An inverse problem with the nonlocal Dirichlet boundary condition

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Abstract

In this study we consider the inverse problem of determining the nonlinear right-hand side of a quasi-linear parabolic equation and prove a uniqueness theorem. Stability estimate for the solution are obtained. A method of representing the nonlinear right-hand side explicitly is proposed for the special case.

1 Introduction

Inverse heat source problems have various important applications in engineering and science. A typical property of this kind of problems is that well-posedness conditions are not always guaranteed such as existence,uniqueness and stability of their solutions. Particularly, the determination of source terms in the quasi linear parabolic problem has been extensively explored. For instance, Bushuyev [1] has shown the uniqueness result for the unknown time-dependent right-hand side with explicitly bounded growth rate determined by one additional final measurement. Choulli [2] has considered the determination of a function p from overspecified data, where the function p appears in an initial-boundary value problem for the equation $u_t - \Delta u - pu + f(u) = 0$. Dehghan [3] has presented several finite-difference schemes concerning diffusion equation with source control parameter. Lorenzi [5] has studied the stability of an unknown non-linear term in a parabolic equation in dependence on over specified Cauchy-Dirichlet data prescribed on the parabolic boundary of the open set under consideration. A uniqueness theorem has been obtained in the semilinear parabolic equation by Isakov [4]. In this paper,we are interested in reconstruction the right hand side in a quasi linear parabolic problem. We prove the uniqueness and stability theorem for the inverse source problem and derive some examples to construct the source term for the special case of quasi linear equation.

2 Main results

We consider the followingsemilinear parabolic for the heat equation:

$$\begin{cases} u_t = u_x + F(x,t,u_x), \quad (x,t) \in \Omega_T := (0,L) \times (0,T], \\ u\left(x,0\right) = u_0\left(x\right), \;\; x \in (0,L), \\ u\left(0,t\right) = g_0\left(t\right), u\left(L,t\right) = g_1\left(t\right), \;\; t \in (0,T], \end{cases} \tag{1}$$

where u_0, g_0, g_1 are given functions, and u represents the temperature,. $F(x,t,u_x)$ is the nonlinear heat source term. Solving the equation (1) to find the unknown function u under the given parameters u_0, g_0, g_1, F is called the direct problem. The uniqueness of the semilinear parabolic problem with a nonlocal Dirichlet boundary condition has been proved in [7]. However, sometimes the heat source term are unknown which means the problem is mathematically underdetermined and additional data must be supplied to fully determine the physical process. In this paper, the overspecified condition is given by the following measurement data:

$$\int_0^L w(x)\,u(x,t)\,dx = h(t). \tag{2}$$

Denote the solution of (1) by $u(x,t,F)$ for a given $F(x,t,u_x)$. Then from the additional condition (2), it follows that the inverse source problem is to solve the following nonlinear integral equation

$$\int_0^L w(x)\,u(x,t;F)\,dx = h(t).$$

Like most inverse problems, the inverse heat source problem is also ill-posed, i.e. the existence, uniqueness and stability of its solution are not always guaranteed. To begin with we prove the uniqueness theorem for the inverse source problem we assume that $F(x,t,u_x)$ is Lipschitz continuous function with respect to u_x :

$$|F(x,t,u_x) - F(x,t,v_x)| \leq M\,|u_x(x,t) - v_x(x,t)|. \tag{3}$$

The variational formulation of the direct problem (1) for a given $F(x,t,u_x)$ as follows:

$$\begin{aligned} \int_0^L u_t\varphi dx + \int_0^L u_x\varphi_x dx &= \int_0^L F(x,t,u_x)\varphi dx, \forall\varphi \in H_0^1(0,L) \\ u(0,t) &= g_0(t), u(L,t) = g_1(t),\ t\in(0,T], \end{aligned} \tag{4}$$

where $u \in C\left([0,T], L^2(0,L)\right) \cap L^\infty\left((0,T), H^1(0,L)\right),\ u_t \in L^2\left((0,T), L^2(0,L)\right)$ with the compatibility conditions $u_0(0) = g_0(0), u_0(L) = g_1(0)$. Next, we give a preparation for the uniqueness and stability result for the inverse problem.

Assume that $\{u_1(x,t), F(x,t,u_{1,x})\}$ and $\{u_2(x,t), F(x,t,u_{2,x})\}$ are two solutions of the inverse problem $(1)-(2)$. For the differences $z = u_1 - u_2$, $\Delta F = F(x,t,u_{1,x}) - F(x,t,u_{2,x})$,and $\Delta h(t) = h_1(t) - h_2(t)$, the direct problem (1) implies that

$$\begin{cases} z_t = z_{xx} + \Delta F, & (x,t)\in(0,L)\times(0,T], \\ z(x,0) = 0,\ x\in(0,L), \\ z(0,t) = z(L,t) = 0,\ t\in(0,T], \end{cases} \tag{5}$$

Lemma 1 *Let $z \in C\left([0,T], L^2(0,L)\right) \cap L^\infty\left((0,T), H_0^1(0,L)\right)$ be a solution of the problem (5) then the following relation is satisfied:*

$$\int_0^L z_x^2 dx \leq \int_0^t \Delta h^2 ds + \frac{M+\sqrt{L}}{\varepsilon}\int_0^t \left(\int_0^s \Delta h^2 du\right) e^{\frac{(M+\sqrt{L})(t-s)}{\varepsilon}} ds, \tag{6}$$

where $\varepsilon < \frac{2}{\sqrt{L}+1}$ is a positive constant.

Proof. Choosing $\varphi := z_t - \Delta h$ in (4) we have

$$\int_0^L z_t\left(z_t - \Delta h\right)dx + \int_0^L z_x z_{tx} dx = \int_0^L \Delta F\left(z_t - \Delta h\right)dx,$$

and integrating for any $t\in[0,T]$,

$$\int_0^t\int_0^L z_t^2 dxdt + \frac{1}{2}\int_0^t\int_0^L \frac{d}{dt}\left(z_x^2\right)dxdt = \int_0^t\int_0^L \Delta F z_t dxdt + \int_0^t\int_0^L z_t\Delta h dxdt - \int_0^t\int_0^L \Delta F\Delta h dxdt,$$

using the initial condition of (5) we obtain

$$\int_0^t\int_0^L z_t^2 dxdt + \frac{1}{2}\int_0^L z_x^2 dx = \int_0^t\int_0^L \Delta F z_t dxdt + \int_0^t \Delta h\int_0^L z_t dxdt - \int_0^t \Delta h\int_0^L \Delta F dxdt$$

Now we apply the Cauchy -ε ineqaulity for the first term of right hand side and the Hölder inequality for the other two terms of the right hand side we have

$$\int_0^t\int_0^L z_t^2dxdt+\frac{1}{2}\int_0^L z_x^2dx\leq\frac{1}{2\varepsilon}\int_0^t\int_0^L \Delta F^2dxdt+\frac{\varepsilon}{2}\int_0^t\int_0^L z_t^2dxdt$$
$$+\left(\int_0^t\Delta h^2dt\right)^{1/2}\left(\int_0^t\left(\int_0^L z_tdx\right)^2dt\right)^{1/2}+\left(\int_0^t\Delta h^2dt\right)^{1/2}\left(\int_0^t\left(\int_0^L \Delta Fdx\right)^2dt\right)^{1/2}\tag{7}$$

Note that

$$\left(\int_0^L z_tdx\right)^2\leq\left(\int_0^L dx\right)\left(\int_0^L z_t^2dx\right)=L\left(\int_0^L z_t^2dx\right)$$
$$\left(\int_0^L \Delta Fdx\right)^2\leq\left(\int_0^L dx\right)\left(\int_0^L \Delta F^2dx\right)=L\left(\int_0^L \Delta F^2dx\right)$$

and from the Lipschitz property of F

$$\Delta F^2\leq M^2z_x^2$$

using these facts in (7) we obtain

$$\int_0^t\int_0^L z_t^2dxdt+\frac{1}{2}\int_0^L z_x^2dx\leq\frac{M}{2\varepsilon}\int_0^t\int_0^L z_x^2dxdt+\frac{\varepsilon}{2}\int_0^t\int_0^L z_t^2dxdt$$
$$+\sqrt{L}\left(\int_0^t\Delta h^2dt\right)^{1/2}\left(\int_0^t\int_0^L z_t^2dxdt\right)^{1/2}+\sqrt{L}\left(\int_0^t\Delta h^2dt\right)^{1/2}\left(\int_0^t\int_0^L z_x^2dxdt\right)^{1/2}$$

Applying the Cauchy-ε ineqaulity for the last two terms of the right hand side of the above inequality with $\varepsilon<\frac{2}{\sqrt{L}+1}$ we have

$$\frac{1}{2}\int_0^L z_x^2dx\leq\frac{M+\sqrt{L}}{2\varepsilon}\int_0^t\int_0^L z_x^2dxdt+\frac{\sqrt{L}}{2}\left(\varepsilon+\varepsilon^{-1}\right)\int_0^t\Delta h^2dt.\tag{8}$$

Now using the Gronwall inequality

$$\int_0^L z_x^2dx\leq\int_0^t\Delta h^2ds+\frac{M+\sqrt{L}}{\varepsilon}\int_0^t\left(\int_0^s\Delta h^2du\right)\exp\left(\frac{\left(M+\sqrt{L}\right)(t-s)}{\varepsilon}\right)ds$$

which is the required result. ■

Theorem 2 *Let* $z\in C\left([0,T],L^2(0,L)\right)\cap L^\infty\left((0,T),H_0^1(0,L)\right)$ *be a solution of the problem* (5) *and* $F(x,t,u_x)$ *satisfy* (3). *It follows that if* $h_1(t)=h_2(t),\forall t\in[0,T]$ *then* $F(x,t,u_{1,x})=F(x,t,u_{2,x})$ *and* $u_1(x,t)=u_2(x,t)$.

Proof. Since $\Delta h=0$, from the inequality (6) we have

$$\int_0^L z_x^2dx=0$$

which implies $z(x,t)=\zeta(t)$. Applying the boundary conditions of (5) we conclude $\zeta(t)=0$ providing $u_1=u_2$ and $u_{1,x}=u_{2,x}$. From assumtion (3) we have $F(x,t,u_{1,x})=F(x,t,u_{2,x})$. ■

Next theorem describes the stability of the inverse problem.

Theorem 3 *Let* $z \in C\left([0,T], L^2(0,L)\right) \cap L^\infty\left((0,T), H_0^1(0,L)\right)$ *be a solution of the problem* (5) *and* $F(x,t,u_x)$ *satisfy* (3). *Then the following stability estimate holds*

$$\int_0^L \left(F(x,t,u_{1,x}) - F(x,t,u_{2,x})\right)^2 dx \leq \frac{M^2L^2}{2}\int_0^t \Delta h^2 ds$$
$$+\frac{M^2L^2\left(M+\sqrt{L}\right)}{2\varepsilon}\int_0^t \left(\int_0^s \Delta h^2 du\right)\exp\left(\frac{\left(M+\sqrt{L}\right)(t-s)}{\varepsilon}\right) ds$$

Proof. Using (3) we have

$$\int_0^L \left(F(x,t,u_{1,x}) - F(x,t,u_{2,x})\right)^2 dx \leq M^2 \int_0^L z^2(x,t)\, dx.$$

Applying Poincare inequality we obtain

$$\int_0^L \left(F(x,t,u_{1,x}) - F(x,t,u_{2,x})\right)^2 dx \leq \frac{M^2L^2}{2}\int_0^L z_x^2(x,t)\, dx$$

and from the inequality (6) we have the required result. ■

3 Structure of the source function

Our interest in the present section is studying reconstruction of the source function, explicitly. For simplicity, we set $F(x,t,u_x) := f(t)\, u_x^2(x,t)$. The analytical solution of (1) is given by,[6]:

$$u(x,t) = \varphi(t)\, x^2 + \psi(t)\, x + \kappa(t) \tag{9}$$

where $\varphi(t), \psi(t)$ and $\kappa(t)$ determined by solving the following system of first-order ordinary differential equations with variable coefficients:

$$\begin{cases} \varphi'(t) = 4f(t)\,\varphi^2(t), \\ \psi'(t) = 4f(t)\,\varphi(t)\,\psi(t), \\ \kappa'(t) = 2\varphi(t) + f(t)\,\psi^2(t). \end{cases} \tag{10}$$

Using the boundary condition of (1) we have

$$g_0(t) = \kappa(t), \tag{11}$$
$$\psi(t) = \frac{g_1(t) - g_0(t)}{L} - L\varphi(t). \tag{12}$$

Next employing the measurement data (2) and parabolic equation (1) we obtain

$$\int_0^L w(x)\left[u_{xx} + f(t)\, u_x^2\right] dx = h'(t),$$

which implies

$$f(t) = \frac{h'(t) - \int_0^L w(x)\, u_{xx}(x,t)\, dx}{\int_0^L w(x)\, u_x^2(x,t)\, dx}.$$

From (9), putting $u_x = 2x\varphi(t) + \psi(t)$, $u_{xx} = 2\varphi(t)$ in the above equality we have

$$f(t) = \frac{h'(t) - 2\varphi(t)\int_0^L w(x)\, dx}{4\varphi^2(t)\int_0^L x^2 w(x)\, dx + 4\varphi(t)\,\psi(t)\int_0^L x w(x)\, dx + \psi^2(t)\int_0^L w(x)\, dx}.$$

Substituting $\psi(t)$ from (12) into the above equality we obtain the explicit formulae to construct the function $f(t)$ in the semilinear parabolic equation:

$$f(t)=\frac{h'(t)-2A_3\varphi(t)}{4A_1\varphi^2(t)+4A_2\varphi(t)\left[\frac{g_1(t)-g_0(t)}{L}-L\varphi(t)\right]+A_3\left[\frac{g_1(t)-g_0(t)}{L}-L\varphi(t)\right]^2}, \tag{13}$$

where

$$A_1=\int_0^L x^2w(x)\,dx,\ A_2=\int_0^L xw(x)\,dx,\ A_3=\int_0^L w(x)\,dx. \tag{14}$$

Now let us find the function $\varphi(t)$. Using the measurement data (2) and the solution (9) we have

$$h(t)=\varphi(t)\int_0^L w(x)\,x^2dx+\psi(t)\int_0^L w(x)\,xdx+\kappa(t)\int_0^L w(x)\,dx,$$

considering (11) and (12) the function $\varphi(t)$ is obtained as follows:

$$\varphi(t)=\frac{h(t)-\left[\frac{g_1(t)-g_0(t)}{L}\right]A_2-g_0(t)A_3}{A_1-LA_2}. \tag{15}$$

Substituting this $\varphi(t)$ value in (15) into the (13) we have the explicit formuale for $f(t)$.

Example 4 *As the first example, let us give the following data for the inverse source problem* $(1)-(2)$:

$$h(t)=\frac{L^4}{4(1+t)}+(1+t)\exp(-t)\frac{L^3}{3}+L^2\ln(1+t)+\frac{L^2e^{-2t}}{64}\left(4t^3+14t^2+18t+9\right)$$
$$u(x,0)=x^2+x+\frac{9}{32}$$
$$g_0(t)=2\ln(t+1)+\frac{1}{32}e^{-2t}\left(4t^3+14t^2+18t+9\right)$$
$$g_1(t)=2\ln(t+1)+\frac{1}{32}e^{-2t}\left(4t^3+14t^2+18t+9\right)+\frac{L^2}{t+1}+Le^{-t}(t+1)$$

Choosing $w(x)=x$, *from* (14) *and* (15) *it can be easily seen that*

$$\varphi(t)=\frac{1}{1+t}$$

and from (13) $f(t)$ *is obtained as:*

$$f(t)=-\frac{t}{4}.$$

Example 5 *For the second example let us consider the following data for the inverse source problem* $(1)-(2)$:

$$h(t)=\frac{32\pi^2L+8\pi^2L^2+\pi^2L^3-4L^3+4\pi^2L\sin t}{4\pi^3(1+\cos t)}$$
$$u(x,0)=\frac{1}{8}x^2+x+2$$
$$g_0(t)=\frac{\sin t+8}{2(\cos t+1)}$$
$$g_1(t)=\frac{L^2+8L+2\sin t+16}{4(\cos t+1)}$$

Choosing $w(x) = \sin\frac{\pi x}{L}$*, from* (14) *and* (15) *computations show that*

$$\varphi(t) = \frac{1}{4(1+\cos t)}$$

and from (13) $f(t)$ *is obtained as:*

$$f(t) = \sin t.$$

Example 6 *For the last example we have the following data for the inverse source problem* $(1)-(2)$:

$$\begin{aligned} h(t) &= \frac{L^3 e^t}{12} + \frac{L^2 e^t}{2} + \frac{3Le^t}{2} \\ u(x,0) &= \frac{1}{4}x^2 + x + \frac{3}{2} \\ g_0(t) &= \frac{3}{2}e^t \\ g_1(t) &= \frac{1}{4}e^t\left(L^2 + 4L + 6\right) \end{aligned}$$

Choosing $w(x) = 1$*, from* (14) *and* (15) *we easily obtain that*

$$\varphi(t) = \frac{e^t}{4}$$

and from (13) $f(t)$ *is obtained as:*

$$f(t) = e^{-t}.$$

4 The finite difference schemes

In order to solve problem (1) numerically, we need the linearization of the nonlinear terms:

$$\begin{cases} u_t^{(k)} = u_{xx}^{(k)} + F(x,t,u_x^{(k-1)}), \quad (x,t) \in \Omega_T := (0,L)\times(0,T], \\ u^{(k)}(x,0) = u_0(x), \ x \in (0,L), \\ u^{(k)}(0,t) = g_0(t), u^{(k)}(L,t) = g_1(t), \ t \in (0,T], \end{cases} \tag{16}$$

For each iteration k, employing the transformation $u^{(k)}(x,t) = v(x,t), F(x,t,u_x^{(k-1)}) = \overline{F}(x,t)$ we obtain the following linear problem:

$$\begin{cases} v_t = v_{xx} + \overline{F}(x,t), \quad (x,t) \in \Omega_T := (0,L)\times(0,T], \\ v(x,0) = u_0(x), \ x \in (0,L), \\ v(0,t) = g_0(t), v(L,t) = g_1(t), \ t \in (0,T], \end{cases} \tag{17}$$

The domain $(0,L)\times[0,T]$ is divided into an $M \times N$ mesh with the spatial step size $h = 1/M$ in x direction and the time step size $k = T/N$, respectively. Grid points (x_i, t_n) are defined

$$\begin{aligned} x_i &= ih, \ i = 0,1,2,...,M \\ t_n &= nk, \ n = 0,1,2,...,N \end{aligned}$$

in which M and N are integers. The notations $v_i^n, \overline{F}_i^n$ are used for the finite difference approximations of $v(ih,nk), \overline{F}(ih,nk)$ .Using the initial condition we get

$$v_i^0 = u_0(x_i), \ i = 0,1,2,...,M$$

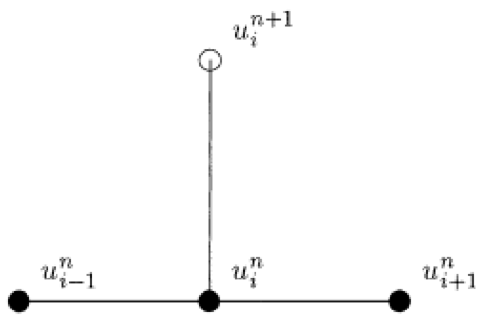
and boundary conditions

$$v_0^{n+1} = g_0(t_{n+1}), v_M^{n+1} = g_1(t_{n+1}), \ n = 0,1,2,...,N$$

A direct simulation to the derivation of the one-dimensional classical forward time centred space (FTCS) finite difference scheme leads to the following difference equation for (17)

$$v_i^{n+1} = v_i^n + s\left(v_{i-1}^n - 2v_i^n + v_{i-1}^n\right) + k\overline{F}_i^n$$

for $1 \leq i \leq M-1, 0 \leq n \leq N-1$ where $s = k/h^2$. The range of stability for this procedure is $0 < s \leq 1/2$.



The modified equivalent partial differential equation of this method shows that the Equation (17) has a truncation error which is $O\left(h^2\right)$, except for $s = 1/6$, where it is $O\left(h^4\right)$. So there is an optimal case, $s = 1/6$ when the formula (17) is fourth-order accurate.

Now, we show some numerical experiments for the reconstruction $F(x, t, u_x)$ in problem (1).

Example 7 *In the first numerical experiment we take the exact solution*

$$\begin{aligned}
u(x,t) &= \left(t^3+1\right)\cos x, (x.t) \in \left(0, \frac{3\pi}{2}\right) \times (0,2) \\
F(x,t,u_x) &= 3t^2\cos x + (\cos x)\left(t^3+1\right) + \underbrace{(\sin x)\left(t^3+1\right)}_{-u_x} \\
u_0(x) &= \cos x \\
g_0(t) &= t^3+1, g_1(t) = 0 \\
w(x) &= 1 \\
& \int_0^{\frac{3\pi}{2}} \left(t^3+1\right)\cos x dx
\end{aligned}$$

Observe that the measurement data :

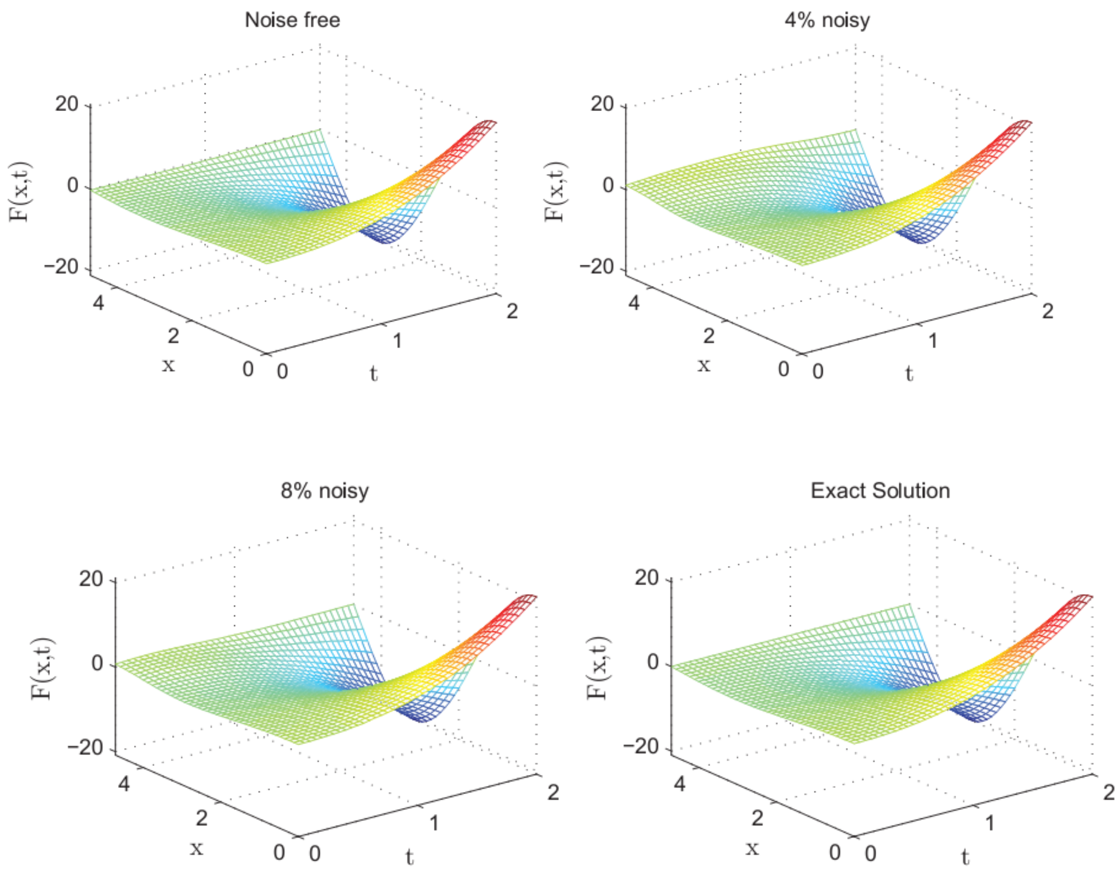
$$h(t) = -t^3 - 1$$

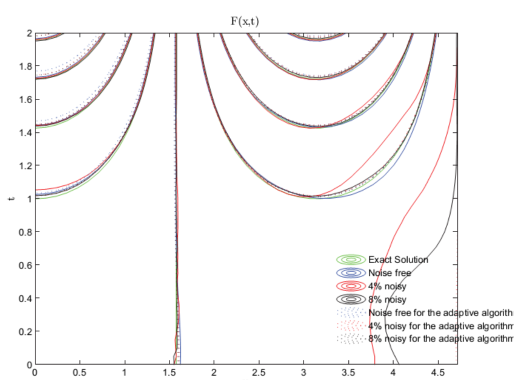
We take the noisy data $h^\delta(t)$ of the following form:

$$h^\delta(t) = h(t) + \delta \frac{\operatorname{rand} n\left(size\left(h(t)\right)\right)}{norm\left(\operatorname{rand} n\left(size\left(h(t)\right)\right)\right)}$$

Here δ is a noise level parameter and "randn" denotes a random number generated by the MATLAB function. The exact solutions $F(x,t)$ with the numerical solutions for various values of the noisy level $\delta \in \{4\%, 8\%\}.N = M = 40$ are shown in the below Figures. We use the stopping criteria as $\left\|u^{(k)}(x,t) - u^{(k-1)}(x,t)\right\| <$

δ , for noisy free data $\delta = 10^{-6}$





In the following Table we present some numerical results for the stopping iteration numbers and the percentage error in $F(x,t,u_x)$ for various amounts of N, M . Here we use the symbol it as the stopping iteration numbers $e_F(\%) := \frac{\|F-\widetilde{F}\|_{C(\Omega_T)}}{\|F\|_{C(\Omega_T)}} * 100$ as the percentage error in $F(x,t,u_x)$. $\widetilde{F}(x,t,u_x)$ is approximate value of $F(x,t,u_x)$.

$N \times M$	δ	it	$e_F(\%)$
20×20	0	51	0.96%
20×20	4%	81	1.91%
20×20	8%	1339	2.89%
30×30	0	52	0.69%
30×30	4%	63	1.26%
30×30	8%	101	1.56%
40×40	0	81	0.56%
40×40	4%	101	1.14%
40×40	8%	139	2.89%

Table 1: Iteration numbers and errors with various noisy levels and $N \times M$ for Example 1.

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