Blowup of smooth solutions to the isentropic compressible quantum hydrodynamic model

Jie Zhang[1] *, Shu Wang[1,2] †, Fan Geng[1] ‡

[1]School of Mathematics and Information Sciences, Guangzhou University, Guangzhou 510006, P.R. China

[2]School of Mathematics, Beijing University of Technology, Beijing 100124, P.R. China

Abstract. In this article, we consider the blowup phenomenon of smooth solutions to the isentropic compressible quantum hydrodynamic model(QHD) with the initial density of compact support in arbitrary space dimensions. This result is an evolution of Xin's work [1].

Keywords: compressible quantum hydrodynamic model; smooth solutions; blowup

Mathematics Subject Classification (2010). 35A01, 35B44, 35Q40

1 Introduction

In this article, we consider the blowup phenomenon of smooth solutions to the isentropic compressible quantum hydrodynamic model(QHD):

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \frac{\varepsilon^2}{2}\rho\nabla(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}), \tag{1.2}$$

with the particle density $\rho(x,t)$, the velocity $u(x,t)$, the pressure $P(\rho)$, the scaled Planck constant ε and the quantum Bohm potential $\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$.

The initial condition of compressible quantum hydrodynamic model

$$(\rho, u)(x, t=0) = (\rho_0(x), u_0(x)) \in H^m(\mathbb{R}^d), \tag{1.3}$$

where $d \geq 1$, $m > [\frac{d}{2}] + 2$, and the pressure can be expressed as ($a > 0$ and $\gamma > 1$ are constants)

$$P(\rho) = a\rho^\gamma. \tag{1.4}$$

*E-mail:zhangjiee@e.gzhu.edu.cn

†Corresponding author. E-mail: wangshu@bjut.edu.cn

‡E-mail: gengfan@e.gzhu.edu.cn

There have been numerous works on the blowup of solutions to the compressible fluid. For the three dimensional compressible Euler equation, Sideris[2] in 1985 firstly presented that the life span of the C^1 solution is bounded if the initial velocity is large enough in some region with compact support. However, for the compressible Navier-Stokes equations without heat conduction, Xin[1] firstly proved the finite time blow up of smooth solution with the initial density has compact support. Then, Cho and Jin[3] obtained the blow up of strong solution of viscous heat-conducting flow when the initial density is compactly supported, which is an extension of Xin's results[1]. Later, in 2008, Rozanova[4] obtained that any smooth solutions to the compressible Navier-Stokes under rapidly decay assumptions will still blow up in finite time. And Du[5] proved that the blow up of smooth solutions to one or two dimensional the compressible isothermal case with the symmetric assumptions. However, Gamba[6] obtained that the smooth solution to compressible quantum hydrodynamic model will blow up in finite time for some boundary condition in a bounded domain. Recently, Guo[7] proved the blow up of smooth solution to the initial value problem of the compressible quantum hydrodynamic model in $\mathbb{R}^d$. Motivated by [4], we get the Theorem 1.1 and we extend Xin's results[1] to the Theorem 1.2.

We introduce the following several physical quantities:

$$m(t) = \int_{\mathbb{R}^d} \rho(x,t)dx, \tag{1.5}$$

$$M(t) = \int_{\mathbb{R}^d} \rho(x,t)|x|^2 dx, \tag{1.6}$$

$$F(t) = \int_{\mathbb{R}^d} \rho(x,t)u(x,t)\cdot x dx, \tag{1.7}$$

$$E(t) = \int_{\mathbb{R}^d} \frac{1}{2}\rho|u|^2 + \frac{1}{\gamma-1}P(\rho) + \frac{\varepsilon^2}{2}|\nabla\sqrt{\rho}|^2 dx = \sum_{i=1}^{3} E_i(t), \tag{1.8}$$

which represent the total mass, momentum of inertia, momentum weight and total energy, respectively. We assume that all the above initial data are finite and assume the initial density ρ_0 has compact support, so there exists a positive constant R_0 for

$$supp\rho_0(x) \subset B_{R_0}, \tag{1.9}$$

where B_{R_0} denotes the ball in $\mathbb{R}^d$ centered at origin with radius R_0. In the space $C^1([0,T], H^m(\mathbb{R}^d))$, we still investigate the solutions to QHD for $T > 0$. And now we know the density $\rho(x,t)$ has compact support from the equation (1.1), and then

$$R(t) = inf\{r | supp\rho(x,t) \subset B_r\} \tag{1.10}$$

is well-defined and finite for $t \in [0,T]$. Our main result in the paper that presents a sufficient condition on the blowup of smooth solutions to the isentropic compressible QHD in arbitrary space dimensions with initial density of compact support. Now, we state main results as follows:

Theorem 1.1 *Assume $(\rho,u)(x,t) \in C^1([0,T], H^m(\mathbb{R}^d))(m > [\frac{d}{2}]+2)$ is a solution to the compressible QHD with (1.3) satisfying (1.9). Furthermore, assume that there exist two constants β, $0 \le \beta < 1$, $C_1 > 0$ independent of T such that*

$$R(t) \le C_1(1+t)^\beta, \forall t \in [0,T]. \tag{1.11}$$

Then, the life span of the solution (ρ,u) is finite.

Theorem 1.2 *Let $T > 0$ and $(\rho, u)(x,t) \in C^1([0,T], H^m(\mathbb{R}^d))$ be a solution to the compressible QHD with initial data (1.3) satisfying (1.9). Assume that $R(t)$ satisfies the condition (1.11). Then*

$$T \le T_1(\gamma), \tag{1.12}$$

where

$$T_1(\gamma) = \begin{cases} \left(\dfrac{M(0) - 2F(0) + 2E(0)}{C_2}\right)^{\dfrac{1}{(\gamma-1)(1-\beta)d}}, & 1 < \gamma < 1 + \dfrac{2}{d}\ , \\ \left(\dfrac{M(0)}{C_2}\right)^{\dfrac{1}{2-(\gamma-1)\beta d}}, & 1 + \dfrac{2}{d} \le \gamma < 1 + \dfrac{2}{\beta d}\ , \end{cases} \tag{1.13}$$

with

$$C_2 = \frac{2}{\gamma - 1} a C_1^{(1-\gamma)d} V_{B_1}^{1-\gamma} m_0^{\gamma}, \qquad V_{B_1} = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}, \tag{1.14}$$

where $E(0)$ denotes the initial total energy. Especially, any smooth solution to the compressible QHD problem satisfies (1.9), will be blowup as long as the (1.11) hold.

2 Proof of Theorem 1.1

The main prove of this subsection contains Lemma 2.1 and lemma 2.2. From the lemma 2.1, we can know that the law of conservation of mass and the law of conservation of energy for the isentropic compressible quantum hydrodynamic model. From the lemma 2.2, we can get the estimate that the bound of the momentum of inertia $M(t)$.

Lemma 2.1 Under the assumptions of Theorem 1.1, it holds that

$$m(t) = m(0), E(t) = E(0). \tag{2.1}$$

Proof. From the continuity equation (1.1), we have

$$\frac{d}{dt} m(t) = \int_{\mathbb{R}^d} \rho_t dx = -\int_{\mathbb{R}^d} div(\rho u) dx = 0, \tag{2.2}$$

which implies $m(t) = m(0)$.

Then, multiplying the equation (1.2) by u and integrating it with respect to x in $\mathbb{R}^d$ lead to

$$\int_{\mathbb{R}^d} (\rho u)_t \cdot u dx + \int_{\mathbb{R}^d} div(\rho u \otimes u) \cdot u dx + \int_{\mathbb{R}^d} \nabla P(\rho) \cdot u dx = \int_{\mathbb{R}^d} \frac{\varepsilon^2}{2} \rho \nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) \cdot u dx. \tag{2.3}$$

We estimate the terms one by one, and from (1.1) and (1.2), we get that

$$\int_{\mathbb{R}^d} (\rho u)_t \cdot u dx + \int_{\mathbb{R}^d} div(\rho u \otimes u) \cdot u dx = \frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} \rho |u|^2 dx. \tag{2.4}$$

Due to (1.1), the pressure $P = a\rho^\gamma$ satisfies

$$P_t + div(Pu) + (\gamma - 1) P divu = 0. \tag{2.5}$$

For the third term, and take (2.5) into consideration, we get that

$$\int_{\mathbb{R}^d} \nabla P(\rho)\cdot u dx = -\int_{\mathbb{R}^d} P(\rho) divu dx = \frac{1}{\gamma-1}\int_{\mathbb{R}^d}[P(\rho)_t + div(Pu)]dx = \frac{1}{\gamma-1}\frac{d}{dt}\int_{\mathbb{R}^d} P(\rho)dx, \tag{2.6}$$

and analogously, it follows from (1.1) and integrating by parts, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{\varepsilon^2}{2}\rho\nabla(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}})\cdot u dx &= -\int_{\mathbb{R}^d}\frac{\varepsilon^2}{2} div(\rho u)\cdot\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}dx = \int_{\mathbb{R}^d}\frac{\varepsilon^2}{2}\rho_t\cdot\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}dx \\ &= \int_{\mathbb{R}^d}\varepsilon^2(\sqrt{\rho})_t\cdot\Delta\sqrt{\rho}dx = -\frac{d}{dt}\int_{\mathbb{R}^d}\frac{\varepsilon^2}{2}|\nabla\sqrt{\rho}|^2 dx.\end{aligned} \tag{2.7}$$

Combining (2.3)-(2.7), we get

$$\frac{d}{dt}\int_{\mathbb{R}^d}\frac{1}{2}\rho|u|^2 + \frac{1}{\gamma-1}P(\rho) + \frac{\varepsilon^2}{2}|\nabla\sqrt{\rho}|^2 dx = 0. \tag{2.8}$$

So by considering (1.8), which complete the proof that

$$E(t) = E(0).$$

Lemma 2.2 Under the assumptions of Theorem 1.1, we have

$$M(t) \geq M(0) + 2F(0)t + min\{2, d(\gamma-1)\}E(0)t^2, \quad M(t) \leq (R(t))^2 m(0). \tag{2.9}$$

Proof. We multiply the momentum equation (1.2) by x and integrate by parts, we have

$$\begin{aligned}\frac{d}{dt}F(t) &= \frac{d}{dt}\int_{\mathbb{R}^d}\rho u\cdot x dx = \int_{\mathbb{R}^d}(\rho u)_t\cdot x dx \\ &= -\int_{\mathbb{R}^d} div(\rho u\otimes u)\cdot x dx - \int_{\mathbb{R}^d}\nabla P\cdot x dx + \int_{\mathbb{R}^d}\frac{\varepsilon^2}{2}\rho\nabla(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}})\cdot x dx \\ &= \sum_{i=1}^{3} I_i(t).\end{aligned} \tag{2.10}$$

Based on integration by parts, we easily deduce that

$$I_1(t) = -\int_{\mathbb{R}^d} div(\rho u\otimes u)\cdot x dx = \int_{\mathbb{R}^d}\rho|u|^2 dx,$$

and

$$I_2(t) = -\int_{\mathbb{R}^d}\nabla P\cdot x dx = \int_{\mathbb{R}^d} P divx dx = d\int_{\mathbb{R}^d} P dx,$$

and similarly, following from (1.1) and integrating by parts, we obtain

$$\begin{aligned}I_3(t) &= \int_{\mathbb{R}^d}\frac{\varepsilon^2}{2}\rho\nabla(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}})\cdot x dx = -\int_{\mathbb{R}^d}\frac{\varepsilon^2}{2} div(\rho x)\cdot(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}})dx \\ &= -\frac{\varepsilon^2}{2}\int_{\mathbb{R}^d}[d\cdot\sqrt{\rho}\Delta\sqrt{\rho} + x\cdot 2\nabla\sqrt{\rho}\Delta\sqrt{\rho}]dx \\ &= \frac{\varepsilon^2}{2}\int_{\mathbb{R}^d}[d|\nabla\sqrt{\rho}|^2 + 2|\nabla\sqrt{\rho}|^2 + x\cdot\nabla|\nabla\sqrt{\rho}|^2]dx \\ &= \frac{\varepsilon^2}{2}\int_{\mathbb{R}^d}[d|\nabla\sqrt{\rho}|^2 + 2|\nabla\sqrt{\rho}|^2 - d|\nabla\sqrt{\rho}|^2]dx \\ &= \varepsilon^2\int_{\mathbb{R}^d}|\nabla\sqrt{\rho}|^2 dx.\end{aligned}$$

Gathering the identities of $I_i(i=1,2,3)$ into (2.10), we obtain

$$\frac{d}{dt}F(t)=\int_{\mathbb{R}^d}\rho|u|^2+dP(\rho)+\varepsilon^2|\nabla\sqrt{\rho}|^2dx. \tag{2.11}$$

Thus, integrating (2.11) with respect to t, we obtain

$$\begin{aligned}F(t)&=F(0)+\int_0^t\int_{\mathbb{R}^d}\rho|u|^2+dP(\rho)+\varepsilon^2|\nabla\sqrt{\rho}|^2dxdt\\&=F(0)+\int_0^t 2E_1+d(\gamma-1)E_2+2E_3dt,\end{aligned}$$

which implies

$$F(t)\geqslant F(0)+min\{2,d(\gamma-1)\}E(t)t. \tag{2.12}$$

By virtue of the continuity equation (1.1) and integrating by parts, we get

$$\frac{d}{dt}M(t)=\frac{d}{dt}\int_{\mathbb{R}^d}\rho|x|^2dx=2F(t), \tag{2.13}$$

next, integrating (2.13) with respect to t, we deduce

$$M(t)=M(0)+\int_0^t 2F(s)ds. \tag{2.14}$$

From (2.12), (2.1) and (2.14), we get that

$$M(t)\geq M(0)+2F(0)t+min\{2,d(\gamma-1)\}E(0)t^2. \tag{2.15}$$

However, because of (2.2), we know that

$$M(t)=\int_{\mathbb{R}^d}\rho|x|^2dx=\int_{B_{R(t)}}\rho|x|^2dx\leq(R(t))^2\int_{B_{R(t)}}\rho dx=(R(t))^2m(t)=(R(t))^2m(0). \tag{2.16}$$

Then, we complete the prove of the lemma 2.1.

Proof of Theorem 1.1. Finally, by (2.15) and (2.16), we get that

$$(R(t))^2m(0)\geq M(0)+2F(0)t+min\{2,d(\gamma-1)\}E(0)t^2. \tag{2.17}$$

Taking (1.11) into consideration, we get that

$$C_1^2m(0)(1+t)^{2\beta}\geq M(0)+2F(0)t+min\{2,d(\gamma-1)\}E(0)t^2. \tag{2.18}$$

Since $\beta<1$,we can deduce from (2.18) that the lifespan of the smooth solution of QHD is finite and we prove the Theorem 1.1.

3 Proof of Theorem 1.2

The solution $(\rho, u) \in C^1([0,T]; H^m(\mathbb{R}^d))$ to the Cauchy problem of compressible QHD through (1.9), as prescribed in Theorem 1.2. The Lemma 3.1 is a key estimate about the total pressure to solve the Theorem 1.2.

Lemma 3.1 Let the pressure $P(x,t)$ associate with the solution $(\rho, u) \in C^1([0,T]; H^m(\mathbb{R}^d))$. Then the following estimates hold:

$$\int_{\mathbb{R}^d} P(x,t)dx \leq \begin{cases} \dfrac{\gamma-1}{2}(1+t)^{-(\gamma-1)d}(M(0) - 2F(0) + 2E(0)), & 1 < \gamma < 1 + \dfrac{2}{d}, \\ \dfrac{\gamma-1}{2}t^{-2}M(0), & 1 + \dfrac{2}{d} \leq \gamma < \infty. \end{cases} \tag{3.1}$$

Remark. The lemma 3.1 holds for the general solution $(\rho, u) \in C^1([0,T]; H^m(\mathbb{R}^d))$ to the QHD and (1.3), without the additional conditions (1.9) and (1.11) as long as $M(0)$ is well-defined.

We study the following function, which can be used to prove lemma 3.1.

$$I_\gamma(t) = \begin{cases} \displaystyle\int_{\mathbb{R}^d} |x - u(t+1)|^2 \rho dx + (t+1)^2 \int_{\mathbb{R}^d} \frac{2}{\gamma-1} P(\rho) + \varepsilon^2 |\nabla \sqrt{\rho}|^2 dx, & 1 < \gamma < 1 + \dfrac{2}{d}, \\ \displaystyle\int_{\mathbb{R}^d} |x - ut|^2 \rho dx + t^2 \int_{\mathbb{R}^d} \frac{2}{\gamma-1} P(\rho) + \varepsilon^2 |\nabla \sqrt{\rho}|^2 dx, & 1 + \dfrac{2}{d} \leq \gamma < \infty. \end{cases} \tag{3.2}$$

When $1 < \gamma < 1 + \frac{2}{d}$,

$$\begin{aligned} I_\gamma(t) &= \int_{\mathbb{R}^d} |x - u(t+1)|^2 \rho dx + (t+1)^2 \int_{\mathbb{R}^d} \frac{2}{\gamma-1} P(\rho) + \varepsilon^2 |\nabla \sqrt{\rho}|^2 dx \\ &= \int_{\mathbb{R}^d} x^2 \rho dx - 2(1+t) \int_{\mathbb{R}^d} x \rho u dx + (1+t)^2 \int_{\mathbb{R}^d} (\rho u^2 + P(\rho) + \varepsilon^2 |\nabla \sqrt{\rho}|^2) dx. \end{aligned} \tag{3.3}$$

So, it follows directly from (3.3) that

$$\begin{aligned} \frac{d}{dt} I_\gamma(t) &= \int_{\mathbb{R}^d} (x^2 \rho_t - 2x\rho u) dx - 2(1+t) \int_{\mathbb{R}^d} [x(\rho u)_t - \rho u^2 - P(\rho) - \varepsilon^2 |\nabla \sqrt{\rho}|^2] dx \\ &\quad + (1+t)^2 \int_{\mathbb{R}^d} (\rho u^2 + P(\rho) + \varepsilon^2 |\nabla \sqrt{\rho}|^2)_t dx \\ &= \sum_{i=1}^{3} J_i(t). \end{aligned} \tag{3.4}$$

Next, we will calculate the right-hand side of (3.4) one by one.
For the first term, taking (1.1) into consideration and integrating by parts, we deduce

$$J_1(t) = \int_{\mathbb{R}^d} (x^2 \rho_t - 2x\rho u) dx = 0.$$

As for the second term, we use (1.2), (2.12), (2.13) and (2.14).

$$
\begin{aligned}
J_2(t) =& -2(1+t)\int_{\mathbb{R}^d}[x(\rho u)_t - \rho u^2 - P(\rho) - \varepsilon^2|\nabla\sqrt{\rho}|^2]dx \\
=& 2(1+t)\int_{\mathbb{R}^d}[\mathrm{div}(\rho u\otimes u) + \nabla P(\rho) - \frac{\varepsilon^2}{2}\rho\nabla(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}})]\cdot xdx \\
&+ 2(1+t)\int_{\mathbb{R}^d}\rho|u|^2 + \frac{2}{\gamma-1}P(\rho) + \varepsilon^2|\nabla\sqrt{\rho}|^2dx \\
=& 2(1+t)\int_{\mathbb{R}^d} -\rho|u|^2 - dP(\rho) - \varepsilon^2|\nabla\sqrt{\rho}|^2dx \\
&+ 2(1+t)\int_{\mathbb{R}^d}\rho|u|^2 + \frac{2}{\gamma-1}P(\rho) + \varepsilon^2|\nabla\sqrt{\rho}|^2dx \\
=& 2(1+t)(\frac{2}{\gamma-1} - d)\int_{\mathbb{R}^d}P(\rho)dx.
\end{aligned}
$$

The third term, we use (1.1) and integrate it by parts that

$$
\begin{aligned}
J_3(t) =& (1+t)^2\int_{\mathbb{R}^d}(\rho u^2 + \frac{2}{\gamma-1}P(\rho) + \varepsilon^2|\nabla\sqrt{\rho}|^2)_t dx \\
=& (1+t)^2\int_{\mathbb{R}^d}[\rho_t u^2 + 2\rho u\cdot u_t + \frac{2}{\gamma-1}P(\rho)_t + 2\varepsilon^2(\nabla\sqrt{\rho})\cdot(\nabla\sqrt{\rho})_t]dx \\
=& (1+t)^2\int_{\mathbb{R}^d}[-div(\rho u)u^2 - 2\rho u^2\cdot\nabla u - 2\nabla P(\rho)u \\
&+ \varepsilon^2\rho\nabla(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}})u + \frac{2}{\gamma-1}P(\rho)_t - 2\varepsilon^2(\Delta\sqrt{\rho})(\sqrt{\rho})_t]dx \\
=& (1+t)^2\int_{\mathbb{R}^d}[2P(\rho)divu - \varepsilon^2(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}})div(\rho u) + \frac{2}{\gamma-1}P(\rho)_t - 2\varepsilon^2(\Delta\sqrt{\rho})(\sqrt{\rho})_t]dx \\
=& (1+t)^2\int_{\mathbb{R}^d}[2P(\rho)divu + \frac{2}{\gamma-1}P(\rho)_t + \varepsilon^2\rho_t(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}) - 2\varepsilon^2(\Delta\sqrt{\rho})(\sqrt{\rho})_t]dx \\
=& (1+t)^2\int_{\mathbb{R}^d}[-\frac{2}{\gamma-1}div(P(\rho)u) + 2\varepsilon^2(\sqrt{\rho})_t\cdot\Delta\sqrt{\rho} - 2\varepsilon^2(\Delta\sqrt{\rho})(\sqrt{\rho})_t]dx \\
=& 0.
\end{aligned}
$$

Plugging the estimates of J_1-J_3 into (3.4), we arrive at

$$
\frac{d}{dt}I_\gamma(t) = \begin{cases} 2(1+t)\dfrac{2-d(\gamma-1)}{\gamma-1}\displaystyle\int_{\mathbb{R}^d}P(\rho)dx, & 1<\gamma<1+\dfrac{2}{d}, \\ 2t\dfrac{2-d(\gamma-1)}{\gamma-1}\displaystyle\int_{\mathbb{R}^d}P(\rho)dx, & 1+\dfrac{2}{d}\le\gamma<\infty. \end{cases}
$$

When $\gamma \ge 1 + \frac{2}{d}$, $2 - d(\gamma-1) \le 0$, thus $I_\gamma(t) \le I_\gamma(0)$ for $t \in [0,T]$, which yields (3.1) in this case immediately.

When $1 + \frac{2}{d} \le \gamma < \infty$, we know that

$$
\frac{d}{dt}I_\gamma(t) < \frac{2-d(\gamma-1)}{1+t}I_\gamma(t), \tag{3.5}
$$

which, together with the Gronwall's inequality, gives

$$I_\gamma(t) \le (1+t)^{2-d(\gamma-1)} I_\gamma(0),$$

which is easily obtained the the estimate (3.1) for $1+\frac{2}{d} \le \gamma < \infty$, and complete the proof of Lemma 3.1.

And when $1<\gamma<1+\frac{2}{d}$, from (1.11) and (3.1), we get that

$$\begin{aligned}
I_\gamma(0) &\ge \frac{2}{\gamma-1}(1+t)^{(\gamma-1)d}\int_{\mathbb{R}^d} P(\rho)dx \\
&\ge \frac{2}{\gamma-1}(1+t)^{(\gamma-1)d} V_{B_{R(t)}} \frac{1}{V_{B_{R(t)}}} a \int_{B_{R(t)}} \rho^\gamma dx \\
&\ge \frac{2}{\gamma-1}(1+t)^{(\gamma-1)d} a V_{B_{R(t)}}^{1-\gamma} m_0^\gamma \\
&\ge \frac{2}{\gamma-1} a C_1^{(1-\gamma)d} V_{B_1}^{1-\gamma} m_0^\gamma (1+t)^{(\gamma-1)d(1-\beta)} \\
&= C_2(1+t)^{(\gamma-1)d(1-\beta)},
\end{aligned}$$

where we have used the fact that

$$\int_{B_{R(t)}} \rho dx = \int_{B_{\mathbb{R}^d}} \rho(x,t)dx = \int_{B_{\mathbb{R}^d}} \rho_0(x)dx = m_0.$$

This gives the desired estimate (1.12) through (1.14) for $1 \le \gamma < 1+\frac{2}{d}$ immediately. The same analysis yields (1.12) through (1.14) for $1+\frac{2}{d} \le \gamma < 1+\frac{2}{\beta d}$. This completes the proof of Theorem 1.2.

4 Acknowledgements

This work was partially supported by the NSFC (No.12171111) of China.

References

[1] Xin ZP. Blow up of smooth solutions to the compressible Navier-Stokes equation with compact density. *Comm Pure Appl Math.* 1998; 51:229-240.

[2] Sideris TC. Formation of singularity in three dimensional compressible fluids. *Comm Math Phys.* 1985; 101:475-485.

[3] Cho Y, Jin BJ. Blow-up of viscous heat-conducting compressible flows. *J Math Anal Appl.* 2006; 320:819-826.

[4] Rozanova O. Blow-up of smooth highly decreasing at infinity solutions to the compressible Navier-Stokes equations. *J Differ Equations.* 2008; 245:1762-1774.

[5] Du DP, Li JY, Zhang KJ. Blow-up of smooth solutions to the Navier-Stokes equations for compressible isothermal fluids. *Commun Math Sci.* 2013; 11:541-546.

[6] Gamba IM, Gualdani MP, Zhang P. On the blowing up of solutions to quantum hydrodynamic models on bounded domains. *Monatsh Math.* 2009; 157:37-54.

[7] Guo BL, Wang GW. Blow-up of the smooth solution to quantum hydrodynamic models in $\mathbb{R}^d$. *J Differential Equations.* 2016; 261:3815-3842.