Turing-Hopf bifurcation of a diffusive Holling-Tanner model with nonlocal effect and digestion time delay

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Abstract

In this paper, we discuss the Turing-Hopf bifurcation of a diffusive Holling-Tanner model with nonlocal effect and digestion time delay. The stability, Turing bifurcation, Hopf bifurcation and Turing-Hopf bifurcation are first researched. Then we derive the algorithm for calculating the normal form of Turing-Hopf bifurcation of a diffusive Holling-Tanner model with nonlocal effect and digestion time delay. At last, we carry out some numerical simulations to verify our theoretical analysis results. The stable positive constant steady state and the stable spatially inhomogeneous periodic solutions are found. Furthermore, the evolution process from unstable spatially inhomogeneous steady states to stable positive constant steady state, the evolution process from unstable spatially inhomogeneous steady states to stable spatially inhomogeneous periodic solutions, the evolution process from one unstable spatially inhomogeneous periodic solution to another stable spatially inhomogeneous periodic solution and the evolution process from unstable spatially inhomogeneous periodic solution to stable positive constant steady state are also found.

Keywords: Diffusive Holling-Tanner model, Nonlocal effect, Digestion time delay, Turing-Hopf bifurcation, Normal form

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1. Introduction

By considering that the self-regulation of prey and by incorporating the Holling type-II functional response function, May [1] has proposed a predator-prey model, which is known as the Holling-Tanner predator-prey model and has the form

$$\begin{cases} \dfrac{du(x,t)}{dt} = r_1 u(x,t)\left(1-\dfrac{u(x,t)}{k}\right) - \dfrac{qu(x,t)v(x,t)}{u(x,t)+m}, \\ \dfrac{dv(x,t)}{dt} = r_2 v(x,t)\left(1-\dfrac{v(x,t)}{\sigma u(x,t)}\right), \end{cases} \tag{1.1}$$

where $u(t)$ and $v(t)$ represent the densities of prey and predator, respectively, r_1 and r_2 are the intrinsic growth rates of the prey and predator, respectively, k is the carrying capacity of the prey, q is the maximum value of prey consumed by per predator per unit time, m is a saturation value, and σ is the conversion or consumption rate of prey to predator. By considering that in the real world, the predator and prey populations may move for many reasons, then by combining with model (1.1), the reaction-diffusion system

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subjects to homogeneous Neumann boundary condition can be written as

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = D_1 \Delta u(x,t) + r_1 u(x,t)\left(1 - \dfrac{u(x,t)}{k}\right) - \dfrac{q u(x,t) v(x,t)}{u(x,t)+m}, & x \in \Omega,\ t > 0, \\ \dfrac{\partial v(x,t)}{\partial t} = D_2 \Delta v(x,t) + r_2 v(x,t)\left(1 - \dfrac{v(x,t)}{\sigma u(x,t)}\right), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \mathbf{n}} = \dfrac{\partial v(x,t)}{\partial \mathbf{n}} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}$ is a smooth and bounded domain, D_1 and D_2 are the diffusion coefficients of $u(x,t)$ and $v(x,t)$, respectively, $\Delta u(x,t) = \partial^2 u(x,t)/\partial x^2$, $\Delta v(x,t) = \partial^2 v(x,t)/\partial x^2$, $\mathbf{n}$ is the outward unit normal vector at the smooth boundary $\partial\Omega$. Moreover, $u_0(x)$ and $v_0(x)$ are the initial functions. For model (1.2), with a non-dimensionalized change of variables

$$u(x,t) \to \frac{u(x,t)}{k},\ v(x,t) \to \frac{v(x,t)}{\sigma k},\ t \to r_1 t,$$

and let

$$d_1 = \frac{D_1}{r_1},\ d_2 = \frac{D_2}{r_1},\ a = \frac{q\sigma}{r_1},\ r = \frac{r_2}{r_1},\ b = \frac{m}{k},$$

An et al. [2] obtained the simplified dimensionless ratio-dependent Holling-Tanner model with diffusion subjects to homogeneous Neumann boundary condition

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d_1 \Delta u(x,t) + u(x,t)(1 - u(x,t)) - \dfrac{a u(x,t) v(x,t)}{u(x,t)+b}, & x \in \Omega,\ t > 0, \\ \dfrac{\partial v(x,t)}{\partial t} = d_2 \Delta v(x,t) = r v(x,t)\left(1 - \dfrac{v(x,t)}{u(x,t)}\right), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \mathbf{n}} = \dfrac{\partial v(x,t)}{\partial \mathbf{n}} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

and they studied the Turing-Hopf bifurcation of this model in the spatial domain $\Omega = (0, \ell\pi)$ with $\ell \in \mathbb{R}^+$. In addition, for the diffusive Holling-Tanner model, Peng et al. [3] studied the local and global stability of the unique positive equilibrium, Banerjee et al. [4] studied the Turing and non-Turing patterns for some fixed parametric values. Furthermore, Ma et al. [5] studied the Hopf bifurcation and steady state bifurcation.

By noticing that the competition between prey for limited resources is often nonlocal in natural environment and the consumption of resources at a spatial location depends not only on the local population density but also on the weighted average of the nearby population, thus Furter et al. [6] and Britton [7] all incorporated the spatial convolution integral to a single-species model. In addition, the nonlocal competition effect has been extensively studied in many different cases, such as single-species models [8], competing population models [9, 10] and predator-prey models [11, 12, 13, 14]. As shown in [6], the most straightforward way of introducing nonlocal effect is to replace the term $1 - u(x,t)$ in model (1.3) by $1 - \widehat{u}$ with

$$\widehat{u} = \int_\Omega G(x,y) u(y,t) dy,$$

where $G(x,y)$ is some reasonable kernels. By taking $\Omega = (0, \ell\pi)$, $\ell \in \mathbb{R}^+$ and $G(x,y) = 1/(\ell\pi)$, the model

(1.3) becomes the nonlocal model

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d_1\Delta u(x,t) + u(x,t)\left(1 - \dfrac{1}{\ell\pi}\displaystyle\int_0^{l\pi} u(y,t)dy\right) - \dfrac{au(x,t)v(x,t)}{u(x,t)+b}, \ x\in(0,\ell\pi), \ t>0, \\ \dfrac{\partial v(x,t)}{\partial t} = d_2\Delta v(x,t) + rv(x,t)\left(1 - \dfrac{v(x,t)}{u(x,t)}\right), \ x\in(0,\ell\pi), \ t>0, \\ u_x(x,t) = v_x(x,t) = 0, \ x = 0,\ell\pi, \ t\geq 0, \\ u(x,0) = u_0(x)\geq 0, \ v(x,0) = v_0(x)\geq 0, \ x\in(0,\ell\pi). \end{cases} \tag{1.4}$$

Here, we take $G(x,y) = 1/|\Omega|$ such that the nonlocal effect does not affect the positive equilibrium of model (1.4). This kernel function means that $\widehat{u}$ is taken as the mean value of $u(x,t)$ in the space interval $(0,\ell\pi)$. Furthermore, the digestion, gestation, hunting, migration and maturation time delays are usually considered in the predator-prey models [15, 16, 17]. Moreover, the model with nonlocal effect involving time delay was first researched by Britton [18], and since then, the predator-prey models involving the nonlocal effect and time delay are consided by many scholars [19, 20, 21, 22]. In this paper, by combining with model (1.4), we study the Turing-Hopf bifurcation of the following diffusive Holling-Tanner model with nonlocal effect and digestion time delay subjects to homogeneous Neumann boundary condition

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d_1\Delta u(x,t) + u(x,t)\left(1 - \dfrac{1}{\ell\pi}\displaystyle\int_0^{l\pi} u(y,t)dy\right) - \dfrac{au(x,t)v(x,t)}{u(x,t)+b}, \ x\in(0,\ell\pi), \ t>0, \\ \dfrac{\partial v(x,t)}{\partial t} = d_2\Delta v(x,t) + rv(x,t)\left(1 - \dfrac{v(x,t)}{u(x,t-\tau)}\right), \ x\in(0,\ell\pi), \ t>0, \\ u_x(x,t) = v_x(x,t) = 0, \ x = 0,\ell\pi, \ t\geq 0, \\ u(x,0) = u_0(x)\geq 0, \ v(x,0) = v_0(x)\geq 0, \ x\in(0,\ell\pi), \ \ -\tau\leq t\leq 0, \end{cases} \tag{1.5}$$

where $\tau\geq 0$ is the digestion time delay. It is necessary to point out that the algorithm for calculating the normal form of Turing-Hopf bifurcation of the general reaction-diffusion systems without time delay or with time delay has been developed in [23, 24, 25, 26, 27] are not applicable to the case with nonlocal effect and time delay.

This paper is organized as follows. In Section 2, we discuss the stability of positive constant steady state of model (1.5), and the Turing bifurcation, Hopf bifurcation, Turing-Hopf bifurcation of model (1.5). In Section 3, we derive an algorithm for calculating the normal form of Turing-Hopf bifurcation of model (1.5). In Section 4, we use the obtained third-order truncated normal form to investigate the classification near the Turing-Hopf bifurcation point, and we carry out some numerical simulations to demonstrate our theoretical analysis results. Finally, we give a brief conclusion and discussion in Section 5. The detailed procedures for calculating the normal form (3.26) are given in Appendix A.

2. Stability and bifurcation analysis

Define the real-valued Sobolev space

$$X := \left\{ (u,v)^T \in \left(W^{2,2}(0,\ell\pi)\right)^2 : \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \text{ at } x = 0,\ell\pi \right\}$$

with the inner product defined by

$$[U_1,U_2] = \int_0^{\ell\pi} U_1^T U_2 \, dx \text{ for } U_1 = (u_1,v_1)^T \in X \text{ and } U_2 = (u_2,v_2)^T \in X,$$

where the symbol T represents the transpose of vector, and let $\mathcal{C} := C([-1,0];X)$ be the Banach space of continuous mappings from $[-1,0]$ to X with the sup norm. It is well known that the eigenvalue problem

$$\begin{cases} \widetilde{\varphi}''(x) = \widetilde{\lambda}\widetilde{\varphi}(x), \ x \in (0, \ell\pi), \\ \widetilde{\varphi}'(0) = \widetilde{\varphi}'(\ell\pi) = 0 \end{cases}$$

has eigenvalues $\widetilde{\lambda}_n = -n^2/\ell^2$ with corresponding normalized eigenfunctions

$$\beta_n^{(j)} = \gamma_n(x)e_j, \ \gamma_n(x) = \frac{\cos(nx/\ell)}{\|\cos(nx/\ell)\|_{L^2}} = \begin{cases} \dfrac{1}{\sqrt{\ell\pi}}, & n = 0, \\ \sqrt{\dfrac{2}{\ell\pi}} \cos\left(\dfrac{nx}{\ell}\right), & n \geq 1, \end{cases} \tag{2.1}$$

where $e_j, \ j = 1,2$ is the unit coordinate vector of $\mathbb{R}^2$, and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is often called wave number, $\mathbb{N}_0$ is the set of all non-negative integers, $\mathbb{N} = \{1, 2, ...\}$ represents the set of all positive integers.

The positive constant steady state of model (1.5) is

$$E_*(u_*, v_*) = \left(\frac{(1-a-b) + \sqrt{(a+b-1)^2 + 4b}}{2}, \frac{(1-a-b) + \sqrt{(a+b-1)^2 + 4b}}{2} \right), \tag{2.2}$$

and the straightforward calculation gives the linearization of (1.5) at $E_*(u_*, v_*)$

$$\begin{pmatrix} \frac{\partial u(x,t)}{\partial t} \\ \frac{\partial v(x,t)}{\partial t} \end{pmatrix} = D \begin{pmatrix} \Delta u(x,t) \\ \Delta v(x,t) \end{pmatrix} + A_1 \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} + A_2 \begin{pmatrix} u(x,t-\tau) \\ v(x,t-\tau) \end{pmatrix} + B_1 \begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix}, \tag{2.3}$$

where

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \ A_1 = \begin{pmatrix} 1 - u_* - \frac{abv_*}{(b+u_*)^2} & -\frac{au_*}{b+u_*} \\ 0 & -r \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}, \ B_1 = \begin{pmatrix} -u_* & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.4}$$

Thus, the characteristic equation of (2.3) is

$$\prod_{n \in \mathbb{N}_0} \Gamma_n(\lambda) = 0,$$

where $\Gamma_n(\lambda) = \det(\mathcal{M}_n(\lambda))$. Moreover, by noticing that

$$\mathcal{M}_n(\lambda) = \begin{cases} \lambda I_2 - A_1 - A_2 e^{-\lambda\tau} - B_1, & n = 0, \\ \lambda I_2 + D\dfrac{n^2}{\ell^2} - A_1 - A_2 e^{-\lambda\tau}, & n \in \mathbb{N} \end{cases} \tag{2.5}$$

with I_2 is the identity matrix of size 2, then from (2.5), we have

$$\begin{aligned} \Gamma_n(\lambda) &= \det(\mathcal{M}_n(\lambda)) \\ &= \begin{cases} \lambda^2 - \left(1 - 2u_* - r - \dfrac{abv_*}{(b+u_*)^2}\right)\lambda + \dfrac{au_* r e^{-\lambda\tau}}{b+u_*} - \left(1 - 2u_* - \dfrac{abv_*}{(b+u_*)^2}\right) r = 0, & n = 0, \\ \lambda^2 + p_n\lambda + r_n + q_n e^{-\lambda\tau} = 0, & n \in \mathbb{N}, \end{cases} \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} p_n &= (d_1 + d_2)\frac{n^2}{\ell^2} + r - \left(1 - u_* - \frac{abv_*}{(b+u_*)^2}\right), \\ r_n &= d_1 d_2 \frac{n^4}{\ell^4} + \left(d_1 r - d_2\left(1 - u_* - \frac{abv_*}{(b+u_*)^2}\right)\right)\frac{n^2}{\ell^2} - r\left(1 - u_* - \frac{abv_*}{(b+u_*)^2}\right), \\ q_n &= \frac{au_* r}{b+u_*}. \end{aligned} \tag{2.7}$$

When $n=0$ and $\tau=0$, the first mathematical expression in (2.6) becomes

$$\lambda^2-\left(1-2u_*-r-\frac{abv_*}{(b+u_*)^2}\right)\lambda+\frac{au_*r}{b+u_*}-\left(1-2u_*-\frac{abv_*}{(b+u_*)^2}\right)r=0, \tag{2.8}$$

and then from (2.8), we know that when

$$\frac{au_*r}{b+u_*}-\left(1-2u_*-\frac{abv_*}{(b+u_*)^2}\right)r>0,\ 1-2u_*-r-\frac{abv_*}{(b+u_*)^2}<0,$$

i.e.,

$$r>\max\left(0,1-2u_*-\frac{abv_*}{(b+u_*)^2}\right),\ \frac{2abu_*+au_*^2}{(b+u_*)^2}-1+2u_*>0, \tag{2.9}$$

the positive constant steady state of model (1.5) is asymptotically stable.

When $n\in\mathbb{N}$ and $\tau=0$, the second mathematical expression in (2.6) becomes

$$\lambda^2-T_n\lambda+D_n=0, \tag{2.10}$$

where

$$\begin{aligned} T_n&=-p_n=-(d_1+d_2)\frac{n^2}{\ell^2}-r+\left(1-u_*-\frac{abv_*}{(b+u_*)^2}\right),\\ D_n&=r_n+q_n=d_1d_2\frac{n^4}{\ell^4}+\left(d_1r-d_2\left(1-u_*-\frac{abv_*}{(b+u_*)^2}\right)\right)\frac{n^2}{\ell^2}-r+ru_*+\frac{2abru_*+aru_*^2}{(b+u_*)^2}. \end{aligned} \tag{2.11}$$

2.1. Turing bifurcation

When $n\in\mathbb{N}$ and $\tau=0$, by combining with (2.9) and the first mathematical expression in (2.11), we have the following Theorem 2.1.

Theorem 2.1. *Assume that*

$$r>\max\left(0,1-2u_*-\frac{abv_*}{(b+u_*)^2}\right),\ \frac{2abu_*+au_*^2}{(b+u_*)^2}-1+2u_*>0$$

holds, then for $n\in\mathbb{N}$ and $\tau=0$,

(i) when $1-u_*<0$, we have $T_n<0$;

(ii) when $1-u_*>0$, we have

$$T_n\begin{cases}>0, & \text{if} \quad r<\widetilde{r}_n,\\ =0, & \text{if} \quad r=\widetilde{r}_n,\\ <0, & \text{if} \quad r>\widetilde{r}_n,\end{cases}$$

where

$$\widetilde{r}_n=\frac{-(d_1+d_2)(b+u_*)^2n^2-abv_*\ell^2+(1-u_*)(b+u_*)^2\ell^2}{\ell^2(b+u_*)^2}. \tag{2.12}$$

Proof. From the first mathematical expression in (2.11), we know that T_n can be rewritten as

$$T_n=\frac{-(d_1+d_2)(b+u_*)^2n^2-r\ell^2(b+u_*)^2-abv_*\ell^2+(1-u_*)(b+u_*)^2\ell^2}{\ell^2(b+u_*)^2}.$$

It is easy to verify that when $1-u_*<0$, we have $T_n<0$ for any $n\in\mathbb{N}$ and $\tau=0$. Thus, the conclusion (i) is proved.

If $1-u_*>0$, we have

$$T_n = \begin{cases} >0, & \text{if} \quad r<\widetilde{r}_n, \\ =0, & \text{if} \quad r=\widetilde{r}_n, \\ <0, & \text{if} \quad r>\widetilde{r}_n, \end{cases}$$

where $\widetilde{r}_n$ is defined by (2.12). Therefore, the conclusion (ii) is proved. ■

In the following, we always assume that

$$(C_0): \ r > \max\left(0, 1-2u_* - \frac{abv_*}{(b+u_*)^2}, \widetilde{r}_n\right), \ \frac{2abu_* + au_*^2}{(b+u_*)^2} - 1 + 2u_* > 0, \ 1-u_*>0$$

for $n \in \mathbb{N}$ and $\tau = 0$. From the second mathematical expression in (2.11), we know that D_n can be expressed as

$$D_n = \frac{A_n}{\ell^4 (b+u_*)^2}, \tag{2.13}$$

where

$$\begin{aligned} A_n = {} & d_1 d_2 (b+u_*)^2 n^4 - \left(d_2\left((1-u_*)(b+u_*)^2 - abv_*\right)\ell^2 - d_1 r (b+u_*)^2 \ell^2\right) n^2 \\ & + \left(2abru_* + aru_*^2 - r(b+u_*)^2 + ru_*(b+u_*)^2\right)\ell^4. \end{aligned} \tag{2.14}$$

Case 2.2. *It is easy to see that if the conditions (C_0) and*

$$\begin{aligned} (C_1): d_2\left((1-u_*)(b+u_*)^2 - abv_*\right)\ell^2 - d_1 r (b+u_*)^2 \ell^2 < 0, \\ \left(2abru_* + aru_*^2 - r(b+u_*)^2 + ru_*(b+u_*)^2\right)\ell^4 > 0, \end{aligned}$$

or

$$(C_{11}): \ \Delta < 0$$

hold, then (2.14) has no positive roots, where

$$\begin{aligned} \Delta := {} & \left(d_2\left((1-u_*)(b+u_*)^2 - abv_*\right)\ell^2 - d_1 r (b+u_*)^2 \ell^2\right)^2 \\ & - 4 d_1 d_2 (b+u_*)^2 \left(2abru_* + aru_*^2 - r(b+u_*)^2 + ru_*(b+u_*)^2\right)\ell^4 < 0. \end{aligned}$$

Hence, all roots of (2.10) have negative real parts under the conditions (C_0) and (C_1) or (C_{11}).

Case 2.3. *If the conditions (C_0) and*

$$\begin{aligned} & (C_2): \ d_2\left((1-u_*)(b+u_*)^2 - abv_*\right)\ell^2 - d_1 r (b+u_*)^2 \ell^2 > 0, \\ & \left(2abru_* + aru_*^2 - r(b+u_*)^2 + ru_*(b+u_*)^2\right)\ell^4 > 0, \\ & \Delta > 0 \end{aligned}$$

hold, then the (2.14) has two positive roots.

Moreover, if we let $\widetilde{x} = n^2$, then the mathematical expression (2.14) can be rewritten as

$$\begin{aligned} \widetilde{f}(\widetilde{x}) = {} & d_1 d_2 (b+u_*)^2 \widetilde{x}^2 - \left(d_2\left((1-u_*)(b+u_*)^2 - abv_*\right)\ell^2 - d_1 r (b+u_*)^2 \ell^2\right)\widetilde{x} \\ & + \left(2abru_* + aru_*^2 - r(b+u_*)^2 + ru_*(b+u_*)^2\right)\ell^4. \end{aligned} \tag{2.15}$$

Without loss of generality, we assume that the two positive roots of (2.15) are $\widetilde{x}_1$ and $\widetilde{x}_2$, i.e.,

$$\widetilde{x}_{1,2} = \frac{-\widetilde{B} \mp \sqrt{\widetilde{B}^2 - 4\widetilde{A}\widetilde{C}}}{2\widetilde{A}} \tag{2.16}$$

under the conditions (C_0) and (C_2), where

$$\begin{aligned}
\widetilde{B} &= -\left(d_2\left((1-u_*)(b+u_*)^2 - abv_*\right)\ell^2 - d_1 r(b+u_*)^2\ell^2\right),\\
\widetilde{A} &= d_1 d_2 (b+u_*)^2,\\
\widetilde{C} &= \left(2abru_* + aru_*^2 - r(b+u_*)^2 + ru_*(b+u_*)^2\right)\ell^4.
\end{aligned} \tag{2.17}$$

Since $\widetilde{x}_1 = \widetilde{n}_1^2$ and $\widetilde{x}_2 = \widetilde{n}_2^2$, then we have $\widetilde{n}_1 = \sqrt{\widetilde{x}_1}$ and $\widetilde{n}_2 = \sqrt{\widetilde{x}_2}$. By using a geometric argument, we can conclude that

$$A_n \begin{cases} < 0, & \widetilde{n}_1 < n < \widetilde{n}_2, \\ \geq 0, & n \leq \widetilde{n}_1 \text{ or } n \geq \widetilde{n}_2, \end{cases}$$

where $n \in \mathbb{N}$.

Furthermore, we have $\mathrm{sign}(D_n) = \mathrm{sign}\,(A_n)$ from (2.13), where $sign(.)$ represents the sign function. In the following, for simplicity, we set

$$\begin{aligned}
&R = \{(r, n, u_*)| \\
&r > \max\left(0, 1 - 2u_* - \frac{abv_*}{(b+u_*)^2}, \widetilde{r}_n\right),\ \frac{2abu_* + au_*^2}{(b+u_*)^2} - 1 + 2u_* > 0,\ \widetilde{n}_1 < n < \widetilde{n}_2,\ 0 < u_* < 1\right\}.
\end{aligned}$$

From the above discussion, we know that R is the region where the Turing bifurcation curve may exist. Moreover, by solving for r in the second mathematical expression of (2.11), we have

$$r = \frac{\widehat{A}}{\widehat{B}}, \tag{2.18}$$

where

$$\begin{aligned}
\widehat{A} &= d_2\left(1 - u_* - \frac{abv_*}{(b+u_*)^2}\right)\frac{n^2}{\ell^2} - d_1 d_2 \frac{n^4}{\ell^4},\\
\widehat{B} &= d_1 \frac{n^2}{\ell^2} - 1 + u_* + \frac{2abu_* + au_*^2}{(b+u_*)^2}.
\end{aligned} \tag{2.19}$$

Furthermore, from the second mathematical expression in (2.11), we know that when $r \geq \widehat{A}/\widehat{B}$, $D_n \geq 0$ and when $r < \widehat{A}/\widehat{B}$, $D_n < 0$.

Theorem 2.4. *From the above discussion, for the case 2.3, and in the region R,*

(1) when $r > \widehat{A}/\widehat{B}$, the positive constant steady state $E_*(u_*, v_*)$ of model (1.5) is asymptotically stable for $\tau = 0$, and if $0 < r < \widehat{A}/\widehat{B}$, the positive constant steady state $E_*(u_*, v_*)$ of model (1.5) is unstable;

(2) the model (1.5) will undergo Turing bifurcation at $r = \widehat{A}/\widehat{B}$.

In the following, we continue to verify the transversality condition.

Lemma 2.5. *In the region R, when $n \in \mathbb{N}$, $\tau = 0$ and $r = \widehat{A}/\widehat{B}$, we have*

$$\left.\frac{d\lambda}{dr}\right|_{\lambda=0,\tau=0,r=\widehat{A}/\widehat{B}} < 0.$$

*Proof.*By differentiating the second mathematical expression in (2.6) with respect to r, we have

$$\left(2\lambda + p_n - \tau q_n e^{-\lambda\tau}\right)\frac{d\lambda}{dr} = -\lambda\frac{dp_n}{dr} - \frac{dr_n}{dr} - e^{-\lambda\tau}\frac{dq_n}{dr},$$

where p_n, q_n and r_n are defined by (2.7). Furthermore, by noticing that when $n \in \mathbb{N}$ and $\tau = 0$, we have $T_n = -p_n < 0$ and

$$\frac{au_*}{b+u_*} - \left(1 - 2u_* - \frac{abv_*}{(b+u_*)^2}\right) > 0,$$

then from Theorem 2.1 and C_0, we have

$$\left.\frac{d\lambda}{dr}\right|_{\lambda=0,\tau=0,r=\widehat{A}/\widehat{B}} = \frac{-d_1\frac{n^2}{\ell^2} + \left(1 - u_* - \frac{abv_*}{(b+u_*)^2}\right) - \frac{au_*}{b+u_*}}{-T_n} < 0.$$

■

2.2. Hopf bifurcation

In the following, the Hopf bifurcation of model (1.5) for $r \geq \widehat{A}/\widehat{B}$ is investigated. Suppose that $\lambda = \pm i\omega_n$ with $\omega_n > 0$ are a pair of purely imaginary roots of the second mathematical expression in (2.6), then substitute $\lambda = \pm i\omega_n$ into the second mathematical expression in (2.6), we have

$$r_n - \omega_n^2 + q_n\cos(\omega_n\tau) + i\left(p_n\omega_n - q_n\sin(\omega_n\tau)\right) = 0 \tag{2.20}$$

for $n \in \mathbb{N}$. By separating the real and imaginary parts of (2.20), we have

$$\sin(\omega_n\tau) = \frac{p_n\omega_n}{q_n}, \;\; \cos(\omega_n\tau) = \frac{\omega_n^2 - r_n}{q_n}. \tag{2.21}$$

Moreover, by combining with $(\sin(\omega_n\tau))^2 + (\cos(\omega_n\tau))^2 = 1$ and (2.21), we have

$$\omega_n^4 + (p_n^2 - 2r_n)w_n^2 + r_n^2 - q_n^2 = 0. \tag{2.22}$$

In the following, we mainly consider the case of $r_n^2 - q_n^2 < 0$, that is (2.22) has only one positive root ω_n. Then we will discuss the case which is used to guarantee $r_n^2 - q_n^2 < 0$. Notice that when $r \geq \widehat{A}/\widehat{B}$, we have $D_n = r_n + q_n \geq 0$. From (2.7), we have

$$r_n - q_n = d_1d_2\frac{n^4}{\ell^4} + \left(d_1r - d_2\left(1 - u_* - \frac{abv_*}{(b+u_*)^2}\right)\right)\frac{n^2}{\ell^2} - r + ru_* - \frac{aru_*^2}{(b+u_*)^2}. \tag{2.23}$$

Since $1 - u_* > 0$, then (2.23) has only one positive root n_*. Moreover, if we let $x = n^2/\ell^2$, then the mathematical expression (2.23) can be rewritten as

$$f(x) = d_1d_2x^2 + \left(d_1r - d_2\left(1 - u_* - \frac{abv_*}{(b+u_*)^2}\right)\right)x - r + ru_* - \frac{aru_*^2}{(b+u_*)^2}, \tag{2.24}$$

and the unique positive root of (2.24) is

$$x_* = \frac{-\widetilde{b} + \sqrt{\widetilde{b}^2 - 4\widetilde{a}\widetilde{c}}}{2\widetilde{a}}, \tag{2.25}$$

where

$$\widetilde{b} = d_1r - d_2\left(1 - u_* - \frac{abv_*}{(b+u_*)^2}\right), \;\; \widetilde{a} = d_1d_2, \;\; c = -r + ru_* - \frac{aru_*^2}{(b+u_*)^2}. \tag{2.26}$$

Since $x_* = n^2/\ell^2$ and by noticing that $r_n - q_n$ is a quadratic polynomial with respect to n^2/ℓ^2 and $r_0 - q_0 < 0$. Therefore, there exists $n_0 = \ell\sqrt{x_*} > 0$ such that $r_{n_0} - q_{n_0} = 0$ and

$$r_n^2 - q_n^2 \begin{cases} < 0, & 0 < n \le n_*, \\ \ge 0, & n \ge n_* + 1, \end{cases}$$

where $n \in \mathbb{N}$, and n_* is defined by

$$n_* = \begin{cases} n_0 - 1, & n_0 \in \mathbb{N}, \\ [n_0], & n_0 \notin \mathbb{N}. \end{cases}$$

Here, $[.]$ stands for the integer part function. Therefore, ω_n is the unique positive root of (2.22) for $0 < n \le n_*$, where

$$\omega_n = \sqrt{\frac{2r_n - p_n^2 + \sqrt{(p_n^2 - 2r_n)^2 - 4(r_n^2 - q_n^2)}}{2}}. \tag{2.27}$$

Furthermore, by noticing that $\sin(\omega_n \tau) > 0$, if we let $\tau_{n,j},\ n \in \mathbb{N},\ j \in \mathbb{N}_0$ be the root of

$$\cos(\omega_n \tau) = \frac{\omega_n^2 - r_n}{q_n} \tag{2.28}$$

with ω_n in $(0, 2\pi]$. According to (2.28), we can define

$$\tau_{n,j} = \frac{1}{\omega_n}\left(\arccos\left(\frac{\omega_n^2 - r_n}{q_n}\right) + 2j\pi\right),\ 0 < n \le n_*, \tag{2.29}$$

then the second mathematical expression in (2.6) has a pair of purely imaginary roots $\pm i\omega_n$ at $\tau = \tau_{n,j}$. If taking τ as a parameter, and let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the pair of roots of the second mathematical expression in (2.6) near $\tau = \tau_{n,j}$ satisfying $\alpha(\tau_{n,j}) = 0$ and $\omega(\tau_{n,j}) = \omega_n$ for $0 < n \le n_*$ with $n \in \mathbb{N}$. In the following, we continue to verify the transversality condition.

Lemma 2.6. *In the region R, when $r \ge \widehat{A}/\widehat{B}$ and $0 < n \le n_*$, we have*

$$\left.\frac{d\,\mathrm{Re}(\lambda(\tau))}{d\tau}\right|_{\tau=\tau_{n,j}} > 0,$$

where $\mathrm{Re}(\lambda(\tau))$ *represents the real part of* $\lambda(\tau)$.

*Proof.*By differentiating the two sides of

$$\Gamma_n(\lambda) = \det\left(\mathcal{M}_n(\lambda)\right) = \lambda^2 + p_n\lambda + r_n + q_n e^{-\lambda\tau} = 0,\ n \in \mathbb{N}$$

with respect to τ, where p_n, r_n and q_n are defined by (2.7), we have

$$\left(\frac{d\lambda(\tau)}{d\tau}\right)^{-1} = \frac{2\lambda + p_n}{\lambda q_n e^{-\lambda\tau}} - \frac{\tau}{\lambda}. \tag{2.30}$$

Therefore, by combining with (2.30), we have

$$\begin{aligned} \mathrm{Re}\left(\left.\frac{d\lambda(\tau)}{d\tau}\right|_{\tau=\tau_{n,j}}\right)^{-1} &= \mathrm{Re}\left(\frac{2i\omega_n + p_n}{i\omega_n q_n e^{-i\omega_n\tau_{n,j}}} - \frac{\tau_{n,j}}{i\omega_n}\right) \\ &= \mathrm{Re}\left(\frac{2i\omega_n + p_n}{i\omega_n q_n(\cos(\omega_n\tau_{n,j}) - i\sin(\omega_n\tau_{n,j}))}\right) \\ &= \mathrm{Re}\left(\frac{(2i\omega_n + p_n)(\cos(\omega_n\tau_{n,j}) + i\sin(\omega_n\tau_{n,j}))}{i\omega_n q_n}\right) \\ &= \frac{2\cos(\omega_n\tau_{n,j})}{q_n} + \frac{p_n\sin(\omega_n\tau_{n,j})}{\omega_n q_n}. \end{aligned} \tag{2.31}$$

From (2.21), we have

$$\sin(\omega_n\tau_{n,j}) = \frac{p_n\omega_n}{q_n}, \ \ \cos(\omega_n\tau_{n,j}) = \frac{\omega_n^2 - r_n}{q_n}. \tag{2.32}$$

Moreover, by combining with (2.31), (2.32) and

$$\omega_n = \sqrt{\frac{2r_n - p_n^2 + \sqrt{(p_n^2 - 2r_n)^2 - 4(r_n^2 - q_n^2)}}{2}} > 0, \ r_n^2 - q_n^2 < 0,$$

we have

$$\begin{aligned}
\operatorname{Re}\left(\left.\frac{d\lambda(\tau)}{d\tau}\right|_{\tau=\tau_{n,j}}\right)^{-1} &= \frac{2\omega_n^2 - 2r_n + p_n^2}{q_n^2} \\
&= \frac{\sqrt{(p_n^2 - 2r_n)^2 - 4(r_n^2 - q_n^2)}}{q_n^2} > 0.
\end{aligned}$$

This, together with the fact that

$$\operatorname{sign}\left\{\left.\frac{d\operatorname{Re}(\lambda(\tau))}{d\tau}\right|_{\tau=\tau_{n,j}}\right\} = \operatorname{sign}\left\{\operatorname{Re}\left(\left.\frac{d\lambda(\tau)}{d\tau}\right|_{\tau=\tau_{n,j}}\right)^{-1}\right\}$$

completes the proof. ■

Moreover, according to the above analysis, we have the following results.

Theorem 2.7. *In the region R, when $r \geq \widehat{A}/\widehat{B}$, if we denote*

$$\tau_* = \min\{\tau_{n,0} : 0 < n \leq n_*, \ n \in \mathbb{N}\},$$

then the positive constant steady state $E_*(u_*, v_*)$ of model (1.5) is asymptotically stable for $0 \leq \tau < \tau_*$ and unstable for $\tau > \tau_*$. Furthermore, model (1.5) undergoes Hopf bifurcation at $\tau = \tau_*$.

2.3. Turing-Hopf bifurcation

By combining with Theorem 2.4 and Theorem 2.7, we have the following lemma.

Lemma 2.8. *In the region R, the model (1.5) undergoes (n_2, n_1)-mode Turing-Hopf bifurcation at (r_*, τ_*), where $r_* = \widehat{A}/\widehat{B}$.*

3. Normal form for Turing-Hopf bifurcation of model (1.5)

Assumption 3.1. *When $(r, \tau) = (r_*, \tau_*)$, there exists $n_2 \in \mathbb{N}$ such that the equation $\Gamma_{n_2}(\lambda) = 0$ has a simple zero root $\lambda = 0$, and there exists $n_1 \in \mathbb{N}$ such that the characteristic equation $\Gamma_{n_1}(\lambda) = 0$ has a pair of simple purely imaginary roots $\pm i\omega_{n_1}$. In addition, all roots of $\Gamma_n(\lambda) = 0$ with $n \neq n_1, n_2$ have negative real parts, and the corresponding transversality conditions hold.*

3.1. Normal form derivation of Turing-Hopf bifurcation for model (1.5)

By the time scaling $t \to t/\tau$ and by setting $r = r_* + \mu_1$, $\tau = \tau_* + \mu_2$ such that $\mu := (\mu_1, \mu_2) = (0, 0)$ is the Turing-Hopf bifurcation point in the perturbation plane of μ_1 and μ_2. Furthermore, by setting

$\widetilde{u}(x,t) = u(x,\tau t) - u_*$, $\widetilde{v}(x,t) = v(x,\tau t) - v_*$ and $U(t) = (\widetilde{u}(x,t), \widetilde{v}(x,t))^T$, then dropping the tildes for simplicity, the model (1.5) can be rewritten as the following model in the space $\mathcal{C}$

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = (\tau_c + \mu_2)\left(d_1 \Delta u(x,t) + (u(x,t)+u_*)\left(1 - (\widehat{u} + u_*)\right) - \dfrac{a(u(x,t)+u_*)(v(x,t)+v_*)}{b + (u(x,t)+u_*)}\right), \\ \dfrac{\partial v(x,t)}{\partial t} = (\tau_c + \mu_2)\left(d_2 \Delta v(x,t) + (r+\mu_1)(v(x,t)+v_*)\left(1 - \dfrac{v(x,t)+v_*}{u(x,t-1)+u_*}\right)\right). \end{cases} \tag{3.1}$$

Then for $U_t(\theta) = U(t+\theta) \in \mathcal{C}$, $\widehat{U}_t(\theta) = \widehat{U}(t+\theta) \in \mathcal{C}$, $-1 \le \theta \le 0$, the model (3.1) becomes the compact form

$$\frac{dU(t)}{dt} = d(\mu)\Delta U_t(0) + L(\mu)U_t(\theta) + \widehat{L}(\mu)\widehat{U}_t(0) + F\left(U_t(\theta), \widehat{U}_t(0), \mu\right), \tag{3.2}$$

where for $\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta))^T \in \mathcal{C}$ and $\widehat{\varphi}(0) = (\widehat{\varphi}_1(0), \widehat{\varphi}_2(0))^T \in \mathcal{C}$, $L(\mu), \widehat{L}(\mu) : \mathcal{C} \to X$, $F : \mathcal{C} \times \mathcal{C} \times \mathbb{R}^2 \to X$ and $d(\mu)\Delta$ are given, respectively, by

$$\begin{aligned} &d(\mu)\Delta\varphi(0) = (\tau_c + \mu_2)D\Delta\varphi(0),\ L(\mu)\varphi(\theta) = (\tau_c+\mu_2)(A_1(\mu_1)\varphi(0) + A_2(\mu_1)\varphi(-1)), \\ &\widehat{L}(\mu)\widehat{\varphi}(0) = (\tau_c + \mu_2)B_1\widehat{\varphi}(0) \end{aligned}$$

and

$$\begin{aligned} F(\varphi(\theta), \widehat{\varphi}(0), \mu) = (\tau_c + \mu_2) &\begin{pmatrix} (\varphi_1(0)+u_*)\left(1 - (\widehat{\varphi}_1(0) + u_*)\right) - \dfrac{a(\varphi_1(0)+u_*)(\varphi_2(0)+v_*)}{b + (\varphi_1(0)+u_*)} \\ (r+\mu_1)(\varphi_2(0)+v_*)\left(1 - \dfrac{\varphi_2(0)+v_*}{\varphi_1(-1)+u_*}\right) \end{pmatrix} \\ &- L(\mu)\varphi(\theta) - \widehat{L}(\mu)\widehat{\varphi}(0). \end{aligned} \tag{3.3}$$

Here, D and B_1 are defined by (2.4),

$$A_1(\mu_1) = \begin{pmatrix} 1 - u_* - \frac{abv_*}{(b+u_*)^2} & -\frac{au_*}{b+u_*} \\ 0 & -(r+\mu_1) \end{pmatrix},\ A_2(\mu_1) = \begin{pmatrix} 0 & 0 \\ r+\mu_1 & 0 \end{pmatrix}.$$

In what follows, we assume that $F(\varphi(\theta), \widehat{\varphi}(0), \mu)$ is C^k, $k \ge 3$, smooth with respect to $\varphi(\theta)$, $\widehat{\varphi}(0)$ and μ. By noticing that μ is the perturbation parameter and is treated as a variable in the calculation of normal form, thus if we denote

$$d_0 := d(0),\ L_0\varphi(\theta) := L(0)\varphi(\theta) = \tau_c(A_1\varphi(0) + A_2\varphi(-1)),\ \widehat{L}_0\widehat{\varphi}(0) := \widehat{L}(0)\widehat{\varphi}(0) = \tau_c B_1\widehat{\varphi}(0),$$

the system (3.2) can be rewritten as

$$\frac{dU(t)}{dt} = d_0\Delta U_t(0) + L_0U_t(\theta) + \widehat{L}_0\widehat{U}_t(0) + \widetilde{F}(U_t(\theta), \widehat{U}_t(0), \mu) \tag{3.4}$$

by separating the linear terms from the nonlinear terms, where

$$\widetilde{F}(U_t(\theta), \widehat{U}_t(0), \mu) = \mu_2 D\Delta U_t(0) + \mu_2 A_1(\mu_1)U_t(0) + \mu_2 A_2(\mu_1)U_t(-1) + \mu_2 B_1\widehat{U_t}(0) + F(U_t(\theta), \widehat{U}_t(0), \mu). \tag{3.5}$$

Therefore, the characteristic equation for the linearized system of (3.4)

$$\frac{dU(t)}{dt} = d_0\Delta U_t(0) + L_0U_t(\theta) + \widehat{L}_0\widehat{U}_t(0) \tag{3.6}$$

is

$$\prod_{n \in \mathbb{N}_0} \widetilde{\Gamma}_n(\lambda) = 0,$$

where $\widetilde{\Gamma}_n(\lambda) = \det\left(\widetilde{\mathcal{M}}_n(\lambda)\right)$ with

$$\widetilde{\mathcal{M}}_n(\lambda) = \begin{cases} \lambda I_2 - \tau_c A_1 - \tau_c A_2 e^{-\lambda} - \tau_c B_1, \ n = 0, \\ \lambda I_2 + \tau_c D \dfrac{n^2}{\ell^2} - \tau_c A_1 - \tau_c A_2 e^{-\lambda}, \ n \in \mathbb{N}. \end{cases} \tag{3.7}$$

Then from (3.7), we have

$$\begin{aligned} &\widetilde{\Gamma}_n(\lambda) = \det\left(\widetilde{\mathcal{M}}_n(\lambda)\right) \\ &= \begin{cases} \lambda^2 - \tau_c\left(1 - 2u_* - r - \dfrac{abv_*}{(b+u_*)^2}\right)\lambda + \dfrac{\tau_c^2 a u_* r e^{-\lambda}}{b+u_*} - \tau_c^2\left(1 - 2u_* - \dfrac{abv_*}{(b+u_*)^2}\right) r = 0, & n = 0, \\ \lambda^2 + \tau_c p_n \lambda + \tau_c^2 r_n + \tau_c^2 q_n e^{-\lambda} = 0, & n \in \mathbb{N}, \end{cases} \end{aligned} \tag{3.8}$$

where p_n, r_n and q_n are defined by (2.7).

By comparing (3.8) with (2.6), we know that (3.8) has a pair of purely imaginary roots $\pm i\omega_c$ for $n = n_1 \in \mathbb{N}$, and all other eigenvalues have negative real parts, where $\omega_c = \tau_c \omega_{n_1}$. In order to write (3.4) as an abstract ordinary differential equation in a Banach space, follows by [28], we can take the enlarged space

$$\mathcal{BC} := \left\{\psi(\theta) : [-1, 0] \to X : \psi(\theta) \text{ is continuous on } [-1, 0), \exists \lim_{\theta \to 0^-} \psi(\theta) \in X\right\},$$

then the system (3.4) is equivalent to an abstract ordinary differential equation on $\mathcal{BC}$

$$\frac{dU_t(\theta)}{dt} = AU_t(\theta) + X_0 \widetilde{F}(U_t(\theta), \widehat{U}_t(0), \mu).$$

Here, A is a operator from $\mathcal{C}_0^1 = \{\varphi(\theta) \in \mathcal{C} : \dot{\varphi}(\theta) \in \mathcal{C}, \varphi(0) \in \mathrm{dom}(\Delta)\}$ to $\mathcal{BC}$, which is defined by

$$A\varphi(\theta) = \dot{\varphi}(\theta) + X_0\left(\tau_c D \Delta \varphi(0) + L_0^n(\varphi(\theta)) - \dot{\varphi}(0)\right),$$

where

$$L_0^n(\varphi(\theta)) = \begin{cases} \tau_c A_1 \varphi(0) + \tau_c A_2 \varphi(-1) + \tau_c B_1 \varphi(0), & n = 0, \\ \tau_c A_1 \varphi(0) + \tau_c A_2 \varphi(-1), & n \in \mathbb{N}, \end{cases} \tag{3.9}$$

and $X_0 = X_0(\theta)$ is given by

$$X_0(\theta) = \begin{cases} 0, & -1 \le \theta < 0, \\ 1, & \theta = 0. \end{cases}$$

In the following, the method given in [28] is used to complete the decomposition of $\mathcal{BC}$. Let $C := C\left([-1, 0], \mathbb{R}^2\right)$, $C^* := C\left([0, 1], \mathbb{R}^{2*}\right)$, where $\mathbb{R}^{2*}$ is the two-dimensional space of row vectors, and define the adjoint bilinear form on $C^* \times C$ as follows

$$\langle \psi(s), \phi(\theta) \rangle = \psi(0)\phi(0) - \int_{-1}^{0} \int_{0}^{\theta} \psi(\xi - \theta) dM_k(\theta) \phi(\xi) d\xi \text{ for } \psi \in C^*, \ \phi \in C \text{ and } \xi \in [-1, 0], \tag{3.10}$$

where $M_k(\theta)$ is a bounded variation function from $[-1, 0]$ to $\mathbb{R}^{2\times2}$, i.e., $M_k(\theta) \in BV\left([-1, 0]; \mathbb{R}^{2\times2}\right)$, such that for $\phi(\theta) \in \mathcal{C}$, we have

$$\mathcal{L}_n(\phi(\theta)) = \int_{-1}^{0} dM_k(\theta)\phi(\theta),$$

where

$$\mathcal{L}_n(\phi(\theta)) = L_n^d(\phi(\theta)) + L_0^n(\phi(\theta)) \tag{3.11}$$

with

$$L_n^d(\phi(\theta)) = -\tau_c \frac{n^2}{\ell^2} D\phi(0),\ L_0^n(\phi(\theta)) = \begin{cases} \tau_c A_1 \phi(0) + \tau_c A_2 \phi(-1) + \tau_c B_1 \phi(0), & n = 0, \\ \tau_c A_1 \phi(0) + \tau_c A_2 \phi(-1), & n \in \mathbb{N}. \end{cases}$$

By choosing

$$\begin{aligned} \Phi_1(\theta) &= \left(\xi_{n_1} e^{i\omega_c \theta}, \overline{\xi}_{n_1} e^{-i\omega_c \theta}\right), \qquad & \Phi_2(\theta) = \xi_{n_2}, \\ \Psi_1(s) &= \mathrm{col}\left(\eta_{n_1}^T e^{-i\omega_c s}, \overline{\eta}_{n_1}^T e^{i\omega_c s}\right), \quad & \Psi_2(s) = \eta_{n_2}^T, \end{aligned} \tag{3.12}$$

where the col(.) represents the column vector, $\xi_{n_1} = \mathrm{col}(\xi_{n_1 1}, \xi_{n_1 2}) \in \mathbb{C}^2$ and $\xi_{n_2} = \mathrm{col}(\xi_{n_2 1}, \xi_{n_2 2}) \in \mathbb{R}^2$ are the eigenvectors of system (3.6) associated with the eigenvalues $i\omega_c$ and 0, respectively, $\eta_{n_1} = \mathrm{col}(\eta_{n_1 1}, \eta_{n_1 2}) \in \mathbb{C}^2$ and $\eta_{n_2} = \mathrm{col}(\eta_{n_2 1}, \eta_{n_2 2}) \in \mathbb{R}^2$ are the corresponding adjoint eigenvectors such that

$$\langle \Psi_1(s), \Phi_1(\theta) \rangle = I_2,\ \ \langle \Psi_2(s), \Phi_2(\theta) \rangle = 1. \tag{3.13}$$

According to [28], the phase space $\mathcal{C}$ can be decomposed as

$$\mathcal{C} = \mathcal{P} \oplus \mathcal{Q},\ \mathcal{P} = \mathrm{Im}\ \pi,\ \mathcal{Q} = \mathrm{Ker}\ \pi,$$

where for $\phi \in \mathcal{C}$, the projection $\pi : \mathcal{C} \to \mathcal{P}$ is defined by

$$\pi(\phi) = \left(\Phi_1 \left\langle \Psi_1, \left(\begin{array}{c} \left[\phi(\cdot), \beta_{n_1}^{(1)}\right] \\ \left[\phi(\cdot), \beta_{n_1}^{(2)}\right] \end{array} \right) \right\rangle \right)^T \beta_{n_1} + \left(\Phi_2 \left\langle \Psi_2, \left(\begin{array}{c} \left[\phi(\cdot), \beta_{n_2}^{(1)}\right] \\ \left[\phi(\cdot), \beta_{n_2}^{(2)}\right] \end{array} \right) \right\rangle \right)^T \beta_{n_2}. \tag{3.14}$$

Next, we will calculate the eigenvectors ξ_{n_1} and ξ_{n_2} as well as the corresponding adjoint eigenvectors η_{n_1} and η_{n_2} associated with the eigenvalues $i\omega_c$ and 0, respectively. Choose

$$M_0(\theta) = \begin{cases} \tau_c A_2, & \theta = -1, \\ 0, & -1 < \theta < 0, \\ \tau_c A_1 + \tau_c B_1, & \theta = 0, \end{cases} \quad M_n(\theta) = \begin{cases} \tau_c A_2, & \theta = -1, \\ 0, & -1 < \theta < 0, \\ -\tau_c \frac{n^2}{\ell^2} D + \tau_c A_1, & \theta = 0, \end{cases} \tag{3.15}$$

then by combining with (2.4), when $n = 0$, we have

$$\begin{aligned} \tau_c A_1 + \tau_c B_1 &= \tau_c \begin{pmatrix} 1 - 2u_* - \frac{abv_*}{(b+u_*)^2} & -\frac{au_*}{b+u_*} \\ 0 & -r \end{pmatrix} := \tau_c \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \\ \tau_c A_2 &= \tau_c \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix} := \tau_c \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \end{aligned}$$

and when $n \in \mathbb{N}$, we have

$$\begin{aligned} \tau_c A_1 &= \tau_c \begin{pmatrix} 1 - u_* - \frac{abv_*}{(b+u_*)^2} & -\frac{au_*}{b+u_*} \\ 0 & -r \end{pmatrix} := \tau_c \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \\ \tau_c A_2 &= \tau_c \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix} := \tau_c \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}. \end{aligned}$$

Furthermore, by combining with (3.10), (3.12), (3.13) and (3.15), after a straightforward calculation, we can obtain

$$\begin{aligned} \xi_{n_1} &= (1, k_1)^{\mathrm{T}},\ \xi_{n_2} = (1, k_3)^{\mathrm{T}}, \\ \eta_{n_1} &= (T_1 (1, k_2))^T,\ \eta_{n_2} = (T_2 (1, k_4))^T, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
k_1 &= \frac{i\omega_c + \tau_c \frac{n_1^2}{\ell^2} d_1 - \tau_c a_{11} - \tau_c b_{11} e^{-i\omega_c}}{\tau_c a_{12} + \tau_c b_{12} e^{-i\omega_c}},\ k_2 = \frac{i\omega_c + \tau_c \frac{n_1^2}{\ell^2} d_1 - \tau_c a_{11} - \tau_c b_{11} e^{-i\omega_c}}{\tau_c a_{21} + \tau_c b_{21} e^{-i\omega_c}}, \\
k_3 &= \frac{\tau_c d_1 \frac{n_2^2}{\ell^2} - \tau_c a_{11} - \tau_c b_{11}}{\tau_c a_{12} + \tau_c b_{12}},\ k_4 = \frac{\tau_c d_1 \frac{n_2^2}{\ell^2} - \tau_c a_{11} - \tau_c b_{11}}{\tau_c a_{21} + \tau_c b_{21}}, \\
T_1 &= \frac{1}{k_1 k_2 + 1 + e^{-i\omega_c} \left(\tau_c b_{11} + \tau_c b_{21} k_2 + k_1 \left(\tau_c b_{12} + \tau_c b_{22} k_2\right)\right)} \\
T_2 &= \frac{1}{\left(\tau_c b_{12} + \tau_c b_{22} k_4 + k_4\right) k_3 + \left(\tau_c b_{11} + \tau_c b_{21} k_4 + 1\right)}.
\end{aligned} \tag{3.17}$$

Therefore, according to the method given in [28], $\mathcal{BC}$ can be divided into a direct sum of center subspace and its complementary space, that is

$$\mathcal{BC} = \mathcal{P} \oplus \ker\ \pi, \tag{3.18}$$

where dim $\mathcal{P} = 3$. It is easy to see that the projection π which is defined by (3.14), is extended to a continuous projection (which is still denoted by π), that is, $\pi : \mathcal{BC} \to \mathcal{P}$. In particular, for $\alpha \in \mathcal{C}$, we have

$$\pi\left(X_0(\theta)\alpha\right) = \left(\Phi_1(\theta)\Psi_1(0)\begin{pmatrix} \left[\alpha, \beta_{n_1}^{(1)}\right] \\ \left[\alpha, \beta_{n_1}^{(2)}\right] \end{pmatrix}\right)^T \beta_{n_1} + \left(\Phi_2(\theta)\Psi_2(0)\begin{pmatrix} \left[\alpha, \beta_{n_2}^{(1)}\right] \\ \left[\alpha, \beta_{n_2}^{(2)}\right] \end{pmatrix}\right)^T \beta_{n_2}. \tag{3.19}$$

By combining with (3.18) and (3.19), $U_t(\theta)$ can be decomposed as

$$\begin{aligned}
U_t(\theta) &= \left(\Phi_1(\theta)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)^T \begin{pmatrix} \beta_{n_1}^{(1)} \\ \beta_{n_1}^{(2)} \end{pmatrix} + \left(z_3 \Phi_2(\theta)\right)^T \begin{pmatrix} \beta_{n_2}^{(1)} \\ \beta_{n_2}^{(2)} \end{pmatrix} + w(\theta) \\
&= \left(z_1 \xi_{n_1} e^{i\omega_c \theta} + z_2 \overline{\xi}_{n_1} e^{-i\omega_c \theta}\right) \gamma_{n_1}(x) + z_3 \xi_{n_2} \gamma_{n_2}(x) + w(\theta) \\
&= \left(\Phi_1(\theta)\ \ \Phi_2(\theta)\right) \begin{pmatrix} z_1 \gamma_{n_1}(x) \\ z_2 \gamma_{n_1}(x) \\ z_3 \gamma_{n_2}(x) \end{pmatrix} + \begin{pmatrix} w_1(\theta) \\ w_2(\theta) \end{pmatrix},
\end{aligned} \tag{3.20}$$

where $w(\theta) = \mathrm{col}(w_1(\theta), w_2(\theta)) \in \mathcal{C}_0^1 \cap \mathrm{Ker}\ \pi := \mathcal{Q}^1$ and

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \left\langle \Psi_1(0), \begin{pmatrix} \left[U_t(\theta), \beta_{n_1}^{(1)}\right] \\ \left[U_t(\theta), \beta_{n_2}^{(2)}\right] \end{pmatrix} \right\rangle,\ z_3 = \left\langle \Psi_2(0), \begin{pmatrix} \left[U_t(\theta), \beta_{n_2}^{(1)}\right] \\ \left[U_t(\theta), \beta_{n_2}^{(2)}\right] \end{pmatrix} \right\rangle.$$

Therefore, if we assume that

$$\Phi(\theta) = (\Phi_1(\theta), \Phi_2(\theta)),\ z_x = \left(z_1 \gamma_{n_1}(x), z_2 \gamma_{n_1}(x), z_3 \gamma_{n_2}(x)\right)^T,$$

then (3.20) can be rewritten as

$$U_t(\theta) = \Phi(\theta) z_x + w(\theta). \tag{3.21}$$

By combining with $\widehat{U}_t(\theta) = \widehat{w}(0)$ and (3.21), and for the simplicity of notation, we let

$$\begin{aligned}
&\begin{pmatrix} \left[\widetilde{F}\left(\Phi(\theta) z_x + w(\theta), \widehat{w}(0), \mu\right), \beta_\nu^{(1)}\right] \\ \left[\widetilde{F}\left(\Phi(\theta) z_x + w(\theta), \widehat{w}(0), \mu\right), \beta_\nu^{(2)}\right] \end{pmatrix}_{\nu=n_1}^{\nu=n_2} \\
&= \mathrm{col}\left(\begin{pmatrix} \left[\widetilde{F}\left(\Phi(\theta) z_x + w(\theta), \widehat{w}(0), \mu\right), \beta_{n_1}^{(1)}\right] \\ \left[\widetilde{F}\left(\Phi(\theta) z_x + w(\theta), \widehat{w}(0), \mu\right), \beta_{n_1}^{(2)}\right] \end{pmatrix}, \begin{pmatrix} \left[\widetilde{F}\left(\Phi(\theta) z_x + w(\theta), \widehat{w}(0), \mu\right), \beta_{n_2}^{(1)}\right] \\ \left[\widetilde{F}\left(\Phi(\theta) z_x + w(\theta), \widehat{w}(0), \mu\right), \beta_{n_2}^{(2)}\right] \end{pmatrix}\right),
\end{aligned}$$

then the system (3.4) is decomposed as a system of abstract ordinary differential equations (ODEs) on $\mathbb{R}^3 \times \mathrm{Ker}\ \pi$, with finite and infinite dimensional variables are separated in the linear term. That is

$$\begin{cases} \dot{z} = Bz + \Psi(0) \begin{pmatrix} \left[\widetilde{F}(\Phi(\theta)z_x + w(\theta), \widehat{w}(0), \mu), \beta_\nu^{(1)}\right] \\ \left[\widetilde{F}(\Phi(\theta)z_x + w(\theta), \widehat{w}(0), \mu), \beta_\nu^{(2)}\right] \end{pmatrix}_{\nu=n_1}^{\nu=n_2}, \\ \dot{w} = A_{\mathcal{Q}^1} w + (I - \pi) X_0(\theta) \widetilde{F}(\Phi(\theta)z_x + w(\theta), \widehat{w}(0), \mu), \end{cases} \tag{3.22}$$

where $\Psi(0) = \mathrm{diag}(\Psi_1(0), \Psi_2(0))$, I is the identity matrix, $z = (z_1, z_2, z_3)^T$, $B = \mathrm{diag}\,\{i\omega_c, -i\omega_c, 0\}$ is the diagonal matrix, and $A_{\mathcal{Q}^1} : \mathcal{Q}^1 \to \mathrm{Ker}\,\pi$ is defined by

$$A_{\mathcal{Q}^1} w = \dot{w} + X_0(\theta)(\tau_c D \Delta w(0) + L_0^n w(\theta) - \dot{w}(0)).$$

By combining with (3.5) and the formal Taylor expansions of $L(\mu)$, $\widehat{L}(\mu)$, $\widetilde{F}(U_t(\theta), \widehat{U}_t(0), \mu)$ as follows

$$\begin{aligned} L(\mu) &= L_0 + \sum_{j \geq 1} \frac{1}{j!} L_j(\mu),\ \widehat{L}(\mu) = \widehat{L}_0 + \sum_{j \geq 1} \frac{1}{j!} \widehat{L}_j(\mu), \\ \widetilde{F}(U_t(\theta), \widehat{U}_t(0), \mu) &= \sum_{j \geqslant 2} \frac{1}{j!} \widetilde{F}_j(U_t(\theta), \widehat{U}_t(0), \mu), \end{aligned}$$

the systems (3.22) can be rewritten as

$$\begin{cases} \dot{z} = Bz + \displaystyle\sum_{j \geq 2} \frac{1}{j!} f_j^1(z, w, \widehat{w}(0), \mu), \\ \dot{w} = A_{\mathcal{Q}^1} w + \displaystyle\sum_{j \geq 2} \frac{1}{j!} f_j^2(z, w, \widehat{w}(0), \mu), \end{cases}$$

where $f_j := \left(f_j^1, f_j^2\right),\ j \geq 2$, are defined by

$$\begin{aligned} & f_j^1(z, w, \widehat{w}(0), \mu) \\ & = \Psi(0) \begin{pmatrix} \left[jL_{j-1}(\mu)(\Phi(\theta)z_x + w(\theta)), \beta_\nu^{(1)}\right] + \left[j\widehat{L}_{j-1}(\mu)(\widehat{w}(0)), \beta_\nu^{(1)}\right] + \left[F_j(\Phi(\theta)z_x + w(\theta), \widehat{w}(0), \mu), \beta_\nu^{(1)}\right] \\ \left[jL_{j-1}(\mu)(\Phi(\theta)z_x + w(\theta)), \beta_\nu^{(2)}\right] + \left[j\widehat{L}_{j-1}(\mu)(\widehat{w}(0)), \beta_\nu^{(2)}\right] + \left[F_j(\Phi(\theta)z_x + w(\theta), \widehat{w}(0), \mu), \beta_\nu^{(2)}\right] \end{pmatrix}_{\nu=n_1}^{\nu=n_2} \\ & f_j^2(z, w, \widehat{w}(0), \mu) = (I - \pi) X_0(\theta) \left(jL_{j-1}(\mu)(\Phi(\theta)z_x + w(\theta)) + j\widehat{L}_{j-1}(\mu)\widehat{w}(0) + F_j(\Phi(\theta)z_x + w(\theta), \widehat{w}(0), \mu)\right). \end{aligned} \tag{3.23}$$

Furthermore, by using (3.3), we obtain

$$\begin{aligned} F_2\left(\Phi(\theta)z_x + w(\theta), \widehat{w}(0), \mu\right) &= F_2\left(\Phi(\theta)z_x + w(\theta), \widehat{w}(0), 0\right) \\ &= \sum_{q_1+q_2+q_3=2} A_{q_1q_2q_3} \gamma_{n_1}^{q_1+q_2}(x) \gamma_{n_2}^{q_3}(x) z_1^{q_1} z_2^{q_2} z_3^{q_3} \\ &\quad + \mathcal{S}_2\left(\Phi(\theta)z_x, w(\theta)\right) + \widehat{\mathcal{S}_2}\left(\Phi(\theta)z_x, \widehat{w}(0)\right) + O\left(|(w(\theta), \widehat{w}(0))|^2\right) \end{aligned}$$

and

$$\begin{aligned} F_3\left(\Phi(\theta)z_x, 0, \mu\right) = & \sum_{q_1+q_2+q_3=3} A_{q_1q_2q_3} z_1^{q_1} z_2^{q_2} z_3^{q_3} \gamma_{n_1}^{q_1+q_2}(x) \gamma_{n_2}^{q_3}(x) \\ & + \sum_{q_1+q_2+q_3=2} \widehat{A}_{q_1q_2q_3}^{(1)} z_1^{q_1} z_2^{q_2} z_3^{q_3} \gamma_{n_1}^{q_1+q_2}(x) \gamma_{n_2}^{q_3}(x) \mu_1 \\ & + \sum_{q_1+q_2+q_3=2} \widehat{A}_{q_1q_2q_3}^{(2)} z_1^{q_1} z_2^{q_2} z_3^{q_3} \gamma_{n_1}^{q_1+q_2}(x) \gamma_{n_2}^{q_3}(x) \mu_2 + O\left(|z||\mu|^2\right), \end{aligned}$$

where $\mathcal{S}_2\left(\Phi(\theta)z_x, w(\theta)\right)$ is the product term of $\Phi(\theta)z_x$ and $w(\theta)$, $\widehat{\mathcal{S}}_2\left(\Phi(\theta)z_x, \widehat{w}(0)\right)$ is the product term of $\Phi(\theta)z_x$ and $\widehat{w}(0)$, $q_1, q_2, q_3 \in \mathbb{N}_0$. Furthermore, $A_{q_1q_2q_3}, \widehat{A}_{q_1q_2q_3}^{(1)}, \widehat{A}_{q_1q_2q_3}^{(2)} \in \mathbb{R}^2$, and $A_{q_1q_2q_3} = \overline{A}_{q_2q_1q_3}$.

As given in [22], we have

$$\begin{aligned}
\mathrm{Ker}(M_2^1) = \mathrm{span}\,\{ & z_1z_3e_1, z_1\mu_1e_1, z_1\mu_2e_1, z_2z_3e_2, z_2\mu_1e_2, z_2\mu_2e_2, z_1z_2e_3, z_3^2e_3, z_3\mu_1e_3, z_3\mu_2e_3, \\
& \mu_1\mu_2e_3, \mu_1^2e_3, \mu_2^2e_3\}, \\
\mathrm{Ker}(M_3^1) = \mathrm{span}\,\{ & z_1^2z_2e_1, z_1z_3^2e_1, z_1z_3\mu_1e_1, z_1z_3\mu_2e_1, z_1\mu_1^2e_1, z_1\mu_2^2e_1, z_1\mu_1\mu_2e_1, z_1z_2^2e_2, z_2z_3^2e_2, \\
& z_2z_3\mu_1e_2, z_2z_3\mu_2e_2, z_2\mu_1^2e_2, z_2\mu_2^2e_2, z_2\mu_1\mu_2e_2, z_1z_2z_3e_3, z_3^3e_3, z_3^2\mu_1e_3, z_3^2\mu_2e_3, \\
& z_1z_2\mu_1e_3, z_1z_2\mu_2e_3, z_3\mu_1^2e_3, z_3\mu_2^2e_3, z_3\mu_1\mu_2e_3, \mu_1^3e_3, \mu_2^3e_3, \mu_1\mu_2^2e_3, \mu_1^2\mu_2e_3\}
\end{aligned}$$

and

$$U_2^1(z,\mu) = \left(M_2^1\right)^{-1}\mathrm{Proj}_{\mathrm{Im}\left(M_2^1\right)} f_2^1(z,0,0,\mu), \ \ \left(M_2^2U_2^2\right)(z,\mu) = f_2^2(z,0,0,\mu),$$

where $e_i (i = 1, 2, 3)$ are the standard basis on $\mathbb{R}^3$. By letting

$$\begin{aligned}
&\widetilde{U}_2^2(z,\mu) = \mathrm{col}(U_2^2(z,\mu), \widehat{U}_2^2(z,\mu), U_{2,xx}^2(z,\mu)), \\
&U_2^2(z,\mu) \triangleq h(z,\mu,\theta) = \sum_{n\in\mathbb{N}_0} h_n(z,\mu,\theta)\gamma_n(x),
\end{aligned}$$

and from (2.1), we have

$$\widehat{U}_2^2(z,\mu) = \sqrt{\frac{1}{\ell\pi}}h_0(z,\mu,\theta), \ U_{2,xx}^2(z,\mu) = \sum_{n\in\mathbb{N}_0}\left(-\frac{n^2}{\ell^2}\right)h_n(z,\mu,\theta)\gamma_n(x),$$

where

$$h_n(z,\mu,\theta) = \begin{pmatrix} h_n^{(1)}(z,\mu,\theta) \\ h_n^{(2)}(z,\mu,\theta) \end{pmatrix} = \sum_{q_1+q_2+q_3+q_4+q_5=2} \begin{pmatrix} h_{n,q_1q_2q_3q_4q_5}^{(1)}(\theta) \\ h_{n,q_1q_2q_3q_4q_5}^{(2)}(\theta) \end{pmatrix} z_1^{q_1}z_2^{q_2}z_3^{q_3}\mu_1^{q_4}\mu_2^{q_5}.$$

Applying the arguments of [22], and by noticing that the term of order 3 obtained after the changes of variables in previous steps is given by

$$\begin{aligned}
\widetilde{f}_3^1(z,0,0,\mu) = f_3^1(z,0,0,\mu) + \frac{3}{2}\Big(& \left(D_zf_2^1\right)(z,0,0,\mu)U_2^1(z,\mu) + \left(D_{w,\widehat{w},\widetilde{w}_{xx}}f_2^1\right)(z,0,0,\mu)\widetilde{U}_2^2(z,\mu) \\
& - \left(D_zU_2^1(z,\mu)\right)g_2^1(z,0,0,\mu)\Big),
\end{aligned}$$

we derive the normal form

$$\dot{z} = Bz + \frac{1}{2!}g_2^1(z,0,0,\mu) + \frac{1}{3!}g_3^1(z,0,0,\mu) + O\left(|z||\mu|^2\right) \tag{3.24}$$

with $g_2^1(z,0,0,\mu) = \mathrm{Proj}_{\ker(M_2^1)} f_2^1(z,0,0,\mu)$ and $g_3^1(z,0,0,\mu) = \mathrm{Proj}_{\ker(M_3^1)} \widetilde{f}_3^1(z,0,0,\mu)$.

Here,

$$\begin{aligned}
\frac{1}{2!}g_2^1(z,0,0,\mu) &= \frac{1}{2!}\mathrm{Proj}_{\mathrm{Ker}\left(M_2^1\right)} f_2^1(z,0,0,\mu) = \begin{pmatrix} \mathcal{H}\left(B_{10100}z_1z_3 + (B_{10010}\mu_1 + B_{10001}\mu_2)z_1\right) \\ B_{11000}z_1z_2 + (B_{00110}\mu_1 + B_{00101}\mu_2)z_3 \end{pmatrix}, \\
\frac{1}{3!}g_3^1(z,0,0,\mu) &= \mathrm{Proj}_{\ker(M_3^1)} \widetilde{f}_3^1(z,0,0,\mu) \\
&= \frac{1}{3!}\mathrm{Proj}_{S_1} \widetilde{f}_3^1(z,0,0,0) + \frac{1}{3!}\mathrm{Proj}_{S_2} \widetilde{f}_3^1(z,0,0,\mu) + O(|z||\mu|^2) \\
&= \begin{pmatrix} \mathcal{H}\left(B_{21000}z_1^2z_2 + B_{10200}z_1z_3^2 + F_{10110}z_1z_3\mu_1 + F_{10101}z_1z_3\mu_2\right) \\ B_{11100}z_1z_2z_3 + B_{00300}z_3^3 + F_{11010}z_1z_2\mu_1 + F_{11001}z_1z_2\mu_2 + F_{00210}z_3^2\mu_1 + F_{00201}z_3^2\mu_2 \end{pmatrix}
\end{aligned} \tag{3.25}$$

with

$$S_1 = \operatorname{span}\left\{z_1^2 z_2 e_1, z_1 z_3^2 e_1, z_1 z_2^2 e_2, z_2 z_3^2 e_2, z_1 z_2 z_3 e_3, z_3^3 e_3\right\},$$
$$S_2 = \operatorname{span}\left\{z_1 z_3 \mu_1 e_1, z_1 z_3 \mu_2 e_1, z_2 z_3 \mu_1 e_2, z_2 z_3 \mu_2 e_2, z_1 z_2 \mu_1 e_3, z_1 z_2 \mu_2 e_3, z_3^2 \mu_1 e_3, z_3^2 \mu_2 e_3\right\},$$
$$\mathcal{H}\left(\alpha z_1^{q_1} z_2^{q_2} z_3^{q_3} \mu_1^{l_1} \mu_2^{l_2}\right) = \begin{pmatrix} \alpha z_1^{q_1} z_2^{q_2} z_3^{q_3} \mu_1^{l_1} \mu_2^{l_2} \\ \overline{\alpha} z_1^{q_2} z_2^{q_1} z_3^{q_3} \mu_1^{l_1} \mu_2^{l_2} \end{pmatrix}, \ \alpha \in \mathbb{C}.$$

Therefore, by combining with (3.24) and (3.25), the normal form for Turing-Hopf bifurcation of model (1.5) can be written as

$$\begin{aligned} \dot{z} = Bz &+ \begin{pmatrix} \mathcal{H}\left(B_{10100} z_1 z_3 + \left(B_{10010}\mu_1 + B_{10001}\mu_2\right) z_1\right) \\ B_{11000} z_1 z_2 + \left(B_{00110}\mu_1 + B_{00101}\mu_2\right) z_3 \end{pmatrix} \\ &+ \begin{pmatrix} \mathcal{H}\left(B_{21000} z_1^2 z_2 + B_{10200} z_1 z_3^2 + F_{10110} z_1 z_3 \mu_1 + F_{10101} z_1 z_3 \mu_2\right) \\ B_{11100} z_1 z_2 z_3 + B_{00300} z_3^3 + F_{11010} z_1 z_2 \mu_1 + F_{11001} z_1 z_2 \mu_2 + F_{00210} z_3^2 \mu_1 + F_{00201} z_3^2 \mu_2 \end{pmatrix} \\ &+ O\left(|z||\mu|^2\right). \end{aligned} \tag{3.26}$$

The mathematical expressions of $B_{10100}, B_{10010}, B_{10001}, B_{11000}, B_{00110}, B_{00101}, B_{21000}, B_{10200}, F_{10110}, F_{10101}, B_{11100}, B_{00300}, F_{11010}, F_{11001}, F_{00210}, F_{00201}$ are given in Appendix A. The normal form (3.26) can now be written in real coordinates through the change of variables $z_1 = v_1 - iv_2, z_2 = v_1 + iv_2, z_3 = v_3$, and then changing to cylindrical coordinates by $v_1 = \rho\cos\Theta, v_2 = \rho\sin\Theta, v_3 = \varsigma$, where Θ is the azimuthal angle. Therefore, we obtain, truncating at third order term and removing the azimuthal term

$$\begin{cases} \dot{\rho} = \alpha_1(\mu)\rho + \kappa_{11}\rho^3 + \kappa_{12}\rho\varsigma^2, \ \rho > 0, \\ \dot{\varsigma} = \alpha_2(\mu)\varsigma + \kappa_{21}\rho^2\varsigma + \kappa_{22}\varsigma^3, \end{cases} \tag{3.27}$$

where

$$\begin{gathered} \alpha_1(\mu) = \operatorname{Re}(B_{10010})\mu_1 + \operatorname{Re}(B_{10001})\mu_2, \ \alpha_2(\mu) = B_{00110}\mu_1 + B_{00101}\mu_2, \\ \kappa_{11} = \operatorname{Re}(B_{21000}), \ \kappa_{12} = \operatorname{Re}(B_{10200}), \ \kappa_{21} = B_{11100}, \ \kappa_{22} = B_{00300}. \end{gathered} \tag{3.28}$$

3.2. Calculations of $A_{q_1q_2q_3}$, $\widehat{A}_{q_1q_2q_3}$, $\mathcal{S}_2(\Phi(\theta)z_x, w(\theta))$, $\widehat{\mathcal{S}}_2(\Phi(\theta)z_x, \widehat{w}(0))$

According to Section 3.3.1 in [25], by letting

$$F(\varphi(\theta), \widehat{\varphi}(0), \mu) = \operatorname{col}\left(F^{(1)}(\varphi(\theta), \widehat{\varphi}(0), \mu), F^{(2)}(\varphi(\theta), \widehat{\varphi}(0), \mu)\right)$$

for any $\varphi(\theta) = \operatorname{col}(\varphi_1(\theta), \varphi_2(\theta)) \in \mathcal{BC}$ and $\widehat{\varphi}(0) = \operatorname{col}(\widehat{\varphi}_1(0), \widehat{\varphi}_2(0)) \in \mathcal{BC}$, and writing the m-th Fréchet derivative $F_m(\varphi(\theta), \widehat{\varphi}(0), \mu), \ m \geq 2$ as

$$\frac{1}{m!}F_m(\varphi(\theta), \widehat{\varphi}(0), \mu) = \sum_{l_1+l_2+l_3+l_4+l_5+l_6=m} \frac{1}{l_1!l_2!l_3!l_4!l_5!l_6!} f_{l_1l_2l_3l_4l_5l_6} \varphi_1^{l_1}(0)\varphi_2^{l_2}(0)\varphi_1^{l_3}(-1)\widehat{\varphi}_1^{l_4}(0)\mu_1^{l_5}\mu_2^{l_6},$$

where $f_{l_1l_2l_3l_4l_5l_6} = \operatorname{col}\left(f^{(1)}_{l_1l_2l_3l_4l_5l_6}, f^{(2)}_{l_1l_2l_3l_4l_5l_6}\right)$ with

$$\begin{aligned} f^{(1)}_{l_1l_2l_3l_4l_5l_6} &= \frac{\partial^{l_1+l_2+l_3+l_4+l_5+l_6} F^{(1)}(0,0,0,0,0,0)}{\partial\varphi_1^{l_1}(0)\partial\varphi_2^{l_2}(0)\partial\varphi_1^{l_3}(-1)\partial\widehat{\varphi}_1^{l_4}(0)\partial\mu_1^{l_5}\partial\mu_2^{l_6}}, \\ f^{(2)}_{l_1l_2l_3l_4l_5l_6} &= \frac{\partial^{l_1+l_2+l_3+l_4+l_5+l_6} F^{(2)}(0,0,0,0,0,0)}{\partial\varphi_1^{l_1}(0)\partial\varphi_2^{l_2}(0)\partial\varphi_1^{l_3}(-1)\partial\widehat{\varphi}_1^{l_4}(0)\partial\mu_1^{l_5}\partial\mu_2^{l_6}}, \end{aligned} \tag{3.29}$$

and $F(\varphi(\theta), \widehat{\varphi}(0), \mu)$ is defined by (3.3).

Furthermore, it follows from (3.3) and (3.29), we can see that $f^{(1)}_{l_1 l_2 l_3 l_4 l_5 l_6} = 0$ for either $l_2 \geq 2$, $l_3 \geq 1$, or $l_4 \geq 2$, and $f^{(2)}_{l_1 l_2 l_3 l_4 l_5 l_6} = 0$ for either $l_1 \geq 1$, or $l_2 \geq 3$, or $l_4 \geq 1$. Moreover, we can verify that $f^{(1)}_{010100} = 0$, $f^{(1)}_{000111} = 0$, $f^{(1)}_{010011} = 0$, $f^{(1)}_{010101} = 0$, $f^{(1)}_{010110} = 0$, $f^{(2)}_{001011} = 0$ and $f^{(2)}_{010011} = 0$, then we have

$$\begin{aligned} F_2^{(1)}(\varphi(\theta), \widehat{\varphi}(0), \mu) &= F_2^{(1)}(\varphi(\theta), \widehat{\varphi}(0), 0) = 2f^{(1)}_{100100}\varphi_1(0)\widehat{\varphi}_1(0) + 2f^{(1)}_{110000}\varphi_1(0)\varphi_2(0) + f^{(1)}_{200000}\varphi_1^2(0), \\ F_2^{(2)}(\varphi(\theta), \widehat{\varphi}(0), \mu) &= F_2^{(2)}(\varphi(\theta), \widehat{\varphi}(0), 0) = f^{(2)}_{002000}\varphi_1^2(-1) + 2f^{(2)}_{011000}\varphi_2(0)\varphi_1(-1) + f^{(2)}_{020000}\varphi_2^2(0), \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} F_3^{(1)}(\varphi(\theta), \widehat{\varphi}(0), \mu) &= 0, \\ F_3^{(2)}(\varphi(\theta), \widehat{\varphi}(0), \mu) &= 3f^{(2)}_{002001}\varphi_1^2(-1)\mu_2 + 3f^{(2)}_{002010}\varphi_1^2(-1)\mu_1 + f^{(2)}_{003000}\varphi_1^3(-1) + 6f^{(2)}_{011001}\varphi_2(0)\varphi_1(-1)\mu_2 \\ &\quad + 6f^{(2)}_{011010}\varphi_2(0)\varphi_1(-1)\mu_1 + 3f^{(2)}_{012000}\varphi_2(0)\varphi_1^2(-1) \\ &\quad + 3f^{(2)}_{020001}\varphi_2^2(0)\mu_2 + 3f^{(2)}_{020010}\varphi_2^2(0)\mu_1 + 3f^{(2)}_{021000}\varphi_2^2(0)\varphi_1(-1). \end{aligned} \tag{3.31}$$

By combining with (3.3), (3.29), (3.30) and (3.31), we can obtain

$$\begin{aligned} & f^{(1)}_{100100} = -\tau_c, \ f^{(1)}_{110000} = -\tau_c a(b+u_*)^{-1} + \tau_c a u_*(b+u_*)^{-2}, \\ & f^{(1)}_{200000} = 2\tau_c a v_*(b+u_*)^{-2} - 2\tau_c a u_* v_*(b+u_*)^{-3}, \\ & f^{(2)}_{002000} = -2\tau_c r v_*^2 u_*^{-3}, \ f^{(2)}_{011000} = 2\tau_c r v_* u_*^{-2}, \ f^{(2)}_{020000} = -2\tau_c r u_*^{-1}, \\ & f^{(2)}_{002001} = -2r v_*^2 u_*^{-3}, \ f^{(2)}_{002010} = -2\tau_c v_*^2 u_*^{-3}, \\ & f^{(2)}_{003000} = 6\tau_c r v_*^2 u_*^{-4}, \ f^{(2)}_{011001} = 2r v_* u_*^{-2}, \\ & f^{(2)}_{011010} = 2\tau_c v_* u_*^{-2}, \ f^{(2)}_{012000} = -4\tau_c r v_* u_*^{-3}, \\ & f^{(2)}_{020001} = -2r u_*^{-1}, \ f^{(2)}_{020010} = -2\tau_c u_*^{-1}, \\ & f^{(2)}_{021000} = 2\tau_c r u_*^{-2}. \end{aligned}$$

By using the decomposition of $\mathcal{BC}$, we know that $\varphi(\theta) = \Phi(\theta)z_x + w(\theta)$ and $\widehat{\varphi}(0) = \widehat{w}(0)$. Furthermore, by noticing that $F(0,0,\mu) = 0$ and $DF(0,0,\mu) = 0$, and according to Section 3.3 in [25], $F_2\left(\Phi(\theta)z_x + w(\theta), \widehat{w}(0), \mu\right)$ can be written as

$$\begin{aligned} F_2\left(\Phi(\theta)z_x + w(\theta), \widehat{w}(0), \mu\right) &= F_2\left(\Phi(\theta)z_x + w(\theta), \widehat{w}(0), 0\right) \\ &= \sum_{q_1+q_2+q_3=2} A_{q_1 q_2 q_3}\gamma_{n_1}^{q_1+q_2}(x)\gamma_{n_2}^{q_3}(x) z_1^{q_1} z_2^{q_2} z_3^{q_3} \\ &\quad + \mathcal{S}_2\left(\Phi(\theta)z_x, w(\theta)\right) + \widehat{\mathcal{S}_2}\left(\Phi(\theta)z_x, \widehat{w}(0)\right) + O\left(|(w(\theta), \widehat{w}(0))|^2\right) \\ &= A_{200}\gamma_{n_1}^2(x)z_1^2 + A_{020}\gamma_{n_1}^2(x)z_2^2 + A_{002}\gamma_{n_2}^2(x)z_3^2 \\ &\quad + A_{110}\gamma_{n_1}^2(x)z_1 z_2 + A_{101}\gamma_{n_1}(x)\gamma_{n_2}(x)z_1 z_3 + A_{011}\gamma_{n_1}(x)\gamma_{n_2}(x)z_2 z_3 \\ &\quad + \mathcal{S}_2\left(\Phi(\theta)z_x, w(\theta)\right) + \widehat{\mathcal{S}_2}\left(\Phi(\theta)z_x, \widehat{w}(0)\right) + O\left(|(w(\theta), \widehat{w}(0))|^2\right), \end{aligned} \tag{3.32}$$

and $F_3\left(\Phi(\theta)z_x, 0, \mu\right)$ can be written as

$$
\begin{aligned}
F_3\left(\Phi(\theta)z_x, 0, \mu\right) = & \sum_{q_1+q_2+q_3=3} A_{q_1q_2q_3} z_1^{q_1} z_2^{q_2} z_3^{q_3} \gamma_{n_1}^{q_1+q_2}(x)\gamma_{n_2}^{q_3}(x) \\
& + \sum_{q_1+q_2+q_3=2} \widehat{A}^{(1)}_{q_1q_2q_3} z_1^{q_1} z_2^{q_2} z_3^{q_3} \gamma_{n_1}^{q_1+q_2}(x)\gamma_{n_2}^{q_3}(x)\mu_1 \\
& + \sum_{q_1+q_2+q_3=2} \widehat{A}^{(2)}_{q_1q_2q_3} z_1^{q_1} z_2^{q_2} z_3^{q_3} \gamma_{n_1}^{q_1+q_2}(x)\gamma_{n_2}^{q_3}(x)\mu_2 + O\left(|z||\mu|^2\right) \\
& = A_{300}\gamma_{n_1}^3(x)z_1^3 + A_{030}\gamma_{n_1}^3(x)z_2^3 + A_{003}\gamma_{n_2}^3(x)z_3^3 + A_{210}\gamma_{n_1}^3(x)z_1^2z_2 \\
& + A_{120}\gamma_{n_1}^3(x)z_1z_2^2 + A_{102}\gamma_{n_1}(x)\gamma_{n_2}^2(x)z_1z_3^2 + A_{012}\gamma_{n_1}(x)\gamma_{n_2}^2(x)z_2z_3^2 + A_{021}\gamma_{n_1}^2(x)\gamma_{n_2}(x)z_2^2z_3 \\
& + A_{201}\gamma_{n_1}^2(x)\gamma_{n_2}(x)z_1^2z_3 + A_{111}\gamma_{n_1}^2(x)\gamma_{n_2}(x)z_1z_2z_3 \\
& + \widehat{A}^{(1)}_{200}\gamma_{n_1}^2(x)z_1^2\mu_1 + \widehat{A}^{(1)}_{020}\gamma_{n_1}^2(x)z_2^2\mu_1 + \widehat{A}^{(1)}_{002}\gamma_{n_2}^2(x)z_3^2\mu_1 \\
& + \widehat{A}^{(1)}_{110}\gamma_{n_1}^2(x)z_1z_2\mu_1 + \widehat{A}^{(1)}_{101}\gamma_{n_1}(x)\gamma_{n_2}(x)z_1z_3\mu_1 + \widehat{A}^{(1)}_{011}\gamma_{n_1}(x)\gamma_{n_2}(x)z_2z_3\mu_1 \\
& + \widehat{A}^{(2)}_{200}\gamma_{n_1}^2(x)z_1^2\mu_2 + \widehat{A}^{(2)}_{020}\gamma_{n_1}^2(x)z_2^2\mu_2 + \widehat{A}^{(2)}_{002}\gamma_{n_2}^2(x)z_3^2\mu_2 \\
& + \widehat{A}^{(2)}_{110}\gamma_{n_1}^2(x)z_1z_2\mu_2 + \widehat{A}^{(2)}_{101}\gamma_{n_1}(x)\gamma_{n_2}(x)z_1z_3\mu_2 + \widehat{A}^{(2)}_{011}\gamma_{n_1}(x)\gamma_{n_2}(x)z_2z_3\mu_2 + O\left(|z||\mu|^2\right).
\end{aligned} \tag{3.33}
$$

More precisely, after a straightforward calculation, we have

$$
\begin{aligned}
\varphi(0) &= \Phi(0)z_x + w(0) = \begin{pmatrix} \xi_{n_11}z_1\gamma_{n_1}(x) + \overline{\xi}_{n_11}z_2\gamma_{n_1}(x) + \xi_{n_21}z_3\gamma_{n_2}(x) + w_1(0) \\ \xi_{n_12}z_1\gamma_{n_1}(x) + \overline{\xi}_{n_12}z_2\gamma_{n_1}(x) + \xi_{n_22}z_3\gamma_{n_2}(x) + w_2(0) \end{pmatrix}, \\
\widehat{\varphi}(0) &= \widehat{w}(0) = \begin{pmatrix} \widehat{w}_1(0) \\ \widehat{w}_2(0) \end{pmatrix}, \\
\varphi(-1) &= \Phi(-1)z_x + w(-1) = \begin{pmatrix} \xi_{n_11}z_1\gamma_{n_1}(x)e^{-i\omega_c} + \overline{\xi}_{n_11}z_2\gamma_{n_1}(x)e^{i\omega_c} + \xi_{n_21}z_3\gamma_{n_2}(x) + w_1(-1) \\ \xi_{n_12}z_1\gamma_{n_1}(x)e^{-i\omega_c} + \overline{\xi}_{n_12}z_2\gamma_{n_1}(x)e^{i\omega_c} + \xi_{n_22}z_3\gamma_{n_2}(x) + w_2(-1) \end{pmatrix}.
\end{aligned} \tag{3.34}
$$

By comparing the corresponding coefficients of (3.30) and (3.32) as well as (3.31) and (3.33), respectively, we have

$$
\begin{aligned}
A_{101} &= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_11}\xi_{n_22} + \xi_{n_21}\xi_{n_12}\right) + 2f^{(1)}_{200000}\xi_{n_11}\xi_{n_21} \\ 2f^{(2)}_{002000}\xi_{n_11}\xi_{n_21}e^{-i\omega_c} + 2f^{(2)}_{011000}\left(\xi_{n_12}\xi_{n_21} + \xi_{n_22}\xi_{n_11}e^{-i\omega_c}\right) + 2f^{(2)}_{020000}\xi_{n_12}\xi_{n_22} \end{pmatrix} = \overline{A}_{011}, \\
A_{110} &= \begin{pmatrix} 4f^{(1)}_{110000}\operatorname{Re}\left(\xi_{n_11}\overline{\xi}_{n_12}\right) + 2f^{(1)}_{200000}|\xi_{n_11}|^2 \\ 2f^{(2)}_{002000}|\xi_{n_11}|^2 + 4f^{(2)}_{011000}\operatorname{Re}\left(\xi_{n_12}\overline{\xi}_{n_11}e^{i\omega_c}\right) + 2f^{(2)}_{020000}|\xi_{n_12}|^2 \end{pmatrix}, \\
A_{210} &= \begin{pmatrix} 0 \\ 3f^{(2)}_{003000}\xi_{n_11}^2\overline{\xi}_{n_11}e^{-i\omega_c} + 3f^{(2)}_{012000}\left(2\xi_{n_12}|\xi_{n_11}|^2 + \overline{\xi}_{n_12}\xi_{n_11}^2e^{-2i\omega_c}\right) \\ +3f^{(2)}_{021000}\left(2|\xi_{n_12}|^2\xi_{n_11}e^{-i\omega_c} + \xi_{n_12}^2\overline{\xi}_{n_11}e^{i\omega_c}\right) \end{pmatrix} = \overline{A}_{120}, \\
A_{102} &= \begin{pmatrix} 0 \\ 3f^{(2)}_{003000}\xi_{n_11}\xi_{n_21}^2e^{-i\omega_c} + 3f^{(2)}_{012000}\left(\xi_{n_12}\xi_{n_21}^2 + 2\xi_{n_22}\xi_{n_11}\xi_{n_21}e^{-i\omega_c}\right) \\ +3f^{(2)}_{021000}\left(\xi_{n_22}^2\xi_{n_11}e^{-i\omega_c} + 2\xi_{n_12}\xi_{n_22}\xi_{n_21}\right) \end{pmatrix} = \overline{A}_{012}, \\
A_{111} &= \begin{pmatrix} 0 \\ 6f^{(2)}_{003000}|\xi_{n_11}|^2\xi_{n_21} \\ +3f^{(2)}_{012000}\left(2\xi_{n_12}\overline{\xi}_{n_11}\xi_{n_21}e^{i\omega_c} + 2\overline{\xi}_{n_12}\xi_{n_11}\xi_{n_21}e^{-i\omega_c} + 2\xi_{n_22}|\xi_{n_11}|^2\right) \\ +3f^{(2)}_{021000}\left(2\overline{\xi}_{n_12}\xi_{n_22}\xi_{n_11}e^{-i\omega_c} + 2\xi_{n_12}\xi_{n_22}\overline{\xi}_{n_11}e^{i\omega_c} + 2|\xi_{n_12}|^2\xi_{n_21}\right) \end{pmatrix},
\end{aligned}
$$

$$A_{003}=\begin{pmatrix} 0 \\ f^{(2)}_{003000}\xi^3_{n_2 1}+3f^{(2)}_{012000}\xi_{n_2 2}\xi^2_{n_2 1}+3f^{(2)}_{021000}\xi^2_{n_2 2}\xi_{n_2 1}\end{pmatrix},$$
$$A_{200}=\begin{pmatrix} 2f^{(1)}_{110000}\xi_{n_1 1}\xi_{n_1 2}+f^{(1)}_{200000}\xi^2_{n_1 1} \\ f^{(2)}_{002000}\xi^2_{n_1 1}e^{-2i\omega_c}+2f^{(2)}_{011000}\xi_{n_1 2}\xi_{n_1 1}e^{-i\omega_c}+f^{(2)}_{020000}\xi^2_{n_1 2}\end{pmatrix}=\overline{A}_{020},$$
$$A_{002}=\begin{pmatrix} 2f^{(1)}_{110000}\xi_{n_2 1}\xi_{n_2 2}+f^{(1)}_{200000}\xi^2_{n_2 1} \\ f^{(2)}_{002000}\xi^2_{n_2 1}+2f^{(2)}_{011000}\xi_{n_2 2}\xi_{n_2 1}+f^{(2)}_{020000}\xi^2_{n_2 2}\end{pmatrix}$$

and

$$\widehat{A}^{(1)}_{101}=\begin{pmatrix} 0 \\ 6f^{(2)}_{002010}\xi_{n_1 1}\xi_{n_2 1}e^{-i\omega_c}+6f^{(2)}_{011010}\left(\xi_{n_1 2}\xi_{n_2 1}+\xi_{n_2 2}\xi_{n_1 1}e^{-i\omega_c}\right)+6f^{(2)}_{020010}\xi_{n_1 2}\xi_{n_2 2}\end{pmatrix},$$
$$\widehat{A}^{(2)}_{101}=\begin{pmatrix} 0 \\ 6f^{(2)}_{002001}\xi_{n_1 1}\xi_{n_2 1}e^{-i\omega_c}+6f^{(2)}_{011001}(\xi_{n_1 2}\xi_{n_2 1}+\xi_{n_2 2}\xi_{n_1 1}e^{-i\omega_c})+6f^{(2)}_{020001}\xi_{n_1 2}\xi_{n_2 2}\end{pmatrix},$$
$$\widehat{A}^{(1)}_{110}=\begin{pmatrix} 0 \\ 6f^{(2)}_{002010}|\xi_{n_1 1}|^2+12f^{(2)}_{011010}\operatorname{Re}\left(\xi_{n_1 2}\overline{\xi}_{n_1 1}e^{i\omega_c}\right)+6f^{(2)}_{020010}|\xi_{n_1 2}|^2\end{pmatrix},$$
$$\widehat{A}^{(2)}_{110}=\begin{pmatrix} 0 \\ 6f^{(2)}_{002001}|\xi_{n_1 1}|^2+12f^{(2)}_{011001}\operatorname{Re}\left(\xi_{n_1 2}\overline{\xi}_{n_1 1}e^{i\omega_c}\right)+6f^{(2)}_{020001}|\xi_{n_1 2}|^2\end{pmatrix}.$$

Moreover, by combining with (3.30) and (3.34), we have

$$\mathcal{S}_2\left(\xi_{n_1}e^{i\omega_c\theta},h_{0,11000}(\theta)\right)$$
$$=\begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_1 1}h^{(2)}_{0,11000}(0)+\xi_{n_1 2}h^{(1)}_{0,11000}(0)\right)+2f^{(1)}_{200000}\xi_{n_1 1}h^{(1)}_{0,11000}(0) \\ 2f^{(2)}_{002000}\xi_{n_1 1}h^{(1)}_{0,11000}(-1)e^{-i\omega_c}+2f^{(2)}_{011000}\left(\xi_{n_1 2}h^{(1)}_{0,11000}(-1)+\xi_{n_1 1}h^{(2)}_{0,11000}(0)e^{-i\omega_c}\right) \\ +2f^{(2)}_{020000}\xi_{n_1 2}h^{(2)}_{0,11000}(0)\end{pmatrix},$$

$$\mathcal{S}_2\left(\overline{\xi}_{n_1}e^{-i\omega_c\theta},h_{0,20000}(\theta)\right)$$
$$=\begin{pmatrix} 2f^{(1)}_{110000}\left(\overline{\xi}_{n_1 1}h^{(2)}_{0,20000}(0)+\overline{\xi}_{n_1 2}h^{(1)}_{0,20000}(0)\right)+2f^{(1)}_{200000}\overline{\xi}_{n_1 1}h^{(1)}_{0,20000}(0) \\ 2f^{(2)}_{002000}\overline{\xi}_{n_1 1}h^{(1)}_{0,20000}(-1)e^{i\omega_c}+2f^{(2)}_{011000}\left(\overline{\xi}_{n_1 2}h^{(1)}_{0,20000}(-1)+\overline{\xi}_{n_1 1}h^{(2)}_{0,20000}(0)e^{i\omega_c}\right) \\ +2f^{(2)}_{020000}\overline{\xi}_{n_1 2}h^{(2)}_{0,20000}(0)\end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_1}e^{i\omega_c\theta},h_{2n_1,11000}(\theta)\right)$$
$$=\begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_1 1}h^{(2)}_{2n_1,11000}(0)+\xi_{n_1 2}h^{(1)}_{2n_1,11000}(0)\right)+2f^{(1)}_{200000}\xi_{n_1 1}h^{(1)}_{2n_1,11000}(0) \\ 2f^{(2)}_{002000}\xi_{n_1 1}h^{(1)}_{2n_1,11000}(-1)e^{-i\omega_c}+2f^{(2)}_{011000}\left(\xi_{n_1 2}h^{(1)}_{2n_1,11000}(-1)+\xi_{n_1 1}h^{(2)}_{2n_1,11000}(0)e^{-i\omega_c}\right) \\ +2f^{(2)}_{020000}\xi_{n_1 2}h^{(2)}_{2n_1,11000}(0)\end{pmatrix},$$

$$\mathcal{S}_2\left(\overline{\xi}_{n_1}e^{-i\omega_c\theta},h_{2n_1,20000}(\theta)\right)$$
$$=\begin{pmatrix} 2f^{(1)}_{110000}\left(\overline{\xi}_{n_1 1}h^{(2)}_{2n_1,20000}(0)+\overline{\xi}_{n_1 2}h^{(1)}_{2n_1,20000}(0)\right)+2f^{(1)}_{200000}\overline{\xi}_{n_1 1}h^{(1)}_{2n_1,20000}(0) \\ 2f^{(2)}_{002000}\overline{\xi}_{n_1 1}h^{(1)}_{2n_1,20000}(-1)e^{i\omega_c}+2f^{(2)}_{011000}\left(\overline{\xi}_{n_1 2}h^{(1)}_{2n_1,20000}(-1)+\overline{\xi}_{n_1 1}h^{(2)}_{2n_1,20000}(0)e^{i\omega_c}\right) \\ +2f^{(2)}_{020000}\overline{\xi}_{n_1 1}h^{(2)}_{2n_1,20000}(0)\end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_1}e^{i\omega_c\theta}, h_{0,00200}(\theta)\right)$$
$$= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_11}h^{(2)}_{0,00200}(0)+\xi_{n_12}h^{(1)}_{0,00200}(0)\right)+2f^{(1)}_{200000}\xi_{n_11}h^{(1)}_{0,00200}(0) \\ 2f^{(2)}_{002000}\xi_{n_11}h^{(1)}_{0,00200}(-1)e^{-i\omega_c}+2f^{(2)}_{011000}\left(\xi_{n_12}h^{(1)}_{0,00200}(-1)+\xi_{n_11}h^{(2)}_{0,00200}(0)e^{-i\omega_c}\right) \\ +2f^{(2)}_{020000}\xi_{n_12}h^{(2)}_{0,00200}(0) \end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_1}e^{i\omega_c\theta}, h_{2n_1,00200}(\theta)\right)$$
$$= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_11}h^{(2)}_{2n_1,00200}(0)+\xi_{n_12}h^{(1)}_{2n_1,00200}(0)\right)+2f^{(1)}_{200000}\xi_{n_11}h^{(1)}_{2n_1,00200}(0) \\ 2f^{(2)}_{002000}\xi_{n_11}h^{(1)}_{2n_1,00200}(-1)e^{-i\omega_c}+2f^{(2)}_{011000}\left(\xi_{n_12}h^{(1)}_{2n_1,00200}(-1)+\xi_{n_11}h^{(2)}_{2n_1,00200}(0)e^{-i\omega_c}\right) \\ +2f^{(2)}_{020000}\xi_{n_12}h^{(2)}_{2n_1,00200}(0) \end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_2}, h_{n_1+n_2,10100}(\theta)\right)$$
$$= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_21}h^{(2)}_{n_1+n_2,10100}(0)+\xi_{n_22}h^{(1)}_{n_1+n_2,10100}(0)\right)+2f^{(1)}_{200000}\xi_{n_21}h^{(1)}_{n_1+n_2,10100}(0) \\ 2f^{(2)}_{002000}\xi_{n_21}h^{(1)}_{n_1+n_2,10100}(-1)+2f^{(2)}_{011000}\left(\xi_{n_22}h^{(1)}_{n_1+n_2,10100}(-1)+\xi_{n_21}h^{(2)}_{n_1+n_2,10100}(0)\right) \\ +2f^{(2)}_{020000}\xi_{n_22}h^{(2)}_{n_1+n_2,10100}(0) \end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_2}, h_{|n_2-n_1|,10100}(\theta)\right)$$
$$= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_21}h^{(2)}_{|n_2-n_1|,10100}(0)+\xi_{n_22}h^{(1)}_{|n_2-n_1|,10100}(0)\right)+2f^{(1)}_{200000}\xi_{n_21}h^{(1)}_{|n_2-n_1|,10100}(0) \\ 2f^{(2)}_{002000}\xi_{n_21}h^{(1)}_{|n_2-n_1|,10100}(-1)+2f^{(2)}_{011000}\left(\xi_{n_22}h^{(1)}_{|n_2-n_1|,10100}(-1)+\xi_{n_21}h^{(2)}_{|n_2-n_1|,10100}(0)\right) \\ +2f^{(2)}_{020000}\xi_{n_22}h^{(2)}_{|n_2-n_1|,10100}(0) \end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_1}e^{i\omega_c\theta}, h_{n_1+n_2,01100}(\theta)\right)$$
$$= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_11}h^{(2)}_{n_1+n_2,01100}(0)+\xi_{n_12}h^{(1)}_{n_1+n_2,01100}(0)\right)+2f^{(1)}_{200000}\xi_{n_11}h^{(1)}_{n_1+n_2,01100}(0) \\ 2f^{(2)}_{002000}\xi_{n_11}h^{(1)}_{n_1+n_2,01100}(-1)e^{-i\omega_c}+2f^{(2)}_{011000}\left(\xi_{n_12}h^{(1)}_{n_1+n_2,01100}(-1)+\xi_{n_11}h^{(2)}_{n_1+n_2,01100}(0)e^{-i\omega_c}\right) \\ +2f^{(2)}_{020000}\xi_{n_12}h^{(2)}_{n_1+n_2,01100}(0) \end{pmatrix},$$

$$\mathcal{S}_2\left(\overline{\xi}_{n_1}e^{-i\omega_c\theta}, h_{n_1+n_2,10100}(\theta)\right)$$
$$= \begin{pmatrix} 2f^{(1)}_{110000}\left(\overline{\xi}_{n_11}h^{(2)}_{n_1+n_2,10100}(0)+\overline{\xi}_{n_12}h^{(1)}_{n_1+n_2,10100}(0)\right)+2f^{(1)}_{200000}\overline{\xi}_{n_11}h^{(1)}_{n_1+n_2,10100}(0) \\ 2f^{(2)}_{002000}\overline{\xi}_{n_11}h^{(1)}_{n_1+n_2,10100}(-1)e^{i\omega_c}+2f^{(2)}_{011000}\left(\overline{\xi}_{n_12}h^{(1)}_{n_1+n_2,10100}(-1)+\overline{\xi}_{n_11}h^{(2)}_{n_1+n_2,10100}(0)e^{i\omega_c}\right) \\ +2f^{(2)}_{020000}\overline{\xi}_{n_12}h^{(2)}_{n_1+n_2,10100}(0) \end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_1}e^{i\omega_c\theta}, h_{|n_2-n_1|,01100}(\theta)\right)$$
$$= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_11}h^{(2)}_{|n_2-n_1|,01100}(0)+\xi_{n_12}h^{(1)}_{|n_2-n_1|,01100}(0)\right)+2f^{(1)}_{200000}\xi_{n_11}h^{(1)}_{|n_2-n_1|,01100}(0) \\ 2f^{(2)}_{002000}\xi_{n_11}h^{(1)}_{|n_2-n_1|,01100}(-1)e^{-i\omega_c}+2f^{(2)}_{011000}\left(\xi_{n_12}h^{(1)}_{|n_2-n_1|,01100}(-1)+\xi_{n_11}h^{(2)}_{|n_2-n_1|,01100}(0)e^{-i\omega_c}\right) \\ +2f^{(2)}_{020000}\xi_{n_12}h^{(2)}_{|n_2-n_1|,01100}(0) \end{pmatrix},$$

$$\mathcal{S}_2\left(\overline{\xi}_{n_1}e^{-i\omega_c\theta}, h_{|n_2-n_1|,10100}(\theta)\right)$$
$$= \begin{pmatrix} 2f^{(1)}_{110000}\left(\overline{\xi}_{n_11}h^{(2)}_{|n_2-n_1|,10100}(0)+\overline{\xi}_{n_12}h^{(1)}_{|n_2-n_1|,10100}(0)\right)+2f^{(1)}_{200000}\overline{\xi}_{n_11}h^{(1)}_{|n_2-n_1|,10100}(0) \\ 2f^{(2)}_{002000}\overline{\xi}_{n_11}h^{(1)}_{|n_2-n_1|,10100}(-1)e^{i\omega_c}+2f^{(2)}_{011000}\left(\overline{\xi}_{n_12}h^{(1)}_{|n_2-n_1|,10100}(-1)+\overline{\xi}_{n_11}h^{(2)}_{|n_2-n_1|,10100}(0)e^{i\omega_c}\right) \\ +2f^{(2)}_{020000}\overline{\xi}_{n_12}h^{(2)}_{|n_2-n_1|,10100}(0) \end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_2}, h_{0,11000}(\theta)\right) = \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_2 1}h^{(2)}_{0,11000}(0) + \xi_{n_2 2}h^{(1)}_{0,11000}(0)\right) + 2f^{(1)}_{200000}\xi_{n_2 1}h^{(1)}_{0,11000}(0) \\ 2f^{(2)}_{002000}\xi_{n_2 1}h^{(1)}_{0,11000}(-1) + 2f^{(2)}_{011000}\left(\xi_{n_2 2}h^{(1)}_{0,11000}(-1) + \xi_{n_2 1}h^{(2)}_{0,11000}(0)\right) \\ +2f^{(2)}_{020000}\xi_{n_2 2}h^{(2)}_{0,11000}(0) \end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_2}, h_{2n_2,11000}(\theta)\right) = \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_2 1}h^{(2)}_{2n_2,11000}(0) + \xi_{n_2 2}h^{(1)}_{2n_2,11000}(0)\right) + 2f^{(1)}_{200000}\xi_{n_2 1}h^{(1)}_{2n_2,11000}(0) \\ 2f^{(2)}_{002000}\xi_{n_2 1}h^{(1)}_{2n_2,11000}(-1) + 2f^{(2)}_{011000}\left(\xi_{n_2 2}h^{(1)}_{2n_2,11000}(-1) + \xi_{n_2 1}h^{(2)}_{2n_2,11000}(0)\right) \\ +2f^{(2)}_{020000}\xi_{n_2 2}h^{(2)}_{2n_2,11000}(0) \end{pmatrix},$$

$$\begin{aligned}&\mathcal{S}_2\left(\xi_{n_2}, h_{0,00200}(\theta)\right) \\ &= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_2 1}h^{(2)}_{0,00200}(0) + \xi_{n_2 2}h^{(1)}_{0,00200}(0)\right) + 2f^{(1)}_{200000}\xi_{n_2 1}h^{(1)}_{0,00200}(0) \\ 2f^{(2)}_{002000}\xi_{n_2 1}h^{(1)}_{0,00200}(-1) + 2f^{(2)}_{011000}\left(\xi_{n_2 2}h^{(1)}_{0,00200}(-1) + \xi_{n_2 1}h^{(2)}_{0,00200}(0)\right) \\ +2f^{(2)}_{020000}\xi_{n_2 2}h^{(2)}_{0,00200}(0) \end{pmatrix},\end{aligned}$$

$$\begin{aligned}&\mathcal{S}_2\left(\xi_{n_2}, h_{2n_2,00200}(\theta)\right) \\ &= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_2 1}h^{(2)}_{2n_2,00200}(0) + \xi_{n_2 2}h^{(1)}_{2n_2,00200}(0)\right) + 2f^{(1)}_{200000}\xi_{n_2 1}h^{(1)}_{2n_2,00200}(0) \\ 2f^{(2)}_{002000}\xi_{n_2 1}h^{(1)}_{2n_2,00200}(-1) + 2f^{(2)}_{011000}\left(\xi_{n_2 2}h^{(1)}_{2n_2,00200}(-1) + \xi_{n_2 1}h^{(2)}_{2n_2,00200}(0)\right) \\ +2f^{(2)}_{020000}\xi_{n_2 2}h^{(2)}_{2n_2,00200}(0) \end{pmatrix},\end{aligned}$$

$$\begin{aligned}&\mathcal{S}_2\left(\xi_{n_1}e^{i\omega_c\theta}, h_{0,00110}(\theta)\right) \\ &= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_1 1}h^{(2)}_{0,00110}(0) + \xi_{n_1 2}h^{(1)}_{0,00110}(0)\right) + 2f^{(1)}_{200000}\xi_{n_1 1}h^{(1)}_{0,00110}(0) \\ 2f^{(2)}_{002000}\xi_{n_1 1}h^{(1)}_{0,00110}(-1)e^{-i\omega_c} + 2f^{(2)}_{011000}\left(\xi_{n_1 2}h^{(1)}_{0,00110}(-1) + \xi_{n_1 1}h^{(2)}_{0,00110}(0)e^{-i\omega_c}\right) \\ +2f^{(2)}_{020000}\xi_{n_1 2}h^{(2)}_{0,00110}(0) \end{pmatrix},\end{aligned}$$

$$\begin{aligned}&\mathcal{S}_2\left(\xi_{n_1}e^{i\omega_c\theta}, h_{2n_1,00110}(\theta)\right) \\ &= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_1 1}h^{(2)}_{2n_1,00110}(0) + \xi_{n_1 2}h^{(1)}_{2n_1,00110}(0)\right) + 2f^{(1)}_{200000}\xi_{n_1 1}h^{(1)}_{2n_1,00110}(0) \\ 2f^{(2)}_{002000}\xi_{n_1 1}h^{(1)}_{2n_1,00110}(-1)e^{-i\omega_c} + 2f^{(2)}_{011000}\left(\xi_{n_1 2}h^{(1)}_{2n_1,00110}(-1) + \xi_{n_1 1}h^{(2)}_{2n_1,00110}(0)e^{-i\omega_c}\right) \\ +2f^{(2)}_{020000}\xi_{n_1 2}h^{(2)}_{2n_1,00110}(0) \end{pmatrix},\end{aligned}$$

$$\begin{aligned}&\mathcal{S}_2\left(\xi_{n_2}, h_{n_1+n_2,10010}(\theta)\right) \\ &= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_2 1}h^{(2)}_{n_1+n_2,10010}(0) + \xi_{n_2 2}h^{(1)}_{n_1+n_2,10010}(0)\right) + 2f^{(1)}_{200000}\xi_{n_2 1}h^{(1)}_{n_1+n_2,10010}(0) \\ 2f^{(2)}_{002000}\xi_{n_2 1}h^{(1)}_{n_1+n_2,10010}(-1) + 2f^{(2)}_{011000}\left(\xi_{n_2 2}h^{(1)}_{n_1+n_2,10010}(-1) + \xi_{n_2 1}h^{(2)}_{n_1+n_2,10010}(0)\right) \\ +2f^{(2)}_{020000}\xi_{n_2 2}h^{(2)}_{n_1+n_2,10010}(0) \end{pmatrix},\end{aligned}$$

$$\begin{aligned}&\mathcal{S}_2\left(\xi_{n_2}, h_{|n_2-n_1|,10010}(\theta)\right) \\ &= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_2 1}h^{(2)}_{|n_2-n_1|,10010}(0) + \xi_{n_2 2}h^{(1)}_{|n_2-n_1|,10010}(0)\right) + 2f^{(1)}_{200000}\xi_{n_2 1}h^{(1)}_{|n_2-n_1|,10010}(0) \\ 2f^{(2)}_{002000}\xi_{n_2 1}h^{(1)}_{|n_2-n_1|,10010}(-1) + 2f^{(2)}_{011000}\left(\xi_{n_2 2}h^{(1)}_{|n_2-n_1|,10010}(-1) + \xi_{n_2 1}h^{(2)}_{|n_2-n_1|,10010}(0)\right) \\ +2f^{(2)}_{020000}\xi_{n_2 2}h^{(2)}_{|n_2-n_1|,10010}(0) \end{pmatrix},\end{aligned}$$

$$\begin{aligned}&\mathcal{S}_2\left(\xi_{n_1}e^{i\omega_c\theta}, h_{0,00101}(\theta)\right) \\ &= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_1 1}h^{(2)}_{0,00101}(0) + \xi_{n_1 2}h^{(1)}_{0,00101}(0)\right) + 2f^{(1)}_{200000}\xi_{n_1 1}h^{(1)}_{0,00101}(0) \\ 2f^{(2)}_{002000}\xi_{n_1 1}h^{(1)}_{0,00101}(-1)e^{-i\omega_c} + 2f^{(2)}_{011000}\left(\xi_{n_1 2}h^{(1)}_{0,00101}(-1) + \xi_{n_1 1}h^{(2)}_{0,00101}(0)e^{-i\omega_c}\right) \\ +2f^{(2)}_{020000}\xi_{n_1 2}h^{(2)}_{0,00101}(0) \end{pmatrix},\end{aligned}$$

$$\mathcal{S}_2\left(\xi_{n_1}e^{i\omega_c\theta}, h_{2n_1,00101}(\theta)\right)$$
$$= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_1 1}h^{(2)}_{2n_1,00101}(0) + \xi_{n_1 2}h^{(1)}_{2n_1,00101}(0)\right) + 2f^{(1)}_{200000}\xi_{n_1 1}h^{(1)}_{2n_1,00101}(0) \\ 2f^{(2)}_{002000}\xi_{n_1 1}h^{(1)}_{2n_1,00101}(-1)e^{-i\omega_c} + 2f^{(2)}_{011000}\left(\xi_{n_1 2}h^{(1)}_{2n_1,00101}(-1) + \xi_{n_1 1}h^{(2)}_{2n_1,00101}(0)e^{-i\omega_c}\right) \\ +2f^{(2)}_{020000}\xi_{n_1 2}h^{(2)}_{2n_1,00101}(0) \end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_2}, h_{n_1+n_2,10001}(\theta)\right)$$
$$= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_2 1}h^{(2)}_{n_1+n_2,10001}(0) + \xi_{n_2 2}h^{(1)}_{n_1+n_2,10001}(0)\right) + 2f^{(1)}_{200000}\xi_{n_2 1}h^{(1)}_{n_1+n_2,10001}(0) \\ 2f^{(2)}_{002000}\xi_{n_2 1}h^{(1)}_{n_1+n_2,10001}(-1) + 2f^{(2)}_{011000}\left(\xi_{n_2 2}h^{(1)}_{n_1+n_2,10001}(-1) + \xi_{n_2 1}h^{(2)}_{n_1+n_2,10001}(0)\right) \\ +2f^{(2)}_{020000}\xi_{n_2 2}h^{(2)}_{n_1+n_2,10001}(0) \end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_2}, h_{|n_2-n_1|,10001}(\theta)\right)$$
$$= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_2 1}h^{(2)}_{|n_2-n_1|,10001}(0) + \xi_{n_2 2}h^{(1)}_{|n_2-n_1|,10001}(0)\right) + 2f^{(1)}_{200000}\xi_{n_2 1}h^{(1)}_{|n_2-n_1|,10001}(0) \\ 2f^{(2)}_{002000}\xi_{n_2 1}h^{(1)}_{|n_2-n_1|,10001}(-1) + 2f^{(2)}_{011000}\left(\xi_{n_2 2}h^{(1)}_{|n_2-n_1|,10001}(-1) + \xi_{n_2 1}h^{(2)}_{|n_2-n_1|,10001}(0)\right) \\ +2f^{(2)}_{020000}\xi_{n_2 2}h^{(2)}_{|n_2-n_1|,10001}(0) \end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_1}e^{i\omega_c\theta}, h_{n_1+n_2,01010}(\theta)\right)$$
$$= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_1 1}h^{(2)}_{n_1+n_2,01010}(0) + \xi_{n_1 2}h^{(1)}_{n_1+n_2,01010}(0)\right) + 2f^{(1)}_{200000}\xi_{n_1 1}h^{(1)}_{n_1+n_2,01010}(0) \\ 2f^{(2)}_{002000}\xi_{n_1 1}h^{(1)}_{n_1+n_2,01010}(-1)e^{-i\omega_c} + 2f^{(2)}_{011000}\left(\xi_{n_1 2}h^{(1)}_{n_1+n_2,01010}(-1) + \xi_{n_1 1}h^{(2)}_{n_1+n_2,01010}(0)e^{-i\omega_c}\right) \\ +2f^{(2)}_{020000}\xi_{n_1 2}h^{(2)}_{n_1+n_2,01010}(0) \end{pmatrix},$$

$$\mathcal{S}_2\left(\overline{\xi}_{n_1}e^{-i\omega_c\theta}, h_{n_1+n_2,10010}(\theta)\right)$$
$$= \begin{pmatrix} 2f^{(1)}_{110000}\left(\overline{\xi}_{n_1 1}h^{(2)}_{n_1+n_2,10010}(0) + \overline{\xi}_{n_1 2}h^{(1)}_{n_1+n_2,10010}(0)\right) + 2f^{(1)}_{200000}\overline{\xi}_{n_1 1}h^{(1)}_{n_1+n_2,10010}(0) \\ 2f^{(2)}_{002000}\overline{\xi}_{n_1 1}h^{(1)}_{n_1+n_2,10010}(-1)e^{i\omega_c} + 2f^{(2)}_{011000}\left(\overline{\xi}_{n_1 2}h^{(1)}_{n_1+n_2,10010}(-1) + \overline{\xi}_{n_1 1}h^{(2)}_{n_1+n_2,10010}(0)e^{i\omega_c}\right) \\ +2f^{(2)}_{020000}\overline{\xi}_{n_1 2}h^{(2)}_{n_1+n_2,10010}(0) \end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_1}e^{i\omega_c\theta}, h_{|n_2-n_1|,01010}(\theta)\right)$$
$$= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_1 1}h^{(2)}_{|n_2-n_1|,01010}(0) + \xi_{n_1 2}h^{(1)}_{|n_2-n_1|,01010}(0)\right) + 2f^{(1)}_{200000}\xi_{n_1 1}h^{(1)}_{|n_2-n_1|,01010}(0) \\ 2f^{(2)}_{002000}\xi_{n_1 1}h^{(1)}_{|n_2-n_1|,01010}(-1)e^{-i\omega_c} + 2f^{(2)}_{011000}\left(\xi_{n_1 2}h^{(1)}_{|n_2-n_1|,01010}(-1) + \xi_{n_1 1}h^{(2)}_{|n_2-n_1|,01010}(0)e^{-i\omega_c}\right) \\ +2f^{(2)}_{020000}\xi_{n_1 2}h^{(2)}_{|n_2-n_1|,01010}(0) \end{pmatrix},$$

$$\mathcal{S}_2\left(\overline{\xi}_{n_1}e^{-i\omega_c\theta}, h_{|n_2-n_1|,10010}(\theta)\right)$$
$$= \begin{pmatrix} 2f^{(1)}_{110000}\left(\overline{\xi}_{n_1 1}h^{(2)}_{|n_2-n_1|,10010}(0) + \overline{\xi}_{n_1 2}h^{(1)}_{|n_2-n_1|,10010}(0)\right) + 2f^{(1)}_{200000}\overline{\xi}_{n_1 1}h^{(1)}_{|n_2-n_1|,10010}(0) \\ 2f^{(2)}_{002000}\overline{\xi}_{n_1 1}h^{(1)}_{|n_2-n_1|,10010}(-1)e^{i\omega_c} + 2f^{(2)}_{011000}\left(\overline{\xi}_{n_1 2}h^{(1)}_{|n_2-n_1|,10010}(-1) + \overline{\xi}_{n_1 1}h^{(2)}_{|n_2-n_1|,10010}(0)e^{i\omega_c}\right) \\ +2f^{(2)}_{020000}\overline{\xi}_{n_1 2}h^{(2)}_{|n_2-n_1|,10010}(0) \end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_1}e^{i\omega_c\theta}, h_{n_1+n_2,01001}(\theta)\right)$$
$$= \begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_1 1}h^{(2)}_{n_1+n_2,01001}(0) + \xi_{n_1 2}h^{(1)}_{n_1+n_2,01001}(0)\right) + 2f^{(1)}_{200000}\xi_{n_1 1}h^{(1)}_{n_1+n_2,01001}(0) \\ 2f^{(2)}_{002000}\xi_{n_1 1}h^{(1)}_{n_1+n_2,01001}(-1)e^{-i\omega_c} + 2f^{(2)}_{011000}\left(\xi_{n_1 2}h^{(1)}_{n_1+n_2,01001}(-1) + \xi_{n_1 1}h^{(2)}_{n_1+n_2,01001}(0)e^{-i\omega_c}\right) \\ +2f^{(2)}_{020000}\xi_{n_1 2}h^{(2)}_{n_1+n_2,01001}(0) \end{pmatrix},$$

$$\mathcal{S}_2\left(\overline{\xi}_{n_1}e^{-i\omega_c\theta},h_{n_1+n_2,10001}(\theta)\right)$$
$$=\begin{pmatrix} 2f^{(1)}_{110000}\left(\overline{\xi}_{n_11}h^{(2)}_{n_1+n_2,10001}(0)+\overline{\xi}_{n_12}h^{(1)}_{n_1+n_2,10001}(0)\right)+2f^{(1)}_{200000}\overline{\xi}_{n_11}h^{(1)}_{n_1+n_2,10001}(0)\\ 2f^{(2)}_{002000}\overline{\xi}_{n_11}h^{(1)}_{n_1+n_2,10001}(-1)e^{i\omega_c}+2f^{(2)}_{011000}\left(\overline{\xi}_{n_12}h^{(1)}_{n_1+n_2,10001}(-1)+\overline{\xi}_{n_11}h^{(2)}_{n_1+n_2,10001}(0)e^{i\omega_c}\right)\\ +2f^{(2)}_{020000}\overline{\xi}_{n_12}h^{(2)}_{n_1+n_2,10001}(0)\end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_1}e^{i\omega_c\theta},h_{|n_2-n_1|,01001}(\theta)\right)$$
$$=\begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_11}h^{(2)}_{|n_2-n_1|,01001}(0)+\xi_{n_12}h^{(1)}_{|n_2-n_1|,01001}(0)\right)+2f^{(1)}_{200000}\xi_{n_11}h^{(1)}_{|n_2-n_1|,01001}(0)\\ 2f^{(2)}_{002000}\xi_{n_11}h^{(1)}_{|n_2-n_1|,01001}(-1)e^{-i\omega_c}+2f^{(2)}_{011000}\left(\xi_{n_12}h^{(1)}_{|n_2-n_1|,01001}(-1)+\xi_{n_11}h^{(2)}_{|n_2-n_1|,01001}(0)e^{-i\omega_c}\right)\\ +2f^{(2)}_{020000}\xi_{n_12}h^{(2)}_{|n_2-n_1|,01001}(0)\end{pmatrix},$$

$$\mathcal{S}_2\left(\overline{\xi}_{n_1}e^{-i\omega_c\theta},h_{|n_2-n_1|,10001}(\theta)\right)$$
$$=\begin{pmatrix} 2f^{(1)}_{110000}\left(\overline{\xi}_{n_11}h^{(2)}_{|n_2-n_1|,10001}(0)+\overline{\xi}_{n_12}h^{(1)}_{|n_2-n_1|,10001}(0)\right)+2f^{(1)}_{200000}\overline{\xi}_{n_11}h^{(1)}_{|n_2-n_1|,10001}(0)\\ 2f^{(2)}_{002000}\overline{\xi}_{n_11}h^{(1)}_{|n_2-n_1|,10001}(-1)e^{i\omega_c}+2f^{(2)}_{011000}\left(\overline{\xi}_{n_12}h^{(1)}_{|n_2-n_1|,10001}(-1)+\overline{\xi}_{n_11}h^{(2)}_{|n_2-n_1|,10001}(0)e^{i\omega_c}\right)\\ +2f^{(2)}_{020000}\overline{\xi}_{n_12}h^{(2)}_{|n_2-n_1|,10001}(0)\end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_2},h_{0,00110}(\theta)\right)$$
$$=\begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_21}h^{(2)}_{0,00110}(0)+\xi_{n_22}h^{(1)}_{0,00110}(0)\right)+2f^{(1)}_{200000}\xi_{n_21}h^{(1)}_{0,00110}(0)\\ 2f^{(2)}_{002000}\xi_{n_21}h^{(1)}_{0,00110}(-1)+2f^{(2)}_{011000}\left(\xi_{n_22}h^{(1)}_{0,00110}(-1)+\xi_{n_21}h^{(2)}_{0,00110}(0)\right)\\ +2f^{(2)}_{020000}\xi_{n_22}h^{(2)}_{0,00110}(0)\end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_2},h_{2n_2,00110}(\theta)\right)$$
$$=\begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_21}h^{(2)}_{2n_2,00110}(0)+\xi_{n_22}h^{(1)}_{2n_2,00110}(0)\right)+2f^{(1)}_{200000}\xi_{n_21}h^{(1)}_{2n_2,00110}(0)\\ 2f^{(2)}_{002000}\xi_{n_21}h^{(1)}_{2n_2,00110}(-1)+2f^{(2)}_{011000}\left(\xi_{n_22}h^{(1)}_{2n_2,00110}(-1)+\xi_{n_21}h^{(2)}_{2n_2,00110}(0)\right)\\ +2f^{(2)}_{020000}\xi_{n_22}h^{(2)}_{2n_2,00110}(0)\end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_2},h_{0,00101}(\theta)\right)$$
$$=\begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_21}h^{(2)}_{0,00101}(0)+\xi_{n_22}h^{(1)}_{0,00101}(0)\right)+2f^{(1)}_{200000}\xi_{n_21}h^{(1)}_{0,00101}(0)\\ 2f^{(2)}_{002000}\xi_{n_21}h^{(1)}_{0,00101}(-1)+2f^{(2)}_{011000}\left(\xi_{n_22}h^{(1)}_{0,00101}(-1)+\xi_{n_21}h^{(2)}_{0,00101}(0)\right)\\ +2f^{(2)}_{020000}\xi_{n_22}h^{(2)}_{0,00101}(0)\end{pmatrix},$$

$$\mathcal{S}_2\left(\xi_{n_2},h_{2n_2,00101}(\theta)\right)$$
$$=\begin{pmatrix} 2f^{(1)}_{110000}\left(\xi_{n_21}h^{(2)}_{2n_2,00101}(0)+\xi_{n_22}h^{(1)}_{2n_2,00101}(0)\right)+2f^{(1)}_{200000}\xi_{n_21}h^{(1)}_{2n_2,00101}(0)\\ 2f^{(2)}_{002000}\xi_{n_21}h^{(1)}_{2n_2,00101}(-1)+2f^{(2)}_{011000}\left(\xi_{n_22}h^{(1)}_{2n_2,00101}(-1)+\xi_{n_21}h^{(2)}_{2n_2,00101}(0)\right)\\ +2f^{(2)}_{020000}\xi_{n_22}h^{(2)}_{2n_2,00101}(0)\end{pmatrix},$$

$$\widehat{\mathcal{S}}_2\left(\xi_{n_1}e^{i\omega_c\theta},h_{0,11000}(\theta)\right)=\begin{pmatrix} 2f^{(1)}_{100100}\xi_{n_11}h^{(1)}_{0,11000}(0)\\ 0\end{pmatrix},$$

$$\widehat{S}_2(\overline{\xi}_{n_1}e^{-i\omega_c\theta},h_{0,20000}(\theta))=\begin{pmatrix} 2f^{(1)}_{100100}\overline{\xi}_{n_11}h^{(1)}_{0,20000}(0)\\ 0\end{pmatrix},$$

$$\widehat{S}_2(\xi_{n_1}e^{i\omega_c\theta},h_{0,00200}(\theta))=\begin{pmatrix} 2f^{(1)}_{100100}\xi_{n_11}h^{(1)}_{0,00200}(0)\\ 0\end{pmatrix},$$

$$\widehat{S}_2(\xi_{n_2},h_{0,11000}(\theta))=\begin{pmatrix} 2f^{(1)}_{100100}\xi_{n_21}h^{(1)}_{0,11000}(0)\\ 0\end{pmatrix},$$

$$\widehat{S}_2(\xi_{n_2}, h_{0,00200}(\theta)) = \begin{pmatrix} 2f^{(1)}_{100100}\xi_{n_21}h^{(1)}_{0,00200}(0) \\ 0 \end{pmatrix},$$

$$\widehat{\mathcal{S}}_2\left(\xi_{n_1}e^{i\omega_c\theta}, h_{0,00110}(\theta)\right) = \begin{pmatrix} 2f^{(1)}_{100100}\xi_{n_11}h^{(1)}_{0,00110}(0) \\ 0 \end{pmatrix},$$

$$\widehat{\mathcal{S}}_2\left(\xi_{n_1}e^{i\omega_c\theta}, h_{0,00101}(\theta)\right) = \begin{pmatrix} 2f^{(1)}_{100100}\xi_{n_11}h^{(1)}_{0,00101}(0) \\ 0 \end{pmatrix},$$

$$\widehat{S}_2(\xi_{n_2}, h_{0,00110}(\theta)) = \begin{pmatrix} 2f^{(1)}_{100100}\xi_{n_21}h^{(1)}_{0,00110}(0) \\ 0 \end{pmatrix},$$

$$\widehat{S}_2(\xi_{n_2}, h_{0,00101}(\theta)) = \begin{pmatrix} 2f^{(1)}_{100100}\xi_{n_21}h^{(1)}_{0,00101}(0) \\ 0 \end{pmatrix}.$$

4. Numerical simulations

For the parameters $d_1 = 0.1, d_2 = 10, \ell = 3, a = 2.5, b = 1, r = 0.8$, according to (2.2), we know that the positive constant steady state of model (1.5) is $E_*(0.3508, 0.3508)$. From (2.16) and (2.17), we have $\widetilde{x}_1 = 2.6353$ and $\widetilde{x}_2 = 11.8182$, then we have $\widetilde{n}_1 = \sqrt{\widetilde{x}_1} = 1.6234$ and $\widetilde{n}_2 = \sqrt{\widetilde{x}_2} = 3.4378$. From (2.25) and (2.26), we have $x_* = 2.0411$, then $n_0 = \ell\sqrt{x_*} = 4.2861$. Therefore, in this paper, we take $n_1 = 1$ and $n_2 = 3$. When $n_2 = 3$, by combining with (2.18) and (2.19), we have $r_* = 1.1814$. Furthermore, according to (2.27), we have $\omega_{n_1} = 0.2921$, and we have $\tau_* = \tau_{n_1,0} = 3.2507$ from (2.29). By combining with (3.16) and (3.17), we have

$$\begin{aligned} &\xi_{n_1} = (1, 0.2426 - 0.4499i)^T, \ \xi_{n_2} = (1, 0.1057)^T, \\ &\eta_{n_1} = (0.3199 - 0.7941i, -0.0609 + 0.2326i)^T, \ \eta_{n_2} = (1.2972, -0.0753)^T. \end{aligned}$$

Furthermore, according to Theorem 2.8, we know that model (1.5) undergoes Turing-Hopf bifurcation at $(r_*, \tau_*) = (1.1814, 3.2507)$. According to (3.27) and (3.28), the third order truncated normal form for Turing-Hopf bifurcation of model (1.5) is

$$\begin{cases} \dot{\rho} = (0.2074\mu_1 + 0.2075\mu_2)\rho + 0.5370\rho^3 - 0.5916\rho\varsigma^2, \ \rho > 0, \\ \dot{\varsigma} = -0.2190\mu_1\varsigma - 0.1178\rho^2\varsigma + 0.0359\varsigma^3. \end{cases} \tag{4.1}$$

The system (4.1) has a zero equilibrium point $A_0(0,0)$ for any $\mu_1, \mu_2 \in \mathbb{R}$, three boundary equilibrium points

$$\begin{aligned} &A_1\left(\sqrt{\frac{-0.2074\mu_1 - 0.2075\mu_2}{0.5370}}, 0\right), \quad \text{for } \mu_2 < -0.9995\mu_1, \\ &A_2^{\pm}\left(0, \pm\sqrt{\frac{0.2190\mu_1}{0.0359}}\right), \quad \text{for } \mu_1 > 0, \end{aligned}$$

and two interior equilibrium points

$$A_3^{\pm}\left(\sqrt{\frac{-0.1221\mu_1 + 0.0074\mu_2}{0.0504}}, \pm\sqrt{\frac{-0.0932\mu_1 + 0.0244\mu_2}{0.0504}}\right)$$

for $\mu_2 > 16.5\mu_1$ and $\mu_2 > 3.8197\mu_1$.

In addition, according to the above four inequalities which are used to ensure the existence of the equilibrium points $A_1, A_2^{\pm}$ and $A_3^{\pm}$, we can define the critical bifurcation lines as follows

$$
\begin{aligned}
&T : \mu_1 = 0, \\
&C_1 : \mu_2 = -0.9995\mu_1, \\
&C_2 : \mu_2 = 16.5\mu_1, \ \ \mu_1 > 0, \\
&C_3 : \mu_2 = 3.8197\mu_1, \ \ \mu_1 < 0.
\end{aligned}
$$

Notice that these four lines divide the $\mu_1 - \mu_2$ parameter plane into six different regions marked as $R_j(j = 1, 2, 3, 4, 5, 6)$, see Fig.1 for details.

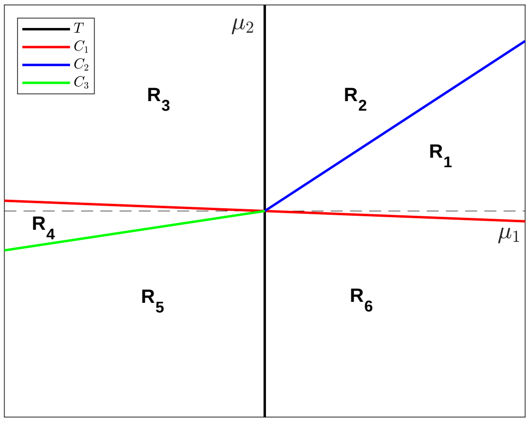


Figure 1: Bifurcation diagram of model (1.5) near the Turing-Hopf bifurcation point $(r_*, \tau_*) = (1.1814, 3.2507)$.

In region R_1, the normal form (4.1) has equilibrium points A_0 and $A_2^{\pm}$. In region R_2, the normal form (4.1) has equilibrium points A_0, $A_2^{\pm}$ and $A_3^{\pm}$. In region R_3, the normal form (4.1) has equilibrium points A_0 and $A_3^{\pm}$. In region R_4, the normal form (4.1) has equilibrium points A_0, A_1 and $A_3^{\pm}$. In region R_5, the normal form (4.1) has equilibrium points A_0 and A_1. In region R_6, the normal form (4.1) has equilibrium points A_0, A_1 and $A_2^{\pm}$.

In region R_1, system (4.1) has two equilibrium points A_0 and $A_2^{\pm}$. Moreover, $A_2^{\pm}$ is unstable, A_0 is asymptotically stable. Figure 2 shows the heteroclinic orbit connecting the unstable spatially inhomogeneous steady states shaped like $\cos(x)$ to the positive constant steady state $E_*(u_*, v_*)$.

In region R_2, system (4.1) has three equilibrium points A_0, $A_2^{\pm}$ and $A_3^{\pm}$. Moreover, A_0 and $A_2^{\pm}$ are unstable, while $A_3^{\pm}$ is asymptotically stable. It therefore follows that model (1.5) possesses two stable spatially inhomogeneous periodic solutions with a spatial shape like the combination of $\cos(x/3)$ and $\cos(x)$, as depicted in Fig.3.

In region R_3, system (4.1) has two equilibrium points A_0 and $A_3^{\pm}$. Moreover, A_0 is unstable, while $A_3^{\pm}$ is asymptotically stable. It therefore follows that model (1.5) possesses two stable spatially inhomogeneous periodic solutions with a spatial shape like the combination of $\cos(x/3)$ and $\cos(x)$, as depicted in Fig.4.

In region R_4, system (4.1) has three equilibrium points A_0, A_1 and $A_3^{\pm}$. Moreover, A_0 and A_1 are unstable, while $A_3^{\pm}$ is asymptotically stable. It therefore follows that model (1.5) possesses two stable spatially inhomogeneous periodic solutions with a spatial shape like the combination of $\cos(x/3)$ and $\cos(x)$.

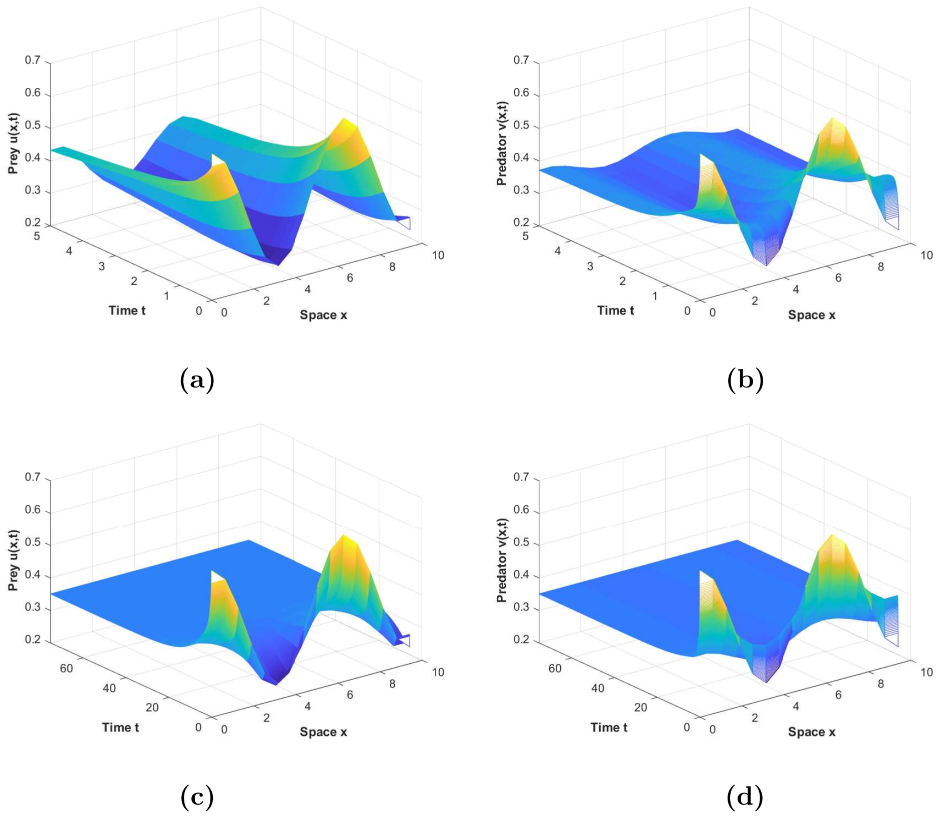


Figure 2: For $(\mu_1, \mu_2) = (3.2, -3) \in R_1$, $r = 1.1814 + \mu_1$ and $\tau = 3.2507 + \mu_2$, the spatially inhomogeneous steady states are unstable, and there exists a heteroclinic orbit connecting the unstable spatially inhomogeneous steady states shaped like $\cos(x)$ to the positive constant steady state $E_*(u_*, v_*) = (0.3508, 0.3508)$. (a) and (b) are transient behaviours for the prey $u(x,t)$ and predator $v(x,t)$ populations, respectively. (c) and (d) are the evolution processes for the prey $u(x,t)$ and predator $v(x,t)$ populations, respectively. The initial values are $u(x, 0) = u_* + 0.1 + 0.2\cos(x)$ and $v(x, 0) = v_* + 0.1 + 0.2\cos(x)$.

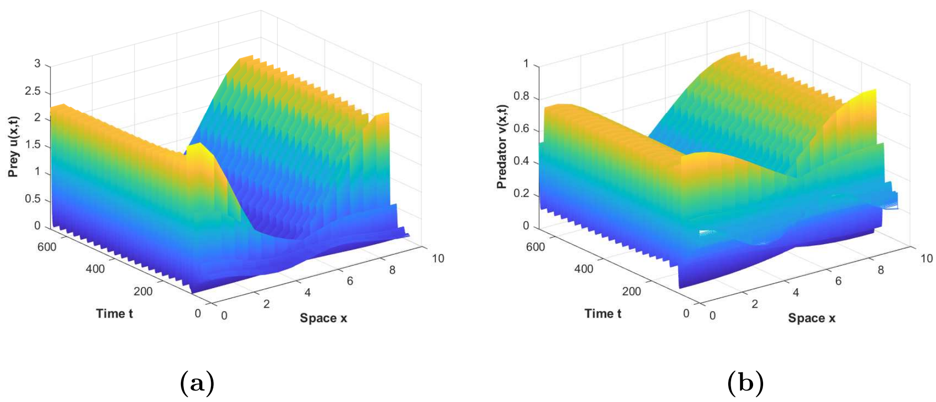
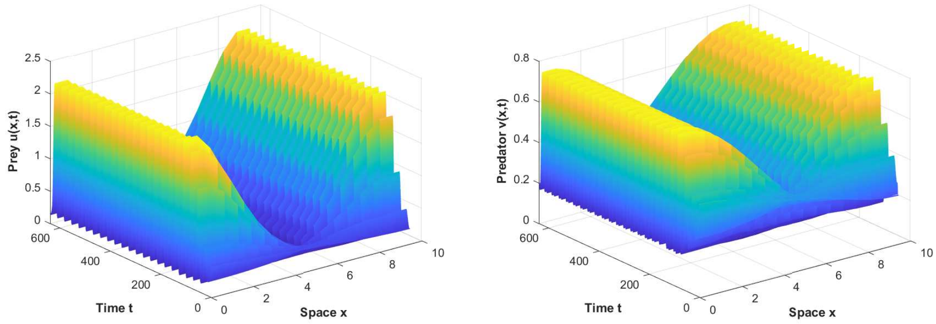


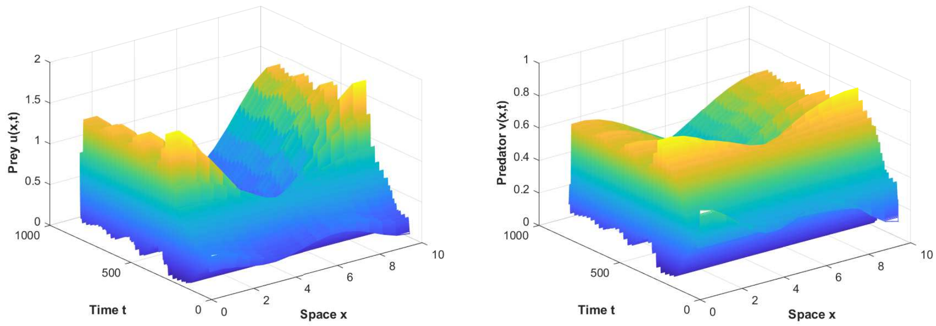
Figure 3: For $(\mu_1, \mu_2) = (0.01, 1.2) \in R_2$, $r = 1.1814 + \mu_1$ and $\tau = 3.2507 + \mu_2$, the positive constant steady state $E_*(u_*, v_*) = (0.3508, 0.3508)$ and the spatially inhomogeneous steady states are unstable, and there are stable spatially inhomogeneous periodic solutions with a spatial shape like the combination of $\cos(x/3)$ and $\cos(x)$. The initial values are $u(x, 0) = u_* + 0.1\cos(x)$ and $v(x, 0) = v_* + 0.1\cos(x)$.



(a) **(b)**

Figure 4: For $(\mu_1, \mu_2) = (-0.01, 1) \in R_3$, $r = 1.1814 + \mu_1$ and $\tau = 3.2507 + \mu_2$, the positive constant steady state $E_*(u_*, v_*) = (0.3508, 0.3508)$ is unstable, and there are stable spatially inhomogeneous periodic solutions with a spatial shape like the combination of $\cos(x/3)$ and $\cos(x)$. The initial values are $u(x, 0) = u_* + 0.1\cos(x/3) + 0.01\cos(x)$ and $v(x, 0) = v_* + 0.1\cos(x/3) + 0.01\cos(x)$.

Figure 5 shows the heteroclinic orbit connecting the unstable spatially inhomogeneous periodic solution shaped like $\cos(x/3)$ to the stable spatially inhomogeneous periodic solutions shaped like the combination of $\cos(x/3)$ and $\cos(x)$.



(a) **(b)**

Figure 5: For $(\mu_1, \mu_2) = (-0.01, 0.009) \in R_4$, $r = 1.1814 + \mu_1$ and $\tau = 3.2507 + \mu_2$, the positive constant steady state $E_*(u_*, v_*) = (0.3508, 0.3508)$ and the spatially inhomogeneous periodic solution shaped like $\cos(x/3)$ are unstable, and there exists a heteroclinic orbit connecting the unstable spatially inhomogeneous periodic solution shaped like $\cos(x/3)$ to the stable spatially inhomogeneous periodic solutions with a spatial shape like the combination of $\cos(x/3)$ and $\cos(x)$. The initial values are $u(x, 0) = u_* + 0.1\cos(x/3) + 0.1\cos(x)$ and $v(x, 0) = v_* + 0.1\cos(x/3) + 0.1\cos(x)$.

In region R_5, system (4.1) has two equilibrium points A_0 and A_1. Moreover, A_0 is unstable, while A_1 is asymptotically stable. It therefore follows that model (1.5) possesses stable spatially inhomogeneous periodic solution with a spatial shape like $\cos(x/3)$, as depicted in Fig.6.

In region R_6, system (4.1) has three equilibrium points A_0, A_1 and $A_2^{\pm}$. A_1 and $A_2^{\pm}$ are unstable, while A_0 is asymptotically stable. Figure 7 shows the heteroclinic orbit connecting the unstable spatially inhomogeneous periodic solution to the positive constant steady state $E_*(u_*, v_*)$. Figure 8 shows the heteroclinic orbit connecting the unstable spatially inhomogeneous steady states shaped like $\cos(x)$ to the positive constant steady state $E_*(u_*, v_*)$.

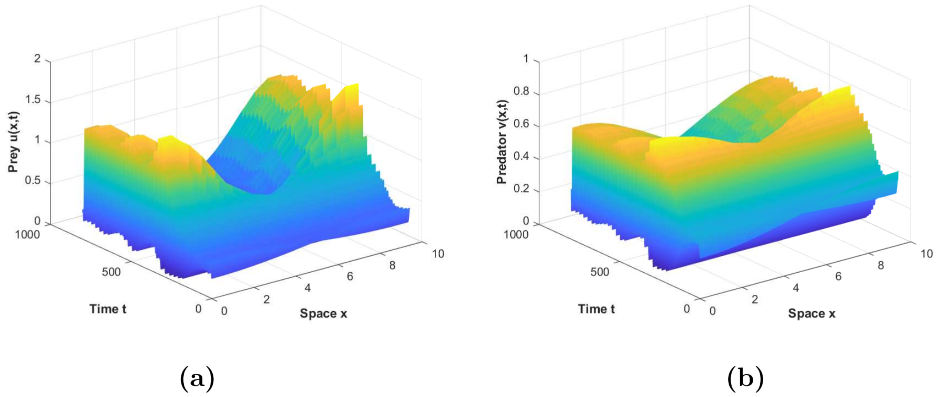


Figure 6: For $(\mu_1, \mu_2) = (-0.01, -0.05) \in R_5$, $r = 1.1814 + \mu_1$ and $\tau = 3.2507 + \mu_2$, the positive constant steady state $E_*(u_*, v_*) = (0.3508, 0.3508)$ is unstable, and there is stable spatially inhomogeneous periodic solution shaped like $\cos(x/3)$. The initial values are $u(x, 0) = u_* - 0.1\cos(x/3)$ and $v(x, 0) = v_* - 0.1\cos(x/3)$.

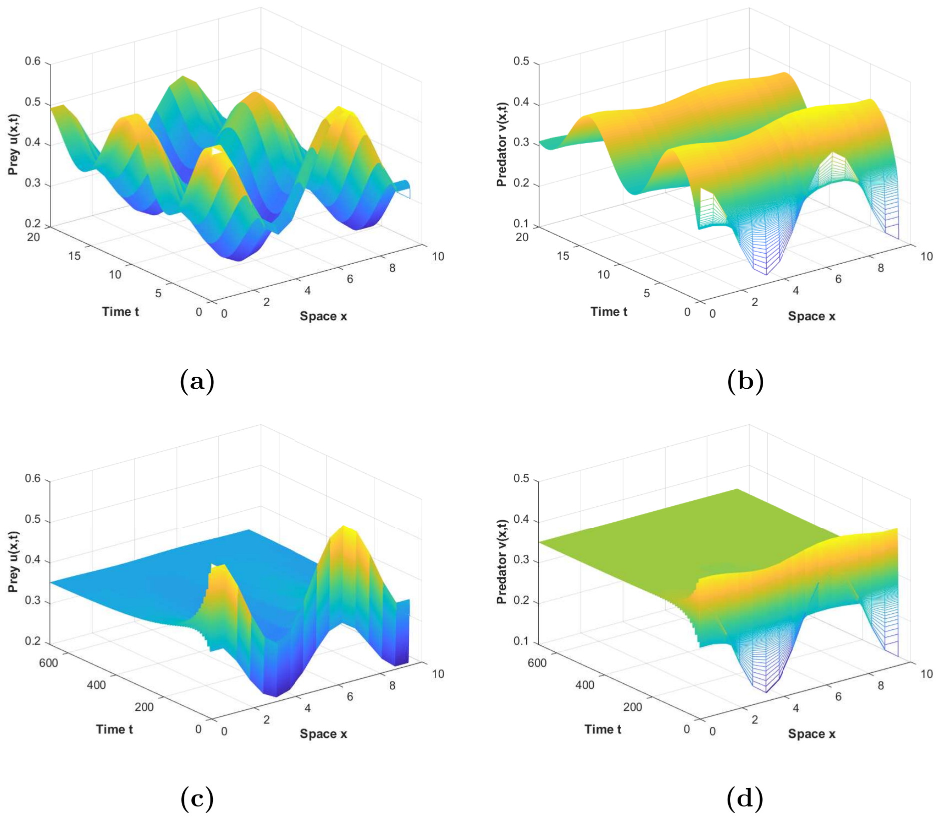


Figure 7: For $(\mu_1, \mu_2) = (0.01, -1) \in R_6$, $r = 1.1814 + \mu_1$ and $\tau = 3.2507 + \mu_2$, the spatially inhomogeneous periodic solution shaped like $\cos(x/3)$ are unstable, and there exists a heteroclinic orbit connecting the unstable spatially inhomogeneous periodic solution shaped like $\cos(x/3)$ to the positive constant steady state $E_*(u_*, v_*) = (0.3508, 0.3508)$. (a) and (b) are transient behaviours for the prey $u(x, t)$ and predator $v(x, t)$ populations, respectively. (c) and (d) are the evolution processes for the prey $u(x, t)$ and predator $v(x, t)$ populations, respectively. The initial values are $u(x, 0) = u_* + 0.1 + 0.13\cos(x)$ and $v(x, 0) = v_* - 0.1 + 0.13\cos(x)$.

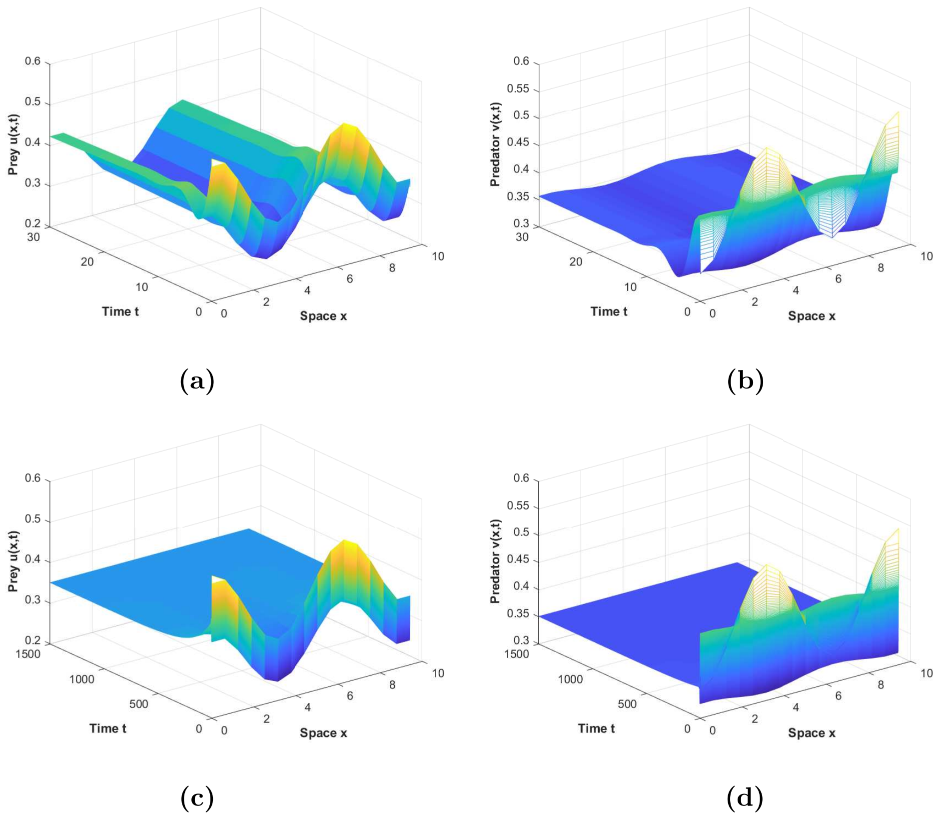


Figure 8: For $(\mu_1, \mu_2) = (0.02, -2.5) \in R_6$, $r = 1.1814 + \mu_1$ and $\tau = 3.2507 + \mu_2$, the spatially inhomogeneous steady states are unstable, and there exists a heteroclinic orbit connecting the unstable spatially inhomogeneous steady states shaped like $\cos(x)$ to the positive constant steady state $E_*(u_*, v_*) = (0.3508, 0.3508)$. (a) and (b) are transient behaviours for the prey $u(x,t)$ and predator $v(x,t)$ populations, respectively. (c) and (d) are the evolution processes for the prey $u(x,t)$ and predator $v(x,t)$ populations, respectively. The initial values are $u(x,0) = u_* + 0.1 + 0.1\cos(x)$ and $v(x,0) = v_* + 0.1 - 0.1\cos(x)$.

5. Conclusion and discussion

In this paper, we investigate a diffusive Holling-Tanner model with nonlocal effect and digestion time delay. We discuss the stability of positive constant steady state, the Turing bifurcation, Hopf bifurcation and Turing-Hopf bifurcation of model (1.5). More specifically, when $\tau = 0,\ n \in \mathbb{N}$, the Turing bifurcation of model (1.5) is researched, when $\tau > 0,\ n \in \mathbb{N}$, the Hopf bifurcation of model (1.5) is researched. In addition, we discuss the Turing-Hopf bifurcation of model (1.5) under the condition of Turing stability.

In order to investigate the dynamical behaviours near the Turing-Hopf bifurcation point, we choose r and τ as bifurcation parameters, and we derive the algorithm for calculating the normal form of Turing-Hopf bifurcation of model (1.5). By choosing some proper parameters and according to the algorithm for calculating the normal form of Turing-Hopf bifurcation of model (1.5), we divide the $\mu_1 - \mu_2$ plane into six regions and illustrate different dynamical behaviours in these regions. Furthermore, we find rich dynamical behaviours of model (1.5) by some numerical simulations.

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Declarations of interest

This research did not involve human participants and animals.

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Conflicts of interest: The author declares that there is not conflict of interest, whether financial or non-financial.

Code availability: The numerical simulations in this paper are carried out by using the MATLAB software.

Authors' contributions: This manuscript is investigated and written by Yehu Lv.

Appendix A: Detailed calculations of the normal form (3.26)

By combining with (2.1), we first give some mathematical expressions that will be useful later on

$$\alpha_1 := \int_0^{\ell\pi} \gamma_{n_1}^2(x)\gamma_{n_2}(x)dx = \begin{cases} \sqrt{\frac{1}{2\ell\pi}}, & \text{for } n_2 = 2n_1, \\ 0, & \text{for } n_2 \neq 2n_1, \end{cases}$$

$$\alpha_2 := \int_0^{\ell\pi} \gamma_{n_1}^2(x)dx = 1, \ \alpha_3 := \int_0^{\ell\pi} \gamma_{n_2}^2(x)dx = 1,$$

$$\alpha_4 := \int_0^{\ell\pi} \gamma_{n_1}^4(x)dx = \frac{3}{2\ell\pi}, \ \alpha_5 := \int_0^{\ell\pi} \gamma_{n_1}^2(x)\gamma_{n_2}^2(x)dx = \frac{1}{\ell\pi},$$

$$\alpha_6 := \int_0^{\ell\pi} \gamma_{n_2}^4(x)dx = \frac{3}{2\ell\pi}, \ \alpha_7 := \int_0^{\ell\pi} \gamma_{n_1}^3(x)dx = 0,$$

$$\alpha_8 := \int_0^{\ell\pi} \gamma_{n_1}(x)\gamma_{n_2}^2(x)dx = \begin{cases} \sqrt{\frac{1}{2\ell\pi}}, & \text{for } n_1 = 2n_2, \\ 0, & \text{for } n_1 \neq 2n_2, \end{cases}$$

$$\alpha_9 := \int_0^{\ell\pi} \gamma_{n_2}^3(x)dx = 0,$$

$$\alpha_{10} := \int_0^{\ell\pi} \gamma_{2n_1}(x)\gamma_{n_2}(x)dx = \begin{cases} 1, & \text{for } n_2 = 2n_1, \\ 0, & \text{for } n_2 \neq 2n_1, \end{cases}$$

$$\alpha_{11} := \int_0^{\ell\pi} \gamma_{n_1}(x)\gamma_{2n_2}(x)dx = \begin{cases} 1, & \text{for } n_1 = 2n_2, \\ 0, & \text{for } n_1 \neq 2n_2, \end{cases}$$

$$\alpha_{12} := \int_0^{\ell\pi} \gamma_{n_1}(x)\gamma_{|n_2-n_1|}(x)dx = \begin{cases} 1, & \text{for } n_1 = |n_2 - n_1|, \\ 0, & \text{for } n_1 \neq |n_2 - n_1|. \end{cases}$$

From the above mathematical expressions, when $n_1 \neq 2n_2$, $n_2 \neq 2n_1$ and $n_1 \neq |n_2 - n_1|$, we have $\alpha_1 = \alpha_8 = \alpha_{10} = \alpha_{11} = \alpha_{12} = 0$. In the following, we give the detail calculation procedures of

$B_{10100}, B_{10010}, B_{10001}, B_{11000}, B_{00110}, B_{00101}, C_{21000}, C_{10200}, C_{11100}, C_{00300}, D_{21000}, D_{10200}, D_{11100}, D_{00300},$
$E_{21000}, E_{10200}, E_{11100}, E_{00300}, B_{21000}, B_{10200}, B_{11100}, B_{00300}, G_{10110}, G_{10101}, G_{11010}, G_{11001}, G_{00210}, G_{00201},$
$H_{10110}, H_{10101}, H_{11010}, H_{11001}, H_{00210}, H_{00201}, I_{10110}, I_{10101}, I_{11010}, I_{11001}, I_{00210}, I_{00201}, F_{10110}, F_{10101},$
$F_{11010}, F_{11001}, F_{00210}, F_{00201}$

steps by steps.

Step 1:

$$\begin{cases} B_{10100} = \frac{1}{2}\eta_{n_1}^T A_{101}\alpha_1, \\ B_{10010} = \eta_{n_1}^T L_1^{(1,0)}\xi_{n_1} e^{i\omega_c\theta}\alpha_2, \\ B_{10001} = \eta_{n_1}^T L_1^{(0,1)}\xi_{n_1} e^{i\omega_c\theta}\alpha_2 - \eta_{n_1}^T D\frac{n_1^2}{\ell^2}\xi_{n_1}\alpha_2, \\ B_{11000} = \frac{1}{2}\eta_{n_2}^T A_{110}\alpha_1, \\ B_{00110} = \eta_{n_2}^T L_1^{(1,0)}\xi_{n_2}\alpha_3, \\ B_{00101} = \eta_{n_2}^T L_1^{(0,1)}\xi_{n_2}\alpha_3 - \eta_{n_2}^T D\frac{n_2^2}{\ell^2}\xi_{n_2}\alpha_3, \end{cases}$$

where

$$L_1^{(1,0)}(\xi_{n_1}e^{i\omega_c\theta}) = \tau_c \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \xi_{n_1} + \tau_c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \xi_{n_1} e^{-i\omega_c},$$

$$L_1^{(0,1)}(\xi_{n_1}e^{i\omega_c\theta}) = \begin{pmatrix} 1 - u_* - \frac{abv_*}{(b+u_*)^2} & -\frac{au_*}{b+u_*} \\ 0 & -\gamma \end{pmatrix} \xi_{n_1} + \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} \xi_{n_1} e^{-i\omega_c},$$

$$L_1^{(1,0)}\xi_{n_2} = \tau_c \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \xi_{n_2} + \tau_c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \xi_{n_2},$$

$$L_1^{(0,1)}\xi_{n_2} = \begin{pmatrix} 1 - u_* - \frac{abv_*}{(b+u_*)^2} & -\frac{au_*}{b+u_*} \\ 0 & -\gamma \end{pmatrix} \xi_{n_2} + \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} \xi_{n_2}.$$

Step 2:

$$\begin{cases} C_{21000} = \dfrac{1}{4\ell\pi}\eta_{n_1}^T A_{210}, \ C_{10200} = \dfrac{1}{6\ell\pi}\eta_{n_1}^T A_{102}, \\ C_{11100} = \dfrac{1}{6\ell\pi}\eta_{n_2}^T A_{111}, \ C_{00300} = \dfrac{1}{4\ell\pi}\eta_{n_2}^T A_{003}, \\ D_{21000} = \dfrac{1}{12i\omega_c}\left(\alpha_1^2 \eta_{n_1}^T A_{011} \eta_{n_2}^T A_{200}\right), \\ D_{10200} = \dfrac{1}{6i\omega_c}\left(\dfrac{1}{2}\alpha_1^2 |\eta_{n_1}^T A_{011}|^2 + 2\alpha_8^2 \eta_{n_1}^T A_{002} \eta_{n_2}^T A_{101}\right), \\ D_{11100} = \dfrac{1}{6i\omega_c}\left(-\alpha_1^2 \eta_{n_2}^T A_{200} \eta_{n_1}^T A_{011} + \alpha_1^2 \eta_{n_2}^T A_{020} \overline{\eta}_{n_1}^T A_{101} - \alpha_8^2 \eta_{n_2}^T A_{101} \eta_{n_2}^T A_{011} + \alpha_8^2 \eta_{n_2}^T A_{011} \eta_{n_2}^T A_{101}\right), \\ D_{00300} = \dfrac{1}{6i\omega_c}\left(-\alpha_8^2 \eta_{n_2}^T A_{101} \eta_{n_1}^T A_{002} + \alpha_8^2 \eta_{n_2}^T A_{011} \overline{\eta}_{n_1}^T A_{002}\right). \end{cases}$$

Step 3:

$$\begin{cases} E_{21000} = \eta_{n_1}^T \left(\dfrac{1}{6\sqrt{\ell\pi}} (S_2(\xi_{n_1}e^{i\omega_c\theta}, h_{0,11000}(\theta)) + S_2(\overline{\xi}_{n_1}e^{-i\omega_c\theta}, h_{0,20000}(\theta))) \right. \\ \qquad + \dfrac{1}{6\sqrt{2\ell\pi}} (S_2(\xi_{n_1}e^{i\omega_c\theta}, h_{2n_1,11000}(\theta)) + S_2(\overline{\xi}_{n_1}e^{-i\omega_c\theta}, h_{2n_1,20000}(\theta))) \\ \qquad \left. + \dfrac{1}{6\sqrt{\ell\pi}} (\widehat{S}_2(\xi_{n_1}e^{i\omega_c\theta}, h_{0,11000}(\theta)) + \widehat{S}_2(\overline{\xi}_{n_1}e^{-i\omega_c\theta}, h_{0,20000}(\theta))) \right), \\ E_{10200} = \eta_{n_1}^T \left(\dfrac{1}{6\sqrt{\ell\pi}} S_2(\xi_{n_1}e^{i\omega_c\theta}, h_{0,00200}(\theta)) + \dfrac{1}{6\sqrt{2\ell\pi}} S_2(\xi_{n_1}e^{i\omega_c\theta}, h_{2n_1,00200}(\theta)) \right. \\ \qquad + \dfrac{1}{6\sqrt{2\ell\pi}} S_2(\xi_{n_2}, h_{n_1+n_2,10100}(\theta)) + \dfrac{1}{6\sqrt{2\ell\pi}} S_2(\xi_{n_2}, h_{|n_2-n_1|,10100}(\theta)) \\ \qquad \left. + \dfrac{1}{6\sqrt{\ell\pi}} \widehat{S}_2(\xi_{n_1}e^{i\omega_c\theta}, h_{0,00200}(\theta)) \right), \\ E_{11100} = \eta_{n_2}^T \left(\dfrac{1}{6\sqrt{2\ell\pi}} (S_2(\xi_{n_1}e^{i\omega_c\theta}, h_{n_1+n_2,01100}(\theta)) + S_2(\overline{\xi}_{n_1}e^{-i\omega_c\theta}, h_{n_1+n_2,10100}(\theta))) \right. \\ \qquad + \dfrac{1}{6\sqrt{2\ell\pi}} (S_2(\xi_{n_1}e^{i\omega_c\theta}, h_{|n_2-n_1|,01100}(\theta)) + S_2(\overline{\xi}_{n_1}e^{-i\omega_c\theta}, h_{|n_2-n_1|,10100}(\theta))) \\ \qquad \left. + \dfrac{1}{6\sqrt{\ell\pi}} S_2(\xi_{n_2}, h_{0,11000}(\theta)) + \dfrac{1}{6\sqrt{2\ell\pi}} S_2(\xi_{n_2}, h_{2n_2,11000}(\theta)) + \dfrac{1}{6\sqrt{\ell\pi}} \widehat{S}_2(\xi_{n_2}, h_{0,11000}(\theta)) \right), \\ E_{00300} = \eta_{n_2}^T \left(\dfrac{1}{6\sqrt{\ell\pi}} S_2(\xi_{n_2}, h_{0,00200}(\theta)) + \dfrac{1}{6\sqrt{2\ell\pi}} S_2(\xi_{n_2}, h_{2n_2,00200}(\theta)) + \dfrac{1}{6\sqrt{\ell\pi}} \widehat{S}_2(\xi_{n_2}, h_{0,00200}(\theta)) \right). \end{cases}$$

Step 4:

$$\begin{cases} B_{21000}=C_{21000}+\dfrac{3}{2}\left(D_{21000}+E_{21000}\right),\ B_{10200}=C_{10200}+\dfrac{3}{2}\left(D_{10200}+E_{10200}\right),\\ B_{11100}=C_{11100}+\dfrac{3}{2}\left(D_{11100}+E_{11100}\right),\ B_{00300}=C_{00300}+\dfrac{3}{2}\left(D_{00300}+E_{00300}\right),\\ G_{10110}=\dfrac{1}{6}\alpha_1\eta_{n_1}^T\widehat{A}_{101}^{(1)},\ G_{10101}=\dfrac{1}{6}\alpha_1\eta_{n_1}^T\widehat{A}_{101}^{(2)},\\ G_{11010}=\dfrac{1}{6}\alpha_1\eta_{n_2}^T\widehat{A}_{110}^{(1)},\ G_{11001}=\dfrac{1}{6}\alpha_1\eta_{n_2}^T\widehat{A}_{110}^{(2)},\\ G_{00210}=0,\ G_{00201}=0. \end{cases}$$

Step 5:

$$\begin{cases} H_{10110}=\dfrac{1}{3i\omega_c}\alpha_1\operatorname{Re}\left(\eta_{n_1}^T L_1^{(1,0)}\overline{\xi}_{n_1}e^{-i\omega_c\theta}\overline{\eta}_{n_1}^T A_{101}\right),\\ H_{10101}=\dfrac{1}{3i\omega_c}\alpha_1\left(\operatorname{Re}\left(\eta_{n_1}^T L_1^{(0,1)}\overline{\xi}_{n_1}e^{-i\omega_c\theta}\overline{\eta}_{n_1}^T A_{101}\right)+\operatorname{Re}\left(\eta_{n_1}^T\left(-D\dfrac{n_1^2}{\ell^2}\right)\overline{\xi}_{n_1}e^{-i\omega_c\theta}\overline{\eta}_{n_1}^T A_{101}\right)\right),\\ H_{11010}=-\dfrac{2}{3i\omega_c}\alpha_1\operatorname{Imag}\left(\eta_{n_2}^T A_{200}\eta_{n_1}^T L_1^{(1,0)}\overline{\xi}_{n_1}e^{-i\omega_c\theta}\right),\\ H_{11001}=-\dfrac{2}{3i\omega_c}\alpha_1\left(\operatorname{Imag}\left(\eta_{n_2}^T A_{200}\eta_{n_1}^T L_1^{(0,1)}\overline{\xi}_{n_1}e^{-i\omega_c\theta}\right)+\operatorname{Imag}\left(\eta_{n_2}^T A_{200}\eta_{n_1}^T\left(-D\dfrac{n_1^2}{\ell^2}\right)\overline{\xi}_{n_1}e^{-i\omega_c\theta}\right)\right),\\ H_{00210}=0,\ H_{00201}=0, \end{cases}$$

where $\operatorname{Imag}(\nu)$ represents the imaginary part of ν.

Step 6:

$$\begin{cases} I_{10110}=\dfrac{1}{6}\eta_{n_1}^T\left(\dfrac{1}{\sqrt{\ell\pi}}S_2(\xi_{n_1}e^{i\omega_c\theta},h_{0,00110}(\theta))+\dfrac{1}{\sqrt{2\ell\pi}}S_2(\xi_{n_1}e^{i\omega_c\theta},h_{2n_1,00110}(\theta))\right.\\ \qquad\left.+\dfrac{1}{\sqrt{2\ell\pi}}S_2(\xi_{n_2},h_{n_1+n_2,10010}(\theta))+\dfrac{1}{\sqrt{2\ell\pi}}S_2(\xi_{n_2},h_{|n_2-n_1|,10010}(\theta))+\dfrac{1}{\sqrt{\ell\pi}}\widehat{S}_2(\xi_{n_1}e^{i\omega_c\theta},h_{0,00110}(\theta))\right),\\ I_{10101}=\eta_{n_1}^T\left(2A_1h_{n_1,101}(0)+2A_2h_{n_1,101}(-1)-D\dfrac{2n_1^2}{\ell^2}h_{n_1,101}(0)\right)\\ \qquad+\dfrac{1}{6}\eta_{n_1}^T\left(\dfrac{1}{\sqrt{\ell\pi}}S_2(\xi_{n_1}e^{i\omega_c\theta},h_{0,00101}(\theta))+\dfrac{1}{\sqrt{2\ell\pi}}S_2(\xi_{n_1}e^{i\omega_c\theta},h_{2n_1,00101}(\theta))\right.\\ \qquad\left.+\dfrac{1}{\sqrt{2\ell\pi}}S_2(\xi_{n_2},h_{n_1+n_2,10001}(\theta))+\dfrac{1}{\sqrt{2\ell\pi}}S_2(\xi_{n_2},h_{|n_2-n_1|,10001}(\theta))+\dfrac{1}{\sqrt{\ell\pi}}\widehat{S}_2(\xi_{n_1}e^{i\omega_c\theta},h_{0,00101}(\theta))\right),\\ I_{11010}=\dfrac{1}{6}\eta_{n_2}^T\left(\dfrac{1}{\sqrt{2\ell\pi}}\left(S_2(\xi_{n_1}e^{i\omega_c\theta},h_{n_1+n_2,01010}(\theta))+S_2(\overline{\xi}_{n_1}e^{-i\omega_c\theta},h_{n_1+n_2,10010}(\theta))\right)\right.\\ \qquad\left.+\dfrac{1}{\sqrt{2\ell\pi}}\left(S_2(\xi_{n_1}e^{i\omega_c\theta},h_{|n_2-n_1|,01010}(\theta))+S_2(\overline{\xi}_{n_1}e^{-i\omega_c\theta},h_{|n_2-n_1|,10010}(\theta))\right)\right),\\ I_{11001}=\eta_{n_2}^T\left(2A_1h_{n_2,110}(0)+2A_2h_{n_2,110}(-1)-D\dfrac{2n_2^2}{\ell^2}h_{n_2,110}(0)\right)\\ \qquad+\dfrac{1}{6}\eta_{n_2}^T\left(\dfrac{1}{\sqrt{2\ell\pi}}\left(S_2(\xi_{n_1}e^{i\omega_c\theta},h_{n_1+n_2,01001}(\theta))+S_2(\overline{\xi}_{n_1}e^{-i\omega_c\theta},h_{n_1+n_2,10001}(\theta))\right)\right.\\ \qquad\left.+\dfrac{1}{\sqrt{2\ell\pi}}\left(S_2(\xi_{n_1}e^{i\omega_c\theta},h_{|n_2-n_1|,01001}(\theta))+S_2(\overline{\xi}_{n_1}e^{-i\omega_c\theta},h_{|n_2-n_1|,10001}(\theta))\right)\right),\\ I_{00210}=\dfrac{1}{6}\eta_{n_2}^T\left(\dfrac{1}{\sqrt{\ell\pi}}S_2(\xi_{n_2},h_{0,00110}(\theta))+\dfrac{1}{\sqrt{2\ell\pi}}S_2(\xi_{n_2},h_{2n_2,00110}(\theta))+\dfrac{1}{\sqrt{\ell\pi}}\widehat{S}_2(\xi_{n_2},h_{0,00110}(\theta))\right),\\ I_{00201}=\eta_{n_2}^T\left(2A_1h_{n_2,002}(0)+2A_2h_{n_2,002}(-1)-D\dfrac{2n_2^2}{\ell^2}h_{n_2,002}(0)\right)\\ \qquad+\dfrac{1}{6}\eta_{n_2}^T\left(\dfrac{1}{\sqrt{\ell\pi}}S_2(\xi_{n_2},h_{0,00101}(\theta))+\dfrac{1}{\sqrt{2\ell\pi}}S_2(\xi_{n_2},h_{2n_2,00101}(\theta))+\dfrac{1}{\sqrt{\ell\pi}}\widehat{S}_2(\xi_{n_2},h_{0,00101}(\theta))\right). \end{cases}$$

Step 7:

$$\begin{cases} F_{10110} = G_{10110} + \dfrac{3}{2}\left(H_{10110} + I_{10110}\right), \; F_{10101} = G_{10101} + \dfrac{3}{2}\left(H_{10101} + I_{10101}\right), \\ F_{11010} = G_{11010} + \dfrac{3}{2}\left(H_{11010} + I_{11010}\right), \; F_{11001} = G_{11001} + \dfrac{3}{2}\left(H_{11001} + I_{11001}\right), \\ F_{00210} = G_{00210} + \dfrac{3}{2}\left(H_{00210} + I_{00210}\right), \; F_{00201} = G_{00201} + \dfrac{3}{2}\left(H_{00201} + I_{00201}\right). \end{cases}$$

Step 8: Clearly, we still need to compute $h_{k,11000}$, $h_{k,20000}$, $h_{k,00200}$ with $k = 0, 2n_1$, $h_{k,10100}(\theta)$, $h_{k,01100}(\theta)$ with $k = n_1 + n_2, |n_2 - n_1|$, $h_{k,00200}(\theta)$, $h_{k,11000}(\theta)$ with $k = 2n_2$, $h_{k,10100}$ with $k = n_1$, $h_{k,11000}$, $h_{k,00200}$ with $k = n_2$, $h_{k,00110}$, $h_{k,00101}$ with $k = 0$, $h_{k,00110}$, $h_{k,00101}$ with $k = 2n_1, 2n_2$, $h_{k,10010}$, $h_{k,10001}$, $h_{k,01010}$ and $h_{k,01001}$ with $k = n_1 + n_2, |n_2 - n_1|$. Similar to [22] and [25], we have

$$M_2^2\left(\sum_{n\in\mathbb{N}_0} h_n(z,\mu,\theta)\gamma_n(x)\right) = D_z\left(\sum_{n\in\mathbb{N}_0} h_n(z,\mu,\theta)\gamma_n(x)\right) Bz - A_{\mathcal{Q}^1}\left(\sum_{n\in\mathbb{N}_0} h_n(z,\mu,\theta)\gamma_n(x)\right)$$

and

$$\begin{aligned} &\begin{pmatrix} \left[M_2^2\left(\sum_{n\in\mathbb{N}_0} h_n(z,\mu,\theta)\gamma_n(x)\right), \beta_n^{(1)}\right] \\ \left[M_2^2\left(\sum_{n\in\mathbb{N}_0} h_n(z,\mu,\theta)\gamma_n(x)\right), \beta_n^{(2)}\right] \end{pmatrix} \\ &= \sum_{n\in\mathbb{N}_0} \Big(i\omega_c\left(2h_{n,20000}(\theta)z_1^2 - 2h_{n,02000}(\theta)z_2^2 + h_{n,10100}(\theta)z_1z_3 - h_{n,01100}(\theta)z_2z_3\right) \\ &+ i\omega_c\left(h_{n,10010}(\theta)z_1\mu_1 + h_{n,10001}(\theta)z_1\mu_2 - h_{n,01010}(\theta)z_2\mu_1 - h_{n,01001}(\theta)z_2\mu_2\right) \\ &- \left(\dot{h}_n(z,\mu,\theta) + X_0(\theta)\left(\tau_c D\Delta(h_n(z,\mu,0)) + L_0^n h_n(z,\mu,\theta) - \dot{h}_n(z,\mu,0)\right)\right)\Big)\gamma_n(x), \end{aligned}$$

where $L_0^n h_n(z,\mu,\theta)$ is defined by (3.9). It follows from (3.19) and the second mathematical expression in (3.23) that

$$\begin{aligned} f_2^2(z,0,0,\mu) &= (I-\pi)X_0(\theta)\widetilde{F}_2\left(\Phi(\theta)z_x, 0, \mu\right) \\ &= X_0(\theta)\widetilde{F}_2\left(\Phi(\theta)z_x, 0, \mu\right) - \Phi_{n_1}(\theta)\Psi_{n_1}(0)\begin{pmatrix} \left[\widetilde{F}_2\left(\Phi(\theta)z_x, 0, \mu\right), \beta_{n_1}^{(1)}(x)\right] \\ \left[\widetilde{F}_2\left(\Phi(\theta)z_x, 0, \mu\right), \beta_{n_1}^{(2)}(x)\right] \end{pmatrix}\gamma_{n_1}(x) \\ &- \Phi_{n_2}(\theta)\Psi_{n_2}(0)\begin{pmatrix} \left[\widetilde{F}_2\left(\Phi(\theta)z_x, 0, \mu\right), \beta_{n_2}^{(1)}(x)\right] \\ \left[\widetilde{F}_2\left(\Phi(\theta)z_x, 0, \mu\right), \beta_{n_2}^{(2)}(x)\right] \end{pmatrix}\gamma_{n_2}(x). \end{aligned}$$

Then from

$$\begin{pmatrix} \left[M_2^2\left(h_n(z,\mu,\theta)\gamma_n(x)\right), \beta_n^{(1)}\right] \\ \left[M_2^2\left(h_n(z,\mu,\theta)\gamma_n(x)\right), \beta_n^{(2)}\right] \end{pmatrix} = \begin{pmatrix} \left[f_2^2(z,0,0,\mu), \beta_n^{(1)}\right] \\ \left[f_2^2(z,0,0,\mu), \beta_n^{(2)}\right] \end{pmatrix},$$

we can obtain

$$
k=0\quad\begin{cases}
z_1z_2\begin{cases}\dot{h}_{k,11000}(\theta)=0,\\[4pt] \dot{h}_{k,11000}(0)-\mathcal{L}_k\left(h_{k,11000}(\theta)\right)=\dfrac{1}{\sqrt{\ell\pi}}A_{110},\end{cases}\\[12pt]
z_1^2\begin{cases}\dot{h}_{k,20000}(\theta)-2i\omega_c h_{k,20000}(\theta)=0,\\[4pt] \dot{h}_{k,20000}(0)-\mathcal{L}_k\left(h_{k,20000}(\theta)\right)=\dfrac{1}{\sqrt{\ell\pi}}A_{200},\end{cases}\\[12pt]
z_3^2\begin{cases}\dot{h}_{k,00200}(\theta)=0,\\[4pt] \dot{h}_{k,00200}(0)-\mathcal{L}_k\left(h_{k,00200}(\theta)\right)=\dfrac{1}{\sqrt{\ell\pi}}A_{002}.\end{cases}
\end{cases}
$$

$$
k=2n_1\quad\begin{cases}
z_1z_2\begin{cases}\dot{h}_{k,11000}(\theta)=0,\\[4pt] \dot{h}_{k,11000}(0)-\mathcal{L}_k\left(h_{k,11000}(\theta)\right)=\dfrac{1}{\sqrt{2\ell\pi}}A_{110},\end{cases}\\[12pt]
z_1^2\begin{cases}\dot{h}_{k,20000}(\theta)-2i\omega_c h_{k,20000}(\theta)=0,\\[4pt] \dot{h}_{k,20000}(0)-\mathcal{L}_k\left(h_{k,20000}(\theta)\right)=\dfrac{1}{\sqrt{2\ell\pi}}A_{200},\end{cases}\\[12pt]
z_3^2\begin{cases}\dot{h}_{k,00200}(\theta)=0,\\[4pt] \dot{h}_{k,00200}(0)-\mathcal{L}_k\left(h_{k,00200}(\theta)\right)=0.\end{cases}
\end{cases}
$$

$$
k=n_1+n_2\quad\begin{cases}
z_1z_3\begin{cases}\dot{h}_{k,10100}(\theta)-i\omega_c h_{k,10100}(\theta)=0,\\[4pt] \dot{h}_{k,10100}(0)-\mathcal{L}_k\left(h_{k,10100}(\theta)\right)=\dfrac{1}{\sqrt{2\ell\pi}}A_{101},\end{cases}\\[12pt]
z_2z_3\begin{cases}\dot{h}_{k,01100}(\theta)+i\omega_c h_{k,01100}(\theta)=0,\\[4pt] \dot{h}_{k,01100}(0)-\mathcal{L}_k\left(h_{k,01100}(\theta)\right)=\dfrac{1}{\sqrt{2\ell\pi}}A_{011}.\end{cases}
\end{cases}
$$

$$
k=|n_2-n_1|\quad\begin{cases}
z_1z_3\begin{cases}\dot{h}_{k,10100}(\theta)-i\omega_c h_{k,10100}(\theta)=0,\\[4pt] \dot{h}_{k,10100}(0)-\mathcal{L}_k\left(h_{k,10100}(\theta)\right)=\dfrac{1}{\sqrt{2\ell\pi}}A_{101},\end{cases}\\[12pt]
z_2z_3\begin{cases}\dot{h}_{k,01100}(\theta)+i\omega_c h_{k,01100}(\theta)=0,\\[4pt] \dot{h}_{k,01100}(0)-\mathcal{L}_k\left(h_{k,01100}(\theta)\right)=\dfrac{1}{\sqrt{2\ell\pi}}A_{011}.\end{cases}
\end{cases}
$$

$$
k=2n_2\quad\begin{cases}
z_3^2\begin{cases}\dot{h}_{k,00200}(\theta)=0,\\[4pt] \dot{h}_{k,00200}(0)-\mathcal{L}_k\left(h_{k,00200}(\theta)\right)=\dfrac{1}{\sqrt{2\ell\pi}}A_{002},\end{cases}\\[12pt]
z_1z_2\begin{cases}\dot{h}_{k,11000}(\theta)=0,\\[4pt] \dot{h}_{k,11000}(0)-\mathcal{L}_k\left(h_{k,11000}(\theta)\right)=0.\end{cases}
\end{cases}
$$

$$
k=n_1,\ \ z_1z_3\begin{cases}\dot{h}_{k,10100}(\theta)-i\omega_c h_{k,10100}(\theta)=\alpha_1\Phi_{n_1}(\theta)\Psi_{n_1}(0)A_{101},\\[4pt] \dot{h}_{k,10100}(0)-\mathcal{L}_k\left(h_{k,10100}(\theta)\right)=\alpha_1A_{101}.\end{cases}
$$

$$
k=n_2\quad\begin{cases}
z_1z_2\begin{cases}\dot{h}_{k,11000}(\theta)=\alpha_1\Phi_{n_2}(\theta)\Psi_{n_2}(0)A_{110},\\[4pt] \dot{h}_{k,11000}(0)-\mathcal{L}_k\left(h_{k,11000}(\theta)\right)=\alpha_1A_{110},\end{cases}\\[12pt]
z_3^2\begin{cases}\dot{h}_{k,00200}(\theta)=0,\\[4pt] \dot{h}_{k,00200}(0)-\mathcal{L}_k\left(h_{k,00200}(\theta)\right)=0.\end{cases}
\end{cases}
$$

$$k=0\begin{cases} z_3\mu_1\begin{cases}\dot{h}_{k,00110}(\theta)=0,\\ \dot{h}_{k,00110}(0)-\mathcal{L}_k\left(h_{k,00110}(\theta)\right)=0,\end{cases}\\ z_3\mu_2\begin{cases}\dot{h}_{k,00101}(\theta)=0,\\ \dot{h}_{k,00101}(0)-\mathcal{L}_k\left(h_{k,00101}(\theta)\right)=0.\end{cases}\end{cases}$$

$$k=2n_1\begin{cases} z_3\mu_1\begin{cases}\dot{h}_{k,00110}(\theta)=\alpha_{10}\Phi_{n_2}(\theta)\Psi_{n_2}(0)L_1^{(1,0)}\xi_{n_2},\\ \dot{h}_{k,00110}(0)-\mathcal{L}_k\left(h_{k,00110}(\theta)\right)=\alpha_{10}L_1^{(1,0)}\xi_{n_2},\end{cases}\\ z_3\mu_2\begin{cases}\dot{h}_{k,00101}(\theta)=\alpha_{10}\Phi_{n_2}(\theta)\Psi_{n_2}(0)\left(L_1^{(0,1)}\xi_{n_2}-D\dfrac{n_2^2}{\ell^2}\xi_{n_2}\right),\\ \dot{h}_{k,00101}(0)-\mathcal{L}_k\left(h_{k,00101}(\theta)\right)=\alpha_{10}\left(L_1^{(0,1)}\xi_{n_2}-D\dfrac{n_2^2}{\ell^2}\xi_{n_2}\right).\end{cases}\end{cases}$$

$$k=2n_2\begin{cases} z_3\mu_1\begin{cases}\dot{h}_{k,00110}(\theta)=0,\\ \dot{h}_{k,00110}(0)-\mathcal{L}_k\left(h_{k,00110}(\theta)\right)=0,\end{cases}\\ z_3\mu_2\begin{cases}\dot{h}_{k,00101}(\theta)=0,\\ \dot{h}_{k,00101}(0)-\mathcal{L}_k\left(h_{k,00101}(\theta)\right)=0.\end{cases}\end{cases}$$

$$k=n_1+n_2\begin{cases} z_1\mu_1\begin{cases}\dot{h}_{k,10010}(\theta)-i\omega_c h_{k,10010}(\theta)=0,\\ \dot{h}_{k,10010}(0)-\mathcal{L}_k\left(h_{k,10010}(\theta)\right)=0,\end{cases}\\ z_1\mu_2\begin{cases}\dot{h}_{k,10001}(\theta)-i\omega_c h_{k,10001}(\theta)=0,\\ \dot{h}_{k,10001}(0)-\mathcal{L}_k\left(h_{k,10001}(\theta)\right)=0,\end{cases}\\ z_2\mu_1\begin{cases}\dot{h}_{k,01010}(\theta)+i\omega_c h_{k,01010}(\theta)=0,\\ \dot{h}_{k,01010}(0)-\mathcal{L}_k\left(h_{k,01010}(\theta)\right)=0,\end{cases}\\ z_2\mu_2\begin{cases}\dot{h}_{k,01001}(\theta)+i\omega_c h_{k,01001}(\theta)=0,\\ \dot{h}_{k,01001}(0)-\mathcal{L}_k\left(h_{k,01001}(\theta)\right)=0.\end{cases}\end{cases}$$

$$k=|n_2-n_1|\begin{cases} z_1\mu_1\begin{cases}\dot{h}_{k,10010}(\theta)-i\omega_c h_{k,10010}(\theta)=\alpha_{12}\Phi_{n_1}(\theta)\Psi_{n_1}(0)L_1^{(1,0)}\xi_{n_1}e^{i\omega_c\theta},\\ \dot{h}_{k,10010}(0)-\mathcal{L}_k\left(h_{k,10010}(\theta)\right)=\alpha_{12}L_1^{(1,0)}\xi_{n_1}e^{i\omega_c\theta},\end{cases}\\ z_1\mu_2\begin{cases}\dot{h}_{k,10001}(\theta)-i\omega_c h_{k,10001}(\theta)=\alpha_{12}\Phi_{n_1}(\theta)\Psi_{n_1}(0)\left(L_1^{(0,1)}\xi_{n_1}e^{i\omega_c\theta}-D\dfrac{n_1^2}{\ell^2}\xi_{n_1}\right),\\ \dot{h}_{k,10001}(0)-\mathcal{L}_k\left(h_{k,10001}(\theta)\right)=\alpha_{12}\left(L_1^{(0,1)}\xi_{n_1}e^{i\omega_c\theta}-D\dfrac{n_1^2}{\ell^2}\xi_{n_1}\right),\end{cases}\\ z_2\mu_1\begin{cases}\dot{h}_{k,01010}(\theta)+i\omega_c h_{k,01010}(\theta)=\alpha_{12}\Phi_{n_1}(\theta)\Psi_{n_1}(0)L_1^{(1,0)}\overline{\xi}_{n_1}e^{-i\omega_c\theta},\\ \dot{h}_{k,01010}(0)-\mathcal{L}_k\left(h_{k,01010}(\theta)\right)=\alpha_{12}L_1^{(1,0)}\overline{\xi}_{n_1}e^{-i\omega_c\theta},\end{cases}\\ z_2\mu_2\begin{cases}\dot{h}_{k,01001}(\theta)+i\omega_c h_{k,01001}(\theta)=\alpha_{12}\Phi_{n_1}(\theta)\Psi_{n_1}(0)\left(L_1^{(0,1)}\overline{\xi}_{n_1}e^{-i\omega_c\theta}-D\dfrac{n_1^2}{\ell^2}\overline{\xi}_{n_1}\right),\\ \dot{h}_{k,01001}(0)-\mathcal{L}_k\left(h_{k,01001}(\theta)\right)=\alpha_{12}\left(L_1^{(0,1)}\overline{\xi}_{n_1}e^{-i\omega_c\theta}-D\dfrac{n_1^2}{\ell^2}\overline{\xi}_{n_1}\right).\end{cases}\end{cases}$$

Here, $\mathcal{L}_k\left(.\right)$ is defined by (3.11).

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