Stabilization of a coupled wave equations with one localized non-regular fractional Kelvin-Voigt damping with non-smooth coefficients

Li Zhang, Wenjun Liu*, Yanning An and Xinxin Cao

School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

Abstract

In this paper, we study the stabilization of a coupled wave system formed by one localized non-regular fractional viscoelastic damping of Kelvin-Voigt type and localized non-smooth coefficients. Our main aim is to prove that the C_0-semigroup associated with this model is strong stability and decays polynomially at a rate of t^{-1}. By introducing a new system to deal with fractional Kelvin-Voigt damping, we obtain a new equivalent augmented system, so as to show the well-posedness of the system based on Lumer-Phillips theorem. We achieve the strong stability for the C_0-semigroup associated with this new model by using a general criteria of Arendt-Batty, and then turn out a polynomial energy decay rate of order t^{-1} with the help of a frequency domain approach.

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1 Introduction

Coupled systems have many applications in manipulation and modeling of structural engineering, such as automotive, spacecraft, turbines, satellites and road traffic (see [18]). A lot of work in coupled systems is to consider the stability of the system under different damping positions, different damping types or different couplers. Due to the wide application of smart materials in modern science and technology, there are more and more studies on viscoelastic damped elastic systems (see [12, 24, 25, 35, 37]). Indeed, when smart materials are applied to elastic structures, their damping coefficient, mass density and Young's modulus will change accordingly. Viscoelastic damping generally comes in two types. One is Kelvin-Voigt damping and the other is Botzmann damping (see [28, 29]). And the Kelvin-Voigt damping is often called internal damping because it is caused by the internal friction of the vibrating structural material. In addition, there are many studies on local and global damping, which we recommend to readers [11, 13, 22, 23, 30].

In recent years, the study of wave equations with different damping types has attracted extensive attention. In [40], Tcheugoue Tebou proved some decay estimates for wave equations with a

*Corresponding author. Email address: wjliu@nuist.edu.cn (W. J. Liu).

nonlinear damping term localized in a neighborhood of a suitable subset of the boundary. In [21], Cavalcati and Martinez studied the existence and uniform decay rate of solutions of nonlinear damped wave equations acting on the boundary. In [42], Zhang investigated the elastic wave equation with local Kelvin-Voigt damping, and analyzed how the dissipation mechanism introduced by local Kelvin-Voigt damping affects the long-term energy behavior of the system, then demonstrated the exponential stability of the system when the coefficient function near the interface is sufficiently smooth. Alabau et al. [8] investigated the coupled waves with partial frictional damping, that is

$$\begin{cases} u_{tt} - \Delta u + \alpha v = 0, & (x,t) \in \Omega \times \mathbb{R}^+, \\ v_{tt} - \Delta v + \alpha u + \beta v_t = 0, & (x,t) \in \Omega \times \mathbb{R}^+, \end{cases} \tag{1.1}$$

which is subjected to Dirichlet boundary conditions. For system (1.1), the energy is not exponentially stable, but decays at a polynomial rate of $t^{-1/2}$. For optimality of the polynomial decay rate of the system, Lobato et al. proved in [32]. Moreover, Oquendo and Pacheco [36] considered a system consisting of two coupled waves in which the partial frictional damping in system (1.1) is substituted by the partial Kelvin-Voigt damping, the system is described by

$$\begin{cases} u_{tt} - \Delta u + \alpha v = 0, & (x,t) \in \Omega \times \mathbb{R}^+, \\ v_{tt} - \Delta v + \alpha u - \beta \Delta v_t = 0, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u = 0, \quad v = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+. \end{cases} \tag{1.2}$$

For system (1.2), although the Kelvin-Voigt damping is stronger than the frictional one, they proved that it decays at a slower polynomial rate of $t^{-1/4}$. They also demonstrated that the decay rate is optimal.

For the coupled viscoelastic model, which is driven by an abundance of physical factors and has been widely used in engineering and mechanics. It has received great attention in recent years. Next we introduce the case where the viscoelastic wave equations is coupled by velocity terms. In [26], Hassine and Souayeh investigated the behavior of coupled wave system with partial Kelvin-Voigt damping. They mainly consider the following system:

$$\begin{cases} u_{tt} - [u_x + a(x)u_{xt}]_x + v_t = 0, & (x,t) \in (-1,1) \times \mathbb{R}^+, \\ v_{tt} - cv_{xx} - u_t = 0, & (x,t) \in (-1,1) \times \mathbb{R}^+, \\ u(1,t) = v(1,t) = 0, \quad u(-1,t) = v(-1,t) = 0, & t \in \mathbb{R}^+, \\ (u, u_t, v, v_t)(x,0) = (u_0, u_1, v_0, v_1)(x), & x \in (-1,1), \end{cases} \tag{1.3}$$

where $c > 0$ and $a \in L^\infty(-1,1)$ is a non-negative function. They supposed that the damping coefficient is a piece-wise function satisfies the form $a = d.\mathbf{1}_{[0,1]}$, where d is a strictly positive constant. Thanks to the Kelvin-Voigt damping $(a(x)u_{xt})_x$ is singular, system (1.3) can be regarded as a coupling of the transmitted wave equation with the conservative wave equation. And they established that the system is lack of the exponential stability and polynomially stable with a slower rate of type $t^{-1/12}$. Moreover, Wehbe et al. in [41] investigated the stability of the Kelvin-Voigt type locally coupled wave equations with only one internal viscoelastic damping through non-smooth coefficients. The system is described as

$$\begin{cases} u_{tt} - (au_x + b(x)u_{xt})_x + c(x)y_t = 0, & (x,t) \in (0,L) \times \mathbb{R}^+, \\ y_{tt} - y_{xx} - c(x)u_t = 0, & (x,t) \in (0,L) \times \mathbb{R}^+, \\ u(0,t) = u(L,t) = y(0,t) = y(L,t) = 0, & t \in \mathbb{R}^+, \\ (u, u_t, v, v_t)(x,0) = (u_0, u_1, v_0, v_1)(x), & x \in (0,L), \end{cases} \tag{1.4}$$

where

$$b(x) = \begin{cases} b_0, & x \in (\alpha_1, \alpha_3), \\ 0, & otherwise, \end{cases} \quad and \quad c(x) = \begin{cases} c_0, & x \in (\alpha_2, \alpha_4), \\ 0, & otherwise, \end{cases} \tag{1.5}$$

and $a > 0, b_0 > 0$ and $c_0 \in \mathbb{R}^*$, and they considered $0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < L$. They demonstrated the system is polynomial stability. In addition, Akil et al. in [2] investigated the stability of coupled wave equations with non-smooth localized viscoelastic damping of Kelvin-Voigt type and localized time delay and proved the polynomial stability of the system.

Over the past several years, another type of damping that has been widely used is fractional damping. Fractional calculus has been successfully applied in a variety of fields and modified many existing models of physical processes, such as heat conduction, wave propagation, diffusion, viscoelasticity and electronics, see [15, 16, 39] and abundant references therein. Caputo and Mainardi established the relationship between fractional derivative and theory of viscoelasticity in [20]. Besides, the application of fractional computation in modeling can improve the capture of complex dynamics of natural systems, and fractional-order control can achieve the performance that could not be achieved in integer-order control. Readers can refer to [27, 33, 38] for further application of fractional calculus.

Fractional calculus contains various expansions of the general definition of integral derivative to real derivative, including Caputo derivative, Riemann-Liouville derivative and Riesz derivative, etc. For the Caputo derivative, there are many applications in different equations. For example, in [1] Achouri et al. considered the Euler-Bernoulli beam equation with a boundary damping of fractional derivative type and studied the polynomial stability of the system using the semigroup theory of linear operators. In [17], Benaissa and Benkhedda considered the wave equation with a dynamic boundary control condition of fractional derivative type and demonstrated the energy of the system is polynomially stable. In [3], Akil et al. investigated the stability results of hyperbolic systems of wave-wave, wave-Euler-Bernoulli beam and beam-beam types. Two major non-smooth local fractional Kelvin-Voigt damping models are coupled by boundary connections. And they established different types of polynomial energy decay rate of these systems.

In our paper, we investigate the stabilization of a system of localized coupled wave equations, which is based on system (1.4). The coupling is via non-smooth coefficients with only one localized non-regular fractional Kelvin-Voigt damping. The system is described by

$$\begin{cases} u_{tt} - (au_x + b(x)\partial_t^{\alpha,\eta} u_x)_x + c(x)y_t = 0, & (x,t) \in (0,L) \times \mathbb{R}^+, \\ y_{tt} - y_{xx} - c(x)u_t = 0, & (x,t) \in (0,L) \times \mathbb{R}^+, \\ u(0,t) = u(L,t) = y(0,t) = y(L,t) = 0, & t \in \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in (0,L), \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), & x \in (0,L), \end{cases} \tag{1.6}$$

where

$$b(x) = \begin{cases} b_0, & x \in (0,\beta_2), \\ 0, & x \in (\beta_2, L), \end{cases} \quad \textit{and} \quad c(x) = \begin{cases} c_0, & x \in (\beta_1,\beta_3), \\ 0, & x \in (0,\beta_1) \cup (\beta_3, L), \end{cases} \tag{1.7}$$

and $a > 0, b_0 > 0$ and $c_0 \in \mathbb{R}^*$, and they considered $0 < \beta_1 < \beta_2 < \beta_3 < L$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order $\alpha \in (0,1)$ is regard to the time variable t and is defined as

$$\left[D^{\alpha,\eta} w\right](t) = \partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)ds, \ \eta \geq 0, \tag{1.8}$$

where Γ is the usual Euler Gamma function. The fractional differentiation $D^{\alpha,\eta}$ is the inverse operation of fractional integration defined by

$$\left[I^{\alpha,\eta} w\right](t) = \int_0^t \frac{(t-s)^{\alpha-1} e^{-\eta(t-s)}}{\Gamma(\alpha)} w(s)ds. \tag{1.9}$$

From (1.8) and (1.9), we know that

$$\left[D^{\alpha,\eta} w\right] = I^{1-\alpha,\eta} Dw. \tag{1.10}$$

The system consists of two coupled wave equations. The Kelvin-Voigt singular local viscoelastic damping is applied to the first equation, and the lack of feedback on the second equation is compensated by the coupling effects. Only one equation in this coupled system is damped, because when coupled systems are involved some undamped equations, they are usually considered to be indirectly damped. Such "indirect" stability problems are studied in [4, 5, 8] and further investigated in [6, 7, 9]. These authors used different methods to investigate whether a single damping term is sufficient to ensure that the energy of the entire system decays to zero at infinity, and to confirm at what rate. In addition, some researchers also noted that in the indirectly damped system of Kelvin-Voigt, the coupling terms would affect the stability of the system. In [10], the author compared his proof results with [8] and showed that different forms of coupling terms (velocity coupling and displacement coupling) lead to different stability of the system.

In this paper, we will further analyze whether the rate of decay is determined by the damping term or the coupling terms in the case of piecewise damping and piecewise velocity couplings. Through calculation, it is found that by changing the damping term of the system to the fractional order, the same stability result as integer order damping is obtained, that is, the system decays at t^{-1}. This shows that the velocity coupling terms have a great influence on the stability of the system. Meanwhile, in the process of proof, we also find that the reason why the rate of decay of the system can reach t^{-1} is indeed caused by the estimation of the coupling terms u_t and y_t.

The organization of this paper is as follows. In Section 2, in order to deal with the fractional damping term, we reformulate system (1.6)-(1.7) into an equivalent augmented system. Based on semigroup approach, we analyze the well-posedness of the system. In Section 3, we investigate the strong stability of system (1.6)-(1.7) in the absence of the compactness of the resolvent through a general criteria of Arendt and Batty. In Section 4, we establish a polynomial decay for solution of type t^{-1}, which based on a frequency domain approach combined with multiplier technique.

2 The augmented model and well-posedness

In this subsection, we analyze the well-posedness of system (1.6)-(1.7) by introducing a new function to transform the system into an augmented model. For this purpose, we will introduce the following claims, which will be used hereinafter.

Theorem 2.1 *(see [34]) Let μ be the function defined by*

$$\mu(\xi) = |\xi|^{\frac{2\alpha-1}{2}}, \qquad \xi \in \mathbb{R},\ \alpha \in (0,1),$$

and $\eta \geq 0$. Then the relation between the "input" V and the "output" O of the following system:

$$\partial_t w(x,\xi,t) + (\xi^2+\eta)w(x,\xi,t) - V(x,t)\mu(\xi) = 0,\ (x,\xi,t) \in (0,L)\times\mathbb{R}\times\mathbb{R}^+, \tag{2.1}$$

$$w(x,\xi,0) = 0,\ (x,\xi) \in (0,L)\times\mathbb{R}, \tag{2.2}$$

$$O(x,t) - k(\alpha)\int_{\mathbb{R}} \mu(\xi)w(x,\xi,t)d\xi = 0,\ (x,t) \in (0,L)\times\mathbb{R}^+, \tag{2.3}$$

is given by

$$O(x,t) = I^{1-\alpha,\eta}V(x,t), \tag{2.4}$$

where $k(\alpha) = \frac{\sin(\alpha\pi)}{\pi}$ and $I^{1-\alpha,\eta}$ is defined as (1.9).

For the above theorem, we take $V(x,t) = \sqrt{b(x)}u_{xt}(x,t)$. Based on (1.8) and (1.10), it is easy to get

$$[D^{\alpha,\eta}w] = \partial_t^{\alpha,\eta}w = I^{1-\alpha,\eta}Dw.$$

Thus, according to Theorem 2.1, we can conclude that the output O is described by

$$O(x,t) = \sqrt{b(x)}I^{1-\alpha,\eta}u_{xt}(x,t) = \frac{\sqrt{b(x)}}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}\partial_s u_x(x,s)ds = \sqrt{b(x)}\partial_t^{\alpha,\eta}u_x(x,t).$$

Therefore, substituting $V(x,t) = \sqrt{b(x)}u_{xt}(x,t)$ and $O(x,t) = \sqrt{b(x)}\partial_t^{\alpha,\eta}u_x(x,t)$ into system (2.1)-(2.3), we can derive

$$\begin{aligned} \partial_t w(x,\xi,t) + (\xi^2+\eta)w(x,\xi,t) - \sqrt{b(x)}u_{xt}(x,t)|\xi|^{\frac{2\alpha-1}{2}} = 0,\ (x,\xi,t) \in (0,L)\times\mathbb{R}\times\mathbb{R}^+,& \\ w(x,\xi,0) = 0,\ (x,\xi) \in (0,L)\times\mathbb{R},& \\ \sqrt{b(x)}\partial_t^{\alpha,\eta}u_x(x,t) - k(\alpha)\int_{\mathbb{R}}|\xi|^{\frac{2\alpha-1}{2}}w(x,\xi,t)d\xi = 0,\ (x,t) \in (0,L)\times\mathbb{R}^+.& \end{aligned} \tag{2.5}$$

Based on system (2.5), we derive system (1.6) into the augmented model

$$u_{tt} - \left(au_x + \sqrt{b(x)}k(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} w(x,\xi,t) d\xi \right)_x + c(x) y_t = 0,\ (x,t) \in (0,L) \times \mathbb{R}^+, \tag{2.6}$$

$$y_{tt} - y_{xx} - c(x) u_t = 0,\ (x,t) \in (0,L) \times \mathbb{R}^+, \tag{2.7}$$

$$w_t(x,\xi,t) + (\xi^2 + \eta) w(x,\xi,t) - \sqrt{b(x)} u_{xt}(x,t) |\xi|^{\frac{2\alpha-1}{2}} = 0,\ (x,\xi,t) \in (0,L) \times \mathbb{R} \times \mathbb{R}^+, \tag{2.8}$$

with the boundary conditions

$$u(0,t) = u(L,t) = y(0,t) = y(L,t) = 0, \quad t \in \mathbb{R}^+, \tag{2.9}$$

and the initial conditions

$$\begin{matrix} u(x,0) = u_0(x), & u_t(x,0) = u_1(x), & w(x,\xi,0) = 0, \\ y(x,0) = y_0(x), & y_t(x,0) = y_1(x), & \end{matrix} \quad x \in (0,L),\ \xi \in \mathbb{R}, \tag{2.10}$$

where $b(x)$ and $c(x)$ are given by (1.7).

Now, let us start to define the Hilbert energy space

$$\mathcal{H} = \left\{ (u,v,y,z,w) \in H_0^1(0,L) \times L^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \times W \right\},$$

where $W = L^2((0,L) \times \mathbb{R})$. The Hilbert space $\mathcal{H}$ is equipped with the inner product

$$\langle U, U_1 \rangle_{\mathcal{H}} = \int_0^L \left(v\overline{v}_1 + au_x \overline{u}_{1,x} + z\overline{z}_1 + y_x \overline{y}_{1,x} \right) dx + k(\alpha) \int_0^L \int_{\mathbb{R}} w(x,\xi) \overline{w}_1(x,\xi) d\xi dx, \tag{2.11}$$

for all $U = (u,v,y,z,w)$ and $U_1 = (u_1, v_1, y_1, z_1, w_1)$ in $\mathcal{H}$. Then the corresponding norm is represented by

$$\|U\|_{\mathcal{H}}^2 = \int_0^L \left(|v|^2 + a|u_x|^2 + |z|^2 + |y_x|^2 \right) dx + k(\alpha) \int_0^L \int_{\mathbb{R}} |w(x,\xi)|^2 d\xi dx.$$

Moreover, the energy of system (2.6)-(2.10) is established by

$$E(t) = \frac{1}{2} \int_0^L \left(|u_t|^2 + a|u_x|^2 + |y_t|^2 + |y_x|^2 \right) dx + \frac{1}{2} k(\alpha) \int_0^L \int_{\mathbb{R}} |w(x,\xi,t)|^2 d\xi dx. \tag{2.12}$$

Multiplying (2.6), (2.7) by $\overline{u}_t$, $\overline{y}_t$ respectively, integrating by parts on $(0,L)$, and multiplying (2.8) by $k(\alpha)\overline{w}$, integrating on $(0,L) \times \mathbb{R}$, then taking the real part respectively, we obtain

$$\frac{d}{dt} E(t) = -k(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |w(x,\xi,t)|^2 d\xi dx. \tag{2.13}$$

Since $\alpha \in (0,1)$, it follows that $k(\alpha) > 0$, then $\frac{d}{dt} E(t) \le 0$. Consequently, system (2.6)-(2.10) is dissipative.

If $U = (u,v,y,z,w)$ is a solution of system (2.6)-(2.10), we can transform the system into the first-order equation

$$\begin{cases} U_t = \mathcal{A} U, \\ U(0) = U_0, \end{cases} \tag{2.14}$$

where $U_0 = (u_0, u_1, y_0, y_1, 0)$ and the unbounded linear operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{bmatrix} u \\ v \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} v \\ \left(au_x + \sqrt{b(x)}k(\alpha)\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} w(x,\xi)d\xi\right)_x - c(x)z \\ z \\ y_{xx} + c(x)v \\ -(\xi^2+\eta)w(x,\xi) + \sqrt{b(x)}v_x|\xi|^{\frac{2\alpha-1}{2}} \end{bmatrix},$$

where the domain of $\mathcal{A}$ is described by

$$\begin{aligned} \mathcal{D}(\mathcal{A}) := \Big\{ & U = (u,v,y,z,w) \in \mathcal{H};\ y \in H^2(0,L) \cap H_0^1(0,L), \quad v, z \in H_0^1(0,L), \\ & \left(au_x + \sqrt{b(x)}k(\alpha)\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} w(x,\xi)d\xi\right)_x \in L^2(0,L),\ |\xi| w(x,\xi) \in W, \\ & -(\xi^2+\eta)w(x,\xi) + \sqrt{b(x)}v_x|\xi|^{\frac{2\alpha-1}{2}} \in W \Big\}. \end{aligned} \tag{2.15}$$

Lemma 2.1 *(see [3]) Let $\alpha \in (0,1)$, $\eta \geq 0$, one has*

$$I_1(\eta,\alpha) = k(\alpha)\int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{1+\xi^2+\eta}d\xi, \quad I_2(\eta,\alpha) = \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{(1+\xi^2+\eta)^2}d\xi$$

$$\text{and} \quad I_3(\eta,\alpha) = k(\alpha)\int_0^{+\infty} \frac{|\xi|^{2\alpha+1}}{(1+\xi^2+\eta)^2}d\xi$$

are well defined.

Proposition 2.1 *The unbounded linear operator $\mathcal{A}$ is m-dissipative in the Hilbert space $\mathcal{H}$.*

Proof. For all $U = (u,v,y,z,w) \in \mathcal{D}(\mathcal{A})$, it is easy to obtain

$$\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -k(\alpha)\int_0^L \int_{\mathbb{R}} (\xi^2+\eta)|w(x,\xi)|^2 d\xi dx \leq 0, \tag{2.16}$$

which states that $\mathcal{A}$ is dissipative. We then prove that $I - \mathcal{A}$ maps $\mathcal{D}(\mathcal{A})$ onto $\mathcal{H}$. Consequently, let $F = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$, there exists $U = (u,v,y,z,w) \in \mathcal{D}(\mathcal{A})$ solution of

$$(I - \mathcal{A})U = F.$$

Equivalently, one has

$$u - v = f_1, \tag{2.17}$$

$$v - \left(au_x + \sqrt{b(x)}k(\alpha)\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} w(x,\xi)d\xi\right)_x + c(x)z = f_2, \tag{2.18}$$

$$y - z = f_3, \tag{2.19}$$

$$z - y_{xx} - c(x)v = f_4, \tag{2.20}$$

$$(1+\xi^2+\eta)w(x,\xi) - \sqrt{b(x)}v_x|\xi|^{\frac{2\alpha-1}{2}} = f_5(x,\xi). \tag{2.21}$$

By applying (2.17), (2.21) and the fact that $\eta \geq 0$, we obtain

$$w(x,\xi) = \frac{f_5(x,\xi)}{1+\xi^2+\eta} + \frac{\sqrt{b(x)}u_x|\xi|^{\frac{2\alpha-1}{2}}}{1+\xi^2+\eta} - \frac{\sqrt{b(x)}f_{1,x}|\xi|^{\frac{2\alpha-1}{2}}}{1+\xi^2+\eta}. \tag{2.22}$$

Substituting (2.22) into (2.18) and (2.17), (2.19) into (2.18), (2.20), that is

$$u - \left(au_x + b(x)I_1(\eta,\alpha)u_x - b(x)I_1(\eta,\alpha)f_{1,x} + \sqrt{b(x)}k(\alpha)\int_{\mathbb{R}} \frac{|\xi|^{\frac{2\alpha-1}{2}} f_5(x,\xi)}{1+\xi^2+\eta} d\xi \right)_x + c(x)y$$
$$= f_1 + f_2 + c(x)f_3, \tag{2.23}$$
$$y - y_{xx} - c(x)u = f_3 + f_4 - c(x)f_1, \tag{2.24}$$

where $I_1(\eta,\alpha)$ is well defined in Lemma 2.1, and with the boundary conditions

$$u(0) = u(L) = 0 \quad and \quad y(0) = y(L) = 0. \tag{2.25}$$

Let $(\varphi,\psi) \in H_0^1(0,L) \times H_0^1(0,L)$, we then multiply (2.23) and (2.24) by $\overline{\varphi}$ and $\overline{\psi}$ respectively and integrate by parts on $(0,L)$ to obtain

$$\int_0^L u\overline{\varphi}dx + \int_0^L au_x\overline{\varphi}_x dx + \int_0^L c(x)y\overline{\varphi}dx + I_1(\eta,\alpha)\int_0^L b(x)u_x\overline{\varphi}_x dx = \int_0^L (f_1+f_2+c(x)f_3)\overline{\varphi}dx$$
$$+ I_1(\eta,\alpha)\int_0^L b(x)f_{1,x}\overline{\varphi}_x dx - k(\alpha)\int_0^L \sqrt{b(x)}\overline{\varphi}_x \left(\int_{\mathbb{R}} \frac{|\xi|^{\frac{2\alpha-1}{2}} f_5(x,\xi)}{1+\xi^2+\eta} d\xi\right) dx, \tag{2.26}$$

$$\int_0^L y\overline{\psi}dx + \int_0^L y_x\overline{\psi}_x dx - \int_0^L c(x)u\overline{\psi}dx = \int_0^L (f_3+f_4-c(x)f_1)\overline{\psi}dx. \tag{2.27}$$

Combining (2.26) and (2.27), we have

$$a((u,y),(\varphi,\psi)) = L(\varphi,\psi), \quad \forall (\varphi,\psi) \in H_0^1(0,L) \times H_0^1(0,L), \tag{2.28}$$

where

$$a((u,y),(\varphi,\psi)) = \int_0^L u\overline{\varphi}dx + \int_0^L au_x\overline{\varphi}_x dx + \int_0^L c(x)y\overline{\varphi}dx + I_1(\eta,\alpha)\int_0^L b(x)u_x\overline{\varphi}_x dx$$
$$+ \int_0^L y\overline{\psi}dx + \int_0^L y_x\overline{\psi}_x dx - \int_0^L c(x)u\overline{\psi}dx$$

and

$$L(\varphi,\psi) = \int_0^L (f_1+f_2+c(x)f_3)\overline{\varphi}dx - k(\alpha)\int_0^L \sqrt{b(x)}\overline{\varphi}_x \left(\int_{\mathbb{R}} \frac{|\xi|^{\frac{2\alpha-1}{2}} f_5(x,\xi)}{1+\xi^2+\eta} d\xi\right) dx$$
$$+ I_1(\eta,\alpha)\int_0^L b(x)f_{1,x}\overline{\varphi}_x dx + \int_0^L (f_3+f_4-c(x)f_1)\overline{\psi}dx.$$

Since $I_1(\eta,\alpha)$ is well defined and the fact that $I_1(\eta,\alpha) > 0$, we can deduce that a is a sesquilinear, continuous coercive form on $(H_0^1(0,L) \times H_0^1(0,L))^2$. Furthermore, based on the definition of $b(x)$ and the Cauchy-Schwartz inequality, we derive

$$\left| \int_0^L \sqrt{b(x)}\overline{\varphi}_x \left(\int_{\mathbb{R}} \frac{|\xi|^{\frac{2\alpha-1}{2}} f_5(x,\xi)}{1+\xi^2+\eta} d\xi\right) dx \right| \leq \sqrt{b_0}\sqrt{I_2(\eta,\alpha)}\|\varphi_x\|_{L^2(0,\beta_2)}\|f_5(x,\xi)\|_W, \tag{2.29}$$

where $I_2(\eta,\alpha)$ is well defined. Consequently, it is easy to check that L is an antilinear continuous form on $H_0^1(0,L)\times H_0^1(0,L)$. Therefore, by applying Lax-Milgram theorem, we infer that for all $(\varphi,\psi)\in H_0^1(0,L)\times H_0^1(0,L)$, problem (2.28) admits a unique solution $(u,y)\in H_0^1(0,L)\times H_0^1(0,L)$. By using the classical elliptic regularity, it follows that $y\in H^2(0,L)\cap H_0^1(0,L)$ and

$$\left(au_x+\sqrt{b(x)}k(\alpha)\int_{\mathbb{R}}|\xi|^{\frac{2\alpha-1}{2}}w(x,\xi)d\xi\right)_x\in L^2(0,L).$$

From (2.17) and (2.19), it is clear to see that $v,z\in H_0^1(0,L)$.

In order to demonstrate the existence of U in $\mathcal{D}(\mathcal{A})$, we need to show that $w(x,\xi)$, $|\xi|w(x,\xi)$ in W. Applying (2.22), one has

$$\int_0^L\int_{\mathbb{R}}|w(x,\xi)|^2d\xi dx\le C\int_0^L\int_{\mathbb{R}}\frac{|f_5(x,\xi)|^2}{(1+\xi^2+\eta)^2}d\xi dx+Cb_0I_2(\eta,\alpha)\int_0^{\beta_2}(|u_x|^2+|f_{1,x}|^2)dx.$$

From Lemma 2.1 and the fact that $(u,f_1)\in H_0^1(0,L)\times H_0^1(0,L)$, it follows that

$$I_2(\eta,\alpha)\int_0^{\beta_2}(|u_x|^2+|f_{1,x}|^2)dx<+\infty.$$

Applying the fact that $f_5(x,\xi)\in W$, we obtain

$$\int_0^L\int_{\mathbb{R}}\frac{|f_5(x,\xi)|^2}{(1+\xi^2+\eta)^2}d\xi dx\le\frac{1}{(1+\eta)^2}\int_0^L\int_{\mathbb{R}}|f_5(x,\xi)|^2d\xi dx<+\infty.$$

Hence, we can conclude that $w(x,\xi)\in W$. Therefore, according to (2.22) one can attain

$$\int_0^L\int_{\mathbb{R}}|\xi w(x,\xi)|^2d\xi dx\le C\int_0^L\int_{\mathbb{R}}\frac{\xi^2|f_5(x,\xi)|^2}{(1+\xi^2+\eta)^2}d\xi dx+Cb_0I_3(\eta,\alpha)\int_0^{\beta_2}(|u_x|^2+|f_{1,x}|^2)dx.$$

In the same manner, we can deduce that

$$I_3(\eta,\alpha)\int_0^{\beta_2}(|u_x|^2+|f_{1,x}|^2)dx<+\infty.$$

Since $f_5(x,\xi)\in W$ and $\max_{\xi\in\mathbb{R}}\dfrac{\xi^2}{(1+\xi^2+\eta)^2}=\dfrac{1}{4(1+\eta)}<\dfrac{1}{4}$, it follows that

$$\begin{aligned}\int_0^L\int_{\mathbb{R}}\frac{\xi^2|f_5(x,\xi)|^2}{(1+\xi^2+\eta)^2}d\xi dx&\le\max_{\xi\in\mathbb{R}}\frac{\xi^2}{(1+\xi^2+\eta)^2}\int_0^L\int_{\mathbb{R}}|f_5(x,\xi)|^2d\xi dx\\&<\frac{1}{4}\int_0^L\int_{\mathbb{R}}|f_5(x,\xi)|^2d\xi dx<+\infty.\end{aligned}$$

So we infer that $|\xi|w(x,\xi)\in W$. Therefore, we have

$$-(\xi^2+\eta)w(x,\xi)+\sqrt{b(x)}v_x|\xi|^{\frac{2\alpha-1}{2}}=w(x,\xi)-f_5(x,\xi)\in W.$$

Consequently, there exists $U=(u,v,y,z,w)\in\mathcal{D}(\mathcal{A})$ unique solution of $(I-\mathcal{A})U=F$. The proof is completed. $\square$

From Proposition 2.1, we deduce that operator $\mathcal{A}$ is m-dissipative on the Hilbert space $\mathcal{H}$. Therefore, it is the infinitesimal generator of a C_0-semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ based on Lumer-Phillips theorem (see [31,34]). Hence, the solution of (2.14) can be denoted by

$$U(t) = e^{t\mathcal{A}}U_0,\ t \geq 0,$$

which states the well-posedness of (2.14). Consequently, we can obtain the following assertion:

Theorem 2.2 *Let $U_0 \in \mathcal{H}$, problem (2.14) exists a unique solution satisfies*

$$U(t) \in C^0(\mathbb{R}^+;\mathcal{H}).$$

Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, problem (2.14) exists a unique solution satisfies

$$U(t) \in C^1(\mathbb{R}^+;\mathcal{H}) \cap C^0(\mathbb{R}^+;\mathcal{D}(\mathcal{A})).$$

3 Strong stability

In this subsection, we will show the strong stability of the C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$. First of all, we introduce the theorem of Arendt and Batty, which will be used hereinafter.

Theorem 3.1 *(see [14]) Assume that $\mathcal{A}$ is the generator of a C_0-semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If*
1. $\mathcal{A}$ has no pure imaginary eigenvalues,
2. $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable,
where $\sigma(\mathcal{A})$ represents the spectrum of $\mathcal{A}$, then the C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable.

Based on the above theorem, the main result of this part is the following theorem:

Theorem 3.2 *Suppose that $\eta \geq 0$, then the C_0-semigroup of contractions $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly stable on the Hilbert space $\mathcal{H}$, that is, for all $U_0 \in \mathcal{H}$, the solution of (2.14) satisfies*

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

According to Theorem 3.1, to complete the proof of Theorem 3.2, we need to prove that the operator $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable. For this purpose, we give the subsequent lemmas.

Lemma 3.1 *(see [3]) Let $\alpha \in (0,1)$, $\eta \geq 0$, $\lambda \in \mathbb{R}$. For ($\eta > 0$ and $\lambda \in \mathbb{R}$) or ($\eta = 0$ and $\lambda \in \mathbb{R}^*$), we obtain*

$$\begin{aligned}
&I_4(\lambda,\eta,\alpha) = i\lambda k(\alpha)\int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{i\lambda+\xi^2+\eta}d\xi, \quad I_5(\lambda,\eta,\alpha) = k(\alpha)\int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{i\lambda+\xi^2+\eta}d\xi,\\
&I_6(x,\lambda,\eta,\alpha) = k(\alpha)\int_{\mathbb{R}} \frac{|\xi|^{\frac{2\alpha-1}{2}}f_5(x,\xi)}{i\lambda+\xi^2+\eta}d\xi, \quad I_7(\lambda,\eta,\alpha) = k(\alpha)\int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{\lambda^2+(\xi^2+\eta)^2}d\xi,\\
&I_8(\lambda,\eta,\alpha) = k(\alpha)\int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}(\xi^2+\eta)}{\lambda^2+(\xi^2+\eta)^2}d\xi \quad \text{and} \quad I_9(\lambda,\eta,\alpha) = \int_{\mathbb{R}} \frac{|\xi|^{2\alpha+1}}{\lambda^2+(\xi^2+\eta)^2}d\xi
\end{aligned}$$

are well defined.

Lemma 3.2 *Suppose that $\eta \geq 0$, then for all $\lambda \in \mathbb{R}$, we get $i\lambda I - \mathcal{A}$ is injective, that is,*

$$\ker(i\lambda I - \mathcal{A}) = \{0\}.$$

Proof. Let us assume that there exists $\lambda \in \mathbb{R}$ and $U = (u, v, y, z, w) \in \mathcal{D}(\mathcal{A})$ such that

$$\mathcal{A}U = i\lambda U.$$

Therefore, one has

$$v = i\lambda u, \tag{3.1}$$

$$\left(au_x + \sqrt{b(x)}k(\alpha)\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} w(x,\xi)d\xi\right)_x - c(x)z = i\lambda v, \tag{3.2}$$

$$z = i\lambda y, \tag{3.3}$$

$$y_{xx} + c(x)v = i\lambda z, \tag{3.4}$$

$$(i\lambda + \xi^2 + \eta)w(x,\xi) = \sqrt{b(x)}v_x|\xi|^{\frac{2\alpha-1}{2}}, \tag{3.5}$$

with the boundary conditions

$$u(0) = u(L) = y(0) = y(L) = 0. \tag{3.6}$$

By a simple computation we can get

$$0 = \Re\langle i\lambda U, U\rangle_{\mathcal{H}} = \Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -k(\alpha)\int_0^L \int_{\mathbb{R}} (\xi^2 + \eta)|w(x,\xi)|^2 d\xi dx.$$

It follows that

$$w(x,\xi) = 0 \quad a.e. \ in \ (0, L) \times \mathbb{R}. \tag{3.7}$$

Bringing (3.7) into (3.5) and applying the definition of $b(x)$, we infer that

$$v_x = 0 \quad in \ (0, \beta_2). \tag{3.8}$$

From (3.1), one gets

$$\lambda u_x = 0 \quad in \ (0, \beta_2). \tag{3.9}$$

Here are two cases.
Case 1. If $\lambda = 0$:
By using (3.1) and (3.3), it follows that

$$v = z = 0 \quad in \quad (0, L).$$

From (3.2), (3.4) and (3.7), we have

$$u_{xx} = y_{xx} = 0.$$

In view of the boundary conditions in (3.6) and the fact that $(u, y) \in C^1([0, L])$, it is easy to obtain that

$$u = y = 0.$$

Consequently, $U = 0$. Then we get

$$\ker(i\lambda I - \mathcal{A}) = \{0\}.$$

Case 2. If $\lambda \neq 0$:
By applying (3.9), one attains

$$u_x = 0 \quad in\ (0, \beta_2). \tag{3.10}$$

According to (3.2), (3.3), (3.7), (3.8), (3.10) and the definition of $c(x)$, we deduce that

$$y_x = 0 \quad in\ (\beta_1, \beta_2). \tag{3.11}$$

Inserting (3.1), (3.3) into (3.2), (3.4), then applying (3.7), we conclude that

$$\lambda^2 u + a u_{xx} - i\lambda c(x) y = 0 \quad in\ (0, L), \tag{3.12}$$

$$\lambda^2 y + y_{xx} + i\lambda c(x) u = 0 \quad in\ (0, L). \tag{3.13}$$

We then show that $u = y = 0$ in $(0, L)$ based on the subsequent three steps.
Step 1. The purpose of the first step is to prove that $u = y = 0$ in $(0, \beta_2)$. From (3.10), we obtain

$$u_x = 0 \quad in\ (0, \beta_1). \tag{3.14}$$

Applying (3.14), (3.12) and the fact that $c(x) = 0$ in $(0, \beta_1)$, we deduce that

$$u = 0 \quad in\ (0, \beta_1). \tag{3.15}$$

According to (3.10), (3.15) and $u \in C^1([0, L])$, we derive

$$u = 0 \quad in\ (\beta_1, \beta_2). \tag{3.16}$$

Thus, we have

$$u = 0 \quad in\ (0, \beta_2). \tag{3.17}$$

By applying (3.10) and $c(x) = c_0$ in (β_1, β_2) in equation (3.12), one gets

$$u = \frac{ic_0}{\lambda} y \quad in\ (\beta_1, \beta_2). \tag{3.18}$$

In view of $c_0 \in \mathbb{R}^*, \lambda \in \mathbb{R}^*$ and (3.17), we infer that

$$u = y = 0 \quad in\ (\beta_1, \beta_2). \tag{3.19}$$

Thanks to $y \in C^1([0, L])$, we attain

$$y(\beta_1) = y_x(\beta_1) = 0. \tag{3.20}$$

From (3.13), (3.20) and the fact that $c(x) = 0$ in $(0, \beta_1)$, one has

$$y = 0 \quad in\ (0, \beta_1). \tag{3.21}$$

Combining (3.17), (3.19) and (3.21), we get $u = y = 0$ in $(0, \beta_2)$. Hence, it follows that

$$U = 0, \ in\ (0, \beta_2).$$

Step 2. The purpose of this step is to prove that $u = y = 0$ in (β_2, β_3). According to (3.19) and $(u, y) \in C^1([0, L])$, we deduce that

$$u(\beta_2) = u_x(\beta_2) = y(\beta_2) = y_x(\beta_2) = 0. \tag{3.22}$$

Based on (3.12), (3.13) and $c(x) = c_0$ in (β_2, β_3), it is easy to derive the expression

$$au_{xxxx} + (a + 1)\lambda^2 u_{xx} + \lambda^2(\lambda^2 - c_0^2)u = 0. \tag{3.23}$$

Therefore, we can obtain the characteristic equation of system (3.23), that is

$$H(r) := ar^4 + (a + 1)\lambda^2 r^2 + \lambda^2(\lambda^2 - c_0^2).$$

Here we take $m = r^2$, then

$$H_0(m) := am^2 + (a + 1)\lambda^2 m + \lambda^2(\lambda^2 - c_0^2).$$

There are two different real roots m_1 and m_2 for the polynomial H_0, described by

$$m_1 = \frac{-\lambda^2(a + 1) - \sqrt{\lambda^4(a - 1)^2 + 4ac_0^2\lambda^2}}{2a}$$

and

$$m_2 = \frac{-\lambda^2(a + 1) + \sqrt{\lambda^4(a - 1)^2 + 4ac_0^2\lambda^2}}{2a}.$$

Obviously, $m_1 < 0$ and the symbol for m_2 depends on the value of λ relative to c_0. For this, we can divide it into three cases: $\lambda^2 > c_0^2, \lambda^2 = c_0^2$ and $\lambda^2 < c_0^2$.
Case 1. If $\lambda^2 > c_0^2$, then $m_2 < 0$. We set

$$r_1 = \sqrt{-m_1}, \ r_2 = \sqrt{-m_2}.$$

Therefore H has four simple roots $ir_1, -ir_1, ir_2$ and $-ir_2$, then the general solution of system (3.12)-(3.13) has the form

$$\begin{cases} u(x) = c_1 \sin(r_1 x) + c_2 \cos(r_1 x) + c_3 \sin(r_2 x) + c_4 \cos(r_2 x), \\ y(x) = \dfrac{(\lambda^2 - ar_1^2)}{i\lambda c_0}(c_1 \sin(r_1 x) + c_2 \cos(r_1 x)) + \dfrac{(\lambda^2 - ar_2^2)}{i\lambda c_0}(c_3 \sin(r_2 x) + c_4 \cos(r_2 x)), \end{cases}$$

where $c_j \in \mathbb{C}$, for $j = 1, 2, 3, 4$. And the boundary condition (3.22) can be described by

$$M_1 \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0,$$

where

$$M_1 = \begin{bmatrix} \sin(r_1\beta_2) & \cos(r_1\beta_2) & \sin(r_2\beta_2) & \cos(r_2\beta_2) \\ r_1\cos(r_1\beta_2) & -r_1\sin(r_1\beta_2) & r_2\cos(r_2\beta_2) & -r_2\sin(r_2\beta_2) \\ \frac{(\lambda^2-ar_1^2)}{i\lambda c_0}\sin(r_1\beta_2) & \frac{(\lambda^2-ar_1^2)}{i\lambda c_0}\cos(r_1\beta_2) & \frac{(\lambda^2-ar_2^2)}{i\lambda c_0}\sin(r_2\beta_2) & \frac{(\lambda^2-ar_2^2)}{i\lambda c_0}\cos(r_2\beta_2) \\ \frac{(\lambda^2-ar_1^2)}{i\lambda c_0}r_1\cos(r_1\beta_2) & -\frac{(\lambda^2-ar_1^2)}{i\lambda c_0}r_1\sin(r_1\beta_2) & \frac{(\lambda^2-ar_2^2)}{i\lambda c_0}r_2\cos(r_2\beta_2) & -\frac{(\lambda^2-ar_2^2)}{i\lambda c_0}r_2\sin(r_2\beta_2) \end{bmatrix}.$$

Let the determinant of M_1 is denoted by $\det(M_1)$, we obtain

$$\det(M_1) = -\frac{r_1 r_2 a^2 (r_1^2 - r_2^2)^2}{\lambda^2 c_0^2}.$$

Owing to $r_1^2 - r_2^2 = m_2 - m_1 \neq 0$, it follows that $\det(M_1) \neq 0$. Hence, system (3.12)-(3.13) with the boundary conditions (3.22) has only a trivial solution $u = y = 0$ in (β_2, β_3).

Case 2. If $\lambda^2 = c_0^2$, then $m_2 = 0$. We set

$$r_1 = \sqrt{-m_1} = \sqrt{\frac{(a+1)c_0^2}{a}}.$$

Therefore H has two simple roots $ir_1, -ir_1$ and 0 is a double root, then the general solution of system (3.12)-(3.13) has the form

$$\begin{cases} u(x) = c_1\sin(r_1 x) + c_2\cos(r_1 x) + c_3 x + c_4, \\ y(x) = \dfrac{(\lambda^2 - ar_1^2)}{i\lambda c_0}(c_1\sin(r_1 x) + c_2\cos(r_1 x)) + \dfrac{\lambda}{ic_0}(c_3 x + c_4), \end{cases}$$

where $c_j \in \mathbb{C}$, for $j = 1, 2, 3, 4$. And the boundary condition (3.22) can be described by

$$M_2 \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0,$$

where

$$M_2 = \begin{bmatrix} \sin(r_1\beta_2) & \cos(r_1\beta_2) & \beta_2 & 1 \\ r_1\cos(r_1\beta_2) & -r_1\sin(r_1\beta_2) & 1 & 0 \\ \frac{(\lambda^2-ar_1^2)}{i\lambda c_0}\sin(r_1\beta_2) & \frac{(\lambda^2-ar_1^2)}{i\lambda c_0}\cos(r_1\beta_2) & \frac{\lambda\beta_2}{ic_0} & \frac{\lambda}{ic_0} \\ \frac{(\lambda^2-ar_1^2)}{i\lambda c_0}r_1\cos(r_1\beta_2) & -\frac{(\lambda^2-ar_1^2)}{i\lambda c_0}r_1\sin(r_1\beta_2) & \frac{\lambda}{ic_0} & 0 \end{bmatrix}.$$

Similarly, we obtain

$$\det(M_2) = -\frac{a^2 r_1^5}{\lambda^2 c_0^2}.$$

Thanks to $r_1 = \sqrt{-m_1} \neq 0$, it follows that $\det(M_2) \neq 0$. Hence, system (3.12)-(3.13) with the boundary conditions (3.22) has only a trivial solution $u = y = 0$ in (β_2, β_3).
Case 3. If $\lambda^2 < c_0^2$, then $m_2 > 0$. We set

$$r_1 = \sqrt{-m_1},\ r_2 = \sqrt{m_2}.$$

Therefore H has four simple roots $ir_1, -ir_1, r_2$ and $-r_2$, then the general solution of system (3.12)-(3.13) has the form

$$\begin{cases} u(x) = c_1 \sin(r_1 x) + c_2 \cos(r_1 x) + c_3 \cosh(r_2 x) + c_4 \sinh(r_2 x), \\ y(x) = \dfrac{(\lambda^2 - a r_1^2)}{i\lambda c_0}(c_1 \sin(r_1 x) + c_2 \cos(r_1 x)) + \dfrac{(\lambda^2 + a r_2^2)}{i\lambda c_0}(c_3 \cosh(r_2 x) + c_4 \sinh(r_2 x)), \end{cases}$$

where $c_j \in \mathbb{C}$, for $j = 1, 2, 3, 4$. And the boundary condition (3.22) can be described by

$$M_3 \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0,$$

where

$$M_3 = \begin{bmatrix} \sin(r_1\beta_2) & \cos(r_1\beta_2) & \cosh(r_2\beta_2) & \sinh(r_2\beta_2) \\ r_1\cos(r_1\beta_2) & -r_1\sin(r_1\beta_2) & r_2\sinh(r_2\beta_2) & r_2\cosh(r_2\beta_2) \\ \frac{(\lambda^2 - a r_1^2)}{i\lambda c_0}\sin(r_1\beta_2) & \frac{(\lambda^2 - a r_1^2)}{i\lambda c_0}\cos(r_1\beta_2) & \frac{(\lambda^2 + a r_2^2)}{i\lambda c_0}\cosh(r_2\beta_2) & \frac{(\lambda^2 + a r_2^2)}{i\lambda c_0}\sinh(r_2\beta_2) \\ \frac{(\lambda^2 - a r_1^2)}{i\lambda c_0} r_1\cos(r_1\beta_2) & -\frac{(\lambda^2 - a r_1^2)}{i\lambda c_0} r_1\sin(r_1\beta_2) & \frac{(\lambda^2 + a r_2^2)}{i\lambda c_0} r_2\sinh(r_2\beta_2) & \frac{(\lambda^2 + a r_2^2)}{i\lambda c_0} r_2\cosh(r_2\beta_2) \end{bmatrix}.$$

Similarly, we obtain

$$\det(M_3) = \frac{r_1 r_2 a^2 (r_1^2 + r_2^2)^2}{\lambda^2 c_0^2}.$$

Because $r_1^2 + r_2^2 = m_2 - m_1 \neq 0$, it follows that $\det(M_3) \neq 0$. Hence, system (3.12)-(3.13) with the boundary conditions (3.22) has only a trivial solution $u = y = 0$ in (β_2, β_3). Therefore,

$$U = 0, \quad in\ (\beta_2, \beta_3).$$

Step 3. The purpose of this step is to prove that $u = y = 0$ in (β_3, L). According to (3.12), (3.13) and the fact that $c(x) = 0$ in (β_3, L), we further deduce

$$\begin{cases} \lambda^2 u + a u_{xx} = 0 \quad in\ (\beta_3, L), \\ \lambda^2 y + y_{xx} = 0 \quad in\ (\beta_3, L). \end{cases} \tag{3.24}$$

Owing to $(u, y) \in C^1([0, L])$ and $u = y = 0$ in (β_2, β_3), we arrive at

$$u(\beta_3) = u_x(\beta_3) = y(\beta_3) = y_x(\beta_3) = 0. \tag{3.25}$$

Obviously, it is clear to derive that system (3.24) with the boundary condition (3.25) has only a trivial solution $u = y = 0$ in (β_3, L).

Accordingly, we showed that $U = 0$ in $(0, L)$. The proof of Lemma 3.2 is completed. $\square$

Lemma 3.3 *Assume that ($\eta > 0$ and $\lambda \in \mathbb{R}$), or ($\eta = 0$ and $\lambda \in \mathbb{R}^*$), we get $i\lambda I - \mathcal{A}$ is surjective, that is*

$$\mathrm{R}(i\lambda I - \mathcal{A}) = \mathcal{H}.$$

Proof. Set $F = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$, we need to show that there exists $U = (u, v, y, z, w) \in \mathcal{D}(\mathcal{A})$ such that solution of

$$(i\lambda I - \mathcal{A})U = F. \tag{3.26}$$

Therefore, we deduce that

$$\begin{aligned}
i\lambda u - v &= f_1, &(3.27)\\
i\lambda v - \left(au_x + \sqrt{b(x)}k(\alpha)\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} w(x,\xi)d\xi\right)_x + c(x)z &= f_2, &(3.28)\\
i\lambda y - z &= f_3, &(3.29)\\
i\lambda z - y_{xx} - c(x)v &= f_4, &(3.30)\\
(i\lambda + \xi^2 + \eta)w(x,\xi) - \sqrt{b(x)}v_x|\xi|^{\frac{2\alpha-1}{2}} &= f_5(x,\xi). &(3.31)
\end{aligned}$$

From (3.27), (3.31) and $\eta \geq 0$, we obtain

$$w(x,\xi) = \frac{f_5(x,\xi)}{i\lambda + \xi^2 + \eta} + \frac{\sqrt{b(x)}i\lambda u_x|\xi|^{\frac{2\alpha-1}{2}}}{i\lambda + \xi^2 + \eta} - \frac{\sqrt{b(x)}f_{1,x}|\xi|^{\frac{2\alpha-1}{2}}}{i\lambda + \xi^2 + \eta}. \tag{3.32}$$

Inserting (3.27) and (3.29) into (3.28) and (3.30), then applying (3.32), we have

$$\begin{aligned}
\lambda^2 u + (au_x + b(x)I_4(\lambda,\eta,\alpha)u_x - h(x,\lambda,\eta,\alpha))_x - i\lambda c(x)y &= F_1, &(3.33)\\
\lambda^2 y + y_{xx} + i\lambda c(x)u &= F_2, &(3.34)
\end{aligned}$$

where

$$\begin{aligned}
&F_1 = -(f_2 + i\lambda f_1 + c(x)f_3), \quad F_2 = -(f_4 + i\lambda f_3 - c(x)f_1),\\
&h(x,\lambda,\eta,\alpha) = b(x)I_5(\lambda,\eta,\alpha)f_{1,x} - \sqrt{b(x)}I_6(x,\lambda,\eta,\alpha),
\end{aligned}$$

and $I_4(\lambda,\eta,\alpha)$, $I_5(\lambda,\eta,\alpha)$ and $I_6(x,\lambda,\eta,\alpha)$ are well defined in Lemma 3.1. In fact, we can divide it into two different cases:

Case 1. If $\eta > 0$ and $\lambda = 0$. System (3.33)-(3.34) can be converted to the following form:

$$\begin{aligned}
(au_x - h(x,0,\eta,\alpha))_x &= -(f_2 + c(x)f_3), &(3.35)\\
y_{xx} &= -(f_4 - c(x)f_1). &(3.36)
\end{aligned}$$

Taking into consideration Lax-Milgram theorem and Lemma 3.1, it follows that system (3.35)-(3.36) admits a unique solution $(u, y) \in H_0^1(0, L) \times H_0^1(0, L)$.

Case 2. If $\eta \geq 0$ and $\lambda \in \mathbb{R}^*$. System (3.33)-(3.34) can be converted to the following form:

$$\lambda^2 u + (au_x + b(x)I_4(\lambda,\eta,\alpha)u_x)_x - i\lambda c(x)y = F_3, \tag{3.37}$$

$$\lambda^2 y + y_{xx} + i\lambda c(x)u = F_2, \tag{3.38}$$

where

$$F_3 = F_1 + h_x(x, \lambda, \eta, \alpha).$$

Define the linear unbounded operator $\mathcal{L} : \mathbb{H} := H_0^1(0, L) \times H_0^1(0, L) \mapsto \mathbb{H}'$ such that $\mathbb{H}'$ is the dual space of $\mathbb{H}$. For all $U \in \mathbb{H}$, it follows that

$$\mathcal{L}U = \begin{bmatrix} -(au_x + b(x)I_4(\lambda, \eta, \alpha)u_x)_x + i\lambda c(x)y \\ -y_{xx} - i\lambda c(x)u \end{bmatrix}.$$

By applying Lax-Milgram theorem, we can find that $\mathcal{L}$ is isomorphism. And system (3.37)-(3.38) can be expressed by

$$(\lambda^2 \mathcal{L}^{-1} - I)U = \mathcal{L}^{-1}\mathcal{F}, \tag{3.39}$$

where $U = (u, y)^T$, $\mathcal{F} = (F_3, F_2)^T$. Because I is a compact operator from $\mathbb{H}$ to $\mathbb{H}'$ and the operator $\mathcal{L}^{-1}$ is isomorphism, we can deduce that $\mathcal{L}^{-1}$ is compact operator from $\mathbb{H}$ to $\mathbb{H}$. Based on Fredholm's alternative theorem, the existence of solution of system (3.39) is equivalent to proving $(\lambda^2\mathcal{L}^{-1} - I)U = 0$ has only zero solution, i.e. $\ker(\lambda^2\mathcal{L}^{-1} - I) = \{0\}$. If $(\widehat{u}, \widehat{y}) \in \ker(\lambda^2\mathcal{L}^{-1} - I)$, it follows that $\lambda^2(\widehat{u}, \widehat{y}) - \mathcal{L}(\widehat{u}, \widehat{y}) = 0$. Hence,

$$\lambda^2\widehat{u} + (a\widehat{u}_x + b(x)I_4(\lambda, \eta, \alpha)\widehat{u}_x)_x - i\lambda c(x)\widehat{y} = 0, \tag{3.40}$$

$$\lambda^2\widehat{y} + \widehat{y}_{xx} + i\lambda c(x)\widehat{u} = 0, \tag{3.41}$$

with the boundary conditions

$$\widehat{u}(0) = \widehat{u}(L) = \widehat{y}(0) = \widehat{y}(L) = 0. \tag{3.42}$$

We multiply (3.40), (3.41) by $\overline{\widehat{u}}, \overline{\widehat{y}}$ respectively, integrate by parts on $(0, L)$ and take the sum, then based on the boundary conditions and take the imaginary part, one obtains

$$b_0\Im(I_4(\lambda, \eta, \alpha)) \int_0^{\beta_2} |\widehat{u}_x|^2 dx = 0.$$

Since $I_4(\lambda, \eta, \alpha) = \lambda^2 I_7(\lambda, \eta, \alpha) + i\lambda I_8(\lambda, \eta, \alpha)$, where $I_7(\lambda, \eta, \alpha), I_8(\lambda, \eta, \alpha)$ are well defined in Lemma 3.1, we conclude that $\Im(I_4(\lambda, \eta, \alpha)) = \lambda I_8(\lambda, \eta, \alpha) \neq 0$. Consequently, we arrive at

$$\widehat{u}_x = 0 \quad in\ (0, \beta_2).$$

Therefore, system (3.40)-(3.42) converts

$$\lambda^2\widehat{u} + a\widehat{u}_{xx} - i\lambda c(x)\widehat{y} = 0 \quad in\ (0, L), \tag{3.43}$$

$$\lambda^2\widehat{y} + \widehat{y}_{xx} + i\lambda c(x)\widehat{u} = 0 \quad in\ (0, L), \tag{3.44}$$

$$\widehat{u}_x = 0 \quad in\ (0, \beta_2). \tag{3.45}$$

If $(\widehat{u}, \widehat{y})$ is a solution of system (3.43)-(3.45), then $\widehat{U} = (\widehat{u}, i\lambda\widehat{u}, \widehat{y}, i\lambda\widehat{y}, 0) \in \mathcal{D}(\mathcal{A})$, moreover $i\lambda\widehat{U} - \mathcal{A}\widehat{U} = 0$. Thus, $\widehat{U} \in \ker(i\lambda I - \mathcal{A})$. In view of Lemma 3.2, we deduce $\widehat{U} = 0$. From Fredholm's alternative theorem, we infer that system (3.39) has a unique solution $(u, y) \in H_0^1(0, L) \times H_0^1(0, L)$. And by applying $v = i\lambda u - f_1,\ z = i\lambda y - f_3$ and $F \in \mathcal{H}$, we arrive at $v, z \in H_0^1(0, L)$, then based on (3.32) and the classical regularity arguments, we get $y \in H^2(0, L) \cap H_0^1(0, L)$ and

$$\left(au_x + \sqrt{b(x)}k(\alpha)\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} w(x, \xi)d\xi\right)_x \in L^2(0, L).$$

Similarly, we next demonstrate $w(x, \xi),\ |\xi|w(x, \xi)$ in W. Applying (3.32), one has

$$\int_0^L \int_{\mathbb{R}} |w(x, \xi)|^2 d\xi dx \leq C \int_0^L \int_{\mathbb{R}} \frac{|f_5(x, \xi)|^2}{\lambda^2 + (\xi^2 + \eta)^2} d\xi dx + Cb_0 \frac{I_7(\lambda, \eta, \alpha)}{k(\alpha)} \int_0^{\beta_2} (|\lambda u_x|^2 + |f_{1,x}|^2)dx.$$

Applying the fact that $f_5(x, \xi) \in W$ and ($\eta > 0$ and $\lambda \in \mathbb{R}$) or ($\eta = 0$ and $\lambda \in \mathbb{R}^*$), we get

$$\int_0^L \int_{\mathbb{R}} \frac{|f_5(x, \xi)|^2}{\lambda^2 + (\xi^2 + \eta)^2} d\xi dx \leq \frac{1}{\lambda^2 + \eta^2} \int_0^L \int_{\mathbb{R}} |f_5(x, \xi)|^2 d\xi dx < +\infty.$$

From Lemma 3.1, we have $w(x, \xi) \in W$. Moreover,

$$\int_0^L \int_{\mathbb{R}} |\xi w(x, \xi)|^2 d\xi dx \leq C \int_0^L \int_{\mathbb{R}} \frac{\xi^2 |f_5(x, \xi)|^2}{\lambda^2 + (\xi^2 + \eta)^2} d\xi dx + Cb_0 I_9(\lambda, \eta, \alpha) \int_0^{\beta_2} (|\lambda u_x|^2 + |f_{1,x}|^2)dx.$$

Since $f_5(x, \xi) \in W$ and $\max\limits_{\xi \in \mathbb{R}} \dfrac{\xi^2}{\lambda^2 + (\xi^2 + \eta)^2} = \dfrac{\sqrt{\eta^2 + \lambda^2}}{\lambda^2 + (\sqrt{\eta^2 + \lambda^2} + \eta)^2} = C(\lambda, \eta)$, it follows that

$$\begin{aligned}\int_0^L \int_{\mathbb{R}} \frac{\xi^2 |f_5(x, \xi)|^2}{\lambda^2 + (\xi^2 + \eta)^2} d\xi dx &\leq \max_{\xi \in \mathbb{R}} \frac{\xi^2}{\lambda^2 + (\xi^2 + \eta)^2} \int_0^L \int_{\mathbb{R}} |f_5(x, \xi)|^2 d\xi dx \\ &= C(\lambda, \eta) \int_0^L \int_{\mathbb{R}} |f_5(x, \xi)|^2 d\xi dx < +\infty.\end{aligned}$$

By using Lemma 3.1, we infer that $|\xi|w(x, \xi) \in W$. Therefore, we have

$$-(\xi^2 + \eta)w(x, \xi) + \sqrt{b(x)}v_x |\xi|^{\frac{2\alpha-1}{2}} = i\lambda w(x, \xi) - f_5(x, \xi) \in W.$$

Consequently, we find that (3.26) admits a unique solution $U = (u, v, y, z, w) \in \mathcal{D}(\mathcal{A})$. The proof is completed. $\square$

Proof of Theorem 3.2. Based on Lemma 3.2, we obtain that $\ker(i\lambda I - \mathcal{A}) = \{0\}$, i.e. the operator $\mathcal{A}$ has no pure imaginary eigenvalues. Then, by applying $\mathrm{R}(i\lambda I - \mathcal{A}) = \mathcal{H}$ for all $\lambda \in \mathbb{R},\ \eta > 0$ and $\lambda \in \mathbb{R}^*,\ \eta = 0$ in Lemma 3.3 and the closed graph theorem of Banach, we arrive at $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{\emptyset\}$ if $\eta > 0$ and $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$ if $\eta = 0$. Accordingly, we complete the proof Theorem 3.2 by using Theorem 3.1. $\square$

4 Polynomial stability

In this subsection, we discuss the polynomial stability of the system (2.6)-(2.10) under the condition $\eta > 0$. To achieve this aim, we rely on the multiplier technique and frequency domain approach, as well as a recent result by Borichev and Tomilov (see [19]). We first give the following theorems and lemmas, which will be used later.

Theorem 4.1 *(see [19]) Let $(e^{t\mathcal{A}})_{t\geq 0}$ be a C_0-semigroup on a Hilbert space $\mathcal{H}$ with generator $\mathcal{A}$ such that $i\mathbb{R}\subset\rho(\mathcal{A})$. Then for any $\ell>0,\ t>0, U_0\in\mathcal{D}(\mathcal{A})$ and for some $C>0$, we have*

$$\sup_{\lambda\in\mathbb{R}}\|(i\lambda I-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}=O(|\lambda|^{\ell})\Longleftrightarrow\|e^{t\mathcal{A}}U_0\|^2_{\mathcal{H}}\leq\frac{C}{t^{\frac{2}{\ell}}}\|U_0\|^2_{\mathcal{D}(\mathcal{A})}.$$

Therefore, our main result in this part is as follows.

Theorem 4.2 *Suppose that $\eta>0$, the C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is polynomial stability, that is, there exists $C>0$ independent of $U_0\in\mathcal{D}(\mathcal{A})$, such that*

$$E(t)\leq\frac{C}{t}\|U_0\|^2_{\mathcal{D}(\mathcal{A})},\quad\forall\ t>0.\tag{4.1}$$

From Theorem 4.1, we take $\ell=2$, the polynomial energy decay (4.1) holds if

$$i\mathbb{R}\subset\rho(\mathcal{A})\tag{H_1}$$

and

$$\sup_{\lambda\in\mathbb{R}}\|(i\lambda I-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}=O(|\lambda|^2)\tag{H_2}$$

are satisfied. Condition (H_1) is already showed in Lemma 3.2. Then we will show that condition (H_2) holds by contradiction. For this, we assume (H_2) is false, then there exists a real sequence (λ_n) with $|\lambda_n|\to+\infty$, and a sequence $(U_n)\subset\mathcal{D}(\mathcal{A})$ with

$$\|U_n\|_{\mathcal{H}}=\|(u_n,v_n,y_n,z_n,w_n)\|_{\mathcal{H}}=1,\tag{4.2}$$

such that

$$\lambda_n^2(i\lambda_n I-\mathcal{A})U_n=F_n:=(f_{1,n},f_{2,n},f_{3,n},f_{4,n},f_{5,n}(\cdot,\xi))\to 0\quad in\ \mathcal{H}.\tag{4.3}$$

Here we will find a contradiction with (4.2) such as $\|U_n\|_{\mathcal{H}}=o(1)$. For simplicity of calculation, we omit the index n. Then, equation (4.3) in terms of its components are described by

$$i\lambda u-v=\frac{f_1}{\lambda^{\ell}},\quad f_1\to 0\quad in\ H_0^1(0,L),\tag{4.4}$$

$$i\lambda v-(S_b)_x+c(x)z=\frac{f_2}{\lambda^{\ell}},\quad f_2\to 0\quad in\ L^2(0,L),\tag{4.5}$$

$$i\lambda y-z=\frac{f_3}{\lambda^{\ell}},\quad f_3\to 0\quad in\ H_0^1(0,L),\tag{4.6}$$

$$i\lambda z-y_{xx}-c(x)v=\frac{f_4}{\lambda^{\ell}},\quad f_4\to 0\quad in\ L^2(0,L),\tag{4.7}$$

$$(i\lambda+\xi^2+\eta)w(x,\xi)-\sqrt{b(x)}v_x|\xi|^{\frac{2\alpha-1}{2}}=\frac{f_5(x,\xi)}{\lambda^{\ell}},\quad f_5(x,\xi)\to 0\quad in\ W,\tag{4.8}$$

where

$$S_b=\begin{cases}S_{b_0}:=au_x+\sqrt{b_0}k(\alpha)\displaystyle\int_{\mathbb{R}}|\xi|^{\frac{2\alpha-1}{2}}w(x,\xi)d\xi, & x\in(0,\beta_2),\\ au_x, & x\in(\beta_2,L).\end{cases}$$

Lemma 4.1 *(see [3]) Let $\alpha \in (0,1),\ \eta \geq 0$ and $\lambda \in \mathbb{R}$, we obtain*

$$\begin{aligned} I_{10}(\lambda,\eta,\alpha) &= \int_{\mathbb{R}} \frac{|\xi|^{\alpha+\frac{1}{2}}}{(|\lambda|+\xi^2+\eta)^2} d\xi = c_1(|\lambda|+\eta)^{\frac{\alpha}{2}-\frac{5}{4}}, \\ I_{11}(\lambda,\eta) &= \left(\int_{\mathbb{R}} \frac{1}{(|\lambda|+\xi^2+\eta)^2} d\xi\right)^{\frac{1}{2}} = \sqrt{\frac{\pi}{2}} \frac{1}{(|\lambda|+\eta)^{\frac{3}{4}}}, \\ I_{12}(\lambda,\eta) &= \left(\int_{\mathbb{R}} \frac{\xi^2}{(|\lambda|+\xi^2+\eta)^4} d\xi\right)^{\frac{1}{2}} = \frac{\sqrt{\pi}}{4} \frac{1}{(|\lambda|+\eta)^{\frac{5}{4}}}, \end{aligned}$$

where $c_1 > 0$ is a positive constant.

Lemma 4.2 *Let $\alpha \in (0,1),\ \eta > 0$. Then the solution $(u, v, y, z, w) \in \mathcal{D}(\mathcal{A})$ of system (4.4)-(4.8) satisfies*

$$\int_0^L \int_{\mathbb{R}} (\xi^2+\eta)|w(x,\xi)|^2 d\xi dx = \frac{o(1)}{\lambda^\ell}, \tag{4.9}$$

$$\int_0^{\beta_2} |v_x|^2 dx = \frac{o(1)}{\lambda^{\ell+\alpha-1}}, \tag{4.10}$$

$$\int_0^{\beta_2} |u_x|^2 dx = \frac{o(1)}{\lambda^{\ell+\alpha+1}}. \tag{4.11}$$

Proof. Taking the inner product of (4.3) with U in $\mathcal{H}$, then applying (4.2) and the fact that U is uniformly bounded in $\mathcal{H}$, we arrive at

$$k(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2+\eta)|w(x,\xi)|^2 d\xi dx = -\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \Re\langle (i\lambda I - \mathcal{A})U, U\rangle_{\mathcal{H}} = o(\lambda^{-\ell}).$$

Moreover, based on (4.8), we find

$$\begin{aligned} \sqrt{b(x)}|\xi|^{\frac{2\alpha-1}{2}}|v_x| &= \left|(i\lambda+\xi^2+\eta)w(x,\xi) - \frac{f_5(x,\xi)}{\lambda^\ell}\right| \\ &\leq (|\lambda|+\xi^2+\eta)|w(x,\xi)| + |\lambda|^{-\ell}|f_5(x,\xi)|. \end{aligned}$$

We multiply the above expression by $\dfrac{|\xi|}{(|\lambda|+\xi^2+\eta)^2}$ and integrate on $\mathbb{R}$, one obtains

$$\sqrt{b(x)} I_{10}(\lambda,\eta,\alpha)|v_x| \leq I_{11}(\lambda,\eta) \left(\int_{\mathbb{R}} |\xi w(x,\xi)|^2 d\xi\right)^{\frac{1}{2}} + |\lambda|^{-\ell} I_{12}(\lambda,\eta) \left(\int_{\mathbb{R}} |f_5(x,\xi)|^2 d\xi\right)^{\frac{1}{2}}, \tag{4.12}$$

where $I_{10}(\lambda,\eta,\alpha),\ I_{11}(\lambda,\eta)$ and $I_{12}(\lambda,\eta)$ are well defined in Lemma 4.1. Integrating (4.12) on $(0, L)$ and according to the definition of $b(x)$ and the Young inequality, we attain

$$\int_0^{\beta_2} |v_x|^2 dx \leq \frac{C I_{11}^2(\lambda,\eta)}{b_0 I_{10}^2(\lambda,\eta,\alpha)} \int_0^L \int_{\mathbb{R}} |\xi w(x,\xi)|^2 d\xi dx + \frac{C I_{12}^2(\lambda,\eta)}{b_0 I_{10}^2(\lambda,\eta,\alpha)} |\lambda|^{-2\ell} \int_0^L \int_{\mathbb{R}} |f_5(x,\xi)|^2 d\xi dx.$$

Thanks to (4.9) and $\|f_5\|_W = o(1)$, we derive

$$\int_0^{\beta_2} |v_x|^2 dx \leq C \frac{I_{11}^2(\lambda,\eta)}{I_{10}^2(\lambda,\eta,\alpha)} \frac{o(1)}{|\lambda|^\ell} + C \frac{I_{12}^2(\lambda,\eta)}{I_{10}^2(\lambda,\eta,\alpha)} \frac{o(1)}{|\lambda|^{2\ell}}.$$

From Lemma 4.1, we can infer that

$$\int_0^{\beta_2} |v_x|^2 dx \leq \frac{C}{(|\lambda|+\eta)^{\alpha-1}} \frac{o(1)}{|\lambda|^{\ell}} + \frac{C}{(|\lambda|+\eta)^{\alpha}} \frac{o(1)}{|\lambda|^{2\ell}}. \tag{4.13}$$

Since $\alpha \in (0,1)$, it follows that $\min\{\ell+\alpha-1, 2\ell+\alpha\} = \ell+\alpha-1$, then we get (4.10). Furthermore, based on (4.4), one sees that

$$i\lambda u_x = v_x + |\lambda|^{-\ell} f_{1,x}.$$

We further deduce

$$\|\lambda u_x\|_{L^2(0,\beta_2)} \leq \|v_x\|_{L^2(0,\beta_2)} + |\lambda|^{-\ell}\|f_{1,x}\|_{L^2(0,\beta_2)} \leq \frac{o(1)}{|\lambda|^{\frac{\ell+\alpha-1}{2}}} + \frac{o(1)}{|\lambda|^{\ell}}.$$

Since $\alpha \in (0,1)$, it follows that $\min\left\{\dfrac{\ell+\alpha+1}{2}, 1+\ell\right\} = \dfrac{\ell+\alpha+1}{2}$. Consequently, the proof of Lemma 4.2 is completed. □

Lemma 4.3 *Let $\alpha \in (0,1)$, $\eta > 0$. Then the solution $(u,v,y,z,w) \in \mathcal{D}(\mathcal{A})$ of system* (4.4)-(4.8) *satisfies*

$$\int_0^{\beta_2} |S_{b_0}|^2\, dx = \frac{o(1)}{\lambda^{\ell}}. \tag{4.14}$$

Proof. Based on the inequality $|a+b|^2 \leq 2a^2 + 2b^2$, we have

$$\begin{aligned}
\int_0^{\beta_2} |S_{b_0}|^2\, dx &= \int_0^{\beta_2} \left| a u_x + \sqrt{b_0} k(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} w(x,\xi) d\xi \right|^2 dx \\
&\leq 2a^2 \int_0^{\beta_2} |u_x|^2 dx + 2b_0 k^2(\alpha) \int_0^{\beta_2} \left(\int_{\mathbb{R}} \frac{|\xi|^{\frac{2\alpha-1}{2}}\sqrt{\xi^2+\eta}}{\sqrt{\xi^2+\eta}} w(x,\xi) d\xi \right)^2 dx \\
&\leq 2a^2 \int_0^{\beta_2} |u_x|^2 dx + M_1 \int_0^{\beta_2} \int_{\mathbb{R}} (\xi^2+\eta) |w(x,\xi)|^2 d\xi dx,
\end{aligned}$$

where $M_1 = 2b_0 k^2(\alpha) I_{13}(\alpha,\eta)$ and $I_{13}(\alpha,\eta) = \displaystyle\int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{\xi^2+\eta} d\xi$. For $I_{13}(\alpha,\eta)$, it follows that

$$\frac{|\xi|^{2\alpha-1}}{\xi^2+\eta} \underset{0}{\sim} \frac{|\xi|^{2\alpha-1}}{\eta} \quad \textit{and} \quad \frac{|\xi|^{2\alpha-1}}{\xi^2+\eta} \underset{+\infty}{\sim} \frac{1}{|\xi|^{3-2\alpha}}.$$

By using the fact that $\alpha \in (0,1)$ and $\eta > 0$, we get $I_{13}(\alpha,\eta)$ is well defined. According to Lemma 4.2, we derive

$$\int_0^{\beta_2} |S_{b_0}|^2\, dx \leq C \frac{o(1)}{\lambda^{\ell+\alpha+1}} + C \frac{o(1)}{\lambda^{\ell}}.$$

It is easy to obtain the desired result. □

Lemma 4.4 *Set $0 < \varepsilon < \min\left\{\dfrac{\beta_1}{2}, \dfrac{\beta_2-\beta_1}{4}\right\}$. Then the solution $(u,v,y,z,w) \in \mathcal{D}(\mathcal{A})$ of system* (4.4)-(4.8) *satisfies*

$$\int_{\varepsilon}^{\beta_2-\varepsilon} |v|^2\, dx = o(1). \tag{4.15}$$

Proof. We take a cut-off function $\rho_1(x) \in C^1([0, L])$ such that

$$\rho_1(x) = \begin{cases} 1 & if \quad x \in [\varepsilon, \beta_2 - \varepsilon], \\ 0 & if \quad x \in \{0\} \cup [\beta_2, L], \\ 0 \leq \rho_1 \leq 1 & otherwise, \end{cases} \qquad and \qquad \max_{x \in [0,L]} |\rho_1'(x)| = M_{\rho_1'},$$

where $M_{\rho_1'}$ is strictly positive constant. Multiplying (4.5) by $\frac{1}{\lambda}\rho_1 \overline{v}$, integrating by parts on $(0, L)$ and taking the imaginary part, we deduce

$$\begin{aligned} \int_0^L \rho_1 |v|^2 dx = & - \Im \left\{ \frac{1}{\lambda} \int_0^L S_b(\rho_1' \overline{v} + \rho_1 \overline{v}_x) dx \right\} - \Im \left\{ \frac{1}{\lambda} \int_0^L c(x) z \rho_1 \overline{v} dx \right\} \\ & + \Im \left\{ \frac{1}{\lambda^{\ell+1}} \int_0^L \rho_1 f_2 \overline{v} dx \right\}. \end{aligned} \tag{4.16}$$

To facilitate subsequent calculations, we first give the following estimates by using Lemma 4.2, Lemma 4.3, the Cauchy-Schwarz inequality, the definition of $c(x), S_b$ and ρ_1, $\|f_2\|_{L^2(0,L)} = o(1)$ and v, z are uniformly bounded in $L^2(0, L)$, we arrive at

$$\left| \Im \left\{ \frac{1}{\lambda} \int_0^L S_b(\rho_1' \overline{v} + \rho_1 \overline{v}_x) dx \right\} \right| = \left| \Im \left\{ \frac{1}{\lambda} \int_0^{\beta_2} S_{b_0}(\rho_1' \overline{v} + \rho_1 \overline{v}_x) dx \right\} \right| = \frac{o(1)}{|\lambda|^{\frac{\ell}{2}+1}}, \tag{4.17}$$

$$\left| \Im \left\{ \frac{1}{\lambda} \int_0^L c(x) z \rho_1 \overline{v} dx \right\} \right| = \left| \Im \left\{ \frac{c_0}{\lambda} \int_{\beta_1}^{\beta_2} z \rho_1 \overline{v} dx \right\} \right| = \frac{O(1)}{|\lambda|} = o(1), \tag{4.18}$$

$$\left| \Im \left\{ \frac{1}{\lambda^{\ell+1}} \int_0^L \rho_1 f_2 \overline{v} dx \right\} \right| = \frac{o(1)}{|\lambda|^{\ell+1}}. \tag{4.19}$$

Substituting (4.17), (4.18) and (4.19) into (4.16), we can see that

$$\int_0^L \rho_1 |v|^2 dx = o(1).$$

Consequently, we derive the expression of (4.15) by using the definition of ρ_1. $\square$

Lemma 4.5 *Set* $0 < \varepsilon < \min \left\{ \frac{\beta_1}{2}, \frac{\beta_2 - \beta_1}{4} \right\}$. *Then the solution* $(u, v, y, z, w) \in \mathcal{D}(\mathcal{A})$ *of system* (4.4)-(4.8) *satisfies*

$$\int_{\beta_1}^{\beta_2 - 2\varepsilon} |z|^2 dx = o(1) \quad and \quad \int_{\beta_1 + \varepsilon}^{\beta_2 - 3\varepsilon} |y_x|^2 dx = o(1). \tag{4.20}$$

Proof. We take a cut-off function $\rho_2(x) \in C^1([0, L])$ such that

$$\rho_2(x) = \begin{cases} 0 & if \quad x \in [0, \varepsilon] \cup [\beta_2 - \varepsilon, L], \\ 1 & if \quad x \in [2\varepsilon, \beta_2 - 2\varepsilon], \\ 0 \leq \rho_2 \leq 1 & otherwise, \end{cases} \qquad and \qquad \max_{x \in [0,L]} |\rho_2'(x)| = M_{\rho_2'},$$

where $M_{\rho_2'}$ is strictly positive constant. Multiplying (4.5) and (4.7) by $\rho_2\overline{z}$ and $\rho_2\overline{v}$ respectively, integrating on $(0,L)$ and taking the real part, then taking the sum and using integration by parts, we deduce

$$\int_0^L c(x)\rho_2|z|^2dx = \int_0^L c(x)\rho_2|v|^2dx - \Re\left\{\int_0^L(\rho_2'\overline{z}+\rho_2\overline{z}_x)S_bdx\right\} - \Re\left\{\int_0^L(\rho_2'\overline{v}+\rho_2\overline{v}_x)y_xdx\right\}$$
$$+\Re\left\{\frac{1}{\lambda^\ell}\int_0^L\rho_2(f_2\overline{z}+f_4\overline{v})dx\right\}. \tag{4.21}$$

In view of (4.6), we derive

$$\overline{z}_x = -i\lambda\overline{y}_x - \lambda^{-\ell}\overline{f}_{3,x}. \tag{4.22}$$

Based on (4.22), Lemma 4.2-4.4, the Cauchy-Schwarz inequality, the definition of S_b and ρ_2, $\|f_{3,x}\|_{L^2(0,L)} = o(1)$ and $y_x,\ z$ are uniformly bounded in $L^2(0,L)$, we arrive at

$$\left|\Re\left\{\int_0^L(\rho_2'\overline{v}+\rho_2\overline{v}_x)y_xdx\right\}\right| = \left|\Re\left\{\int_\varepsilon^{\beta_2-\varepsilon}(\rho_2'\overline{v}+\rho_2\overline{v}_x)y_xdx\right\}\right| = o(1), \tag{4.23}$$

$$\left|\Re\left\{\int_0^L(\rho_2'\overline{z}+\rho_2\overline{z}_x)S_bdx\right\}\right| = \left|\Re\left\{\int_\varepsilon^{\beta_2-\varepsilon}\left[\rho_2'\overline{z}+\rho_2(-i\lambda\overline{y}_x-\lambda^{-\ell}\overline{f}_{3,x})\right]S_{b_0}dx\right\}\right| = \frac{o(1)}{|\lambda|^{\frac{\ell}{2}-1}}. \tag{4.24}$$

From the fact that $\|f_2\|_{L^2(0,L)} = o(1), \|f_4\|_{L^2(0,L)} = o(1), \ell = 2$ and v, z are uniformly bounded in $L^2(0,L)$, then substituting (4.23) and (4.24) into (4.21), it is easy to check that

$$\int_0^L c(x)\rho_2|z|^2dx = \int_0^L c(x)\rho_2|v|^2dx + o(1).$$

In view of Lemma 4.4 and the definition of $c(x)$ and ρ_2, we get

$$\int_{\beta_1}^{\beta_2-2\varepsilon}|z|^2dx = o(1). \tag{4.25}$$

We then take a cut-off function $\rho_3(x) \in C^1([0,L])$ such that

$$\rho_3(x) = \begin{cases} 0 & if \quad x\in[0,\beta_1]\cup[\beta_2-2\varepsilon,L], \\ 1 & if \quad x\in[\beta_1+\varepsilon,\beta_2-3\varepsilon], \\ 0\le\rho_3\le 1 & otherwise. \end{cases}$$

Multiplying (4.7) by $-\dfrac{1}{\lambda}\rho_3\overline{z}$, integrating by parts on $(0,L)$, taking the imaginary part and applying (4.22), we deduce

$$\int_0^L\rho_3|y_x|^2dx = \int_0^L\rho_3|z|^2dx + \Im\left\{\frac{1}{\lambda}\int_0^L\rho_3'y_x\overline{z}dx\right\} - \Im\left\{\frac{1}{\lambda}\int_0^L c(x)v\rho_3\overline{z}dx\right\}$$
$$-\Im\left\{\frac{1}{\lambda^{\ell+1}}\int_0^L\rho_3(\overline{f}_{3,x}y_x+f_4\overline{z})dx\right\}. \tag{4.26}$$

Thanks to (4.25) and $y_x,\ v$ are uniformly bounded in $L^2(0,L)$, as well as the definition of $c(x)$ and ρ_3, we infer

$$\Im\left\{\frac{1}{\lambda}\int_0^L \rho_3' y_x \overline{z} dx\right\} = \Im\left\{\frac{1}{\lambda}\int_{\beta_1}^{\beta_2-2\varepsilon} \rho_3' y_x \overline{z} dx\right\} = \frac{o(1)}{|\lambda|}, \tag{4.27}$$

$$\Im\left\{\frac{1}{\lambda}\int_0^L c(x) v \rho_3 \overline{z} dx\right\} = \Im\left\{\frac{c_0}{\lambda}\int_{\beta_1}^{\beta_2-2\varepsilon} v \rho_3 \overline{z} dx\right\} = \frac{o(1)}{|\lambda|}. \tag{4.28}$$

From the fact that $\|f_{3,x}\|_{L^2(0,L)} = o(1), \|f_4\|_{L^2(0,L)} = o(1)$ and y_x, z are uniformly bounded in $L^2(0,L)$, then substituting (4.27) and (4.28) into (4.26), it is easy to check that

$$\int_0^L \rho_3 |y_x|^2 dx = \int_0^L \rho_3 |z|^2 dx + \frac{o(1)}{|\lambda|}.$$

Based on (4.25) and the definition of ρ_3, we obtain

$$\int_{\beta_1+\varepsilon}^{\beta_2-3\varepsilon} |y_x|^2 dx = o(1). \tag{4.29}$$

From (4.25) and (4.29), we complete the proof of Lemma 4.5. $\square$

Lemma 4.6 *Set* $0 < \varepsilon < \min\left\{\frac{\beta_1}{2}, \frac{\beta_2-\beta_1}{4}\right\}$. *Then the solution* $(u,v,y,z,w) \in \mathcal{D}(\mathcal{A})$ *of system* (4.4)-(4.8) *satisfies*

$$|v(\beta_2-3\varepsilon)|^2 + |v(\beta_3)|^2 + \frac{1}{a}|S_b(\beta_2-3\varepsilon)|^2 + \frac{1}{a}|S_b(\beta_3)|^2 = O(1), \tag{4.30}$$

$$|z(\beta_2-3\varepsilon)|^2 + |z(\beta_3)|^2 + |y_x(\beta_2-3\varepsilon)|^2 + |y_x(\beta_3)|^2 = O(1). \tag{4.31}$$

Proof. We first take $p \in C^1([\beta_2-3\varepsilon, \beta_3])$ such that

$$p(\beta_2-3\varepsilon) = -p(\beta_3) = 1$$

and

$$\max_{x\in[\beta_2-3\varepsilon,\beta_3]} |p(x)| = M_p, \qquad \max_{x\in[\beta_2-3\varepsilon,\beta_3]} \left|p'(x)\right| = M_{p'}.$$

According to (4.4), it is easy to get

$$i\lambda u_x - v_x = \frac{f_{1,x}}{\lambda^\ell}. \tag{4.32}$$

Then, multiplying (4.32) and (4.5) by $2p\overline{v}$ and $2a^{-1}p\overline{S}_b$ respectively, integrating by parts on $(\beta_2-3\varepsilon, \beta_3)$, then taking the real part and applying the definition of S_b and $c(x)$, we infer that

$$\Re\left\{2i\lambda\int_{\beta_2-3\varepsilon}^{\beta_3} p u_x \overline{v} dx\right\} + \int_{\beta_2-3\varepsilon}^{\beta_3} p'|v|^2 dx - \left[p|v|^2\right]_{\beta_2-3\varepsilon}^{\beta_3} = \Re\left\{\frac{2}{\lambda^\ell}\int_{\beta_2-3\varepsilon}^{\beta_3} p f_{1,x} \overline{v} dx\right\} \tag{4.33}$$

and

$$\Re\left\{2i\lambda\int_{\beta_2-3\varepsilon}^{\beta_3} p v \overline{u}_x dx\right\} + \Re\left\{\frac{2i\lambda}{a}\int_{\beta_2-3\varepsilon}^{\beta_2} p v \left(\sqrt{b_0} k(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \overline{w}(x,\xi) d\xi\right) dx\right\}$$

$$+\frac{1}{a}\int_{\beta_2-3\varepsilon}^{\beta_2}p'|S_{b_0}|^2dx+a\int_{\beta_2}^{\beta_3}p'|u_x|^2dx-\left[\frac{1}{a}p|S_b|^2\right]_{\beta_2-3\varepsilon}^{\beta_3}+\Re\left\{\frac{2c_0}{a}\int_{\beta_2-3\varepsilon}^{\beta_2}pz\overline{S}_{b_0}dx\right\}$$
$$+\Re\left\{2c_0\int_{\beta_2}^{\beta_3}pz\overline{u}_xdx\right\}=\Re\left\{\frac{2}{a\lambda^\ell}\int_{\beta_2-3\varepsilon}^{\beta_2}pf_2\overline{S}_{b_0}dx\right\}+\Re\left\{\frac{2}{\lambda^\ell}\int_{\beta_2}^{\beta_3}pf_2\overline{u}_xdx\right\}. \tag{4.34}$$

Combining (4.33) and (4.34), then applying the definition of p, we arrive at

$$\begin{aligned}
&|v(\beta_2-3\varepsilon)|^2+|v(\beta_3)|^2+\frac{1}{a}|S_b(\beta_2-3\varepsilon)|^2+\frac{1}{a}|S_b(\beta_3)|^2\\
=&-\int_{\beta_2-3\varepsilon}^{\beta_3}p'|v|^2dx-\frac{1}{a}\int_{\beta_2-3\varepsilon}^{\beta_2}p'|S_{b_0}|^2dx-a\int_{\beta_2}^{\beta_3}p'|u_x|^2dx-\Re\left\{\frac{2c_0}{a}\int_{\beta_2-3\varepsilon}^{\beta_2}pz\overline{S}_{b_0}dx\right\}\\
&-\Re\left\{\frac{2i\lambda}{a}\int_{\beta_2-3\varepsilon}^{\beta_2}pv\left(\sqrt{b_0}k(\alpha)\int_{\mathbb{R}}|\xi|^{\frac{2\alpha-1}{2}}\overline{w}(x,\xi)d\xi\right)dx\right\}-\Re\left\{2c_0\int_{\beta_2}^{\beta_3}pz\overline{u}_xdx\right\}\\
&+\Re\left\{\frac{2}{a\lambda^\ell}\int_{\beta_2-3\varepsilon}^{\beta_2}pf_2\overline{S}_{b_0}dx\right\}+\Re\left\{\frac{2}{\lambda^\ell}\int_{\beta_2}^{\beta_3}pf_2\overline{u}_xdx\right\}+\Re\left\{\frac{2}{\lambda^\ell}\int_{\beta_2-3\varepsilon}^{\beta_3}pf_{1,x}\overline{v}dx\right\}.
\end{aligned}$$

Based on Lemma 4.3, the Cauchy-Schwarz inequality and v,z,u_x are uniformly bounded in $L^2(0,L)$, as well as the fact that $\ell=2,\|f_2\|_{L^2(0,L)}=o(1),\|f_{1,x}\|_{L^2(0,L)}=o(1)$, we can conclude that

$$|v(\beta_2-3\varepsilon)|^2+|v(\beta_3)|^2+\frac{1}{a}|S_b(\beta_2-3\varepsilon)|^2+\frac{1}{a}|S_b(\beta_3)|^2=O(1).$$

Similarly, it follows from (4.6), one has

$$i\lambda y_x-z_x=\frac{f_{3,x}}{\lambda^\ell}. \tag{4.35}$$

Similarly, we multiply (4.35) and (4.7) by $2p\overline{z}$ and $2p\overline{y}_x$ respectively, integrate by parts on $(\beta_2-3\varepsilon,\beta_3)$ and take the real part, then take the sum and apply the definition of p and $c(x)$, we can derive

$$\begin{aligned}
&|z(\beta_2-3\varepsilon)|^2+|z(\beta_3)|^2+|y_x(\beta_2-3\varepsilon)|^2+|y_x(\beta_3)|^2\\
=&-\int_{\beta_2-3\varepsilon}^{\beta_3}p'(|z|^2+|y_x|^2)dx+\Re\left\{2c_0\int_{\beta_2-3\varepsilon}^{\beta_3}pv\overline{y}_xdx\right\}+\Re\left\{\frac{2}{\lambda^\ell}\int_{\beta_2-3\varepsilon}^{\beta_3}p(f_4\overline{y}_x+f_{3,x}\overline{z}dx\right\}.
\end{aligned}$$

Thanks to v,z,y_x are uniformly bounded in $L^2(0,L)$, the Cauchy-Schwarz inequality and the fact that $\ell=2,\|f_4\|_{L^2(0,L)}=o(1),\|f_{3,x}\|_{L^2(0,L)}=o(1)$, we can conclude that

$$|z(\beta_2-3\varepsilon)|^2+|z(\beta_3)|^2+|y_x(\beta_2-3\varepsilon)|^2+|y_x(\beta_3)|^2=O(1).$$

Consequently, we complete the proof of Lemma 4.6. □

Lemma 4.7 *Set $h\in C^1([0,L])$, then the solution $(u,v,y,z,w)\in\mathcal{D}(\mathcal{A})$ of system (4.4)-(4.8) satisfies*

$$\begin{aligned}
&\int_0^L h'\left(a^{-1}|S_b|^2+|v|^2+|z|^2+|y_x|^2\right)dx-\left[h(a^{-1}|S_b|^2+|y_x|^2)\right]_0^L-\Re\left\{2\int_0^L c(x)hv\overline{y}_xdx\right\}\\
&+\Re\left\{\frac{2}{a}\int_0^L c(x)hz\overline{S}_bdx\right\}+\Re\left\{\frac{2i\lambda}{a}\int_0^{\beta_2}hv\sqrt{b_0}k(\alpha)\int_{\mathbb{R}}|\xi|^{\frac{2\alpha-1}{2}}\overline{w}(x,\xi)d\xi dx\right\}\\
=&\Re\left\{\frac{2}{\lambda^\ell}\int_0^L h\left(\overline{f}_{1,x}v+\overline{f}_{3,x}z+f_4\overline{y}_x\right)dx\right\}+\Re\left\{\frac{2}{a\lambda^\ell}\int_0^L hf_2\overline{S}_bdx\right\}.
\end{aligned} \tag{4.36}$$

Proof. Multiplying (4.5) and (4.7) by $2a^{-1}h\overline{S}_b$ and $2h\overline{y}_x$ respectively, integrating in the interval $(0,L)$ and taking the real part, we infer that

$$\Re\left\{\frac{2i\lambda}{a}\int_0^L hv\overline{S}_b dx\right\}-\frac{1}{a}\int_0^L h\left(|S_b|^2\right)_x dx+\Re\left\{\frac{2}{a}\int_0^L c(x)hz\overline{S}_b dx\right\}=\Re\left\{\frac{2}{a\lambda^\ell}\int_0^L hf_2\overline{S}_b dx\right\} \tag{4.37}$$

and

$$\Re\left\{2i\lambda\int_0^L hz\overline{y}_x dx\right\}-\int_0^L h\left(|y_x|^2\right)_x dx-\Re\left\{2\int_0^L c(x)hv\overline{y}_x dx\right\}=\Re\left\{\frac{2}{\lambda^\ell}\int_0^L hf_4\overline{y}_x dx\right\}. \tag{4.38}$$

Based on (4.4) and (4.6), it is easy to check that

$$i\lambda\overline{u}_x=-\overline{v}_x-\lambda^{-\ell}\overline{f}_{1,x}, \tag{4.39}$$

$$i\lambda\overline{y}_x=-\overline{z}_x-\lambda^{-\ell}\overline{f}_{3,x}. \tag{4.40}$$

From the definition of S_b and (4.39), we arrive at

$$i\lambda\overline{S}_b=\begin{cases}-a(\overline{v}_x+\lambda^{-\ell}\overline{f}_{1,x})+i\lambda\sqrt{b_0}k(\alpha)\displaystyle\int_{\mathbb{R}}|\xi|^{\frac{2\alpha-1}{2}}\overline{w}(x,\xi)d\xi, & x\in(0,\beta_2),\\ -a(\overline{v}_x+\lambda^{-\ell}\overline{f}_{1,x}), & x\in(\beta_2,L).\end{cases} \tag{4.41}$$

Accordingly, substituting (4.40) and (4.41) into (4.38) and (4.37) respectively, then using integration by parts, we derive

$$\begin{aligned}&\int_0^L h'\left(a^{-1}|S_b|^2+|v|^2\right)dx-\left[ha^{-1}|S_b|^2\right]_0^L+\Re\left\{\frac{2i\lambda}{a}\int_0^{\beta_2}hv\sqrt{b_0}k(\alpha)\int_{\mathbb{R}}|\xi|^{\frac{2\alpha-1}{2}}\overline{w}(x,\xi)d\xi dx\right\}\\&+\Re\left\{\frac{2}{a}\int_0^L c(x)hz\overline{S}_b dx\right\}=\Re\left\{\frac{2}{\lambda^\ell}\int_0^L h\overline{f}_{1,x}v dx\right\}+\Re\left\{\frac{2}{a\lambda^\ell}\int_0^L hf_2\overline{S}_b dx\right\}\end{aligned} \tag{4.42}$$

and

$$\begin{aligned}&\int_0^L h'\left(|z|^2+|y_x|^2\right)dx-\left[h|y_x|^2\right]_0^L-\Re\left\{2\int_0^L c(x)hv\overline{y}_x dx\right\}\\=&\Re\left\{\frac{2}{\lambda^\ell}\int_0^L hf_4\overline{y}_x dx\right\}+\Re\left\{\frac{2}{\lambda^\ell}\int_0^L h\overline{f}_{3,x}z dx\right\}.\end{aligned} \tag{4.43}$$

Consequently, combining (4.42) and (4.43), we get the desired result (4.36). The proof of Lemma 4.7 is complete. □

For further proof, we set $0<\varepsilon<\min\left\{\frac{\beta_1}{2},\frac{\beta_2-\beta_1}{4}\right\}$, and take the cut-off function $\rho_4(x),\rho_5(x)\in C^1([0,L])$ such that

$$\rho_4(x)=\begin{cases}1 & if\quad x\in[0,\beta_1+\varepsilon],\\ 0 & if\quad x\in[\beta_2-3\varepsilon,L],\\ 0\le\rho_4\le 1 & otherwise,\end{cases}\quad and\quad \rho_5(x)=\begin{cases}0 & if\quad x\in[0,\beta_1+\varepsilon],\\ 1 & if\quad x\in[\beta_2-3\varepsilon,L],\\ 0\le\rho_5\le 1 & otherwise.\end{cases}$$

Lemma 4.8 *Set* $0<\varepsilon<\min\left\{\frac{\beta_1}{2},\frac{\beta_2-\beta_1}{4}\right\}$*. Then the solution* $(u,v,y,z,w)\in\mathcal{D}(\mathcal{A})$ *of system* (4.4)-(4.8) *satisfies*

$$\int_0^{\beta_1+\varepsilon}|v|^2dx+\int_0^{\beta_1+\varepsilon}|y_x|^2dx+\int_0^{\beta_1+\varepsilon}|z|^2dx=o(1), \tag{4.44}$$

$$a\int_{\beta_2}^{L}|u_x|^2dx+\int_{\beta_2-3\varepsilon}^{L}|v|^2dx+\int_{\beta_2-3\varepsilon}^{L}|y_x|^2dx+\int_{\beta_2-3\varepsilon}^{L}|z|^2dx=o(1). \tag{4.45}$$

Proof. In view of Lemma 4.7, here we take $h=x\rho_4$. Therefore, form the definition of ρ_4 and S_b, it follows that

$$\begin{aligned}
&\int_0^{\beta_1+\varepsilon}|v|^2dx+\int_0^{\beta_1+\varepsilon}|y_x|^2dx+\int_0^{\beta_1+\varepsilon}|z|^2dx\\
=&-\frac{1}{a}\int_0^{\beta_1+\varepsilon}|S_{b_0}|^2dx-\int_{\beta_1+\varepsilon}^{\beta_2-3\varepsilon}(\rho_4+x\rho_4')\left(a^{-1}|S_{b_0}|^2+|v|^2+|z|^2+|y_x|^2\right)dx\\
&+\Re\left\{2\int_0^L c(x)x\rho_4 v\overline{y}_xdx\right\}-\Re\left\{\frac{2}{a}\int_0^L c(x)x\rho_4 z\overline{S}_bdx\right\}\\
&-\Re\left\{\frac{2i\lambda}{a}\int_0^{\beta_2}x\rho_4 v\sqrt{b_0}k(\alpha)\int_{\mathbb{R}}|\xi|^{\frac{2\alpha-1}{2}}\overline{w}(x,\xi)d\xi dx\right\}\\
&+\Re\left\{\frac{2}{\lambda^\ell}\int_0^L x\rho_4\left(\overline{f}_{1,x}v+\overline{f}_{3,x}z+f_4\overline{y}_x\right)dx\right\}+\Re\left\{\frac{2}{a\lambda^\ell}\int_0^L x\rho_4 f_2\overline{S}_bdx\right\}.
\end{aligned}\tag{4.46}$$

Owing to Lemma 4.3-4.5 and the definition of ρ_4, we arrive at

$$-\frac{1}{a}\int_0^{\beta_1+\varepsilon}|S_{b_0}|^2dx-\int_{\beta_1+\varepsilon}^{\beta_2-3\varepsilon}(\rho_4+x\rho_4')\left(a^{-1}|S_{b_0}|^2+|v|^2+|z|^2+|y_x|^2\right)dx=o(1). \tag{4.47}$$

Based on the definition of ρ_4, the Cauchy-Schwarz inequality and v,z,y_x are uniformly bounded in $L^2(0,L)$, as well as $\|f_{1,x}\|_{L^2(0,L)}=o(1),\|f_4\|_{L^2(0,L)}=o(1),\|f_{3,x}\|_{L^2(0,L)}=o(1)$, we get

$$\Re\left\{\frac{2}{\lambda^\ell}\int_0^L x\rho_4\left(\overline{f}_{1,x}v+\overline{f}_{3,x}z+f_4\overline{y}_x\right)dx\right\}=\frac{o(1)}{\lambda^\ell}. \tag{4.48}$$

Similarly, from Lemma 4.2-4.5, the definition of $\rho_4, c(x)$ and S_b, the Cauchy-Schwarz inequality, $\|f_2\|_{L^2(0,L)}=o(1)$ and the fact that v and y_x are uniformly bounded in $L^2(0,L)$, we refer that

$$\left|\Re\left\{2\int_0^L c(x)x\rho_4 v\overline{y}_xdx\right\}\right|=\left|\Re\left\{2c_0\int_{\beta_1}^{\beta_2-3\varepsilon}x\rho_4 v\overline{y}_xdx\right\}\right|=o(1), \tag{4.49}$$

$$\left|\Re\left\{\frac{2}{a}\int_0^L c(x)x\rho_4 z\overline{S}_bdx\right\}\right|=\left|\Re\left\{\frac{2c_0}{a}\int_{\beta_1}^{\beta_2-3\varepsilon}x\rho_4 z\overline{S}_{b_0}dx\right\}\right|=\frac{o(1)}{|\lambda|^{\frac{\ell}{2}}}, \tag{4.50}$$

$$\begin{aligned}
&\left|\Re\left\{\frac{2i\lambda}{a}\int_0^{\beta_2}x\rho_4 v\sqrt{b_0}k(\alpha)\int_{\mathbb{R}}|\xi|^{\frac{2\alpha-1}{2}}\overline{w}(x,\xi)d\xi dx\right\}\right|\\
=&\left|\Re\left\{\frac{2i\lambda}{a}\int_0^{\beta_2-3\varepsilon}x\rho_4 v\sqrt{b_0}k(\alpha)\int_{\mathbb{R}}|\xi|^{\frac{2\alpha-1}{2}}\overline{w}(x,\xi)d\xi dx\right\}\right|=\frac{o(1)}{|\lambda|^{\frac{\ell}{2}-1}},
\end{aligned}\tag{4.51}$$

$$\left|\Re\left\{\frac{2}{a\lambda^{\ell}}\int_0^L x\rho_4 f_2\overline{S}_b dx\right\}\right| = \left|\Re\left\{\frac{2}{a\lambda^{\ell}}\int_0^{\beta_2-3\varepsilon} x\rho_4 f_2\overline{S}_{b_0} dx\right\}\right| = \frac{o(1)}{|\lambda|^{\frac{3\ell}{2}}}. \tag{4.52}$$

From the fact that $\ell = 2$, then substituting (4.47)-(4.52) into (4.46), we get (4.44). Moreover, by using Lemma 4.7 and take $h = (x-L)\rho_5$. In the same manner, by applying Lemma 4.2-4.5, the definition of ρ_5 and S_b, the Cauchy-Schwarz inequality, $\|f_{1,x}\|_{L^2(0,L)} = o(1), \|f_2\|_{L^2(0,L)} = o(1), \|f_{3,x}\|_{L^2(0,L)} = o(1), \|f_4\|_{L^2(0,L)} = o(1), \ell = 2$ and u_x, y_x, v, z are uniformly bounded in $L^2(0,L)$, we refer that

$$\begin{aligned}
&a\int_{\beta_2}^L |u_x|^2 dx + \int_{\beta_2-3\varepsilon}^L |v|^2 dx + \int_{\beta_2-3\varepsilon}^L |y_x|^2 dx + \int_{\beta_2-3\varepsilon}^L |z|^2 dx \\
=&\Re\left\{2\int_0^L c(x)(x-L)\rho_5 v\overline{y}_x dx\right\} - \Re\left\{\frac{2}{a}\int_0^L c(x)(x-L)\rho_5 z\overline{S}_b dx\right\} - \frac{1}{a}\int_{\beta_1+\varepsilon}^{\beta_2} |S_{b_0}|^2 dx \\
&-\int_{\beta_1+\varepsilon}^{\beta_2-3\varepsilon} \left(\rho_5 + (x-L)\rho_5'\right)\left(a^{-1}|S_{b_0}|^2 + |v|^2 + |z|^2 + |y_x|^2\right) dx \\
&-\Re\left\{\frac{2i\lambda}{a}\int_{\beta_1+\varepsilon}^{\beta_2} (x-L)\rho_5 v\sqrt{b_0}k(\alpha)\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}}\overline{w}(x,\xi) d\xi dx\right\} \\
&+\Re\left\{\frac{2}{\lambda^{\ell}}\int_0^L (x-L)\rho_5\left(\overline{f}_{1,x}v + \overline{f}_{3,x}z + f_4\overline{y}_x\right) dx\right\} + \Re\left\{\frac{2}{a\lambda^{\ell}}\int_0^L (x-L)\rho_5 f_2\overline{S}_b dx\right\} \\
=&\underbrace{\Re\left\{2\int_0^L c(x)(x-L)\rho_5 v\overline{y}_x dx\right\} - \Re\left\{\frac{2}{a}\int_0^L c(x)(x-L)\rho_5 z\overline{S}_b dx\right\}}_{J} + o(1).
\end{aligned} \tag{4.53}$$

Then, from the definition of $c(x), \rho_5$ and S_b, Lemma 4.2-4.5, the Cauchy-Schwarz inequality and z is uniformly bounded in $L^2(0,L)$, we conclude that

$$\begin{aligned}
J =&\Re\left\{2c_0\int_{\beta_1+\varepsilon}^{\beta_2-3\varepsilon} (x-L)\rho_5 v\overline{y}_x dx\right\} + \Re\left\{2c_0\int_{\beta_2-3\varepsilon}^{\beta_3} (x-L)v\overline{y}_x dx\right\} \\
&-\Re\left\{\frac{2c_0}{a}\int_{\beta_1+\varepsilon}^{\beta_2-3\varepsilon} (x-L)\rho_5 z\overline{S}_{b_0} dx\right\} - \Re\left\{2c_0\int_{\beta_2-3\varepsilon}^{\beta_3} (x-L)z\overline{u}_x dx\right\} \\
&-\Re\left\{\frac{2c_0}{a}\int_{\beta_2-3\varepsilon}^{\beta_2} (x-L)z\sqrt{b_0}k(\alpha)\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}}\overline{w}(x,\xi) d\xi dx\right\} \\
=&\Re\left\{2c_0\int_{\beta_2-3\varepsilon}^{\beta_3} (x-L)v\overline{y}_x dx\right\} - \Re\left\{2c_0\int_{\beta_2-3\varepsilon}^{\beta_3} (x-L)z\overline{u}_x dx\right\} + o(1).
\end{aligned} \tag{4.54}$$

By using (4.4) and (4.6), one can obtain

$$\overline{u}_x = i\lambda^{-1}\overline{v}_x + i\lambda^{-\ell-1}\overline{f}_{1,x} \quad and \quad \overline{y}_x = i\lambda^{-1}\overline{z}_x + i\lambda^{-\ell-1}\overline{f}_{3,x}. \tag{4.55}$$

Inserting (4.55) in (4.54), then applying $\|f_{1,x}\|_{L^2(0,L)} = o(1), \|f_{3,x}\|_{L^2(0,L)} = o(1)$ and v, z are uniformly bounded in $L^2(0,L)$, one states that

$$J = \Re\left\{\frac{2c_0 i}{\lambda}\int_{\beta_2-3\varepsilon}^{\beta_3} (x-L)v\overline{z}_x dx\right\} - \Re\left\{\frac{2c_0 i}{\lambda}\int_{\beta_2-3\varepsilon}^{\beta_3} (x-L)z\overline{v}_x dx\right\} + o(1). \tag{4.56}$$

Then, after integrating by parts in the second term of (4.56), it follows that

$$J = \Re\left\{\frac{2c_0 i}{\lambda}\int_{\beta_2-3\varepsilon}^{\beta_3} z\overline{v}dx\right\} - \Re\left\{\frac{2c_0 i}{\lambda}\left[(x-L)z\overline{v}\right]_{\beta_2-3\varepsilon}^{\beta_3}\right\} + o(1). \tag{4.57}$$

By applying the Cauchy-Schwarz inequality and v, z are uniformly bounded in $L^2(0, L)$, then taking into account the fact that $|v(\beta_2 - 3\varepsilon)| = O(1), |v(\beta_3)| = O(1), |z(\beta_2 - 3\varepsilon)| = O(1)$ and $|z(\beta_3)| = O(1)$ in Lemma 4.6, we derive

$$\left|\Re\left\{\frac{2c_0 i}{\lambda}\int_{\beta_2-3\varepsilon}^{\beta_3} z\overline{v}dx\right\}\right| = \frac{O(1)}{|\lambda|} = o(1) \tag{4.58}$$

and

$$\left|\Re\left\{\frac{2c_0 i}{\lambda}\left[(x-L)z\overline{v}\right]_{\beta_2-3\varepsilon}^{\beta_3}\right\}\right| = \frac{O(1)}{|\lambda|} = o(1). \tag{4.59}$$

Combining (4.57), (4.58) and (4.59), we arrive at

$$J = o(1).$$

Thanks to (4.53), it is easy to get (4.45). Thus, we complete the proof of Lemma 4.8. □

Proof of Theorem 4.2. By using Lemma 4.2, Lemma 4.4, Lemma 4.5 and the fact that $\ell = 2$, we derive

$$\begin{aligned}&\int_0^L\int_{\mathbb{R}}(\xi^2+\eta)|w(x,\xi)|^2 d\xi dx = \frac{o(1)}{\lambda^\ell}, \quad \int_0^{\beta_2}|u_x|^2dx = \frac{o(1)}{\lambda^{\ell+\alpha+1}}, \quad \int_\varepsilon^{\beta_2-\varepsilon}|v|^2dx = o(1),\\ &\int_{\beta_1}^{\beta_2-2\varepsilon}|z|^2dx = o(1), \quad \int_{\beta_1+\varepsilon}^{\beta_2-3\varepsilon}|y_x|^2dx = o(1).\end{aligned} \tag{4.60}$$

Then from Lemma 4.8 and (4.60), we arrive at

$$\begin{aligned}&\int_0^\varepsilon|v|^2dx = o(1), \quad \int_0^{\beta_1+\varepsilon}|y_x|^2dx = o(1), \quad \int_0^{\beta_1}|z|^2dx = o(1), \quad \int_{\beta_2}^L|u_x|^2dx = o(1),\\ &\int_{\beta_2-\varepsilon}^L|v|^2dx = o(1), \quad \int_{\beta_2-3\varepsilon}^L|y_x|^2dx = o(1), \quad \int_{\beta_2-2\varepsilon}^L|z|^2dx = o(1).\end{aligned} \tag{4.61}$$

Combining (4.60) and (4.61), we conclude that $\|U\|_{\mathcal{H}} = o(1)$, which contradicts with (4.2). Accordingly, we show that condition (H_2) holds. Therefore, based on Theorem 4.1, we obtain the desired conclusion (4.1), which completes the proof of Theorem 4.2. □

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