Global well posedness for the fourth-order Schrödinger equation with Hartree-type nonlinearity for Cauchy data in L^p

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Abstract This paper is concerned with the Cauchy problem for the nonlinear fourth-order Schrödinger equation on $\mathbb{R}^n$, with the nonlinearity of Hartree-type $(|\cdot|^{-\gamma} * |u|^2)u$. It is shown that a global solution exists for initial data in the spaces $L^p(p < 2)$ under some suitable conditions on γ, n and p. The solution is established by using a data-decomposition argument, two kinds of generalized Strichartz estimates and a interpolation theorem.

Keywords Fourth-order Schrödinger equation; Hartree-type nonlinearity; Global solution; Data-decomposition; Strichartz estimates.
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1 Introduction and main results

The fourth-order Schrödinger equation has been introduced by Karpman and Shagalov [17] and Karpman [18] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium. Such fourth-order Schrödinger equation has the following form

$$iu_t + \Delta^2 u + \varepsilon \Delta u + f(|u|^2)u = 0,\ u(0) = u_0,$$

where $\varepsilon \in \{0, \pm 1\}$, and $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ is a complex valued function. In this paper, we will investigate the Cauchy problem for the fourth-order Hartree equation with $\varepsilon = 0$

$$\begin{cases} iu_t + \Delta^2 u + (|\cdot|^{-\gamma} * |u|^2)u = 0, \quad t > 0,\ x \in \mathbb{R}^n, \\ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

The Hartree type nonlinearity is relevant to describing several physical phenomena, as for instance, the dynamics of the mean-field limits of many-body quantum systems such as coherent states and condensates, the quantum transport in semiconductors superlattices, the study of mesoscopic structures in Chemistry, among others (cf. [9, 10, 18, 19]).

The works addressing fourth-order Schrödinger equation include [4], Ben-Artzi, Koch and

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Saut discussed the sharp space-time decay properties of fundamental solutions to the linear equation

$$iu_t + \Delta^2 u + \varepsilon \Delta u = 0,\ \varepsilon \in \{0, \pm 1\}.$$

Then, thanks to these space-time decay properties, there exist a few literatures treating the fourth-order Schrödinger equation. In [3], Banquent and Villamizar-Roa considered the following problem

$$\begin{cases} iu_t + \alpha(t)\Delta u + \beta(t)\Delta^2 u + \theta(|\cdot|^{-\gamma} * |u|^2)u = 0, & t \in \mathbb{R},\, x \in \mathbb{R}^n, \\ u(x,t_0) = u_0(x), \quad t_0 \in \mathbb{R} \quad x \in \mathbb{R}^n, \end{cases}$$

where the coefficients α, β are real-valued functions which represent the variable dispersion coefficients, and $\theta \neq 0$ is a real coefficient. The main results in [3] is summarized as the following aspects: For initial data $u_0 \in H^s(\mathbb{R}^n)$, $s \geq max(0, \gamma/2-2)$ and $0 < \gamma < n$, they prove the existence of local in time solution $u \in C([-T-t_0, T+t_0]; H^s(\mathbb{R}^n))$. The main ideas of the proof is based on Strichartz estimates for the linear semigroup propagator $e^{it\Delta^2}$, as well as the Hardy-Littlewood-Sobolev inequality which allows us to control the Hartree nonlinearity. For initial data in L^2, by using the conserved quantity $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ it is able to extend the local solution globally. They also proved the existence of global solution in H^1 by combining the energy conservation law and the local well-posedness in H^1. Moreover, in [21] a sharp threshold of global well-posedness and scattering of energy solutions versus finite time blow-up dichotomy were given in the mass-super-critical and energy-critical regimes. For relevant results to the stationary case, one can see [5].

It is known that from the previous considerations [3, 4, 5], in the study of nonlinear dispersive equations, initial data is assumed to be in a suitable function space whose norm is characterized by some kind of quare integrability. Examples of such data spaces are L^2-space, L^2-based Sobolev spaces H^s and so on. However, when the initial data u_0 are not characterized by any kind of square integrability, much less is known about the solvability of (1.1), and most popular examples of such data spaces is L^p spaces ($p \neq 2$). So, the aim of this paper is to discussing the existence of the solutions for the Cauchy problem (1.1) with data $u_0 \in L^p(\mathbb{R}^n)$, where $p \neq 2$.

Zhou [24] made an important breakthrough in this regard. He established a result about the second-order Schrödinger equation with cubic nonlinearity for L^p initial data. In [24], he considered the following Cauchy problem

$$\begin{cases} iu_t - u_{xx} \pm |u|^2 u = 0, & t > 0,\, x \in \mathbb{R}, \\ u(x,0) = u_0(x), & u_0(x) \in L^p(\mathbb{R}), \end{cases} \tag{1.2}$$

where $1 < p < 2$. In fact, after a linear transformation $v(x,t) = e^{it\Delta}u(x,t)$ and applying to the integral equation of (1.2)

$$u(t) = e^{-it\Delta}u_0 \pm i\int_0^t e^{-i(t-\tau)\Delta}(|u(\tau)|^2 u(\tau))d\tau,$$

we can obtain

$$v(t) = u_0 \pm i\int_0^t e^{i\tau\Delta}[e^{i\tau\Delta}\bar{v}(\tau)(e^{-i\tau\Delta}v(\tau))^2]d\tau. \tag{1.3}$$

Afterwards, by a key factorization formula (see [6]) for $e^{it\Delta}$ and $e^{-it\Delta}$

$$e^{it\Delta}\varphi = M_t D_t \mathcal{F} M_t \varphi, \quad e^{-it\Delta}\varphi = M_t^{-1}\mathcal{F}^{-1}D_t^{-1}M_t^{-1}\varphi, \tag{1.4}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transformation and its inverse respectively, the operators M_t and D_t are defined by

$$M_t:\omega\mapsto e^{i\frac{|x|^2}{4t}}\omega,\quad D_t:\omega\mapsto(4\pi it)^{-\frac{n}{2}}\omega(\frac{x}{4\pi it}),\quad t\neq 0,$$

then Zhou showed that (1.3) is locally well-posed in L^p for any $1<p<2$, his work relies on the following trilinear estimates

$$\|e^{i\tau\Delta}[e^{i\tau\Delta}v_1(\tau)e^{-i\tau\Delta}v_2(\tau)e^{-i\tau\Delta}v_3(\tau)]\|_{L^1(\mathbb{R})}\leq C\tau^{-1}\|v_1(\tau)\|_{L^1(\mathbb{R})}\|v_2(\tau)\|_{L^1(\mathbb{R})}\|v_3(\tau)\|_{L^1(\mathbb{R})}.$$

Unfortunately, we can't obtain the factorization formula that is similar to (1.4) about the semigroup operator $e^{it\Delta^2}$ of the linear fouth-order Schrödinger equation. So this idea can't directly apply to our Cauchy problem (1.1). From another perspective, Hyakuna and Tsutsmi [11, 12, 13, 14, 15, 16], adopted a different method to solve the nonlinear second-order Schrödinger equation. They split the data ϕ into the sum of a large L^2 function φ_N and a small remainder function ψ_N to obtain global solution of the original equation. The method is motivated by the work of Vargas and Vega [23], which is known as data-decomposition argument, and this approach can be traced back to Bourgain [2]. In this paper, we adopt this generalization split technique (see definition 1.1) to prove our results.

Before stating our main results, we firstly introduce several notations and definitions. For $p\in[1,2]$, denote that

$$r_0=\frac{4n}{2n-\gamma},\qquad \rho_0=\frac{4n}{2n+\gamma},$$

and let $q(p)$, $\sigma(p)$ are two functions which be defined by the relation

$$\frac{4}{q(p)}+\frac{n}{r_0}=\frac{n}{p},\qquad 4+\frac{n}{p}=\frac{4}{\sigma(p)}+\frac{n}{\rho_0}.$$

We also introduce a function space $\mathcal{A}_{p,\alpha}(\mathbb{R}^n)$, which is related to the decomposition of initial values.

Definition 1.1 *Let $\alpha>0$, $1\leq p\leq 2$, then $\phi(x)\in\mathcal{A}_{p,\alpha}(\mathbb{R}^n)$ if and only if these exist two sequences of functions $(\varphi_N)_{N>1}\subset L^2$ and $(\psi_N)_{N>1}\subset\mathcal{S}'$ satisfying*

$$\phi=\varphi_N+\psi_N,\qquad \forall N>1, \tag{1.5}$$

$$C_0^{-1}N^\alpha\leq\|\varphi_N\|_{L^2}\leq C_0N^\alpha, \tag{1.6}$$

$$\|e^{it\Delta^2}\psi_N\|_{L^{q(p)}L^{r_0}}\leq C_0N^{-1}, \tag{1.7}$$

for some $C_0>1$ independent of N.

In order to prove the generalized Strichartz-type estimates for homogeneous equations, we need give a definition about the weak L^p-space, noted by L^p_*, $1\leq p<\infty$.

Definition 1.2 (see [1]) *For $f\in L^{(p,\infty)}$, which is the general Lorentz spaces with the norm*

$$\|f\|_{L^{(p,\infty)}}=\sup_t t^{1/p}f^*(t)<\infty,$$

where $f^*(t)=\frac{1}{t}\int_0^t f(s)ds$, we denote $f\in L^p_*$, which means

$$\|f\|_{L^p_*}=\|f\|_{L^{(p,\infty)}}<\infty.$$

Remark 1.1 $\|f\|_{L^p_*}$ is not a norm if $1 \le p < \infty$. Indeed, L^p_* is a quasi-normed vector space, because we only conclude that

$$\|f+g\|_{L^p_*} \le 2(\|f\|_{L^p_*} + \|g\|_{L^p_*}).$$

Moreover, we can see $L^p \subset L^p_*$ by [1].

Now our main result can be stated as follows.

Theorem 1.1 *Let $0 < \gamma < min(4,n)$, $\alpha > 0$. And assume that*

$$max(\frac{6n}{3n-2\gamma+8}, \frac{4n}{2n+\gamma}) < p_0 < 2.$$

(i)For any $u_0 \in \mathcal{A}_{p_0,\alpha}$, and $N > 1$, there exists a unique local solution u of (1.1) *in the form of*

$$u = v + w \in C([0,T_N]; L^2(\mathbb{R}^n)) \bigcap L^{q(2)}([0,T_N]; L^{r_0}(\mathbb{R}^n)) + L^{q(p_0)}([0,T_N]; L^{r_0}(\mathbb{R}^n)).$$

Moreover, T_N can be written as

$$T_N = cN^{1-\frac{4}{4-\gamma}[2-\frac{n}{2}(\frac{n+4-\gamma}{2n}-\frac{1}{p_0})]\alpha} \tag{1.8}$$

where $c > 0$ is a constant independent of N.

(ii) Assume moreover that α, n, γ and p_0 satisfy

$$-\infty < \frac{4}{4-\gamma}[2-\frac{n}{2}(\frac{n+4-\gamma}{2n}-\frac{1}{p_0})]\alpha < 1. \tag{1.9}$$

Then, for any $u_0 \in \mathcal{A}_{p_0,\alpha}$, the initial value problem (1.1) *has a unique global solution u in the form of*

$$u = v + w \in C(\mathbb{R}; L^2(\mathbb{R}^n)) \bigcap L^{q(2)}(\mathbb{R}; L^{r_0}(\mathbb{R}^n)) + L^{q(p_0)}(\mathbb{R}; L^{r_0}(\mathbb{R}^n)). \tag{1.10}$$

For initial data $u_0 \in L^p(\mathbb{R}^n)$, our global well-posedness results are gathered in the following theorem.

Theorem 1.2 *Let $0 < \gamma < min(4,n)$, and put p lies in the region*

$$max(\frac{2n(16+\gamma)}{-\gamma^2+(n+4)\gamma+16n}, \frac{2n(16-\gamma)}{\gamma^2-(n+8)\gamma+16n+16}) < p < 2. \tag{1.11}$$

Then, for any $u_0 \in L^p(\mathbb{R}^n)$, there exists a unique global solution of (1.1) *in the form of*

$$u = v + w \in C(\mathbb{R}; L^2(\mathbb{R}^n)) \bigcap L^{q(2)}(\mathbb{R}; L^{r_0}(\mathbb{R}^n)) + L^{q(p_0)}(\mathbb{R}; L^{r_0}(\mathbb{R}^n)),$$

where p_0 is a constnat and satisfies

$$max(\frac{6n}{3n-2\gamma+8}, \frac{4n}{2n+\gamma}) < p_0 < p. \tag{1.12}$$

Finally, we give some notations which are used throughout this paper.

(i) For an arbitrary $a \in [1,\infty]$, a' is the conjugate of a, namely

$$\frac{1}{a} + \frac{1}{a'} = 1.$$

(ii) Let $I \subseteq \mathbb{R}$ and $q, r \in [1, \infty]$, we defined the mixed norm

$$\|u\|_{L_I^q L_x^r} = \|u\|_{L^q(I,L^r)} := (\int_I (\int_{\mathbb{R}} |u(t,x)|^r dx)^{\frac{q}{r}} dt)^{\frac{1}{q}}.$$

In particular, we write $L^q L^r = L^q_{\mathbb{R}} L^r$, when $I = \mathbb{R}$.

(iii) We introduce a trilinear form associated with the Hartree-type nonlinearity: for these space variable functioins f, g, h we define

$$H_\gamma(f, g, h) = (|\cdot|^{-\gamma} * (f \times \bar{g})) \times h.$$

we also write $H_\gamma(f) = H_\gamma(f, f, f)$. Using this, the nonlinearity in (1.1) is expressed as $H_\gamma(u(t))$.

(iv) We say that a pair (q, r) is biharmonic admissible if it satisfy

$$\frac{4}{q} + \frac{n}{r} = \frac{n}{2}.$$

and

$$2 \leq r \begin{cases} \leq \infty, & if \ n \leq 3 \\ < \infty, & if \ n = 4 \\ \leq \frac{2n}{n-4}, & if \ n \geq 5 \end{cases}$$

(v) Let $P, Q \in \mathbb{R}^2$, then the segment connecting P, Q is defined by $[PQ]$, $]PQ[$, $[PQ[$, $]PQ]$, according as it is closed, open, left closed right open and left open right closed.

(vi) In this paper, C, c are positive constants which may vary from line to line and independent of N.

2 Preliminaries

In this section, we first introduce some geometric notations. Consider the unit square R in $\mathbb{R}^2$:

$$R := \{(x, y) \in \mathbb{R}^2, 0 \leq x, y \leq 1\}.$$

Next, we defined some special points in R:

$$\begin{aligned} &O = (0,0),\ B = (\frac{1}{2}, 0),\ C = (\frac{1}{2} - \frac{2}{n}, \frac{1}{2}),\ E = (\frac{1}{2} - \frac{2}{n}, 1),\ F = (\frac{1}{2} - \frac{2}{n}, 0), \\ &D = (\frac{n-4}{2(n-2)}, \frac{n}{2(n-2)}),\ (C = (0, \frac{n}{8}),\ D = E = (0, \frac{n}{4}),\ F = (0,0), \quad if \quad n \leq 3), \\ &O' = (1,1),\ B' = (\frac{1}{2}, 1),\ C' = (\frac{1}{2} + \frac{2}{n}, \frac{1}{2}),\ E' = (\frac{1}{2} + \frac{2}{n}, 0),\ F' = (\frac{1}{2} + \frac{2}{n}, 1), \\ &(C' = (1, 1 - \frac{n}{8}),\ E' = (1, 1 - \frac{n}{4}),\ F' = (1,1), \quad if \quad n \leq 3). \end{aligned}$$

Note that O, C, D are colinear, $D \in [BE[$, $D' \in [B'E'[$, and defined the triangles $G := \triangle BEF$, $G' := \triangle B'E'F'$ and $\widehat{G} := \triangle BCD$, where $\widehat{G}$ is supposed to include the side $]CD[$ (except for n=4) but no other boundary points, and G, G' are open expect that B and B' are included. Obviously, we have

$$BC := \{(x, y) \in R;\ (\frac{1}{y}, \frac{1}{x}) \text{ is biharmonic admissible}\},$$

$$B'C' := \{(x, y) \in R;\ (\frac{1}{y'}, \frac{1}{x'}) \text{ is biharmonic admissible}\}.$$

2.1 Strichartz-type estimates

The classical Strichartz estimates for the free Schrödinger group are inequalities of the form

$$\|e^{it\Delta^2}\phi\|_{L^qL^r} \leq C\|\phi\|_{L^2}.$$

It is well known (see [20]) that the above estimate holds true if and only if (q,r) is a biharmonic admissible pair, it's also to say $(\frac{1}{r},\frac{1}{q}) \in [BC[$. However, in this paper, we need a generalization of these estimates of the form

$$\|e^{it\Delta^2}\phi\|_{L^qL^r} \leq C\|\phi\|_{L^p}. \tag{2.1}$$

We say that, by a scaling argument, q,r,p satisfy

$$\frac{4}{q}+\frac{n}{r}=\frac{n}{p}, \tag{2.2}$$

if (2.2) holds true. For the estimate of type (2.2) we present the following result.

Lemma 2.1 *Let $\frac{1}{2}<\frac{1}{p}<\frac{n}{2(n-2)}$ ($\frac{1}{2}<\frac{1}{p}\leq 1$ if $n\leq 3$), $(\frac{1}{r},\frac{1}{q})$ lie in the closed triangle $\widehat{G}$ and satisfy*

$$\frac{4}{q}+\frac{n}{r}=\frac{n}{p}.$$

Then, the estimate (2.1) *holds true for $\phi \in L^p(\mathbb{R}^n)$.*

Proof For any r satisfies

$$\frac{1}{2}-\frac{2}{n}<\frac{1}{r}<\frac{1}{2}\ (0\leq\frac{1}{r}<\frac{1}{2}\ if\ n\leq 3).$$

Let $Q\in[BC[$ with $x(Q)=\frac{1}{r}$, and denote $\frac{1}{q_1}=y(Q)$, then

$$\|e^{it\Delta^2}\phi\|_{L^{q_1}L^r} \leq C\|\phi\|_{L^2}. \tag{2.3}$$

On the other hand, we have

$$\|e^{it\Delta^2}\phi\|_{L^r} \leq C|t|^{-\frac{n}{4}(1-\frac{2}{r})}\|\phi\|_{L^{r'}}$$

by dispersion estimate (see [8]). Now we define a point $R=(\frac{1}{r},\frac{1}{q_2})=(\frac{1}{r},\frac{n}{4}(1-\frac{2}{r}))\in[BE[$, thus get the another estimate

$$\|e^{it\Delta^2}\phi\|_{L^{q_2}_{*}L^r} \leq C\|\phi\|_{L^{r'}}. \tag{2.4}$$

Since Q and R are on the same vertical line $x=\frac{1}{r}$, it follows from Marcinkewitz's interpolation theorem (see [1]) between (2.3) and (2.4) that if

$$\frac{1}{2}<\frac{1}{p}<\frac{1}{r'},\ \frac{1}{p}=\frac{1-\theta}{2}+\frac{\theta}{r'},\ \frac{1}{q}=\frac{1-\theta}{q_1}+\frac{\theta}{q_2},\ \frac{1}{q}\leq\frac{1}{p} \tag{2.5}$$

then

$$\|e^{it\Delta^2}\phi\|_{L^qL^r} \leq C\|\phi\|_{L^p}.$$

By some easily calculations with (2.5), then

$$\frac{4}{q}+\frac{n}{r}=\frac{n}{p}. \tag{2.6}$$

We now only need to seek the conditions of p and $(\frac{1}{r},\frac{1}{q})$ that satisfy (2.5). From (2.5), we get

$$\frac{1}{q} \le \frac{4}{nq} + \frac{1}{r},$$

when $n \le 4$, it is automatically satisfied. In addition, if $n > 4$ there will introduces a new restriction, it requires that $\frac{1/q}{1/r} \le \frac{n}{n-4}$, this means that $(\frac{1}{r},\frac{1}{q})$ must be below the ray extending $[OD[$. Thus $(\frac{1}{r},\frac{1}{q})$ must belong to $\widehat{G}$. Moreover put $(\frac{1}{r},\frac{1}{q}) = D = (\frac{n-4}{2(n-2)},\frac{n}{2(n-2)})$ into the above equality we get

$$\frac{1}{p} < \frac{4}{nq} + \frac{1}{r} = \frac{n}{2(n-2)}.$$

Summing up, we have proved. $\square$

Remark 2.1 Observe that $(q(p), r_0)$ satisfies the assumption of lemma 2.1 if

$$\frac{4n}{2n+\gamma} < p < 2.$$

Next, we review estimates for the solution of the inhomogenous equation

$$iu_t + \Delta^2 u = F,\ \ u(x,0) = 0.$$

It is konwn that it's solution u is given by

$$i\int_0^t e^{i(t-\tau)\Delta^2} F(\tau)d\tau.$$

Then in this paper we need estimates of the form

$$\left\| \int_0^t e^{i(t-\tau)\Delta^2} F(\tau)d\tau \right\|_{L^q L^r} \le C\|F\|_{L^\sigma L^\rho}. \tag{2.7}$$

It is well known that (2.7) is valid if both (q,r) and (σ',ρ') are biharmonic admissible (see [7]). However, in this paper we need estimate of type (2.7) under some other suitable conditions on q, r, σ, ρ.

Lemma 2.2 (see [8]) *Suppose that $(\frac{1}{r},\frac{1}{q}) \in G, (\frac{1}{\rho},\frac{1}{\sigma}) \in G'$ and*

$$\frac{4}{n\sigma} + \frac{1}{\rho} = \frac{4}{n} + \frac{4}{nq} + \frac{1}{r},\ \ \frac{1}{r} + \frac{1}{\rho} = 1.$$

Then, the estimate (2.7) is valid for any $F \in L^\sigma L^\rho$.

For the nonlinear term in Cauchy problem (1.1) , we have the following proposition.

Proposition 2.1 *Assume that $\gamma < min(4,n)$ and (q,r) is a biharmonic admissible pair satisfy*

$$\begin{cases} \frac{1}{2} - \frac{\gamma}{3n} \le \frac{1}{r} < min(\frac{1}{2}, \frac{1}{2} + \frac{2-\gamma}{3n}), & if \quad n \ge 4, \\ \frac{1}{2} - \frac{\gamma}{3n} \le \frac{1}{r} < min(\frac{1}{2}, \frac{2}{3} - \frac{\gamma}{3n}), & if \quad n \le 3. \end{cases} \tag{2.8}$$

Then there exist a constant C, such that

$$\left\| \int_0^t e^{i(t-\tau)\Delta^2} H_\gamma(u(\tau))d\tau \right\|_{L^q_{[0,T]} L^r} \le CT^{1-\frac{\gamma}{4}} \|u\|^3_{L^q_{[0,T]} L^r}$$

is vald for any $T > 0$ and $u \in L^q_{[0,T]} L^r$.

Proof Let $(\frac{1}{\rho}, \frac{1}{\sigma}) = (\frac{3}{r} + \frac{\gamma}{n} - 1, \frac{3}{q} - \frac{\gamma}{4} + 1)$, then we can easily konwn that $(\frac{1}{\rho}, \frac{1}{\sigma}) \in [B'C'[$ by (2.7). So we have

$$\left\| \int_0^t e^{i(t-\tau)\Delta^2} H_\gamma(u(\tau)) d\tau \right\|_{L^q_{[0,T]} L^r} \leq C \|H_\gamma(u)\|_{L^\sigma_{[0,T]} L^\rho}.$$

Moreover, from Hölder's inequality and Hardy-Littlewood-Sobolev inequality, we can get

$$\begin{aligned} \|H_\gamma(u)\|_{L^\rho} \leq & \||x|^{-\gamma} * |u|^2\|_{L^A} \|u\|_{L^r} \\ \leq & C \||u|^2\|_{L^{\frac{r}{2}}} \|u\|_{L^r} \\ = & C \|u\|_{L^r}^3, \end{aligned}$$

where $\frac{1}{\rho} = \frac{1}{A} + \frac{1}{r}$ and $\frac{2}{r} = 1 + \frac{1}{A} - \frac{\gamma}{n}$. Thus, we have

$$\|H_\gamma(u)\|_{L^\sigma_{[0,T]} L^\rho} \leq C T^B \| \|u\|_{L^r}^3 \|_{L^{\frac{q}{3}}_{[0,T]}},$$

where $\frac{1}{B} = \frac{1}{\sigma} - \frac{3}{q} = 1 - \frac{\gamma}{4}$. Summing up, we have completed the proof. □

2.2 Estimates of L^2-solutions.

If the initial data $u_0 \in L^2$, then the Cauchy problem (1.1) exists a unique global solution $u \in C(\mathbb{R}; L^2(\mathbb{R}^n)) \bigcap L^q(\mathbb{R}; L^p(\mathbb{R}^n))$ by [3], where (p, q) ia a biharmonic admissible pair. Now in this subsection, we will study the $L^q_{[0,T]} L^p$-estimate of the L^2-solution for the Cauchy problem (1.1).

Lemma 2.3 *Suppose that $\gamma < min(4, n)$ and $v : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ with $v(0, x) = \varphi(x) \in L^2$ solves*

$$i v_t + \Delta^2 v + (|\cdot|^{-\gamma} * |v|^2) v = 0, \quad t > 0,\, x \in \mathbb{R}^n.$$

Then, there are positive constants K_1, K_2 depending only on n, γ such that

$$\|v\|_{L^{q(2)}_{[0,\delta]} L^{r_0}} \leq K_1 \|\varphi\|_{L^2}$$

for any $\delta \in [0, (K_2 \|\varphi\|_{L^2})^{-\frac{8}{4-\gamma}}]$.

Proof Put $\delta^* = (2 C_1 C^{\frac{1}{2}} \|\varphi\|_{L^2})^{-\frac{8}{4-\gamma}}$, and

$$I = \{\delta \in [0, \delta^*] : \ \|v\|_{L^{q(2)}_{[0,\delta]} L^{r_0}} \leq C_1 \|\varphi\|_{L^2}\},$$

where C is the constant of proposition 2.1, C_1 is the constant which makes true for the following inequality

$$\|e^{it\Delta^2} \varphi\|_{L^{q(2)}_{[0,T]} L^{r_0}} \leq \frac{C_1}{2} \|\varphi\|_{L^2}, \quad T > 0.$$

Obviously $(q(2), r_0)$ is a biharmonic admissible pair, and $I \neq \varnothing$ because of $0 \in I$. So that is to say $I = [0, \delta^*]$ only if I is both open and closed. Since $\delta \mapsto \|v\|_{L^{q(2)}_{[0,\delta]} L^{r_0}}$ is continuous, so I is closed. Then we only need to show that I is open. For a fixed $\delta \in I$ with $\delta < \delta^*$, take $\varepsilon > 0$ satisfying $\delta + \varepsilon \leq \delta^*$, then for the integral equation

$$v(t) = e^{it\Delta^2} \varphi + i \int_0^t e^{i(t-\tau)\Delta^2} H_\gamma(v(\tau)) d\tau,$$

using proposition 2.1, since r_0 satisfies the condition of (2.8), then we have

$$\begin{aligned}\|v\|_{L^{q(2)}_{[0,\delta+\varepsilon]}L^{r_0}} \leq& \|e^{it\Delta^2}\varphi\|_{L^{q(2)}_{[0,\delta+\varepsilon]}L^{r_0}} + \left\| \int_0^t e^{i(t-\tau)\Delta^2} H_\gamma(v(\tau))d\tau \right\|_{L^{q(2)}_{[0,\delta+\varepsilon]}L^{r_0}} \\ \leq& \frac{C_1}{2}\|\varphi\|_{L^2} + \left\| \int_0^t e^{i(t-\tau)\Delta^2} H_\gamma(v(\tau))d\tau \right\|_{L^{q(2)}_{[0,\delta]}L^{r_0}} + \left\| \int_0^t e^{i(t-\tau)\Delta^2} H_\gamma(v(\tau))d\tau \right\|_{L^{q(2)}_{[\delta,\delta+\varepsilon]}L^{r_0}} \\ \leq& \frac{C_1}{2}\|\varphi\|_{L^2} + C\delta^{1-\frac{\gamma}{4}}\|v\|^3_{L^{q(2)}_{[0,\delta]}L^{r_0}} + C\varepsilon^{1-\frac{\gamma}{4}}\|v\|^3_{L^{q(2)}_{[\delta,\delta+\varepsilon]}L^{r_0}}.\end{aligned}$$

Owing to $\gamma < 4$, then if ε is taken sufficiently small, we have

$$C\varepsilon^{1-\frac{\gamma}{4}}\|v\|^3_{L^{q(2)}_{[\delta,\delta+\varepsilon]}L^{r_0}} \leq \frac{C_1}{4}\|\varphi\|_{L^2}.$$

Since $\delta \in I$ $\delta \leq \delta^*$, so we have

$$\begin{aligned}\|v\|_{L^{q(2)}_{[0,\delta+\varepsilon]}L^{r_0}} \leq& \frac{3C_1}{4}\|\varphi\|_{L^2} + C(\delta^*)^{1-\frac{\gamma}{4}}(C_1\|\varphi\|_{L^2})^3 \\ =& \frac{3C_1}{4}\|\varphi\|_{L^2} + \frac{C_1}{4}\|\varphi\|_{L^2} \\ =& C_1\|\varphi\|_{L^2}.\end{aligned}$$

So $\delta+\varepsilon \in I \;\Rightarrow\; I$ is open $\Rightarrow\; I = [0,\delta^*]$. Then we have completed the proof by let $K_1 = C_1$ and $K_2 = 2C_1C^{\frac{1}{2}}$. $\square$

3 Proof of Theorem 1.1

Since $u_0 \in \mathcal{A}_{p_0,\alpha}$, these exist $C_0 > 1$ and $(\varphi_N) \subset L^2$, $(\psi_N) \subset \mathcal{S}'$ such that

$$\phi = \varphi_N + \psi_N$$

$$C_0^{-1}N^\alpha \leq \|\varphi_N\|_{L^2} \leq C_0 N^\alpha, \tag{3.1}$$

$$\|e^{it\Delta^2}\psi_N\|_{L^{q(p_0)}L^{r_0}} \leq C_0 N^{-1}, \tag{3.2}$$

for all $N > 1$. Now, fixed $N > 1$ we consider the following two Cauchy problem:

$$\begin{cases} iv_t + \Delta^2 v + H_\gamma(v) = 0, \\ v(x,0) = \varphi_N(x), \end{cases} \tag{3.3}$$

$$\begin{cases} iw_t + \Delta^2 w + H_\gamma(w+v) - H_\gamma(v) = 0, \\ w(x,0) = \psi_N(x). \end{cases} \tag{3.4}$$

Then, the solution $u(t)$ of (1.1) can be present in the form of $u(t) = v(t) + w(t)$.

3.1 Existence of local solutions

In this subsection we will construct a local solution on a small interval $[0,\delta_N]$ using a fixed point argument, where $\delta_N > 0$ will be defined below. For later use in the next subsection (see (3.18) below), it is convenient to construct a solution under the weaker assumption

$$(2C_0)^{-1}N^\alpha \leq \|\varphi_N\|_{L^2} \leq 2C_0 N^\alpha \tag{3.5}$$

instead of (3.1). Let us introduce a positive real number

$$\delta_N := (M(2C_0N^\alpha))^{-\frac{8}{4-\gamma}},$$

where M is a sufficiently large constant independent of N.

We first discuss the solution of (3.3), since $\varphi_N \in L^2$,we see there exists a unique global solution $v(t)$ with the L^2-conservation law (see [3]):

$$\|v(t,\cdot)\|_{L^2} = \|\varphi_N\|_{L^2}, \qquad \forall t \in \mathbb{R}. \tag{3.6}$$

Moreover, we observe that, by lemma 2.3, and (3.5)

$$\|v\|_{L^{q(2)}_{[0,\delta_N]}L^{r_0}} \leq K_1\|\varphi_N\|_{L^2}, \tag{3.7}$$

if $M > K_2$.

We use a fixed point argument to obtain a solution of (3.4), consider the corresponding integral equation

$$w(t) = e^{it\Delta^2}\psi_N + i\int_0^t e^{i(t-\tau)\Delta^2}(H_\gamma(v+w) - H_\gamma(v))d\tau \tag{3.8}$$

in the complete metric space

$$\mathcal{V}_N^\delta := \{w \in L^{q(p_0)}_{[0,\delta_N]}L^{r_0} :\ \|w\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}} \leq \frac{8C_0}{N}\},$$

we will show that the operator given by

$$(Tw)(t) := e^{it\Delta^2}\psi_N + i\int_0^t e^{i(t-\tau)\Delta^2}(H_\gamma(v+w) - H_\gamma(v))d\tau$$

is well defined and is a contraction mapping from $\mathcal{V}_N^\delta \to \mathcal{V}_N^\delta$. For any $w \in \mathcal{V}_N^\delta$, note that $(q(p_0), r_0, \sigma(p_0), \rho_0)$ will satisfy the assumption of lemma 2.2 if $p_0 > \frac{4n}{2n+\gamma}$, so by lemma 2.2 and (3.2), we have

$$\begin{aligned}\|Tw\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}} \leq & \|e^{it\Delta^2}\psi_N\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}} + \left\|\int_0^t e^{i(t-\tau)\Delta^2}(H_\gamma(v+w) - H_\gamma(v))d\tau\right\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}} \\ \leq & \frac{C_0}{N} + C\|H_\gamma(v+w) - H_\gamma(v)\|_{L^{\sigma(p_0)}_{[0,\delta_N]}L^{\rho_0}}\end{aligned}$$

Note that $\|H_\gamma(v+w) - H_\gamma(v)\|_{L^{\sigma(p_0)}_{[0,\delta_N]}L^{\rho_0}}$ is only associated with the norm of terms such as

$$H_\gamma(v,v,w),\ H_\gamma(v,w,w),\ H_\gamma(w),$$

this is owing to

$$\begin{aligned}H_\gamma(v+w) - H_\gamma(v) = & H_\gamma(v,v,w) + H_\gamma(v,w,v) + H_\gamma(w,v,v) \\ & + H_\gamma(w,w,v) + H_\gamma(v,w,w) + H_\gamma(w,v,w) \\ & + H_\gamma(w).\end{aligned}$$

By Hölder's inequality and Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}
\|H_\gamma(v,v,w)\|_{L^{\sigma(p_0)}_{[0,\delta_N]}L^{\rho_0}} =&\|(|x|^{-\gamma}*|v|^2)w\|_{L^{\sigma(p_0)}_{[0,\delta_N]}L^{\rho_0}}\\
\leq&\Big\|\||x|^{-\gamma}*|v|^2\|_{L^A_x}\cdot\|w\|_{L^{r_0}_x}\Big\|_{L^{\sigma(p_0)}_t}\\
\leq&C\Big\|\||v|^2\|_{L_x^{\frac{r_0}{2}}}\cdot\|w\|_{L^{r_0}_x}\Big\|_{L^{\sigma(p_0)}_t}\\
\leq&C\|1\|_{L^B_{[0,\delta_N]}}\cdot\|v\|^2_{L^{q(2)}_{[0,\delta_N]}L^{r_0}}\|w\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}},
\end{aligned}\tag{3.9}$$

where A and B satisfy

$$\frac{1}{\rho_0}=\frac{1}{A}+\frac{1}{r_0},\frac{2}{r_0}=1+\frac{1}{A}-\frac{\gamma}{n},\frac{1}{\sigma(p_0)}=\frac{1}{B}+\frac{2}{q(2)}+\frac{1}{q(p_0)}.$$

And $\frac{1}{B}=1-\frac{\gamma}{4}$ by calculation combining with the definition 1.1. Then

$$\|H_\gamma(v,v,w)\|_{L^{\sigma(p_0)}_{[0,\delta_N]}L^{\rho_0}}\leq C(\delta_N)^{1-\frac{\gamma}{4}}\|v\|^2_{L^{q(2)}_{[0,\delta_N]}L^{r_0}}\|w\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}}=I_1.$$

In a similar way, we can get

$$\|H_\gamma(v,w,w)\|_{L^{\sigma(p_0)}_{[0,\delta_N]}L^{\rho_0}}\leq C(\delta_N)^{1-\frac{\gamma}{4}-\frac{n}{4}(\frac{1}{p_0}-\frac{1}{2})}\|v\|_{L^{q(2)}_{[0,\delta_N]}L^{r_0}}\|w\|^2_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}}=I_2,$$

$$\|H_\gamma(w)\|_{L^{\sigma(p_0)}_{[0,\delta_N]}L^{\rho_0}}\leq C(\delta_N)^{1-\frac{\gamma}{4}-\frac{n}{2}(\frac{1}{p_0}-\frac{1}{2})}\|w\|^3_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}}=I_3,$$

where we have used the inequality

$$1-\frac{\gamma}{4}-\frac{n}{2}(\frac{1}{p_0}-\frac{1}{2})>0,\tag{3.10}$$

which follows from the assumption

$$p_0>\frac{6n}{3n-2\gamma+8}>\frac{2n}{n-\gamma+4}.$$

For I_1 we have

$$\begin{aligned}
I_1\leq&C(M2C_0N^\alpha)^{-\frac{8}{4-\gamma}\cdot(1-\frac{\gamma}{4})}\cdot(K_12C_0N^\alpha)^2\times\frac{8C_0}{N}\\
=&CM^{-2}K_1^2\frac{C_0}{N}.
\end{aligned}$$

Since $(-\frac{8}{4-\gamma})\times(-\frac{n}{4}(\frac{1}{p_0}-\frac{1}{2}))=\frac{2n}{4-\gamma}(\frac{1}{p_0}-\frac{1}{2})<1$ by (3.10) and $N>1$, by the same way

$$I_2\leq CM^{-1}K_1C_0\frac{C_0}{N^2}\leq CM^{-1}K_1C_0\frac{C_0}{N},$$

$$I_3\leq CM^{-2+\frac{4n}{4-\gamma}(\frac{1}{p_0}-\frac{1}{2})}C_0^2\frac{C_0}{N}.$$

Obviously, it holds that

$$I_j\leq\frac{C_0}{N},\qquad j=1,2,3,$$

if M is sufficiently large. Collecting these estimates, we get

$$\|Tw\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}} \le \frac{C_0}{N} + 3I_1 + 3I_2 + I_3 \le \frac{8C_0}{N}.$$

Next, we check that T is a contraction mapping. Assume that $w_1, w_2 \in \mathcal{V}_N^\delta$, then arguing similarly as above, we have

$$\|Tw_1 - Tw_2\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}} \le C\|H_\gamma(v+w_1) - H_\gamma(v+w_2)\|_{L^{\sigma(p_0)}_{[0,\delta_N]}L^{\rho_0}}.$$

Note that

$$\begin{aligned} H_\gamma(v+w_1) - H_\gamma(v+w_2) =& H_\gamma(w_1-w_2, v, v) + H_\gamma(v, w_1-w_2, v) + H_\gamma(v, v, w_1-w_2) \\ &+ H_\gamma(w_1, w_1, v) + H_\gamma(w_1, v, w_1) + H_\gamma(v, w_1, w_1) \\ &- H_\gamma(w_2, w_2, v) - H_\gamma(w_2, v, w_2) - H_\gamma(v, w_2, w_2) \\ &+ H_\gamma(w_1) - H_\gamma(w_2), \end{aligned}$$

and

$$\begin{aligned} H_\gamma(v, w_1, w_1) - H_\gamma(v, w_2, w_2) =& H_\gamma(v, w_1, w_1) - H_\gamma(v, w_1, w_2) + H_\gamma(v, w_1, w_2) - H_\gamma(v, w_2, w_2) \\ =& H_\gamma(v, w_1, w_1-w_2) + H_\gamma(v, w_2, w_1-w_2) \\ =& H_\gamma(v, w_j, w_1-w_2), \qquad j \in \{1,2\}, \end{aligned}$$

$$\begin{aligned} H_\gamma(w_1) - H_\gamma(w_2) =& H_\gamma(w_1, w_1, w_1) - H_\gamma(w_1, w_1, w_2) + H_\gamma(w_1, w_1, w_2) - H_\gamma(w_1, w_2, w_2) \\ &+ H_\gamma(w_1, w_2, w_2) - H_\gamma(w_2, w_2, w_2) \\ =& H_\gamma(w_1, w_1, w_1-w_2) + H_\gamma(w_1, w_2, w_1-w_2) + H_\gamma(w_2, w_2, w_1-w_2) \\ =& H_\gamma(w_k, w_j, w_1-w_2), \qquad j,k \in \{1,2\}. \end{aligned}$$

Then it is enough to estimate the norm of the following terms by remark 3.1:

$$H_\gamma(v, v, w_1-w_2),\ H_\gamma(v, w_j, w_1-w_2),\ H_\gamma(w_j, w_k, w_1-w_2), \qquad j,k \in \{1,2\}$$

Arguing similarly as the above, we have

$$\begin{aligned} \|H_\gamma(v, v, w_1-w_2)\|_{L^{\sigma(p_0)}_{[0,\delta_N]}L^{\rho_0}} \le& C(\delta_N)^{1-\frac{\gamma}{4}}\|v\|^2_{L^{q(2)}_{[0,\delta_N]}L^{r_0}}\|w_1-w_2\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}} \\ \le& CM^{-2}K_1^2\|w_1-w_2\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}} = I_4, \end{aligned}$$

$$\begin{aligned} \|H_\gamma(v, w_j, w_1-w_2)\|_{L^{\sigma(p_0)}_{[0,\delta_N]}L^{\rho_0}} \le& C(\delta_N)^{1-\frac{\gamma}{4}-\frac{n}{4}(\frac{1}{p_0}-\frac{1}{2})}\|v\|_{L^{q(2)}_{[0,\delta_N]}L^{r_0}}\|w_j\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}}\|w_1-w_2\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}} \\ \le& CM^{-1}K_1\frac{C_0}{N}\|w_1-w_2\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}} \\ \le& CM^{-1}K_1C_0\|w_1-w_2\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}} = I_5, \end{aligned}$$

$$\begin{aligned} \|H_\gamma(w_j, w_k, w_1-w_2)\|_{L^{\sigma(p_0)}_{[0,\delta_N]}L^{\rho_0}} \le& C(\delta_N)^{1-\frac{\gamma}{4}-\frac{n}{2}(\frac{1}{p_0}-\frac{1}{2})}\|w_j\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}}\|w_k\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}}\|w_1-w_2\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}} \\ \le& CM^{-2+\frac{4n}{4-\gamma}(\frac{1}{p_0}-\frac{1}{2})}\frac{C_0^2}{N^2}\|w_1-w_2\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}} \\ \le& CM^{-2+\frac{4n}{4-\gamma}(\frac{1}{p_0}-\frac{1}{2})}C_0^2\|w_1-w_2\|_{L^{q(p_0)}_{[0,\delta_N]}L^{r_0}} = I_6. \end{aligned}$$

Then we will get

$$I_j \leq \frac{1}{12}\|w_1 - w_2\|_{L^{q(p_0)}_{[0,\delta_N]} L^{r_0}}, \qquad j = 4, 5, 6,$$

if M is sufficiently large. Collecting these estimates, we get

$$\|Tw_1 - Tw_2\|_{L^{q(p_0)}_{[0,\delta_N]} L^{r_0}} \leq 3I_4 + 6I_5 + 3I_6 \leq \|w_1 - w_2\|_{L^{q(p_0)}_{[0,\delta_N]} L^{r_0}}.$$

Thus, we get a unique local solution of the original Cauchy problem (1.1) of the form

$$\begin{aligned} u(t) =& v(t) + w(t) \\ =& v(t) + e^{it\Delta^2}\psi_N + i\int_0^t e^{i(t-\tau)\Delta^2}[H_\gamma(v+w) - H_\gamma(v)]d\tau \end{aligned} \tag{3.11}$$

on the time interval $[0,\delta_N]$.

3.2 Continuation of local solution

In this subsection, we try to extend the local solution to the time T_N. We first observe that the third term in the righthand side of (3.11) is in L^2. To see this, we use the inhomogenous Strichartz estimate for the biharmonic admissible pair to obtain

$$\sup_{t\in[0,\delta_N]} \left\| \int_0^t e^{i(t-\tau)\Delta^2}[H_\gamma(v+w) - H_\gamma(v)]d\tau \right\|_{L^2} \leq C\|H_\gamma(v+w) - H_\gamma(v)\|_{L^{\sigma(2)}_{[0,\delta_N]} L^{\rho_0}}.$$

Taking the same process as in the previous subsection, it is enough to consider the norm of three particular terms, and they are estimated as follows:

$$\begin{aligned} \|H_\gamma(v,v,w)\|_{L^{\sigma(2)}_{[0,\delta_N]} L^{\rho_0}} \leq& (\delta_N)^{1-\frac{\gamma}{4}-\frac{n}{4}(\frac{1}{p_0}-\frac{1}{2})}\|v\|^2_{L^{q(2)}_{[0,\delta_N]} L^{r_0}}\|w\|_{L^{q(p_0)}_{[0,\delta_N]} L^{r_0}} \\ \leq& CN^{-1+\frac{2n}{4-\gamma}(\frac{1}{p_0}-\frac{1}{2})\alpha}. \end{aligned} \tag{3.12}$$

$$\begin{aligned} \|H_\gamma(v,w,w)\|_{L^{\sigma(2)}_{[0,\delta_N]} L^{\rho_0}} \leq& (\delta_N)^{1-\frac{\gamma}{4}-\frac{n}{2}(\frac{1}{p_0}-\frac{1}{2})}\|v\|_{L^{q(2)}_{[0,\delta_N]} L^{r_0}}\|w\|^2_{L^{q(p_0)}_{[0,\delta_N]} L^{r_0}} \\ \leq& CN^{-2+\frac{4n}{4-\gamma}(\frac{1}{p_0}-\frac{1}{2})\alpha-\alpha}. \end{aligned} \tag{3.13}$$

$$\begin{aligned} \|H_\gamma(w)\|_{L^{\sigma(2)}_{[0,\delta_N]} L^{\rho_0}} \leq& (\delta_N)^{1-\frac{\gamma}{4}-\frac{3n}{4}(\frac{1}{p_0}-\frac{1}{2})}\|w\|^3_{L^{q(p_0)}_{[0,\delta_N]} L^{r_0}} \\ \leq& CN^{-3+\frac{6n}{4-\gamma}(\frac{1}{p_0}-\frac{1}{2})\alpha-2\alpha}, \end{aligned} \tag{3.14}$$

where we also have used the inequality

$$1 - \frac{\gamma}{4} - \frac{3n}{4}(\frac{1}{p_0} - \frac{1}{2}) > 0, \tag{3.15}$$

which follows from the assumption

$$p_0 > \frac{6n}{3n - 2\gamma + 8}.$$

Put $a = \frac{2n}{4-\gamma}(\frac{1}{p_0} - \frac{1}{2})$, then $a < 1$ by (3.10), and we have

$$\begin{aligned} -1 + \frac{2n}{4-\gamma}(\frac{1}{p_0} - \frac{1}{2})\alpha &= -1 + a\alpha, \\ -2 + \frac{4n}{4-\gamma}(\frac{1}{p_0} - \frac{1}{2})\alpha - \alpha &= -1 + a\alpha + a\alpha - 1 - \alpha, \\ -3 + \frac{6n}{4-\gamma}(\frac{1}{p_0} - \frac{1}{2})\alpha - 2\alpha &= -1 + a\alpha + 2(a\alpha - 1 - \alpha). \end{aligned}$$

Note that $a\alpha - 1 - \alpha = -1 - (1-a)\alpha < -1 < 0$, so when $N > 1$, we have

$$\sup_{t\in[0,\delta_N]} \left\| \int_0^t e^{i(t-\tau)\Delta^2}[H_\gamma(v+w) - H_\gamma(v)]d\tau \right\|_{L^2} \le CN^{-1+\frac{2n}{4-\gamma}(\frac{1}{p_0}-\frac{1}{2})\alpha}. \tag{3.16}$$

By (3.11) we write

$$u(\delta_N, x) = \varphi_N^1(x) + \psi_N^1(x)$$

with

$$\varphi_N^1(x) := v(\delta_N, x) + i\int_0^{\delta_N} e^{i(\delta_N-\tau)\Delta^2}[H_\gamma(v+w) - H_\gamma(w)]d\tau$$

and

$$\psi_N^1 := e^{i\delta_N\Delta^2}\psi_N.$$

Obviously, we have $\varphi_N^1(x) \in L^2$ and

$$\|e^{it\Delta^2}\psi_N^1\|_{L^{q(p_0)}L^{r_0}} = \|e^{i(t+\delta_N)\Delta^2}\psi_N\|_{L^{q(p_0)}L^{r_0}} \le \frac{C_0}{N}.$$

Therefore, if the estimate

$$(2C_0)^{-1}N^\alpha \le \|\varphi_N^1(x)\|_{L^2} \le 2C_0N^\alpha$$

is valid, that is to say $u(\delta_N) \in \mathcal{A}_{p_0,\alpha}$, then we obtain the solution of

$$\begin{cases} iu_t + \Delta^2 u + (|\cdot|^{-\gamma} * |u|^2)u = 0, & t > \delta_N, \\ u(x, \delta_N) = \varphi_N^1(x) + \psi_N^1, & x \in \mathbb{R}^n, \end{cases}$$

on $[\delta_N, 2\delta_N]$, repeating the calculation of subsection 3.1. In this way, we can construct a solution of (1.1) to the time $2\delta_N$, $3\delta_N$, $\cdots$, $k_0\delta_N$ inductively as

$$(2C_0)^{-1}N^\alpha \le \|\varphi_N^k(x)\|_{L^2} \le 2C_0N^\alpha, \qquad \forall k \le k_0, \tag{3.17}$$

where

$$\varphi_N^k(x) := v(k\delta_N, x) + i\int_o^{k\delta_N} e^{i(k\delta_N-\tau)\Delta^2}[H_\gamma(v+w) - H_\gamma(v)]d\tau$$

and

$$\psi_N^k := e^{ik\delta_N\Delta^2}\psi_N.$$

We seek the largest k_0 for which (3.17) holds. By (3.16) and the conservation law (3.6), if

$$\begin{cases} C_0N^\alpha + Ck_0N^{-1+\frac{2n}{4-\gamma}(\frac{1}{p_0}-\frac{1}{2})\alpha} \le 2C_0N^\alpha, \\ C_0^{-1}N^\alpha - Ck_0N^{-1+\frac{2n}{4-\gamma}(\frac{1}{p_0}-\frac{1}{2})\alpha} \ge (2C_0)^{-1}N^\alpha, \end{cases} \tag{3.18}$$

we obtain (3.17). Solving (3.18), we have

$$k_0 \le CN^{1+[1-\frac{2n}{4-\gamma}(\frac{1}{p_0}-\frac{1}{2})]\alpha}.$$

So, we can extend the solution of (1.1) to the time

$$T_N := ck_0\delta_N = cN^{1-\frac{4}{4-\gamma}[2-\frac{n}{2}(\frac{n-\gamma+4}{2n}-\frac{1}{p_0})]\alpha}, \tag{3.19}$$

where $c > 0$ is a constant independent of N. This proves Theorem (1.1) (i). The statement (ii) follows easily from (i). Indeed, (1.9) implies that the exponent of T in (3.19) is strictly positive. Thus, T_N also can be arbitrarily large as N can be taken arbitrarily large.

4 Proof of Theorem 1.2

We present sufficient conditions on the initial function u_0 under which Theorem 1.1 can be applied. Before giving our result, we state an useful interpolation lemma.

Lemma 4.1 (see [22]) *Suppose that $0 < p_1 < p_2 \leq \infty$, and let $p \in (p_1, p_2)$ be given by*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2},$$

for some $\theta \in (0,1)$, then

$$L^p \hookrightarrow L^{p_1} + L^{p_2}.$$

Moreover, for any $f \in L^p$ there are sequences of functions $(\varphi_t)_{t>0} \subset L^{p_1}$ and $(\psi_t)_{t>0} \subset L^{p_2}$ such that $f = \varphi_t + \psi_t$, $\forall t > 0$ and

$$ct^{-\theta} max(\|\varphi_t\|_{L^{p_1}}, t\|\psi_t\|_{L^{p_2}}) \leq \|f\|_{L^p},$$

where $c > 0$ is independent of t.

Proposition 4.1 *Suppose that p lies in the region*

$$\frac{4n}{2n+\gamma} < p < 2.$$

Then, for any p_0 satisfying

$$\frac{4n}{2n+\gamma} < p_0 < p, \tag{4.1}$$

we have the inclusion $L^p \backslash L^2 \subset \mathcal{A}_{p_0,\alpha}$ with

$$\alpha = \frac{1/p - 1/2}{1/p_0 - 1/p}.$$

Proof Let $\phi \in L^p \backslash L^2$. By lemma 4.1, for $\forall t > 0$, there exist $g_t \in L^2$ and $h_t \in L^{p_0}$ satisfying

$$\phi = g_t + h_t$$

and

$$ct^{-\theta} max(t\|g_t\|_{L^2}, \|h_t\|_{L^{p_0}}) \leq \|\phi\|_{L^p}, \tag{4.2}$$

where $c > 0$ is independent of t and $\theta \in (0,1)$ satisfies

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_0}.$$

Note that $\|g_t\|_{L^2} \to \infty$ as $t \to 0$, since $\phi \notin L^2$. Therefore, we can choose $t_N > 0$ for any $N > 1$ such that

$$\|g_{t_N}\|_{L^2} = N^{\alpha} = N^{\frac{1-\theta}{\theta}}.$$

Then, by (4.2) we get

$$\|h_{t_N}\|_{L^{p_0}} \leq ct_N^{\theta}\|\phi\|_{L^p} \qquad \textit{and} \qquad t_N^{1-\theta} \cdot N^{\frac{1-\theta}{\theta}} \leq \|\phi\|_{L^p}.$$

From these inequalities, we get $\|h_{t_N}\|_{L^{p_0}} \leq c\|\phi\|_{L^p}^{\frac{1}{1-\theta}} N^{-1}$. Now, let us define

$$\varphi_N = g_{t_N},\ \psi_N = h_{t_N}.$$

Then, we see that (3.1) is fulfilled with $C_0 = max(1, cC\|\phi\|_{L^p}^{\frac{1}{1-\theta}})$, where C is the constant of (2.1). Furthermore, (3.2) is also satisfied by the generalized Strichartz inequality lemma 2.1 if we assume (4.1). □

Then the Theorem 1.2 is immediate consequences of Theorem 1.1 and Proposition 4.1.

Remark 4.1 By Theorem 1.1 (ii), T_N can get arbitrarily large if

$$\alpha = \frac{1/p - 1/2}{1/p_0 - 1/p} < \{\frac{4}{4-\gamma}[2 - \frac{n}{2}(\frac{n+4-\gamma}{2n} - \frac{1}{p_0})]\}^{-1},$$

by solving this inequality after the formally letting

$$max(\frac{6n}{3n-2\gamma+8}, \frac{4n}{2n+\gamma}) < p_0 < 2,$$

we see that p lies in the range in (1.11), and the condition (1.11) isn't an empty set with (1.12), because

$$-\gamma^2 + (n+4)\gamma + 16n = n(16+\gamma) + \gamma(4-\gamma),$$
$$\gamma^2 - (n+8)\gamma + 16n + 16 = n(16-\gamma) + (4-\gamma)^2.$$

So it is obviously that $\frac{2n(16+\gamma)}{-\gamma^2+(n+4)\gamma+16n} < 2$ and $\frac{2n(16-\gamma)}{\gamma^2-(n+8)\gamma+16n+16} < 2$. Inaddtion, we can see

$$\frac{2n(16-\gamma)}{\gamma^2 - (n+8)\gamma + 16n + 16} > \frac{6n}{3n-2\gamma+8},$$
$$\frac{2n(16+\gamma)}{-\gamma^2 + (n+4)\gamma + 16n} > \frac{4n}{2n+\gamma} > 1.$$

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Data availability The data that support the findings of this study are available within the article.

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