_{(\xi_i,\xi_{i+1})} \in C^1(\xi_i, \xi_{i+1})$. Therefore, there exists a strictly decreasing inverse function $U^{-1} : (0,1) \to (z_0, z_1)$, $z = U^{-1}(U)$, such that $U^{-1}|_{(s_i,s_{i+1})} \in C^1(s_i, s_{i+1})$, $i = 0, 1, \ldots, n$ and the limits

$$\lim_{U\to s_i-} \frac{\mathrm{d}z}{\mathrm{d}U} = \left(\lim_{z\to\xi_i+} \frac{\mathrm{d}U}{\mathrm{d}z}\right)^{-1}, \quad \lim_{U\to s_i+} \frac{\mathrm{d}z}{\mathrm{d}U} = \left(\lim_{z\to\xi_i-} \frac{\mathrm{d}U}{\mathrm{d}z}\right)^{-1}$$

exist finite, $i = 1, 2, \ldots, n$. Set

$$w(U) = v(z(U)), \quad U \in (0,1). \tag{3.4}$$

Then $w = w(U)$ is a piecewise C^1-function in $(0,1)$,

$$w|_{(s_i,s_{i+1})} \in C^1(s_i, s_{i+1}), \quad i = 0, 1, \ldots, n,$$

with finite limits $\lim_{U\to s_i-} w'(U)$, $\lim_{U\to s_i+} w'(U)$, $i = 1, 2, \ldots, n$. Therefore, for any $z \in (\xi_i, \xi_{i+1})$ and $U \in (s_i, s_{i+1})$, $i = 0, 1, \ldots, n$, we have

$$\frac{\mathrm{d}}{\mathrm{d}z} v(z) = \frac{\mathrm{d}}{\mathrm{d}z} w(U(z)) = \frac{\mathrm{d}w}{\mathrm{d}U}\,(U(z))\, U'(z). \tag{3.5}$$

From $v(z) = -d(U(z))\, |U'(z)|^{p-1}$ we deduce that

$$U'(z) = -\left|\frac{v(z)}{d(U(z))}\right|^{p'-1}, \qquad p' = \frac{p}{p-1}. \tag{3.6}$$

From (3.4), (3.5) and (3.6),

$$\frac{\mathrm{d}v}{\mathrm{d}z} = -\frac{\mathrm{d}w}{\mathrm{d}U}\,(U(z)) \left|\frac{v(z)}{d(U(z))}\right|^{p'-1} = -\frac{\mathrm{d}w}{\mathrm{d}U} \left|\frac{w(U)}{d(U)}\right|^{p'-1}.$$

Therefore, the equation (2.3) for $z \in (\xi_i, \xi_{i+1})$ becomes

$$-\frac{\mathrm{d}w}{\mathrm{d}U} \left|\frac{w(U)}{d(U)}\right|^{p'-1} - c \left|\frac{w(U)}{d(U)}\right|^{p'-1} + g(U) = 0, \quad U \in (s_i, s_{i+1}),$$

$i = 0, 1, \ldots, n$. This is equivalent to

$$|w|^{p'-1} \frac{\mathrm{d}w}{\mathrm{d}U} = -c\, |w|^{p'-1} + (d(U))^{p'-1}\, g(U), \tag{3.7}$$

or

$$\frac{1}{p'} \frac{\mathrm{d}}{\mathrm{d}U} |w|^{p'} = c\, |w|^{p'-1} - (d(U))^{p'-1}\, g(U). \tag{3.8}$$

Set $f(U) = (d(U))^{\frac{1}{p-1}} g(U)$ and write t instead of U and $y(t) = |w(t)|^{p'}$. Then (3.8) becomes

$$y'(t) = p' \left[c\,(y(t))^{\frac{1}{p}} - f(t) \right], \quad t \in (0,1) \setminus \bigcup_{i=1}^{n} \{s_i\}. \tag{3.9}$$

From (3.1) and Definition 2.1 (c) we deduce that $v(z) \to 0$ as $z \to z_0+$ or $z \to z_1-$ which is equivalent to $\lim_{U\to 0+} w(U) = \lim_{U\to 1-} w(U) = 0$. Therefore, $y = y(t)$ satisfies the boundary conditions

$$y(0) = y(1) = 0. \tag{3.10}$$

On the other hand, let us suppose that $y = y(t)$, $y \in C[0,1]$, is a positive solution of (3.9), (3.10). Set $w(s) := -(y(s))^{\frac{1}{p'}}$. Then w satisfies (3.7) and (3.8). For $U \in (0,1)$ set

$$z(U) = -\int_{\frac{1}{2}}^{U} \left| \frac{d(s)}{w(s)} \right|^{\frac{1}{p-1}} \mathrm{d}s, \tag{3.11}$$

where $w(s) = -(y(s))^{\frac{1}{p'}}$. Then the function $z = z(U)$ is continuous strictly decreasing in $(0,1)$, $z(\frac{1}{2}) = 0$ and maps the interval $(0,1)$ onto (z_0, z_1), where $-\infty \le z_0 < z_1 \le +\infty$. Let us denote by $U : (z_0, z_1) \to (0,1)$ the inverse function to $z = z(U)$. Then $U(0) = \frac{1}{2}$, U is continuous strictly decreasing,

$$\lim_{z\to z_0+} U(z) = 1 \quad \text{and} \quad \lim_{z\to z_1-} U(z) = 0.$$

Let $z \in (\xi_i, \xi_{i+1})$, $i = 0, 1, \dots, n$, where $U(\xi_i) = s_i$, $i = 0, 1, \dots, n, n+1$. Then from (3.11) we deduce

$$\frac{\mathrm{d}U}{\mathrm{d}z} = \frac{1}{\frac{\mathrm{d}z(U)}{\mathrm{d}U}} = -\left| \frac{w(U)}{d(U)} \right|^{\frac{1}{p-1}}, \quad U \in (s_i, s_{i+1}), \tag{3.12}$$

i.e., $U \in C^1(\xi_i, \xi_{i+1})$, $U'(z) < 0$ and

$$-\,d(U(z)) \left| \frac{\mathrm{d}U(z)}{\mathrm{d}z} \right|^{p-1} = w(U(z)) =: v(z), \tag{3.13}$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}z} \left[d(U(z)) \left| \frac{\mathrm{d}U}{\mathrm{d}z} \right|^{p-2} \frac{\mathrm{d}U}{\mathrm{d}z} \right] = \frac{\mathrm{d}}{\mathrm{d}z} w(U(z)) = \frac{\mathrm{d}w}{\mathrm{d}U} \frac{\mathrm{d}U(z)}{\mathrm{d}z}. \tag{3.14}$$

From (3.7), (3.13) we deduce

$$\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}U} &= -\,|w(U)|^{-(p'-1)} \left(-c\,|w(U)|^{p'-1} + (d(U))^{p'-1} g(U) \right) \\
&= -c + |w(U)|^{-(p'-1)} (d(U))^{p'-1} g(U) \\
&= -c + (d(U(z)))^{-(p'-1)} \left| \frac{\mathrm{d}U(z)}{\mathrm{d}z} \right|^{-(p-1)(p'-1)} (d(U(z)))^{p'-1} g(U(z)) \\
&= -c + \left| \frac{\mathrm{d}U(z)}{\mathrm{d}z} \right|^{-1} g(U(z)).
\end{aligned}$$

Let us substitute this into (3.14):

$$\frac{\mathrm{d}}{\mathrm{d}z} \left[d(U(z)) \left| \frac{\mathrm{d}U}{\mathrm{d}z} \right|^{p-2} \frac{\mathrm{d}U}{\mathrm{d}z} \right] = \left[-c + \left| \frac{\mathrm{d}U(z)}{\mathrm{d}z} \right|^{-1} g(U(z)) \right] \frac{\mathrm{d}U(z)}{\mathrm{d}z} = -c \frac{\mathrm{d}U(z)}{\mathrm{d}z} - g(U(z)),$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[d(U(z))\left|\frac{\mathrm{d}U}{\mathrm{d}z}\right|^{p-2}\frac{\mathrm{d}U}{\mathrm{d}z}\right]+c\frac{\mathrm{d}U(z)}{\mathrm{d}z}+g(U(z))=0, \quad z\in(\xi_i,\xi_{i+1}), \quad i=0,1,\dots,n.$$

It follows from (3.12) that

$$\lim_{z\to\xi_i\pm}U'(z)=-\left|\frac{w(s_i)}{\lim\limits_{s\to s_i\pm}d(s)}\right|^{\frac{1}{p-1}}, \quad i=1,2,\dots,n.$$

From (3.13) and the continuity of U we then have

$$\lim_{z\to z_0+}d(U(z))\left|U'(z)\right|^{p-2}U'(z)=\lim_{z\to z_1-}d(U(z))\left|U'(z)\right|^{p-2}U'(z)=0$$

and the following one sided limits are finite

$$\lim_{z\to\xi_i-}d(U(z))\left|U'(z)\right|^{p-2}U'(z)=\lim_{z\to\xi_i+}d(U(z))\left|U'(z)\right|^{p-2}U'(z), \quad i=1,2,\dots,n.$$

Since U is monotone decreasing in (z_0,z_1), we have

$$\lim_{z\to\xi_i-}d(U(z))=\lim_{s\to s_i+}d(s) \qquad \text{and} \qquad \lim_{z\to\xi_i+}d(U(z))=\lim_{s\to s_i-}d(s), \quad i=1,2,\dots,n.$$

Therefore, U satisfies the transition condition

$$\left|U'(\xi_i-)\right|^{p-2}U'(\xi_i-)\lim_{s\to s_i+}d(s)=\left|U'(\xi_i+)\right|^{p-2}U'(\xi_i+)\lim_{s\to s_i-}d(s), \quad i=1,2,\dots,n.$$

We may summarize the above reasoning in the following equivalence.

Proposition 3.3. *A function $U:\mathbb{R}\to[0,1]$, $U\in\widehat{C}^1(\mathbb{R})$ is a solution of* (2.1), (3.1) *if and only if $y:[0,1]\to\mathbb{R}$, $y\in C[0,1]$, is a positive solution of* (3.9), (3.10)*. In particular, solution U of* (2.1), (3.1) *is uniquely determined (up to translation) by the solution y of* (3.9), (3.10) *and vice versa.*

Thanks to this proposition we can study the first order problem (3.9), (3.10) in order to derive the existence and uniqueness of solution for (2.1), (3.1). Let us recall that there are two "unknowns" in this problem. Indeed, besides the positive solution $y=y(t)$ we also look for unknown speed of propagation $c>0$. Therefore, (3.9), (3.10) is not overdetermined.

4 The first order ODE, existence, uniqueness and nonexistence

In this section we concentrate on the existence and uniqueness result for the boundary value problem

$$\begin{cases} y'(t)=p'\left[c\left(y^+(t)\right)^{\frac{1}{p}}-f(t)\right], & t\in(0,1),\\ y(0)=y(1)=0. \end{cases} \tag{4.1}$$

Here $y^+(t)=\max\{y(t),0\}$ denotes the positive part of y, $p>1$ and $p'>1$ are conjugate numbers and $f\in L^1(0,1)$. We employ the concept of solution of the first order ODE in the sense of Carathéodory. For $(t,y,c)\in[0,1]\times\mathbb{R}^2$ we set

$$h(t,y,c):=p'\left[c\left(y^+\right)^{\frac{1}{p}}-f(t)\right]$$

and consider the following two initial value problems which depend on a parameter $c \in \mathbb{R}$:

$$y'(t) = h(t, y(t), c), \quad y(0) = 0 \tag{4.2}$$

and

$$y'(t) = h(t, y(t), c), \quad y(1) = 0. \tag{4.3}$$

In both cases we look for a solution $y = y(t), t \in [0,1]$. Therefore, (4.2) is referred to as a *forward initial value problem*, while (4.3) is referred to as a *backward initial value problem*. Note that $f \in L^1(0,1)$ implies that $h = h(t, y, c)$ satisfies Carathéodory's conditions, i.e., for almost every $t \in [0,1]$ fixed, $h(t, \cdot, \cdot)$ is continuous with respect to y and c and for every $y \in \mathbb{R}$ and $c \in \mathbb{R}$ fixed, $h(\cdot, y, c)$ is measurable with respect to t. In what follows, for any fixed $c \in \mathbb{R}$, $y_c = y_c(t)$ denotes the solution in the sense of Carathéodory of the forward and backward initial value problem (4.2) and (4.3), respectively. In particular, y_c is absolutely continuous in $[0,1]$ and the equation holds a.e. in $[0,1]$. We first mention the following global existence result.

Lemma 4.1. *Let $f \in L^1(0,1)$, $c \in \mathbb{R}$. Then there exists at least one global solution $y_c = y_c(t)$ of* (4.2) *defined on the entire interval* $[0,1]$*. The same holds for* (4.3)*.*

Proof. Let $c \in \mathbb{R}$ and $f \in L^1(0,1)$ be fixed. Integrating (4.2) we get

$$y(\sigma) = p'\left(c\int_0^\sigma \left(y^+(\tau)\right)^{\frac{1}{p}}\,\mathrm{d}\tau - \int_0^\sigma f(\tau)\,\mathrm{d}\tau\right), \quad \sigma \in (0,1). \tag{4.4}$$

For $t \in (0,1)$ set

$$\varrho(t) := \max_{\sigma\in[0,t]} |y(\sigma)|.$$

It follows from (4.4) that for $\sigma \in [0,t]$

$$|y(\sigma)| \le p'\left(|c|\int_0^\sigma \left(y^+(\tau)\right)^{\frac{1}{p}}\,\mathrm{d}\tau + \|f\|_{L^1(0,1)}\right)$$

and therefore

$$\begin{aligned}
\varrho(t) &\le p'\left(|c|\max_{\sigma\in[0,t]}\int_0^\sigma \left(y^+(\tau)\right)^{\frac{1}{p}}\,\mathrm{d}\tau + \|f\|_{L^1(0,1)}\right)\\
&\le p'\left(|c|\int_0^1 \max_{\sigma\in[0,t]}\left(y^+(\sigma)\right)^{\frac{1}{p}}\,\mathrm{d}\tau + \|f\|_{L^1(0,1)}\right)\\
&\le p'\left(|c|\left(\max_{\sigma\in[0,t]}|y(\sigma)|\right)^{\frac{1}{p}} + \|f\|_{L^1(0,1)}\right) = p'\left(|c|\left(\varrho(t)\right)^{\frac{1}{p}} + \|f\|_{L^1(0,1)}\right).
\end{aligned}$$

Since $\frac{1}{p} < 1$ the last inequality yields that there is $K > 0$ such that $\varrho(t) \le K$ for all $t \in [0,1]$, i.e., all solutions of (4.2) are a priori bounded by a constant $K > 0$. Setting

$$\widetilde{h}(t, y, c) = \begin{cases} h(t, y, c) & \text{for } |y| \le K,\\ p'\left(c\,K^{\frac{1}{p}} - f(t)\right) & \text{for } |y| > K,\end{cases}$$

the set of solutions of (4.2) coincides with the set of solutions of the modified problem

$$y'(t) = \widetilde{h}(t, y(t), c), \quad y(0) = 0. \tag{4.5}$$

But $\widetilde{h}$ satisfies Carathéodory's conditions and there is a function $m \in L^1(0,1)$ such that $|\widetilde{h}(t, y, c)| \le m(t)$ for $(t, y) \in [0,1] \times \mathbb{R}$. Therefore (4.5) (and thus also (4.2)) has a global solution in $[0,1]$ according to [21, Theorem 10.XVIII]. Similarly we proceed in case of (4.3). $\square$

Remark 4.2. The uniqueness of the solution in the above lemma does not hold in general due to the fact that the function $y \mapsto c\,(y^+)^{\frac{1}{p}}, y \in \mathbb{R}$, does not satisfy the Lipschitz condition at 0. However, it is nondecreasing for $c \geq 0$ and nonincreasing for $c \leq 0$. Therefore, it satisfies one-sided Lipschitz condition in either case and we have the following uniqueness results separately for the forward and backward initial value problems.

Lemma 4.3. *Let $f \in L^1(0,1)$. If $c \leq 0$ then* (4.2) *has exactly one solution $y_c = y_c(t), t \in [0,1]$. If $c \geq 0$ then* (4.3) *has exactly one solution $y_c = y_c(t), t \in [0,1]$.*

Proof. Since the idea of the proof is the same for both alternatives, we only prove the second part of the statement. Let $c \geq 0$ and $y_1 = y_1(t)$, $y_2 = y_2(t)$ be two solutions of (4.3) in $[0,1]$. Set

$$\delta(t) = (y_1(t) - y_2(t))^2 .$$

Then $\delta(1) = 0$, $\delta(t) \geq 0$ and

$$\begin{aligned}\delta'(t) &= 2\left(y_1'(t) - y_2'(t)\right)(y_1(t) - y_2(t))\\ &= 2p'c\left[\left(y_1^+(t)\right)^{\frac{1}{p}} - \left(y_2^+(t)\right)^{\frac{1}{p}}\right](y_1(t) - y_2(t)) \geq 0.\end{aligned}$$

Hence $\delta(t) = 0$ for a.e. $t \in [0,1]$ and $y_1(t) = y_2(t)$, $t \in [0,1]$. □

Thanks to the uniqueness result we also have continuous dependence of solutions on the parameter c.

Lemma 4.4. *Let $f \in L^1(0,1)$, $c_0 \geq 0$. Then $c \to c_0 > 0$ or $c \to 0+$ if $c_0 = 0$ implies that solutions $y_c = y_c(t)$ of the backward initial value problem* (4.3) *converge uniformly in* $[0,1]$ *(i.e., in the topology of $C[0,1]$) to y_{c_0}. Similar result holds for $c_0 \leq 0$ and the forward initial value problem* (4.2).

Proof. The proof follows from the uniqueness result in Lemma 4.3 and [5, Theorems 4.1 and 4.2]. □

As we already observed at the beginning of Section 3, the assumption $g > 0$ in $(0,1)$ yields $c > 0$. For this reason, we further focus on parameters $c \geq 0$ and the backward initial value problem (4.3) rather than on $c \leq 0$ and the forward initial value problem (4.2).

Let us introduce the notion of the *defect* $P_c\varphi$ of a function φ with respect to the differential equation $y' = h(t, y, c)$, see e.g. [21, §9.II]:

$$P_c\varphi := \varphi'(t) - h(t, \varphi(t), c).$$

The following comparison argument is one of our basic tools.

Lemma 4.5. *Let $f \in L^1(0,1)$, $c \geq 0$, $\varphi(1) \leq \psi(1)$, $P_c\varphi \geq P_c\psi$ a.e. in $[0,1]$. Then $\varphi \leq \psi$ in $[0,1]$.*

Proof. Let $w = \psi - \varphi$. Then

$$w' = \psi' - \varphi' = P_c\psi + p'c(\psi^+)^{\frac{1}{p}} - P_c\varphi - p'c(\varphi^+)^{\frac{1}{p}} \leq p'c\left((\psi^+)^{\frac{1}{p}} - (\varphi^+)^{\frac{1}{p}}\right) \tag{4.6}$$

a.e. in $[0,1]$. Assume that there is $t_0 \in (0,1)$ such that $w(t_0) < 0$. Let $t_1 \in (t_0, 1]$ be such that $w(t) \le 0$, $t \in (t_0, t_1]$. It follows from (4.6) that $w' \le p'c[(\psi^+)^{\frac{1}{p}} - (\varphi^+)^{\frac{1}{p}}] \le 0$ a.e. in $(t_0, t_1]$, i.e., $w(t_1) \le w(t_0) < 0$. Using the same argument repeatedly if necessary we conclude that $w(1) < 0$, a contradiction with $\varphi(1) \le \psi(1)$. □

Corollary 4.6. *Let $f \in L^1(0,1)$, $0 \le c_1 \le c_2$. Then*

$$y_{c_1}(t) \ge y_{c_2}(t)\,, \quad t \in [0,1].$$

In particular, $y_{c_1}(0) \ge y_{c_2}(0)$.

Proof. We have

$$\begin{aligned} P_{c_2} y_{c_1} &= y'_{c_1} - h(t, y_{c_1}, c_2) = \underbrace{y'_{c_1} - h(t, y_{c_1}, c_1)}_{=0} + h(t, y_{c_1}, c_1) - h(t, y_{c_1}, c_2) \\ &= p'(c_1 - c_2)\left(y^+_{c_1}\right)^{\frac{1}{p}} \le 0 = y'_{c_2} - h(t, y_{c_2}, c_2) = P_{c_2} y_{c_2} \quad \text{a.e. in } [0,1]. \end{aligned}$$

Then Lemma 4.5 yields that $y_{c_1} \ge y_{c_2}$ in $[0,1]$. □

Theorem 4.7 (Existence)**.** *Let f be lower semicontinuous, $f(t) > 0$, $t \in (0,1)$,*

$$0 < \mu := \sup_{t \in (0,1)} \frac{f(t)}{t^{p'-1}} < +\infty. \tag{4.7}$$

Then there exists a number $c^* \in (0, (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \mu^{\frac{1}{p'}}]$ such that the problem (4.1) has a unique positive solution if and only if $c \ge c^*$.

Proof. It follows from (4.7) that f is bounded in $(0,1)$. In particular, $f \in L^1(0,1)$. For a solution $y_c = y_c(t)$ of (4.3) with $c \ge 0$ we have

$$P_c 0 = 0 - h(t, 0, c) = p' f(t) \ge y'_c - h(t, y_c, c) = P_c y_c \quad \text{a.e. in } [0,1].$$

Then by Lemma 4.5 we have $y_c \ge 0$ in $[0,1]$. We prove that $y_c > 0$ in $(0,1)$. Indeed, let $t_0 \in (0,1)$ be such that $y_c(t_0) = 0$. Since $f > 0$ in $(0,1)$ and f is lower semicontinuous, given $\varepsilon > 0$ arbitrarily small, there is a constant $\eta > 0$ such that $f(t) \ge \eta > 0$ for all $t \in [t_0, 1-\varepsilon]$. Integrating (4.3) from t_0 to $t \in (t_0, 1-\varepsilon]$ and using $y_c(t_0) = 0$ we get

$$y_c(t) = p' \left(c \int_{t_0}^{t} \left(y^+_c(\tau)\right)^{\frac{1}{p}} \,\mathrm{d}\tau - \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau \right),$$

$$\frac{y_c(t)}{t - t_0} = p' \left(c\, \frac{\int_{t_0}^{t} \left(y^+_c(\tau)\right)^{\frac{1}{p}} \,\mathrm{d}\tau}{t - t_0} - \frac{\int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau}{t - t_0} \right). \tag{4.8}$$

We have

$$\frac{\int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau}{t - t_0} \ge \eta \tag{4.9}$$

and since y_c is continuous at t_0 and $y_c(t_0) = 0$,

$$\lim_{t \to t_0+} \frac{\int_{t_0}^{t} \left(y^+_c(\tau)\right)^{\frac{1}{p}} \,\mathrm{d}\tau}{t - t_0} = 0. \tag{4.10}$$

From (4.8)–(4.10) we conclude that for t close enough to t_0, $t > t_0$,

$$\frac{y_c(t)}{t - t_0} < 0,$$

a contradiction. Thus $y_c > 0$ in $(0, 1)$.

Next, using the estimates similar to [11, p. 176], we prove that for c large enough equality $y_c(0) = 0$ must hold. Set $\phi_c(s) = cs^{\frac{1}{p}} - s$, $c > 0$, $s \in (0, c^{p'})$. Then $\phi_c > 0$ in $(0, c^{p'})$, $\phi_c(0) = \phi_c(c^{p'}) = 0$, and ϕ_c attains maximum value $M_c := (\frac{c}{p})^{p'}(p-1)$ at the point $k := (\frac{c}{p})^{p'} \in (0, c^{p'})$. Elementary calculation yields that $c \geq (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \mu^{\frac{1}{p'}}$ if and only if $M_c \geq \mu$, i.e., $\phi_c(k) \geq \mu$, or equivalently we have

$$ck^{\frac{1}{p}} - \mu \geq k\,. \tag{4.11}$$

Let $s(t) := kt^{p'}$. Then $s(1) > 0$ and thanks to (4.11),

$$\begin{aligned} P_c s = s'(t) - h(t, s, c) = kp't^{p'-1} - h(t, s, c) &\leq \left(ck^{\frac{1}{p}} - \mu\right) p't^{p'-1} - p'\left[c(s(t))^{\frac{1}{p}} - f(t)\right] \\ &\leq \left(ck^{\frac{1}{p}} - \mu\right) p't^{p'-1} - p'\left[c(s(t))^{\frac{1}{p}} - \mu t^{p'-1}\right] = 0 = P_c y_c \quad \text{a.e. in} \quad [0, 1]. \end{aligned}$$

Then again by Lemma 4.5 we have

$$0 \leq y_c(t) \leq s(t), \quad t \in [0, 1].$$

In particular,

$$0 = y_c(0) = s(0).$$

To summarize, we have proved that for any $c \geq (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \mu^{\frac{1}{p'}}$ there exists unique positive solution $y_c = y_c(t)$ of the backward initial value problem (4.3) satisfying $y_c(0) = 0$. In particular, $y_c = y_c(t)$ is a positive solution of (4.1).

By Corollary 4.6, $y_{c_1}(t) \geq y_{c_2}(t)$, $t \in (0, 1)$, $y_{c_i}(0) = y_{c_i}(1) = 0$, $i = 1, 2$, if $c_1 < c_2$. Set

$$c^* := \inf\{c > 0 : \text{ (4.1) has a unique positive solution}\}.$$

Then from above we get $c^* \leq (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \mu^{\frac{1}{p'}}$. Let $c_n \to c^*+$, $y_{c_n} = y_{c_n}(t)$, $t \in [0, 1]$, be solutions of (4.1) with $c = c_n$. Then according to Lemma 4.4 solutions y_{c_n} converge uniformly to a solution y_{c^*} of (4.1) with $c = c^*$. Since $c^* \geq 0$, we have $y_{c^*}(t) > 0$, $t \in (0, 1)$. Hence (4.1) has a unique positive solution if and only if $c \geq c^*$. For $c = 0$ we have

$$y_0(t) = p' \int_t^1 f(\tau)\, \mathrm{d}\tau, \quad t \in [0, 1].$$

In particular, $y_0(0) > 0$ and therefore $c^* > 0$. □

Theorem 4.8 (Nonexistence). *Let $f(t) > 0$, $t \in (0, 1)$,*

$$0 < \nu := \liminf_{t \to 0+} \frac{f(t)}{t^{p'-1}}\,. \tag{4.12}$$

If

$$0 < c < (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \nu^{\frac{1}{p'}} \tag{4.13}$$

then the BVP (4.1) *has no positive solution. In particular, if*

$$\lim_{t\to 0+}\frac{f(t)}{t^{p'-1}}=+\infty, \tag{4.14}$$

then (4.1) *has no positive solution for any* $c>0$.

Proof. We proceed by contradiction. Let c be fixed and satisfy (4.13). Assume that (4.1) has a positive solution $y_c=y_c(t)>0$, $t\in(0,1)$. Since y_c is also a solution of the backward initial value problem (4.3) and $c>0$, by Lemma 4.3 function y_c is also a unique solution of (4.1). For $v\in C[0,1]$ fixed let $u\in C[0,1]$ be such that

$$u(t)=p'\int_0^t\left[c\left(v^+(\tau)\right)^{\frac{1}{p}}-f(\tau)\right]\,\mathrm{d}\tau.$$

Then $u=T(v)$ defines a monotone increasing operator from $C[0,1]$ into $C[0,1]$ with a unique fixed point y_c. Indeed, let $v_1,v_2\in C[0,1]$, $v_1(t)\le v_2(t)$, $t\in[0,1]$. Then

$$T(v_1)(t)-T(v_2)(t)=p'\int_0^t c\left[\left(v_1^+(\tau)\right)^{\frac{1}{p}}-\left(v_2^+(\tau)\right)^{\frac{1}{p}}\right]\,\mathrm{d}\tau\le 0.$$

Set $y_0(t)=c^{p'}t^{p'}$, $t\in[0,1]$. Then

$$T(y_0)(t)=y_0(t)-p'\int_0^t f(\tau)\,\mathrm{d}\tau\le y_0(t),\quad t\in[0,1],$$

i.e., y_0 is a supersolution of T (see e.g. [7, Def. 6.3.15]). We consider the following successive approximations

$$y_{n+1}=T(y_n),\quad n=0,1,2,\dots$$

Since T is monotone increasing, we have

$$y_0(t)\ge y_1(t)\ge\cdots\ge y_n(t)\ge\dots \tag{4.15}$$

For any $n\in\mathbb{N}$,

$$y_n(t)=T(y_{n-1})(t)\ge -p'\int_0^t f(\tau)\,\mathrm{d}\tau,$$

i.e., the sequence $\{y_n\}_{n=0}^{\infty}$ is bounded below in $C[0,1]$. By [7, Thm. 6.3.16] this sequence converges to the greatest fixed point of T. Since the solution $y_c=y_c(t)$ of (4.1) is a unique fixed point of T, we get from (4.15) that

$$y_0(t)\ge y_1(t)\ge\cdots\ge y_n(t)\ge\cdots\ge y_c(t)>0,\quad t\in(0,1). \tag{4.16}$$

It follows from (4.12), (4.13) that there exists $\delta\in(0,1]$ and $\tilde\nu\in\left(\frac{1}{p'p^{p'-1}},1\right)$ such that

$$f(t)\ge\tilde\nu c^{p'}t^{p'-1}\ \text{ for all }\ t\in(0,\delta). \tag{4.17}$$

Now, using (4.17) we deduce

$$\begin{aligned}y_1(t)&=p'\left[c\int_0^t\left(y_0^+(\tau)\right)^{\frac{1}{p}}\,\mathrm{d}\tau-\int_0^t f(\tau)\,\mathrm{d}\tau\right]=p'\left[c\int_0^t c^{\frac{p'}{p}}\tau^{\frac{p'}{p}}\,\mathrm{d}\tau-\int_0^t f(\tau)\,\mathrm{d}\tau\right]\\&\le p'c^{p'}\left[\frac{\tau^{p'}}{p'}\right]_0^t-p'\int_0^t\tilde\nu c^{p'}\tau^{p'-1}\,\mathrm{d}\tau=c^{p'}t^{p'}-\tilde\nu c^{p'}t^{p'}=ct^{p'}(1-\tilde\nu),\quad t\in(0,\delta),\end{aligned}$$

$$
\begin{aligned}
y_2(t) &= p'\left[c\int_0^t \left(y_1^+(\tau)\right)^{\frac{1}{p}}\,\mathrm{d}\tau - \int_0^t f(\tau)\,\mathrm{d}\tau\right] \\
&\le p'c\int_0^t c^{\frac{p'}{p}}\tau^{\frac{p'}{p}}(1-\tilde\nu)^{\frac{1}{p}}\,\mathrm{d}\tau - p'\int_0^t \tilde\nu c^{p'}\tau^{p'-1}\,\mathrm{d}\tau = c^{p'}t^{p'}(1-\tilde\nu)^{\frac{1}{p}} - \tilde\nu c^{p'}t^{p'} \\
&= c^{p'}t^{p'}\left[(1-\tilde\nu)^{\frac{1}{p}} - \tilde\nu\right], \quad t\in(0,\delta).
\end{aligned}
$$

Performing the iterative process, we get for $k = 1, 2, \dots$ that

$$
y_k(t) \le a_k c^{p'} t^{p'} \quad \text{for} \quad t\in(0,\delta), \tag{4.18}
$$

$$
a_0 = 1, \;\; a_k = (a_{k-1})^{\frac{1}{p}} - \tilde\nu. \tag{4.19}
$$

It follows from (4.16), (4.18) that

$$
0 < y_c(t) \le \cdots \le a_k c^{p'}t^{p'} \le a_{k-1}c^{p'}t^{p'} \le \cdots \le a_1 c^{p'}t^{p'} \le c^{p'}t^{p'} \tag{4.20}
$$

for $t\in(0,\delta)$. Hence $\{a_k\}_{k=1}^\infty$ is bounded and monotone decreasing and therefore there exists finite limit $a_\infty := \lim_{k\to\infty} a_k$. Then obviously $a_\infty < 1$ and due to (4.20) we infer $a_\infty > 0$. Passing to the limit for $k\to\infty$ in (4.19), we get

$$
a_\infty = a_\infty^{\frac{1}{p}} - \tilde\nu, \;\; \text{i.e.,} \;\; \tilde\nu = a_\infty^{\frac{1}{p}}(1 - a_\infty^{\frac{1}{p'}}).
$$

Since the function $x \mapsto x^{\frac{1}{p}}(1-x^{\frac{1}{p'}})$, $x\in(0,1)$, attains its maximum $\frac{1}{p'p^{p'-1}}$ at the point $x = \frac{1}{p^{p'}}$, we necessarily have $\tilde\nu \le \frac{1}{p'p^{p'-1}}$, a contradiction with the fact $\tilde\nu \in \left(\frac{1}{p'p^{p'-1}}, 1\right)$. Therefore (4.1) cannot have a positive solution. In particular, if (4.14) holds then $\nu = +\infty$ and (4.13) yields that (4.1) has no positive solution for any $c\in\mathbb{R}$. $\square$

Remark 4.9. Let μ and ν be defined as in Theorems 4.7 and 4.8, respectively. Then we conclude from the existence and nonexistence results above that the minimal value of the "critical" speed $c^* > 0$ must satisfy

$$
(p')^{\frac{1}{p'}} p^{\frac{1}{p}} \nu^{\frac{1}{p'}} \le c^* \le (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \mu^{\frac{1}{p'}}.
$$

5 Existence, uniqueness and nonexistence of the travelling wave

The results from Sections 3 and 4 imply the following existence and uniqueness result for the second order boundary value problem

$$
\begin{cases}
\left(d(U(z))\,|U'(z)|^{p-2}\,U'(z)\right)' + cU'(z) + g(U(z)) = 0, & z\in\mathbb{R}, \\
\lim\limits_{z\to-\infty} U(z) = 1, \quad \lim\limits_{z\to+\infty} U(z) = 0.
\end{cases} \tag{5.1}
$$

Theorem 5.1 (Existence)**.** *Let d and g be as in Section 2 and*

$$
0 < \mu := \sup_{t\in(0,1)} \frac{(d(t))^{\frac{1}{p-1}}\, g(t)}{t^{p'-1}} < +\infty. \tag{5.2}
$$

Then there exists a number $c^* \in (0, (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \mu^{\frac{1}{p'}}]$ such that a unique nonincreasing travelling wave profile $U = U(z)$, $z\in\mathbb{R}$, satisfying (5.1) exists if and only if $c \ge c^*$. Moreover, U has the following properties:

(i) *There exist* $z_0, z_1 \in \mathbb{R} \cup \{\pm\infty\}$ *such that* $-\infty \le z_0 < 0 < z_1 \le +\infty$, $U(z) = 1$ *for* $z \in (-\infty, z_0]$ *and* $U(z) = 0$ *for* $z \in [z_1, +\infty)$.

(ii) U *is strictly decreasing in* (z_0, z_1), $U(0) = \frac{1}{2}$.

(iii) *For* $i = 1, 2, \dots, n$ *let* $\xi_i \in (z_0, z_1)$ *be such that* $U(\xi_i) = s_i$, $\xi_0 = z_0$, $\xi_{n+1} = z_1$. *Then* $U \in \widehat{C}^1(\mathbb{R})$,

$$U|_{(\xi_i, \xi_{i+1})} \in C^1(\xi_i, \xi_{i+1}), \quad i = 0, 1, \dots, n$$

and the limits $U'(\xi_i-) = \lim\limits_{z \to \xi_i -} U'(z)$ *and* $U'(\xi_i+) = \lim\limits_{z \to \xi_i +} U'(z)$, $i = 1, 2, \dots, n$, *exist and are finite.*

(iv) *For any* $i = 1, 2, \dots, n$ *transition condition*

$$\left|U'(\xi_i-)\right|^{p-2} U'(\xi_i-) \lim_{s \to s_i+} d(s) = \left|U'(\xi_i+)\right|^{p-2} U'(\xi_i+) \lim_{s \to s_i-} d(s)$$

holds.

(v) U *satisfies*

$$\lim_{z \to z_0+} d(U(z)) \left|U'(z)\right|^{p-2} U'(z) = \lim_{z \to z_1-} d(U(z)) \left|U'(z)\right|^{p-2} U'(z) = 0.$$

(vi) *Equation in* (5.1) *holds pointwise (in a classical sense) in* (ξ_i, ξ_{i+1}), $i = 0, 1, \dots, n$.

The proof of the above theorem follows from Proposition 3.3 and Theorem 4.7. The properties of U follow from the reasoning in Section 3. Notice that condition $U(0) = \frac{1}{2}$ has just a normalizing character. Indeed, since the equation (2.1) is autonomous, then if $U = U(z)$ is a solution of (2.1), given any fixed $\xi \in \mathbb{R}$, the function $z \mapsto U(\xi + z)$ is also a solution of (2.1).

Remark 5.2. Let $\liminf_{s \to 1-} d(s) > 0$ ($\liminf_{s \to 0+} d(s) > 0$). It follows from Theorem 5.1 (v) that $\lim_{z \to z_0+} U'(z) = 0$ ($\lim_{z \to z_1-} U'(z) = 0$). In particular, if $z_0 > -\infty$ ($z_1 < +\infty$), then U is a C^1-function in the neighbourhood of $z_0 \in \mathbb{R}$ ($z_1 \in \mathbb{R}$).

Theorem 5.3 (Nonexistence)**.** *Let* d *and* g *be as in Section 2 and*

$$0 < \nu := \liminf_{t \to 0+} \frac{(d(t))^{\frac{1}{p-1}} g(t)}{t^{p'-1}}.$$

If

$$0 < c < (p')^{\frac{1}{p'}} p^{\frac{1}{p}} \nu^{\frac{1}{p'}}$$

then there is no solution $U = U(z)$, $z \in \mathbb{R}$, *of* (5.1). *In particular,* (5.1) *has no solution for any* $c > 0$ *if*

$$\lim_{t \to 0+} \frac{(d(t))^{\frac{1}{p-1}} g(t)}{t^{p'-1}} = +\infty.$$

The proof follows from Proposition 3.3 and Theorem 4.8.

6 Asymptotic analysis of the travelling wave

In this section we focus on the asymptotic behaviour of the travelling wave profile $U = U(z)$ as $z \to \pm\infty$. In particular, we are interested in conditions on d and g near 0 and 1 which guarantee either $z_0 = -\infty$ ($z_1 = +\infty$) or else $z_0 \in \mathbb{R}$ ($z_1 \in \mathbb{R}$). Following the classification from [15, Definition 1.1], a travelling wave profile U is said to be of *front-type* if $z_0 = -\infty$ and $z_1 = +\infty$, *sharp of type (I)* if $z_0 = -\infty$ and $z_1 \in \mathbb{R}$, *sharp of type (II)* if $z_0 \in \mathbb{R}$ and $z_1 = +\infty$, *sharp of type (III)* if $z_0 \in \mathbb{R}$ and $z_1 \in \mathbb{R}$. In what follows in this section, we assume that the assumptions of Theorem 5.1 hold.

For technical reasons we restrict ourselves to the power-type behaviour of d and g near equilibrium points 0 and 1. For the sake of brevity, for $t_0 \in \mathbb{R}$ we write

$$h_1(t) \sim h_2(t) \;\text{ as } t \to t_0 \quad \text{if and only if} \quad \lim_{t\to t_0} \frac{h_1(t)}{h_2(t)} \in (0, +\infty).$$

6.1 Asymptotics near 1

Let us assume that $g(t) \sim (1-t)^\gamma$ and $d(t) \sim (1-t)^\delta$ as $t \to 1-$ for some $\gamma > 0$ and $\delta \in \mathbb{R}$. The assumption (5.2) yields the following necessary condition for parameters γ, δ and p:

$$\gamma + \frac{\delta}{p-1} \ge 0. \tag{6.1}$$

In this subsection we assume that $c \in [c^*, +\infty)$ is arbitrary but fixed.

First, we establish the asymptotic behaviour of $y_c = y_c(t)$ as $t \to 1-$. Then we get from (3.11) that

$$z_0 = -\int_{\frac{1}{2}}^{1} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}}\, \mathrm{d}t \tag{6.2}$$

and derive either $z_0 > -\infty$ or $z_0 = -\infty$. The asymptotics of $U = U(z)$ as $z \to -\infty$ then follows applying the inverse point of view to $z = z(U)$ as $U \to 1-$. Namely, we have the following results.

Theorem 6.1. *Let us assume* $\gamma > 0$*,*

$$0 \le \gamma + \frac{\delta}{p-1} \le \frac{1}{p-1}, \tag{6.3}$$

$$\frac{\gamma - \delta + 1}{p} < 1. \tag{6.4}$$

Then $z_0 > -\infty$*. If*

$$\frac{\gamma - \delta + 1}{p} \ge 1 \tag{6.5}$$

instead of (6.4)*, then* $z_0 = -\infty$*.*

Proof. The assumptions on d and g yield the existence of $\theta > 0$ such that $f(t) = (d(t))^{\frac{1}{p-1}} g(t)$ is continuous in $(1-\theta, 1)$. Hence $f(t) \sim (1-t)^{\gamma + \frac{\delta}{p-1}}$ is equivalent to $f(t) = \eta(t)(1-t)^{\gamma + \frac{\delta}{p-1}}$, $t \in (1-\theta, 1)$, where $\eta = \eta(t)$ is a continuous function in $(1-\theta, 1)$, $\lim_{t\to 1-} \eta(t) \in (0, +\infty)$.

Let (6.3) hold. For $\kappa > 0$ set

$$y_\kappa(t) = \kappa(1-t)^{\gamma + \frac{\delta}{p-1} + 1}, \quad t \in [1-\theta, 1].$$

Let $y_c = y_c(t)$, $t \in [0,1]$ be a solution of (4.1). Then

$$\begin{aligned} P_c y_\kappa &= y_\kappa' - p' \left[c\,(y_\kappa)^{\frac{1}{p}} - f(t) \right] \\ &= (1-t)^{\gamma + \frac{\delta}{p-1}} \left[-\kappa \left(\gamma + \frac{\delta}{p-1} + 1 \right) + p'\eta(t) \right] - (1-t)^{\frac{\gamma + \frac{\delta}{p-1} + 1}{p}} p' c \kappa^{\frac{1}{p}}, \end{aligned} \tag{6.6}$$

$t \in (1-\theta, 1)$. It follows from (6.3) that the power $(1-t)^{\gamma + \frac{\delta}{p-1}}$ dominates the power $(1-t)^{\frac{\gamma + \frac{\delta}{p-1} + 1}{p}}$ near 1. It then follows from (6.6) that we distinguish between two cases:

(i) There exists $\underline{\kappa} \ll 1$ so small that $P_c y_{\underline{\kappa}} > 0 = P_c y_c$ a.e. in $[1-\theta, 1]$.

(ii) There exists $\overline{\kappa} \gg 1$ so large that $P_c y_{\overline{\kappa}} < 0 = P_c y_c$ a.e. in $[1-\theta, 1]$.

Case (i). Let (6.4) hold. Then it follows from Lemma 4.5 with $[0,1]$ replaced by $[1-\theta, 1]$ that

$$y_c(t) \geq y_{\underline{\kappa}}(t) \quad \text{for } t \in [1-\theta, 1].$$

From here and from (6.2) we conclude that there is a constant $c_1 > 0$ such that

$$\begin{aligned} z_0 = -\int_{\frac{1}{2}}^{1} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}}\,\mathrm{d}t &\geq -\int_{\frac{1}{2}}^{1-\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}}\,\mathrm{d}t - \int_{1-\theta}^{1} \frac{(d(t))^{\frac{1}{p-1}}}{\left(y_{\underline{\kappa}}(t)\right)^{\frac{1}{p}}}\,\mathrm{d}t \\ &\geq -\int_{\frac{1}{2}}^{1-\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}}\,\mathrm{d}t - c_1 \int_{1-\theta}^{1} \frac{\mathrm{d}t}{(1-t)^{\frac{\gamma-\delta+1}{p}}} > -\infty. \end{aligned}$$

Case (ii). Let (6.5) hold. Then it follows from Lemma 4.5 with $[0,1]$ replaced by $[1-\theta, 1]$ that

$$y_c(t) \leq y_{\overline{\kappa}}(t) \quad \text{for } t \in [1-\theta, 1].$$

From here and from (6.2) we conclude that there is a constant $c_2 > 0$ such that

$$\begin{aligned} z_0 = -\int_{\frac{1}{2}}^{1} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}}\,\mathrm{d}t &\leq -\int_{\frac{1}{2}}^{1-\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}}\,\mathrm{d}t - \int_{1-\theta}^{1} \frac{(d(t))^{\frac{1}{p-1}}}{(y_{\overline{\kappa}}(t))^{\frac{1}{p}}}\,\mathrm{d}t \\ &\leq -\int_{\frac{1}{2}}^{1-\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}}\,\mathrm{d}t - c_2 \int_{1-\theta}^{1} \frac{\mathrm{d}t}{(1-t)^{\frac{\gamma-\delta+1}{p}}} = -\infty. \end{aligned}$$

□

Theorem 6.2. *Let us assume* $\gamma > 0$,

$$\gamma + \frac{\delta}{p-1} > \frac{1}{p-1}, \tag{6.7}$$

$$\gamma < 1. \tag{6.8}$$

Then $z_0 > -\infty$. *If*

$$\gamma \geq 1 \tag{6.9}$$

instead of (6.8), *then* $z_0 = -\infty$.

Proof. Let (6.7) hold. For $\kappa > 0$ set

$$y_\kappa(t) = \kappa(1-t)^{p\left(\gamma+\frac{\delta}{p-1}\right)}, \quad t \in [1-\theta, 1].$$

If $y_c = y_c(t)$, $t \in [0,1]$, is a solution of (4.1) then

$$\begin{aligned} P_c y_\kappa &= y'_\kappa - p'\left[c\,(y_\kappa)^{\frac{1}{p}} - f(t)\right] \\ &= -\kappa p\left(\gamma + \frac{\delta}{p-1}\right)(1-t)^{p(\gamma+\frac{\delta}{p-1})-1} - p'\left(c\kappa^{\frac{1}{p}} - \eta(t)\right)(1-t)^{\gamma+\frac{\delta}{p-1}}, \end{aligned} \tag{6.10}$$

$t \in (1-\theta, 1)$. It follows from (6.7) that the power $(1-t)^{\gamma+\frac{\delta}{p-1}}$ dominates the power $(1-t)^{p(\gamma+\frac{\delta}{p-1})-1}$ near 1. It then follows from (6.10) that we distinguish between two cases:

(i) There exists $\underline{\kappa} \ll 1$ so small that $P_c y_{\underline{\kappa}} > 0 = P_c y_c$ a.e. in $[1-\theta, 1]$.

(ii) There exists $\overline{\kappa} \gg 1$ so large that $P_c y_{\overline{\kappa}} < 0 = P_c y_c$ a.e. in $[1-\theta, 1]$.

Case (i). Let (6.8) hold. Then it follows from Lemma 4.5 with $[0,1]$ replaced by $[1-\theta, 1]$ that

$$y_c(t) \geq y_{\underline{\kappa}}(t) \quad \text{for } t \in [1-\theta, 1].$$

From here and from (6.2) we conclude that there is a constant $c_3 > 0$ such that

$$z_0 = -\int_{\frac{1}{2}}^{1} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}}\,\mathrm{d}t \geq -\int_{\frac{1}{2}}^{1-\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}}\,\mathrm{d}t - c_3\int_{1-\theta}^{1} \frac{\mathrm{d}t}{(1-t)^\gamma} > -\infty.$$

Case (ii). Let (6.9) hold. Then it follows from Lemma 4.5 with $[0,1]$ replaced by $[1-\theta, 1]$ that

$$y_c(t) \leq y_{\overline{\kappa}}(t) \quad \text{for } t \in [1-\theta, 1].$$

From here and from (6.2) we conclude that there is a constant $c_4 > 0$ such that

$$z_0 = -\int_{\frac{1}{2}}^{1} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}}\,\mathrm{d}t \leq -\int_{\frac{1}{2}}^{1-\theta} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}}\,\mathrm{d}t - c_4\int_{1-\theta}^{1} \frac{\mathrm{d}t}{(1-t)^\gamma} = -\infty.$$

$\square$

Remark 6.3. To visualize conditions from Theorems 6.1, 6.2, we introduce the following sets:

$$\begin{aligned} \mathcal{M}_1^1 &:= \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, 0 \leq \gamma + \frac{\delta}{p-1} \leq \frac{1}{p-1}, \gamma - \delta + 1 < p\}, \\ \mathcal{M}_1^2 &:= \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, 0 \leq \gamma + \frac{\delta}{p-1} \leq \frac{1}{p-1}, \gamma - \delta + 1 \geq p\}, \\ \mathcal{M}_1^3 &:= \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, \gamma + \frac{\delta}{p-1} > \frac{1}{p-1}, \gamma < 1\}, \\ \mathcal{M}_1^4 &:= \{(\gamma,\delta) \in \mathbb{R}^2 : \gamma > 0, \gamma + \frac{\delta}{p-1} > \frac{1}{p-1}, \gamma \geq 1\}. \end{aligned}$$

Then $z_0 > -\infty$ if and only if $(\gamma,\delta) \in \mathcal{M}_1^1 \cup \mathcal{M}_1^3$ and $z_0 = -\infty$ if and only if $(\gamma,\delta) \in \mathcal{M}_1^2 \cup \mathcal{M}_1^4$. See Figure 6.1 for geometric interpretation.

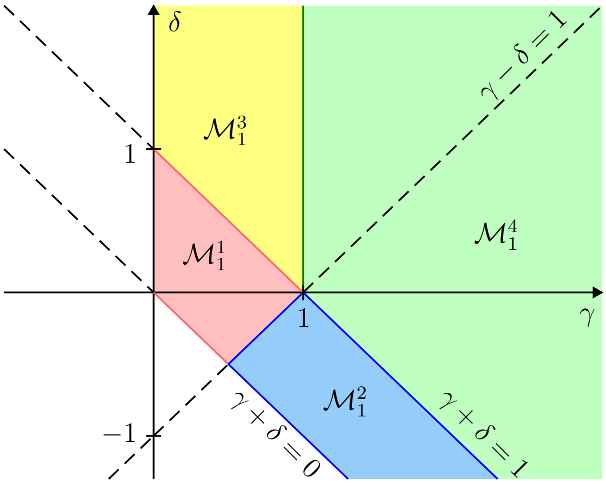

Figure 6.1: Visualization of the sets $\mathcal{M}_1^1$, $\mathcal{M}_1^2$, $\mathcal{M}_1^3$ and $\mathcal{M}_1^4$ for $p = 2$

Remark 6.4. Let $z_0 > -\infty$, i.e., $(\gamma, \delta) \in \mathcal{M}_1^1 \cup \mathcal{M}_1^3$. Then it follows from Remark 5.2 that for $\delta \leq 0$ we have $\lim_{z \to z_0+} U'(z) = 0$, i.e., the travelling wave profile U is a C^1-function in a neighbourhood of $z_0 \in \mathbb{R}$. The above estimates in proofs of Theorems 6.1 and 6.2 allows us to specify this result as follows. Let $(\gamma, \delta) \in \mathcal{M}_1^1$. Then

$$y_c(t) \leq \overline{\kappa}(1-t)^{\gamma + \frac{\delta}{p-1} + 1}, \quad t \in [1-\theta, 1],$$

and, therefore, there is a constant $c_5 > 0$ such that

$$\begin{aligned} \left.\frac{\mathrm{d}z}{\mathrm{d}U}\right|_{U=1} &= \lim_{U \to 1-} \frac{\mathrm{d}z}{\mathrm{d}U} = \lim_{U \to 1-} -\frac{(d(U))^{\frac{1}{p-1}}}{(y_c(U))^{\frac{1}{p}}} \leq -c_5 \lim_{U \to 1-} \frac{(1-U)^{\frac{\delta}{p-1}}}{(1-U)^{\frac{\gamma + \frac{\delta}{p-1} + 1}{p}}} \\ &= -c_5 \lim_{U \to 1-} (1-U)^{-\frac{\gamma - \delta + 1}{p}} = -\infty, \end{aligned}$$

i.e., $U'(z_0+) = 0$ if $\delta < \gamma + 1$.

Let $(\gamma, \delta) \in \mathcal{M}_1^3$. Then

$$y_c(t) \leq \overline{\kappa}(1-t)^{p(\gamma + \frac{\delta}{p-1})}, \quad t \in [1-\theta, 1],$$

and, therefore, there is a constant $c_6 > 0$ such that

$$\begin{aligned} \left.\frac{\mathrm{d}z}{\mathrm{d}U}\right|_{U=1} &= \lim_{U \to 1-} \frac{\mathrm{d}z}{\mathrm{d}U} = \lim_{U \to 1-} -\frac{(d(U))^{\frac{1}{p-1}}}{(y_c(U))^{\frac{1}{p}}} \leq -c_6 \lim_{U \to 1-} \frac{(1-U)^{\frac{\delta}{p-1}}}{(1-U)^{\gamma + \frac{\delta}{p-1}}} \\ &= -c_6 \lim_{U \to 1-} (1-U)^{-\gamma} = -\infty, \end{aligned}$$

i.e., $U'(z_0+) = 0$ if $\gamma > 0$.

To sum up the above discussion, the travelling wave profile U is a C^1-function in a neighbourhood of $z_0 \in \mathbb{R}$ for any $(\gamma, \delta) \in \mathcal{M}_1^1 \cup \mathcal{M}_1^3$.

6.2 Asymptotics near 0

Let us assume that $g(t) \sim t^\alpha$ and $d(t) \sim t^\beta$ as $t \to 0+$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. It follows from Theorem 5.3 that

$$\alpha + \frac{\beta}{p-1} \geq \frac{1}{p-1} \tag{6.11}$$

for otherwise there is no solution of (5.1). At first, we use formula

$$z_1 = \int_0^{\frac{1}{2}} \frac{(d(t))^{\frac{1}{p-1}}}{(y_c(t))^{\frac{1}{p}}} \,\mathrm{d}t \tag{6.12}$$

to prove whether $z_1 = +\infty$. In fact, the proof of Theorem 4.7 offers the method how to prove it. Indeed, inequality

$$0 < y_c(t) \leq k t^{p'}, \quad t \in (0,1)$$

combined with (6.12) yields that there exists a constant $c_7 > 0$ such that

$$z_1 \geq \int_0^{\frac{1}{2}} \frac{(d(t))^{\frac{1}{p-1}}}{k^{\frac{1}{p}} t^{\frac{p'}{p}}} \,\mathrm{d}t \geq c_7 \int_0^{\frac{1}{2}} \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{1}{p-1}}} \,\mathrm{d}t = c_7 \int_0^{\frac{1}{2}} t^{\frac{\beta-1}{p-1}} \,\mathrm{d}t = +\infty \tag{6.13}$$

if and only if $\beta + p \leq 2$. The values of α, β for which this case occurs are for $p = 2$ shown in Figure 6.2. However, the above estimate is far from being optimal. Indeed, we can refine

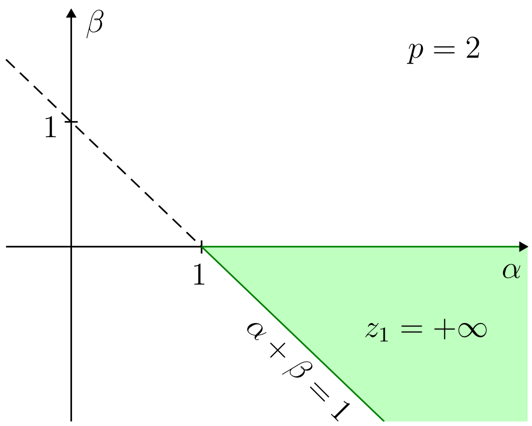

Figure 6.2: Visualization of conditions leading to z_1 infinite

the asymptotics of y_c near 0 in the case of power-type behaviour of g and d near 0 and prove $z_1 = +\infty$ under more general assumptions on α and β.

Notice that (6.11) is equivalent to $p\alpha + p'\beta \geq p'$ and set $\omega := p\alpha + p'\beta$, $y_\kappa(t) := \kappa t^\omega$, $t \in (0,1)$, with $\kappa > 0$. Let

$$f_1 := \sup_{t \in (0,1)} \frac{(d(t))^{\frac{1}{p-1}} g(t)}{t^{\alpha + \frac{\beta}{p-1}}}. \tag{6.14}$$

It follows from (6.11) and (5.2) that $\mu \leq f_1 < +\infty$. In particular, (6.14) yields

$$f(t) \leq f_1 t^{\alpha + \frac{\beta}{p-1}} = f_1 t^{\frac{\omega}{p}}, \quad t \in [0,1].$$

Therefore, we have

$$\begin{aligned}P_c y_\kappa &= y_\kappa'(t) - p'\left[c\left(y_\kappa(t)\right)^{\frac{1}{p}} - f(t)\right] \le \omega\kappa t^{\omega-1} - p' c\kappa^{\frac{1}{p}} t^{\frac{\omega}{p}} + p' f_1 t^{\frac{\omega}{p}}\\ &= t^{\frac{\omega}{p}}\left(\omega\kappa t^{\varepsilon} - p' c\kappa^{\frac{1}{p}} + p' f_1\right), \quad t \in [0,1],\end{aligned}$$

with $\varepsilon = \omega - 1 - \frac{\omega}{p} \ge 0$. Since $t \in [0,1]$, inequality

$$\omega\kappa - p' c\kappa^{\frac{1}{p}} + p' f_1 \le 0 \tag{6.15}$$

would imply that $P_c y_\kappa \le 0$ a.e. in $[0,1]$. Notice that (6.15) is equivalent to

$$c \ge \frac{\omega\kappa + p' f_1}{p'\kappa^{\frac{1}{p}}} =: H(\kappa),\ \ \kappa > 0. \tag{6.16}$$

Obviously, $H(\kappa) > 0$, $\kappa \in (0, +\infty)$ and $\lim_{\kappa\to 0+} H(\kappa) = \lim_{\kappa\to+\infty} H(\kappa) = +\infty$. The first semester calculus yields that the global minimum of H over $(0, +\infty)$ is achieved at the value

$$\kappa_{\min} = \frac{\frac{(p')^2}{p} f_1}{\omega}$$

and, due to $\omega \ge p'$,

$$H(\kappa_{\min}) = (p')^{1-\frac{2}{p}}\, p^{\frac{1}{p}}\, f_1^{\frac{1}{p'}}\, \omega^{\frac{1}{p}} \ge (p')^{\frac{1}{p'}}\, p^{\frac{1}{p}}\, f_1^{\frac{1}{p'}}. \tag{6.17}$$

It follows from (6.15)–(6.17) that for $\overline{\kappa} = \kappa_{\min}$ and all $c \ge (p')^{\frac{1}{p'}}\, p^{\frac{1}{p}}\, f_1^{\frac{1}{p'}}$ we have that $P_c y_{\overline{\kappa}} \le 0 = P_c y_c$ a.e. in $[0,1]$ and since $y_{\overline{\kappa}}(1) > 0$, by Lemma 4.5, we get $y_{\overline{\kappa}}(t) \ge y_c(t)$ for $t \in [0,1]$. In particular, due to (6.12) we have

$$z_1 \ge \int_0^{\frac{1}{2}} \frac{(d(t))^{\frac{1}{p-1}}}{\overline{\kappa}^{\frac{1}{p}} t^{\frac{\omega}{p}}}\,\mathrm{d}t \ge c_8 \int_0^{\frac{1}{2}} \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{\omega}{p}}}\,\mathrm{d}t = +\infty$$

with some $c_8 > 0$ if and only if

$$\frac{\beta}{p-1} - \frac{\omega}{p} \le -1$$

which is equivalent with $\alpha \ge 1$.

On the other hand, let $c \in [c^*, +\infty)$ be fixed. Since d and g are strictly positive in $(0,1)$ and $f \sim t^{\alpha+\frac{\beta}{p-1}}$ as $t \to 0+$, there exists $0 < f_2 < +\infty$ such that

$$f(t) \ge f_2 t^{\alpha+\frac{\beta}{p-1}} = f_2 t^{\frac{\omega}{p}}, \quad t \in \left[0, \frac{1}{2}\right].$$

We set $y_{\underline{\kappa}}(t) = \underline{\kappa} t^{\omega}$, $t \in [0, \frac{1}{2}]$, where

$$\underline{\kappa} := \min\left\{2^{\omega} y_c\left(\frac{1}{2}\right), \left(\frac{f_2}{c}\right)^{p}\right\}.$$

Then $y_{\underline{\kappa}}(\frac{1}{2}) \le y_c(\frac{1}{2})$ and

$$\begin{aligned}P_c y_{\underline{\kappa}} &= y_{\underline{\kappa}}'(t) - p'\left[c\left(y_{\underline{\kappa}}(t)\right)^{\frac{1}{p}} - f(t)\right] \ge \omega\underline{\kappa} t^{\omega-1} - p' c\underline{\kappa}^{\frac{1}{p}} t^{\frac{\omega}{p}} + p' f_2 t^{\frac{\omega}{p}}\\ &\ge p' t^{\frac{\omega}{p}}\left(f_2 - c\underline{\kappa}^{\frac{1}{p}}\right) \ge 0 = P_c y_c \ \text{ in } \ \left[0, \frac{1}{2}\right].\end{aligned}$$

By Lemma 4.5 we conclude $y_{\underline{\kappa}}(t) \le y_c(t)$, $t \in [0, \frac{1}{2}]$. In particular, thanks to (6.12) we have

$$z_1 \le \int_0^{\frac{1}{2}} \frac{(d(t))^{\frac{1}{p-1}}}{\underline{\kappa}^{\frac{1}{p}} t^{\frac{\omega}{p}}} \, \mathrm{d}t \le c_9 \int_0^{\frac{1}{2}} \frac{t^{\frac{\beta}{p-1}}}{t^{\frac{\omega}{p}}} \, \mathrm{d}t < +\infty$$

with some $c_9 > 0$ if and only if

$$\frac{\beta}{p-1} - \frac{\omega}{p} > -1$$

which is equivalent with $\alpha < 1$.

We can summarize the asymptotics of y_c near 0 as follows.

Theorem 6.5. *Let f_1 be as in* (6.14)*,*

$$c \ge (p')^{\frac{1}{p'}} p^{\frac{1}{p}} f_1^{\frac{1}{p'}}, \quad \alpha + \frac{\beta}{p-1} \ge \frac{1}{p-1}, \quad \alpha \ge 1.$$

Then $z_1 = +\infty$.

Let

$$c \ge c^*, \quad \alpha + \frac{\beta}{p-1} \ge \frac{1}{p-1}, \quad 0 < \alpha < 1.$$

Then $z_1 < +\infty$.

Remark 6.6. To visualize conditions from Theorem 6.5, we introduce the following sets

$$\mathcal{M}_0^1 := \{(\alpha, \beta) \in \mathbb{R}^2 : 0 < \alpha < 1, \alpha + \frac{\beta}{p-1} \ge \frac{1}{p-1}\},$$
$$\mathcal{M}_0^2 := \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \ge 1, \alpha + \frac{\beta}{p-1} \ge \frac{1}{p-1}\}.$$

see Figure 6.3 for geometric interpretation.

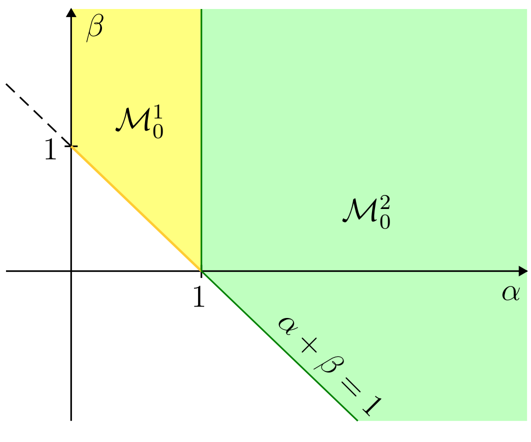

Figure 6.3: Visualization of the sets $\mathcal{M}_0^1$, $\mathcal{M}_0^2$ for $p = 2$

If $(\alpha, \beta) \in \mathcal{M}_0^1$ and $c \ge c^*$, then $z_1 < +\infty$. On the other hand, if $(\alpha, \beta) \in \mathcal{M}_0^2$ and $c \ge (p')^{\frac{1}{p'}} p^{\frac{1}{p}} f_1^{\frac{1}{p'}}$, then $z_1 = +\infty$. Notice that the last inequality for c is more restrictive due to the fact $f_1 \ge \mu$.

Remark 6.7. Let $z_1 < +\infty$, i.e., $(\alpha, \beta) \in \mathcal{M}_0^1$. Then it follows from Remark 5.2 that for $\beta \le 0$ we have $\lim_{z\to z_1-} U'(z) = 0$, i.e., the travelling wave profile U is a C^1-function in a neighbourhood of $z_1 \in \mathbb{R}$. However, this result can be specified as follows.

Let $c \ge (p')^{\frac{1}{p'}} p^{\frac{1}{p}} f_1^{\frac{1}{p'}}$ and $y_{\overline{\kappa}}(t) = \overline{\kappa} t^{\omega}$ be as above. Then $y_c(t) \le \overline{\kappa} t^{\frac{\omega}{p}}$, $t \in [0,1]$, and there exists c_{10} such that

$$\left.\frac{\mathrm{d}z}{\mathrm{d}U}\right|_{U=0} = \lim_{U\to 0+} \frac{\mathrm{d}z}{\mathrm{d}U} = \lim_{U\to 0+} -\frac{(d(U))^{\frac{1}{p-1}}}{(y_c(U))^{\frac{1}{p}}} \le -\lim_{U\to 0+} \frac{(d(U))^{\frac{1}{p-1}}}{\overline{\kappa}^{\frac{1}{p}} U^{\frac{\omega}{p}}} \le -c_{10} \lim_{U\to 0+} \frac{U^{\frac{\beta}{p-1}}}{U^{\frac{\omega}{p}}}$$
$$= -c_{10} \lim_{U\to 0+} U^{-\alpha} = -\infty,$$

i.e., $U'(z_1-) = 0$ if $\alpha > 0$. Therefore, if c is large enough ($c \ge (p')^{\frac{1}{p'}} p^{\frac{1}{p}} f_1^{\frac{1}{p'}}$), the travelling wave profile U is a C^1-function in a neighbourhood of $z_1 \in \mathbb{R}$ for any $(\alpha, \beta) \in \mathcal{M}_0^1$.

7 Concluding remarks and open problems

We emphasize that our asymptotic analysis providing the description of $\mathcal{M}_0^2$ as well as the smoothness of travelling wave profile U in the neighbourhood of $z_1 \in \mathbb{R}$ in Remark 6.7 holds for c large enough, namely, for c satisfying inequality $c \ge (p')^{\frac{1}{p'}} p^{\frac{1}{p}} f_1^{\frac{1}{p'}}$. In this respect, for $c \in [c^*, (p')^{\frac{1}{p'}} p^{\frac{1}{p}} f_1^{\frac{1}{p'}})$ the type as well as the smoothness of the travelling wave profile U might be very different. In view of the results in [14, Theorem 2] one should expect that, e.g., for $p = 2$, $\alpha > 0$, $\beta = 1$ and $c = c^*$, we have $z_1 \in \mathbb{R}$ and $U'(z_1-) < 0$. Since $U'(z_1+) = 0$, U is not a C^1-function at the point $z_1 \in \mathbb{R}$.

Combining Remarks 6.3, 6.4, 6.6 and 6.7 we arrive at the following classification:

(a) Let $(\alpha, \beta) \in \mathcal{M}_0^2$, $(\gamma, \delta) \in \mathcal{M}_1^2 \cup \mathcal{M}_1^4$, $c \ge (p')^{\frac{1}{p'}} p^{\frac{1}{p}} f_1^{\frac{1}{p'}}$. Then the travelling wave profile is of front-type.

(b) Let $(\alpha, \beta) \in \mathcal{M}_0^1$, $(\gamma, \delta) \in \mathcal{M}_1^2 \cup \mathcal{M}_1^4$, $c \ge c^*$. Then the travelling wave profile is sharp of type (I) and it is smooth at $z_1 \in \mathbb{R}$ if $c \ge (p')^{\frac{1}{p'}} p^{\frac{1}{p}} f_1^{\frac{1}{p'}}$.

(c) Let $(\alpha, \beta) \in \mathcal{M}_0^2$, $(\gamma, \delta) \in \mathcal{M}_1^1 \cup \mathcal{M}_1^3$, $c \ge (p')^{\frac{1}{p'}} p^{\frac{1}{p}} f_1^{\frac{1}{p'}}$. Then the travelling wave profile is sharp of type (II) and it is smooth at $z_0 \in \mathbb{R}$.

(d) Let $(\alpha, \beta) \in \mathcal{M}_0^1$, $(\gamma, \delta) \in \mathcal{M}_1^1 \cup \mathcal{M}_1^3$, $c \ge c^*$. Then the travelling wave profile is sharp of type (III) and it is smooth at $z_1 \in \mathbb{R}$ if $c \ge (p')^{\frac{1}{p'}} p^{\frac{1}{p}} f_1^{\frac{1}{p'}}$.

Provided that $d \in C(0,1)$, the corresponding smooth profiles for $c \ge (p')^{\frac{1}{p'}} p^{\frac{1}{p}} f_1^{\frac{1}{p'}}$ are sketched in Figure 7.1.

It is an interesting open problem to study the type of the travelling wave profile U for "small" values of c, i.e., for $c \in [c^*, (p')^{\frac{1}{p'}} p^{\frac{1}{p}} f_1^{\frac{1}{p'}})$ for general $p > 1$ and $(\alpha, \beta) \in \mathcal{M}_0^1 \cup \mathcal{M}_0^2$. We can ask the following questions:

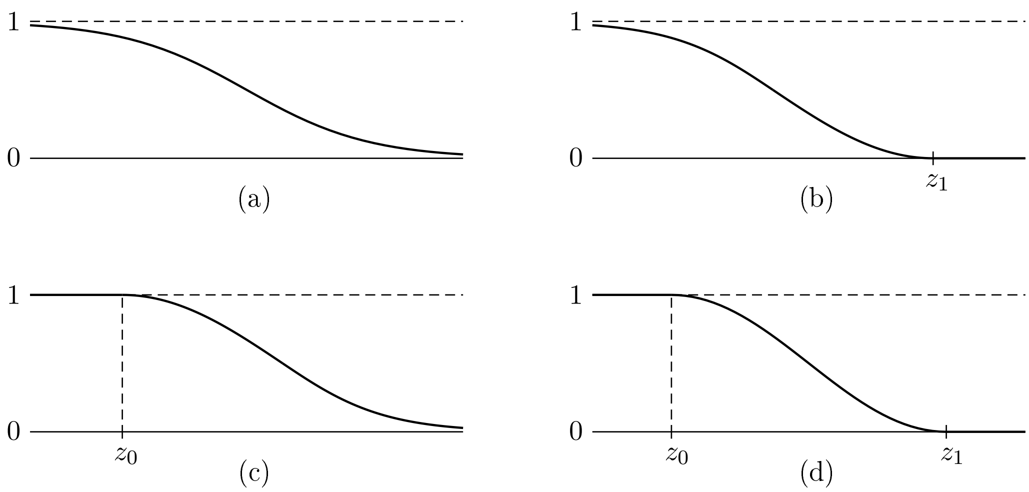

Figure 7.1: Wave profiles $U = U(z)$: (a) front-type; sharp of type (b) I; (c) II; (d) III

"Let $(\alpha, \beta) \in \mathcal{M}_0^1$. For which $c \in [c^*, (p')^{\frac{1}{p'}} p^{\frac{1}{p}} f_1^{\frac{1}{p'}})$ the travelling wave profile satisfies $U'(z_1) = 0$ and for which $c \in [c^*, (p')^{\frac{1}{p'}} p^{\frac{1}{p}} f_1^{\frac{1}{p'}})$ we have $U'(z_1-) < 0$ (including the case $U'(z_1-) = -\infty$)?"
"Let $(\alpha, \beta) \in \mathcal{M}_0^2$. For which $c \in [c^*, (p')^{\frac{1}{p'}} p^{\frac{1}{p}} f_1^{\frac{1}{p'}})$ we have $z_1 = +\infty$ and for which $c \in [c^*, (p')^{\frac{1}{p'}} p^{\frac{1}{p}} f_1^{\frac{1}{p'}})$ we have $z_1 \in \mathbb{R}$? If the latter case occurs, do we have $U'(z_1) = 0$ or $U'(z_1-) < 0$ (including the case $U'(z_1-) = -\infty$)?"

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