Hyperbolic inverse mean curvature flow with forced term: evolution of plane curves

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Abstract: The motion of plane curves specified by hyperbolic inverse mean curvature with a constant force term is considered. We proved that this flow remains the convexity for any forced term. Furthermore, we give an example to understand how the constant forced term c affects this hyperbolic inverse mean flow. Particularly, the asymptotic behavior of the flow under different initial conditions is discussed.

MSC:58J45, 58J47

Keywords: evolution of plane curves, forced term, hyperbolic inverse mean curvature flow

1 Introduction

We consider the evolution of plane curves in $\mathbb{R}^2$. Let $\gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be a family of closed curves which is specified by hyperbolic inverse mean curvature flow with forced term

$$\begin{cases} \dfrac{\partial^2 \gamma}{\partial t^2} = \left(\dfrac{1}{k} - c\right) \vec{\mathbf{n}} - \left\langle \dfrac{\partial^2 \gamma}{\partial s \partial t}, \dfrac{\partial \gamma}{\partial t} \right\rangle \vec{\mathbf{t}}, \\ \gamma\Big|_{t=0} = \gamma_0, \quad \dfrac{\partial \gamma}{\partial t}\Big|_{t=0} = f \vec{\mathbf{n}}_0, \end{cases} \tag{1.1}$$

where $\dfrac{1}{k}$ is the inverse mean curvature of the curve γ, $\vec{\mathbf{n}}$ and $\vec{\mathbf{t}}$ are respectively the unit outer normal vector and tangential vector of γ, $<,>$ means the standard Euclidean metric in $\mathbb{R}^2$, the initial normal velocity of the curve γ is f, c is a constant, $s = s(\cdot, t)$ denotes the arc-length parameter. By the Frenet formula

$$\frac{\partial \vec{\mathbf{t}}}{\partial s} = -k\vec{\mathbf{n}}, \quad \frac{\partial \vec{\mathbf{n}}}{\partial s} = k\vec{\mathbf{t}},$$

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where $\vec{\mathbf{t}} = \dfrac{\partial \gamma}{\partial s}$. So

$$\frac{\partial^2 \gamma}{\partial t^2} = -\left(\frac{1-ck}{k^2}\right)\frac{\partial^2 \gamma}{\partial s^2} - \left\langle \frac{\partial^2 \gamma}{\partial s \partial t}, \frac{\partial \gamma}{\partial t} \right\rangle \frac{\partial \gamma}{\partial s}. \tag{1.2}$$

When $c = 0$, Zhou, Mao and Wu [1] considered hyperbolic inverse mean curvature flow, they proved the local existence of this geometry flow. Some hyperbolic evolution equations of geometric quantities were given. Under different assumption of initial velocity, the asymptotic behavior for an evolving convex closed curve under hyperbolic inverse mean curvature flow with forced term.

Geometric evolution equations are powerful tools in studying mathematical problems and receive more and more attentions in the past few decades. Perelman successfully solve the 3-dimensional Poincar*é* conjecture using Ricci flow. The Ricci flow is an intrinsic flow, however, the mean curvature flow is an extrinsic flow which is the most important geometric flow in the geometry of submanifolds and the image processing (see, e.g., [2]). Huisken and Ilmanen [3] proved the Riemannian Penrose inequality in terms of the weak solution of the inverse mean flow. Brendle etal. [4] proved a sharp Minkowski inequality for mean convex and star-shaped hypersurface in anti-de Sitter-Schwarzschild manifold ($n \geq 3$). Recently, Xia [5] studied inverse anisotropic mean curvature flow and prove a Minkowski type inequality. Liu [6] introduced the inverse mean curvature flow with a constant forced term, she proved the convexity of the flow for any external forced field c . Furthermore, if $c < \dfrac{1}{H_0}$, the solution of the flow expands for all time, and the hypersurface converges to a sphere after rescaling the time. If $c > \dfrac{1}{H_0}$, the hypersurface converges to a point in a finite time.

Hyperbolic mean curvature flow with forced term describes the motion of melting and crystallization interface of helium crystals, one can refer to [7], [8]-[13]. Wang [10] proved the lifespan of classical solution to Cauchy problem for hyperbolic mean curvature flow with a linear forcing term. In [11], under hyperbolic mean curvature flow with different constant external forced term, the evolution of plane curves was studied. Furthermore, Wang [12] investigated symmetry reduction and invariant solutions to a hyperbolic Monge-Amp*è*re equation. They are also some other hyperbolic curvature flows. See [14]-[23]. Based on hyperbolic inverse mean curvature flow, we consider the hyperbolic inverse mean curvature flow with forced term.

This paper consists of the following. Preliminaries and hyperbolic evolution equations that the geometry quantities of the curves satisfied are given in Section 2. Furthermore, the local existence of the flow (1.1) is given. In Section 3, we given an example to analysis the motion of circles specified by the hyperbolic inverse mean curvature flow with forced term c. Finally, the main result is proved.

2 Preliminaries

In this section, we investigate the short-time existence and give some propositions and proof these propositions.

Definition 2.1. A flow is a normal flow if and only if the tangential velocity of the curve is zero for all time.

Lemma 2.1 The hyperbolic inverse curvature flow with forced term (1.1) is a normal flow.

Proof.

$$\begin{aligned}\frac{d}{dt}\left\langle\frac{\partial\gamma}{\partial t},\frac{\partial\gamma}{\partial u}\right\rangle=&\left\langle\frac{\partial^2\gamma}{\partial t^2},\frac{\partial\gamma}{\partial u}\right\rangle+\left\langle\frac{\partial\gamma}{\partial t},\frac{\partial^2\gamma}{\partial t\partial u}\right\rangle\\=&-\left\langle\frac{\partial^2\gamma}{\partial s\partial t},\frac{\partial\gamma}{\partial t}\right\rangle\left\langle\vec{\mathbf{t}},\frac{\partial\gamma}{\partial u}\right\rangle+\left\langle\frac{\partial\gamma}{\partial t},\frac{\partial^2\gamma}{\partial t\partial u}\right\rangle\\=&-\left\langle\frac{\partial^2\gamma}{\partial s\partial t},\frac{\partial\gamma}{\partial t}\right\rangle\left|\frac{\partial\gamma}{\partial u}\right|+\left\langle\frac{\partial\gamma}{\partial t},\frac{\partial^2\gamma}{\partial t\partial u}\right\rangle\\=&-\left\langle\frac{\partial^2\gamma}{\partial u\partial t},\frac{\partial\gamma}{\partial t}\right\rangle+\left\langle\frac{\partial\gamma}{\partial t},\frac{\partial^2\gamma}{\partial t\partial u}\right\rangle\\=&0,\end{aligned}$$

which means that

$$\left\langle\frac{\partial\gamma}{\partial t},\frac{\partial\gamma}{\partial u}\right\rangle=\left\langle\frac{\partial\gamma}{\partial t},\frac{\partial\gamma}{\partial u}\right\rangle\Big|_{t=0}=0.$$

The proof of Lemma 2.1 is finished.

In fact, the flow (1.1) is rewritten as

$$\begin{cases}\dfrac{\partial\gamma}{\partial t}=\sigma\vec{\mathbf{n}}\\ \gamma\Big|_{t=0}=\gamma_0,\end{cases}\tag{2.1}$$

in which σ satisfies

$$\frac{\partial\sigma}{\partial t}=k^{-1}-c,\quad \sigma\frac{\partial\sigma}{\partial s}=\left\langle\frac{\partial^2\gamma}{\partial s\partial t},\frac{\partial\gamma}{\partial t}\right\rangle.$$

By the definition of arc-length,

$$ds=g(u,t)du=\left|\frac{\partial\gamma}{\partial u}\right|du.$$

Assume that the unit outer normal angle of γ is denoted by θ, then $\vec{\mathbf{t}}$ and $\vec{\mathbf{n}}$ are given by

$$\vec{\mathbf{t}}=(-\sin\theta,\cos\theta),\quad \vec{\mathbf{n}}=(\cos\theta,\sin\theta).$$

Then we have $\frac{\partial\theta}{\partial s}=k$ and

$$\frac{\partial\vec{\mathbf{t}}}{\partial t}=\frac{\partial\vec{\mathbf{t}}}{\partial\theta}\frac{\partial\theta}{\partial t}=-\frac{\partial\theta}{\partial t}\vec{\mathbf{n}},\quad \frac{\partial\vec{\mathbf{n}}}{\partial t}=\frac{\partial\vec{\mathbf{n}}}{\partial\theta}\frac{\partial\theta}{\partial t}=\frac{\partial\theta}{\partial t}\vec{\mathbf{t}}. \tag{2.2}$$

Lemma 2.2 $\frac{\partial g}{\partial t}=k\sigma g.$

Proof. Derive g with respect to t, we have

$$\begin{aligned}\frac{\partial}{\partial t}(g^2)=&\frac{\partial}{\partial t}\left\langle\frac{\partial\gamma}{\partial u},\frac{\partial\gamma}{\partial u}\right\rangle=2\left\langle\frac{\partial\gamma}{\partial u},\frac{\partial^2\gamma}{\partial t\partial u}\right\rangle=2\left\langle g\vec{\mathbf{t}},\frac{\partial}{\partial u}(\sigma\vec{\mathbf{n}})\right\rangle\\ =&2\left\langle g\vec{\mathbf{t}},\sigma\frac{\partial\vec{\mathbf{n}}}{\partial u}\right\rangle=2\left\langle g\vec{\mathbf{t}},\sigma\frac{\partial\vec{\mathbf{n}}}{\partial s}\frac{\partial s}{\partial u}\right\rangle\\ =&2\left\langle g\vec{t},g\sigma k\vec{\mathbf{t}}\right\rangle\\ =&2g^2k\sigma,\end{aligned}$$

Lemma 2.2 is proved. ■

In terms the above Lemma, we can deduce

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial}{\partial s}=&\frac{\partial}{\partial t}\left(\frac{1}{g}\frac{\partial}{\partial u}\right)\\ =&-\frac{1}{g^2}\frac{\partial g}{\partial t}\frac{\partial}{\partial u}+\frac{1}{g}\frac{\partial}{\partial t}\frac{\partial}{\partial u}\\ =&-\frac{1}{g^2}(k\sigma g)\frac{\partial}{\partial u}+\frac{1}{g}\frac{\partial}{\partial u}\frac{\partial}{\partial t}\\ =&-k\sigma\frac{\partial}{\partial s}+\frac{\partial}{\partial s}\frac{\partial}{\partial t}.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial\vec{\mathbf{t}}}{\partial t}=&\frac{\partial}{\partial t}\frac{\partial\gamma}{\partial s}=-k\sigma\frac{\partial\gamma}{\partial s}+\frac{\partial}{\partial s}\frac{\partial\gamma}{\partial t}\\ =&-k\sigma\frac{\partial\gamma}{\partial s}+\frac{\partial}{\partial s}(\sigma\vec{\mathbf{n}})\\ =&-k\sigma\vec{\mathbf{t}}+\frac{\partial\sigma}{\partial s}\vec{\mathbf{n}}+\sigma\frac{\partial\vec{\mathbf{n}}}{\partial s}\\ =&-k\sigma\vec{\mathbf{t}}+\frac{\partial\sigma}{\partial s}\vec{\mathbf{n}}+k\sigma\vec{\mathbf{t}}=\frac{\partial\sigma}{\partial s}\vec{\mathbf{n}},\end{aligned}$$

furthermore by (2.2),

$$-\frac{\partial\theta}{\partial t}=\frac{\partial\sigma}{\partial s},\quad \frac{\partial\vec{\mathbf{n}}}{\partial t}=-\frac{\partial\sigma}{\partial s}\vec{\mathbf{t}}.$$

Let us parameterize the evolving curve $\gamma(\cdot,t)$ by the normal angle θ, denote

$$\widetilde{\gamma}(\theta,\tau)=\gamma(u(\theta,\tau),t(\theta,\tau)),$$

in which $t(\theta,\tau)=\tau$. Since

$$\frac{\partial\theta}{\partial\tau}=\frac{\partial\theta}{\partial t}+\frac{\partial\theta}{\partial u}\frac{\partial u}{\partial\tau}=0,$$

we have

$$\frac{\partial\theta}{\partial t}+\frac{\partial\theta}{\partial u}\frac{\partial u}{\partial\tau}=\frac{\partial\theta}{\partial t}+\frac{\partial\theta}{\partial s}\frac{\partial s}{\partial u}\frac{\partial u}{\partial\tau}=\frac{\partial\theta}{\partial t}+kg\frac{\partial u}{\partial\tau}=0.$$

Hence,

$$\frac{\partial\vec{\mathbf{t}}}{\partial\tau}=-\left(\frac{\partial\theta}{\partial t}+kg\frac{\partial u}{\partial\tau}\right)\vec{\mathbf{n}}=0,$$

similarly, we obtain $\frac{\partial\vec{\mathbf{n}}}{\partial\tau}=0$.

The support function of the closed convex curve $\widetilde{\gamma}$ is given by

$$h=\langle\widetilde{\gamma},\vec{\mathbf{n}}\rangle.$$

Its derivative satisfies

$$h_\theta=\langle\widetilde{\gamma},\vec{\mathbf{t}}\rangle.$$

$$h_{\theta\theta}=\langle\widetilde{\gamma}_\theta,\vec{\mathbf{t}}\rangle+\langle\widetilde{\gamma},-\vec{\mathbf{n}}\rangle=\langle\widetilde{\gamma}_\theta,\vec{\mathbf{t}}\rangle-h,$$

and the evolving curve can be depicted by

$$\widetilde{\gamma}=h\vec{\mathbf{n}}+h_\theta\vec{\mathbf{t}}.$$

By a direct computation, we have

$$h_{\theta\theta}+h=\langle\widetilde{\gamma}_\theta,\vec{\mathbf{t}}\rangle=\langle\widetilde{\gamma}_s\cdot\frac{\partial s}{\partial\theta},\vec{\mathbf{t}}\rangle=\frac{\partial s}{\partial\theta}=\frac{1}{k}.$$

Since

$$\widetilde{\gamma}_\tau=h_\tau\vec{\mathbf{n}}+h_{\theta\tau}\vec{\mathbf{t}},$$

we have

$$h_\tau=\langle\widetilde{\gamma}_\tau,\vec{\mathbf{n}}\rangle=\langle\gamma_u\frac{\partial u}{\partial\tau}+\gamma_t,\vec{\mathbf{n}}\rangle=\langle\gamma_t,\vec{\mathbf{n}}\rangle$$

and

$$\begin{aligned}h_{\tau\tau}&=\left\langle\gamma_{ut}\frac{\partial u}{\partial\tau}+\gamma_{tt},\vec{\mathbf{n}}\right\rangle\\&=\left\langle\gamma_{ut}\frac{\partial u}{\partial\tau},\vec{\mathbf{n}}\right\rangle+k^{-1}-c.\end{aligned}$$

By $\dfrac{\partial g}{\partial t}=k\sigma g$, we get

$$h_{\tau\theta}=\left\langle\frac{\partial\widetilde{\gamma}}{\partial\tau},\vec{\mathbf{t}}\right\rangle=\left\langle\frac{\partial\gamma}{\partial t}+\frac{\partial\gamma}{\partial u}\frac{\partial u}{\partial\tau},\vec{\mathbf{t}}\right\rangle=g\frac{\partial u}{\partial\tau}$$

and

$$h_{\theta\tau}=\frac{\partial}{\partial\theta}\langle\gamma_t,\vec{\mathbf{n}}\rangle=\left\langle\gamma_{ut}\frac{\partial u}{\partial\theta},\vec{\mathbf{n}}\right\rangle=\frac{1}{kg}\langle\gamma_{ut},\vec{\mathbf{n}}\rangle.$$

Therefore, the evolution equation of the support function h is given by

$$\begin{aligned} h_{\tau\tau} =& \left\langle \gamma_{ut}\frac{\partial u}{\partial \tau}, \vec{\mathbf{n}} \right\rangle + k^{-1} - c \\ =& k h_{\theta\tau} g \frac{\partial u}{\partial \tau} + k^{-1} - c \\ =& k h_{\theta\tau}^2 + k^{-1} - c, \end{aligned}$$

which is equivalent to

$$h_{\tau\tau} = \frac{h_{\theta\tau}^2}{h_{\theta\theta} + h} + h_{\theta\theta} + h - c.$$

Hence, the flow (1.1) can be rewritten by

$$\begin{cases} h h_{\tau\tau} + h_{\tau\tau} h_{\theta\theta} - h_{\theta\tau}^2 - (h_{\theta\theta} + h)^2 + c h_{\theta\theta} + ch = 0, \\ h(\theta, 0) = l(\theta), \\ h_\tau(\theta, 0) = \widetilde{f}(\theta), \end{cases} \tag{2.3}$$

in which $l(\theta) = \langle \widetilde{\gamma}_0, \vec{\mathbf{n}_0} \rangle$. This is a nonlinear hyperbolic equation, by the linearization method we can get the local existence of the IVP(2.3). Denote

$$A(h_{\theta\theta}, h_{\theta\tau}, h) := \frac{h_{\theta\tau}^2}{h_{\theta\theta} + h} + h_{\theta\theta} + h,$$

then we have

$$h_{\tau\tau} = \frac{\partial A}{\partial h_{\theta\theta}} h_{\theta\theta} + \frac{\partial A}{\partial h_{\theta\tau}} h_{\theta\tau} + \frac{\partial A}{\partial h} h - c \tag{2.4}$$

where

$$\frac{\partial A}{\partial h_{\theta\theta}} = 1 - \frac{h_{\theta\tau}^2}{(h_{\theta\theta} + h)^2}, \quad \frac{\partial A}{\partial h_{\theta\tau}} = \frac{2h_{\theta\tau}}{h_{\theta\theta} + h}, \quad \frac{\partial A}{\partial h} = 1 - \frac{h_{\theta\tau}^2}{(h_{\theta\theta} + h)^2}.$$

The coefficient matrix of terms in (2.4) is

$$\begin{pmatrix} -1 & \dfrac{h_{\theta\tau}}{h_{\theta\theta} + h} \\ \dfrac{h_{\theta\tau}}{h_{\theta\theta} + h} & 1 - \dfrac{h_{\theta\tau}^2}{(h_{\theta\theta} + h)^2} \end{pmatrix}.$$

According to a linear transformation, we have

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

According to the standard theory of linear hyperbolic partial differential equations (c.f. [24]), we get the following result.

Theorem 2.1 *(Local existences and uniqueness) Suppose γ_0 is a strictly convex closed curve, the smooth function $f(u)$ is the initial velocity of γ_0. Then there is a family of strictly convex closed curves $\gamma(\cdot, t)$ which satisfy the flow* (1.1) *for all $t \in [0, T)$ with $T > 0$.*

3 Expansion and Convergence

In this section, We will give an example to try to analysis the exact solutions of the flow (1.1). Without loss of generality, τ is replaced by t.

Example 3.1. Assume that $\gamma(\theta, t)$ is a family of circles centered at the origin in $\mathbb{R}^2$ with the radius $r(t)$, i.e.,

$$\gamma(\theta, t) = r(t)(\cos\theta, \sin\theta).$$

Then the support function and curvature k of $\gamma(\theta, t)$ are given respectively by

$$h(\theta, t) = r(t), \quad k(\theta, t) = \frac{1}{r(t)},$$

which means that the flow (1.1) is rewritten as

$$\begin{cases} r''(t) = r(t) - c, \\ r(0) = r_0 > 0, r'(0) = r_1. \end{cases} \tag{3.1}$$

Solving the equation (3.1) directly yields

$$r(t) = (r_0 - c)\cosh t + r_1 \sinh t + c.$$

Then we have the following

• *if* $r_0 + r_1 - c > 0$, *then* $r''(t) < 0$, *the flow exists for all* $t \in [0, \infty)$. *Furthermore, if* $r_0 - r_1 - c \leq 0$, *the solution increases to the infinity (see Figure 1), if* $r_0 - c > r_1 \geq 0$, *the solution increases to the infinity (see Figure 2), if* $c - r_0 < r_1 < 0 < r_0 - c$, *the solution shrinks firstly to the minimum and then expands to the infinity (see Figure 3);*

• *if* $r_0 = c - r_1$, *then* $r(t) = (r_0 - c)e^{-t} + c$, *there are four subcases:*

if $r_0 > c \geq 0$, *which means that the solution exists for all time, the solution shrinks to a circle with radius* $r(t) = c$ *as* $t \to \infty$ *(see Figure 4);*

if $r_0 > 0 > c$, *which implies the solution shrinks to a point as* $t \to \ln \frac{c - r_0}{c}$ *(see Figure 5);*

if $r_0 = c$, *the evolving curves are in a stable condition which radius is* $r(t) = c$*;*

if $r_0 < c$, *then the solution expands to a circle with radius* $r(t) = c$ *as* $t \to \infty$*(see Figure 6).*

• *if* $r_0 + r_1 - c < 0$, *there are three subcases:*

if $r_0 - r_1 - c > 0$, $r'(t) < 0$, *then the solution shrinks to a point as* $t \to \ln\left(\frac{-c+\sqrt{r_1^2+2cr_0-r_0^2}}{r_0+r_1-c}\right)$ *(see Figure 7);*

if $r_0 - r_1 - c < 0$, *then the evolving curves expand firstly to the maximum and then converge to a point as* $t \to \ln\left(\frac{-c-\sqrt{r_1^2+2cr_0-r_0^2}}{r_0+r_1-c}\right)$ *(see Figure 8);*

if $r_0 - r_1 - c = 0$,*then the evolving curves shrink to a point as* $t \to \ln\left(\frac{-2c}{r_0+r_1-c}\right)$ *(see Figure 9).*

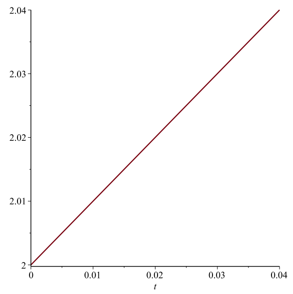

Figure 1: $r_0 = 2, r_1 = 1, c = 2$

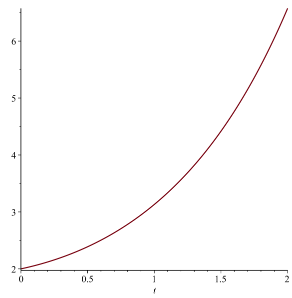

Figure 2: $r_0 = 2, r_1 = \frac{1}{2}, c = 1$

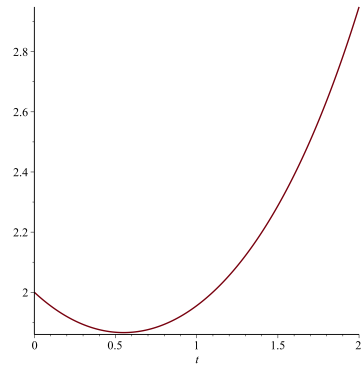

Figure 3: $r_0 = 2, r_1 = -\frac{1}{2}, c = 1$

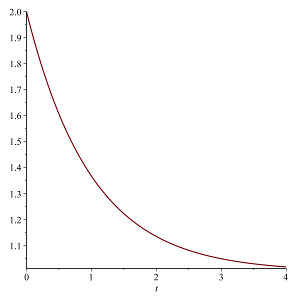

Figure 4: $r_0 = 2, r_1 = -1, c = 1$

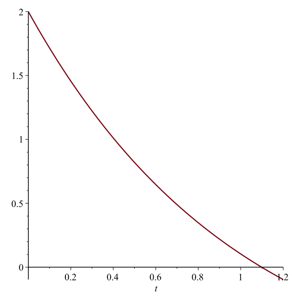

Figure 5: $r_0 = 2, r_1 = -3, c = -1$

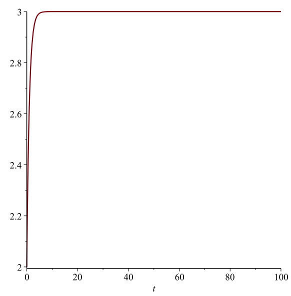

Figure 6: $r_0 = 2, r_1 = 1, c = 3$

Remark 3.1 when the forced term c vanishes, we can get the same results in [1]. As shown in Example 3.1, the flow is affected by the forced term c, hence, we generalized the results in [1]. The inverse mean curvature flow with forced term c, we can get the following equation

$$\begin{cases} r'(t) = r - c, \\ r(0) = r_0 > 0. \end{cases} \tag{3.2}$$

Then the solution of the equation (3.2) is

$$r(t) = (r_0 - c)e^t + c,$$

on $[0, \Omega)$ for some $0 < \Omega \leq \infty$.

- *when* $r_0 - c > 0$, *the solution exists for all* $t \in [0, \infty)$, *the solution expands exponentially to the infinity. The normalized curves converges to a circle.*

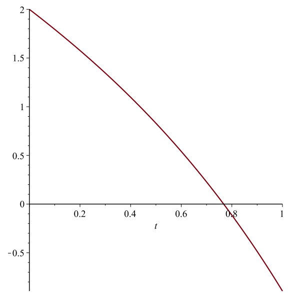

Figure 7: $r_0 = 2, r_1 = -2, c = 3$

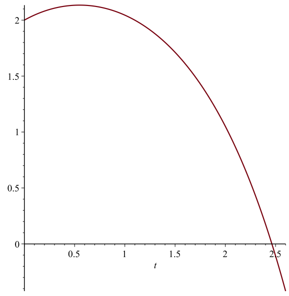

Figure 8: $r_0 = 2, r_1 = \frac{1}{2}, c = 3$

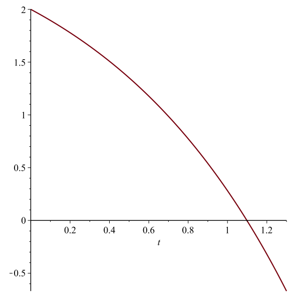

Figure 9: $r_0 = 2, r_1 = -1, c = 3$

- *when $r_0 - c < 0$, the solution shrinks to a point as $t \to \ln\left(\dfrac{c}{c - r_0}\right)$.*

Remark 3.2 In terms of Example 3.1 and Remark 3.1, we know that hyperbolic flow is different from heat flow even if the initial curve is circles, the solution of the flow (1.1) depends on r_0, r_1 and the forced term c. It is not easy to describe the asymptotic behavior of the flow (1.2) as t tends to the maximum time. However, we can overcome this difficulty by the following proposition.

Proposition 3.1 *(Containment principle) Assume that $\gamma_1(u,t)$ and $\gamma_2(u,t)$ are two strictly closed convex solutions of the flow* (1.1)*. Assume that $\gamma_2(u,0)$ is contained in the domain enclosed by $\gamma_1(u,0)$ with $f_1(u) \geq f_2(u)$. Then $\gamma_2(u,t)$ lies in the domain enclosed by $\gamma_1(u,t)$ for all $t \in [0,T)$.*

Proof. Suppose the support function of $\gamma_i(u,t)$ is $h_i(\theta,t)$ $(i = 1,2)$ which satisfies (2.5), and they satisfy $h_2(\theta,0) \leq h_1(\theta,0)$ and $\tilde{f}_2(\theta) \leq \tilde{f}_2(\theta)$.

Denote

$$\nu(\theta,t) := h_2(\theta,t) - h_1(\theta,t),$$

derivative ν with t twice,

$$\begin{aligned} \nu_{tt} = h_{2tt} - h_{1tt} &= \frac{h_{2\theta t}^2 + k_2^{-2}}{h_2 + h_{2\theta\theta}} - \frac{h_{1\theta t}^2 + k_1^{-2}}{h_1 + h_{1\theta\theta}} \\ &= k_1 k_2 \left(\frac{1}{k_1 k_2} - h_{1\theta t} h_{2\theta t}\right) \nu_{\theta\theta} + (k_1 h_{1\theta t} + k_2 h_{2\theta t}) \nu_{\theta t} + k_1 k_2 \left(\frac{1}{k_1 k_2} - h_{1\theta t} h_{2\theta t}\right) \nu, \end{aligned}$$

which means that ν satisfies the following linear hyperbolic equation

$$\begin{cases} \nu_{tt} = k_1 k_2 \left(\dfrac{1}{k_1 k_2} - h_{1\theta t} h_{2\theta t}\right) \nu_{\theta\theta} + (k_1 h_{1\theta t} + k_2 h_{2\theta t}) \omega_{\theta t} + k_1 k_2 \left(\dfrac{1}{k_1 k_2} - h_{1\theta t} h_{2\theta t}\right) \nu, \\ \nu_t\Big|_{t=0} = \tilde{f}_2(\theta) - \tilde{f}_1(\theta) = \nu_1(\theta), \\ \nu\Big|_{t=0} = l_2(\theta) - l_1(\theta) = \nu_0(\theta). \end{cases} \tag{3.3}$$

Denote the operator L by

$$L[\nu] := k_1k_2\left(\frac{1}{k_1k_2} - h_{1\theta t}h_{2\theta t}\right)\nu_{\theta\theta} + (k_1h_{1\theta t} + k_2h_{2\theta t})\nu_{\theta t} - \nu_{tt},$$

then which implies

$$a = k_1k_2\left(\frac{1}{k_1k_2} - h_{1\theta t}h_{2\theta t}\right), b = \frac{1}{2}(k_1h_{1\theta t} + k_2h_{2\theta t}), c = -1,$$

we get

$$\begin{aligned} b^2 - ac &= \frac{1}{4}(k_1h_{1\theta t} + k_2h_{2\theta t})^2 - k_1k_2\left(\frac{1}{k_1k_2} - h_{1\theta t}h_{2\theta t}\right)\cdot(-1) \\ &= \frac{1}{4}(k_1h_{1\theta t} + k_2h_{2\theta t})^2 + 1 > 1. \end{aligned}$$

Therefore, the operator L is uniformly hyperbolic for all $(\theta, t) \in [0, 2\pi] \times [0, T)$. By the maximum principle for one-dimensional wave equation, see chapter 4 of Protter-Weinbergrt [25], we know that $h_2(\theta, t) \le h_1(\theta, t)$ for all $(\theta, t) \in [0, 2\pi] \times [0, T)$. The proof is finished.

Proposition 3.2 *(Preserving convexity)* Assume the curvature of γ_0 satisfies

$$\xi = \min_{\theta\in[0,2\pi]} k_0(\theta) > 0.$$

Then for a fourth continuously differentiable solution h of (2.5), the curvature of $(\gamma(\cdot, t)$ satisfies

$$k(\theta, t) \ge \xi,$$

for all $t \in [0, \Omega)$ which is the maximal interval for the solution of the flow (1.1).

Proof. By Theorem 2.1 and the strictly convexity of γ_0, we know that $\gamma(u, t)$ is strictly convex for any $t \in [0, T)$, where $T \le \Omega$ and h satisfies

$$h_{tt} = kh_{\theta t}^2 + k^{-1} - c.$$

By $k = (h_{\theta\theta} + h)^{-1}$, we get

$$k_t = (h_{\theta\theta} + h)_t^{-1} = -(h_{\theta\theta} + h)^{-2}(h_{\theta\theta t} + h_t) = -k^2(h_{\theta\theta t} + h_t).$$

Therefore, we have

$$\begin{aligned} h_{\theta\theta t} + h_t &= -(h_{\theta\theta} + h)^2 k_t = -\frac{1}{k^2}k_t, \\ h_{\theta\theta tt} + h_{tt} &= (-\frac{1}{k^2}k_t)_t = \frac{2}{k^3}k_t^2 - \frac{1}{k^2}k_{tt}, \\ h_{\theta\theta\theta t} + h_{\theta t} &= (-\frac{1}{k^2}k_t)_\theta = \frac{2}{k^3}k_tk_\theta - \frac{1}{k^2}k_{\theta t}, \end{aligned}$$

and

$$
\begin{aligned}
h_{tt\theta\theta}+h_{tt}=&(kh_{\theta t}^2+k^{-1}-c)_{\theta\theta}+kh_{\theta t}^2+k^{-1}-c\\
=&((h_{\theta t}^2-1)_\theta k+(h_{\theta t}^2-1)k_\theta)_\theta+(k^{-1}+k)_{\theta\theta}+(h_{\theta t}^2-1)k+(k^{-1}+k)-c\\
=&(h_{\theta t}^2-1)(k_{\theta\theta}+k)-[(2h_{\theta t}h_{\theta\theta t})_\theta k+4h_{\theta t}h_{\theta\theta t}k_\theta]-\left[k_{\theta\theta}-\frac{1}{k^2}k_{\theta\theta}+\frac{2}{k^3}k_\theta^2+(k^{-1}+k)-c\right]\\
=&(h_{\theta t}^2-1)(k_{\theta\theta}+k)-\left[2((h_{\theta\theta t}+h_t)^2-2h_{\theta\theta t}h_t-h_t^2+h_{\theta t}(h_{\theta\theta}+h)_{\theta t}-h_{\theta t}^2)k+4k_\theta h_{\theta t}(\frac{1}{k}-h)_t\right],\\
&-k^2\left[\frac{2}{k^3}k_\theta^2+(1-\frac{1}{k^2})k_{\theta\theta}+(k^{-1}+k)-c\right]\\
=&(h_{\theta t}^2-1)(k_{\theta\theta}+k)-2k\left[(-\frac{1}{k^2}k_t)^2-2(-\frac{1}{k^2}k_t)h_t+h_t^2-h_{\theta t}^2+h_{\theta t}(\frac{2}{k^3}k_tk_\theta-\frac{1}{k^2}k_{\theta t})\right]\\
&+4(k_\theta h_{\theta t}\frac{1}{k^2}k_t+k_\theta h_{\theta t}h_t)-\left[\frac{2}{k^3}k_\theta^2+(1-\frac{1}{k^2})k_{\theta\theta}+(k^{-1}+k)-c\right]
\end{aligned}
$$

$$
\begin{aligned}
k_{tt}=&\frac{2}{k}k_t^2-(h_{tt}+h_{\theta\theta tt})k^2\\
=&\frac{2}{k}k_t^2-k^2(h_{\theta t}^2-1)(k_{\theta\theta}+k)-2k^3\left[(-\frac{1}{k^2}k_t)^2+h_t^2-2(-\frac{1}{k^2}k_t)h_t-h_{\theta t}^2+h_{\theta t}(\frac{2}{k^3}k_tk_\theta-\frac{1}{k^2}k_{\theta t})\right].\\
&+4k^2(k_\theta h_{\theta t}\frac{1}{k^2}k_t+k_\theta h_{\theta t}h_t)-k^2\left[\frac{2}{k^3}k_\theta^2+(1-\frac{1}{k^2})k_{\theta\theta}+(k^{-1}+k)-c\right]\\
=&(1-k^2h_{\theta t}^2)k_{\theta\theta}+2kh_{\theta t}k_{\theta t}+4k^2h_{\theta t}h_tk_\theta-\frac{2}{k^3}k_\theta^2-4kh_tk_t+k^3(h_{\theta t}^2-2h_t^2-k^{-2})+ck^2
\end{aligned}
$$

Therefore, the curvature k satisfies

$$k_{tt}=\left(1-k^2h_{\theta t}^2\right)k_{\theta\theta}+2kh_{\theta t}k_{\theta t}+4k^2h_{\theta t}h_tk_\theta-\frac{2}{k^3}k_\theta^2-4kh_tk_t+k^3(h_{\theta t}^2-2h_t^2-k^{-2})+ck^2.$$

Denote the operator $\overline{L}$ by

$$\bar{L}[k]:=\left(1-k^2h_{\theta t}^2\right)k_{\theta\theta}+2kh_{\theta t}k_{\theta t}-k_{tt}+4k^2h_{\theta t}h_tk_\theta-\frac{2}{k^3}k_\theta^2-4kh_tk_t.$$

Then

$$\bar{a}=\left(1-k^2h_{\theta t}^2\right),\bar{b}=kh_{\theta t},\bar{c}=-1.$$

Therefore, we get

$$\bar{b}^2-\bar{a}\bar{c}=(kh_{\theta t})^2-k^2\left(\frac{1}{k^2}-h_{\theta t}^2\right)\cdot(-1)=1>0.$$

Therefore, the operator $\overline{L}$ is uniformly hyperbolic for all $(\theta,t)\in[0,2\pi]\times[0,T)$.

The curvature $k(\theta,t)$ of the curve $\gamma(\cdot,t)$ satisfies the evolution equation

$$
\begin{cases}
(\overline{L}+\bar{l})[k]=0 & in\quad [0,2\pi]\times[0,T),\\
k(\theta,0)=k_0(\theta),\\
(-\bar{b}k_\theta-\bar{c}k_t)\Big|_{t=0}=\beta(\theta)\geq 0,
\end{cases}
$$

in which the operator $\bar{l}$ is denoted by

$$\bar{l}[k] := (h_{\theta t}^2 - 2h_t^2)k^3 - k - k^2.$$

Denote the function $\overline{k}(\theta, t) = \min_{\theta \in [0,2\pi]} k_0(\theta) = \xi$, then it satisfies

$$\begin{cases} (\overline{L} + \bar{l})[\overline{k}] = 0, & in \quad [0, 2\pi] \times [0, T), \\ \overline{k}(\theta, 0) \leq k_0(\theta), \\ -M\overline{k} \leq \beta(\theta) - Mk_0(\theta), \end{cases}$$

in which M is a so large constant that

$$M \geq -\bar{b}_\theta(\theta, 0).$$

Applying the maximum principles in hyperbolic differential equations in [25] to $\overline{k} - k$, we have

$$k(\theta, t) - \overline{k} \geq 0 \qquad in \quad [0, 2\pi] \times [0, T_0),$$

in which $T_0 \leq T$. We can deduce that the curve $\gamma(u, t)$ is convex for all $t \in [0, \Omega)$ and $k \geq \xi = \min_{\mathbb{S}^1} k_0(\theta)$. The proof is completed.

Next we will give the evolution equation for the length of the curve.

Lemma 3.1. The evolution equation of the length $\mathbf{L}(t)$ of $\gamma(u, t)$ is

$$\frac{d\mathbf{L}(t)}{dt} = \int_0^{2\pi} h_t d\theta,$$

and

$$\frac{d^2\mathbf{L}(t)}{dt^2} = \int_0^{2\pi} \left[kh_{\theta t}^2 + k^{-1} - c\right] d\theta.$$

Proof. Since

$$\mathbf{L}(t) = \int_0^{2\pi} g(\theta, t) d\theta,$$

and $\frac{\partial g}{\partial t} = kg\widetilde{\sigma}$, by a direct calculation, we get

$$\frac{d\mathbf{L}(t)}{dt} = \int_0^{2\pi} \frac{\partial g}{\partial t} d\theta = \int_0^{2\pi} kg\widetilde{\sigma} d\theta = \int_0^{2\pi} \widetilde{\sigma}(\theta, t) d\theta = \int_0^{2\pi} h_t d\theta,$$

and

$$\frac{d^2\mathbf{L}(t)}{dt^2} = \int_0^{2\pi} h_{tt} d\theta = \int_0^{2\pi} (kh_{\theta t}^2 + k^{-1} - c) d\theta,$$

The proof is completed.

By Example 3.1, the behavior for an evolving convex closed curves under (1.1) is complicated. However, we can obtain the limit behavior of the flow (1.1) by Proposition 3.1, 3.2 and Lemma 3.1.

Theorem 3.1 *Assume that γ_0 is a strictly convex closed curve, its curvature $k_0(\theta)$ satisfies*

$$0 < \xi = \min_{\theta\in[0,2\pi]} k_0(\theta) \leq k_0(\theta) \leq \varsigma = \max_{\theta\in[0,2\pi]} k_0(\theta).$$

Then the solution $\gamma(u,t)$ of the flow (1.1) *exists for all $t \in [0,\Omega)$, in which $0 < \Omega \leq \infty$. Furthermore, we get*

(I) *if $\varsigma^{-1} + \min\limits_{u\in\mathbb{S}^1} f(u) > c$, then $\Omega = \infty$;*

(II) *if $\xi^{-1} + \max\limits_{u\in\mathbb{S}^1} f(u) < c < \varsigma^{-1}$, then $\Omega < \infty$. Moreover, if $\xi^{-1}\Omega + \max\limits_{u\in\mathbb{S}^1} f(u) \leq c\Omega$, there are two subcases:*

- *the evolving curve $\gamma(\cdot,t)$ shrinks to a point as $t \to \Omega$, which means the curvature $k(u,\Omega)$ is unbounded;*
- *the curvature $k(u,\Omega)$ is discontinuous, i.e., the evolving curve $\gamma(u,t)$ shrinks to a piecewise smooth curve.*

Proof. Suppose that $[0,\Omega)$ is the maximal time interval of the flow (1.1). According to Proposition 3.2, the strictly convexity of the curve $\gamma(\cdot,t)$ is preserved for all $t \in [0,\Omega)$ and the curvature of $k \geq \xi > 0$. There are two cases:

(I) $\varsigma^{-1} + \min\limits_{u\in\mathbb{S}^1} f(u) > c$.

In terms of $\varsigma = \max\limits_{\theta\in[0,2\pi]} k_0(\theta) \geq \xi > 0$, the initial γ_0 is contained by a circle ℓ_0 with radius ς^{-1}. Assume the initial normal velocity of ℓ_0 is $\min\limits_{u\in\mathbb{S}^1} f(u)$, then the circle ℓ_0 specified by the flow (1.1) has a unique solution $\ell(u,t)$. In terms of Example 3.1, the curve $\ell(u,t)$ exists for all $t \in [0,\infty)$. Applying Proposition 3.1, the curve $\ell(u,t)$ always lies in the curve $\gamma(u,t)$ for all $t \in [0,\Omega)$, which means $\Omega = \infty$.

(II) $\xi^{-1} + \max\limits_{u\in\mathbb{S}^1} f(u) < c < \varsigma^{-1}$.

In terms of $\xi = \min\limits_{\theta\in[0,2\pi]} k_0(\theta) > 0$, enclose the initial γ_0 by a large enough circle ℓ_1 with radius ξ^{-1}. Evolving ℓ_1 with initial velocity $\max\limits_{u\in\mathbb{S}^1} f(u)$ specified by the flow (1.1), we get a solution $\widetilde{\ell}_1(u,t)$. According to Example 3.1, the circle $\widetilde{\ell}_1(u,t)$ exists for a finite time interval $[0,\omega)$ and $\widetilde{\ell}_1(u,t)$ converges to a single point as $t \to \omega$. By Proposition 3.1, e$\gamma(u,t)$ lies in the domain enclosed by $\widetilde{\ell}_1(u,t)$ for all $t \in [0,\omega)$. Therefore, we know that evolving curve $\gamma(u,t)$ must be singular at some time $\Omega \leq \omega$.

According to containment principle, if $t_1 < t_2$, then $\gamma(u,t_1)$ always enclose $\gamma(u,t_2)$ under the flow (1.1). In other words, $\gamma(u,t)$ is shrinking. By the Blaschke Selection Theorem in the convex geometry [26], we deduce that $\gamma(u,t)$ converges to a weakly convex curve $\gamma(\cdot,\Omega)$ (maybe degenerate and nonsmooth) in the Hausdorff metric.

We deduce that $\gamma(u,\Omega)$ is either a single point or a limit curve with the discontinuous curvature. According to Lemma 3.1, for all $t \in [0, \Omega)$, we get

$$\frac{d^2\mathbf{L}(t)}{dt^2} = \int_0^{2\pi} \left[k \left(\frac{\partial \widetilde{\sigma}}{\partial \theta} \right)^2 + k^{-1} - c \right] d\theta \geq (\varsigma^{-1} - c)2\pi > 0.$$

Furthermore,

$$\begin{aligned} \widetilde{\sigma}(\theta, t) &= f(u) + \int_0^t [k^{-1}(u, \iota) - c] d\iota \\ &\leq (\xi^{-1} - c)t + \max_{u \in \mathbb{S}^1} f(u) \leq \xi^{-1}\Omega + \max_{u \in \mathbb{S}^1} f(u) - c\Omega < 0, \end{aligned}$$

which means

$$\frac{d\mathbf{L}(t)}{dt} = \int_0^{2\pi} \widetilde{\sigma} d\theta < 0.$$

Therefore we get

$$\frac{d\mathbf{L}(t)}{dt} < 0, \qquad \frac{d^2\mathbf{L}(t)}{dt^2} > 0,$$

Then $\mathbf{L}(t)$ decreases to zero as $t \to T_* < \infty$, i.e., $\mathbf{L}(T_*) = 0$. There are two cases as follows:

• $T_* \leq \Omega$. According to Theorem 2.1, the flow (1.1) exists a unique classical solution $\gamma(\cdot, t)$ on $[0, T_*)$. However, According to $\mathbf{L}(T_*) = 0$, we have

$$\lim_{t \to T_*} k(u, t) = \infty,$$

which means that the solution blows up at T_*. Hence, we deduce that $T_* = \Omega$. So, the solution $\gamma(\cdot, t)$ converges to a point as $t \to \Omega$.

• $T_* > \Omega$. In this case, $\mathbf{L}(\Omega) > 0$, the solution $\gamma(\cdot, \Omega)$ is a non-smooth curve. There are three cases:

(1) $\|\gamma(u, \Omega)\| = \sup |\gamma(u, \Omega)| = \infty$. But by Proposition 3.1, $\gamma(\cdot, \Omega)$ lies in the circle ℓ_1, we deduce that $\|\gamma(u, \Omega)\| < \infty$. This is impossible.

(2) $\|\gamma_u(u, \Omega)\| = \infty$. If so, the length $\mathbf{L}(\Omega)$ satisfies

$$\begin{aligned} \mathbf{L}(\Omega) &= \lim_{t \to \Omega} \int_{\gamma(u,t)} ds \\ &= \lim_{t \to \Omega} \int_{\gamma} |\gamma_u| du \\ &= \int_{\gamma} \lim_{t \to \Omega} |\gamma_u(u, t)| du \\ &= \infty \end{aligned}$$

which is a contradiction, because $\mathbf{L}(\Omega) < \mathcal{L}_0$. Therefore, this is not true.

(3) The curvature of the limit curve $\gamma(\cdot, \Omega)$ is discontinuous. Because the above two possibilities are impossible, then case (3) is true. The proof is completed.

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