TWO-GRID WEAK GALERKIN METHOD FOR SEMILINEAR ELLIPTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we investigate a two-grid weak Galerkin method for semilinear elliptic differential equations. The method mainly contains two steps. First, we solve the semi-linear elliptic equation on the coarse mesh with mesh size H, then, we use the coarse mesh solution as a initial guess to linearize the semilinear equation on the fine mesh, i.e., on the fine mesh (with mesh size h), we only need to solve a linearized system. Theoretical analysis shows that when the exact solution u has sufficient regularity and $h = H^2$, the two-grid weak Galerkin method achieves the same convergence accuracy as weak Galerkin method. Several examples are given to verify the theoretical results.

1. INTRODUCTION

Among numerous methods for solving partial differential equations, weak Galerkin method has attracted extensive attention in the past several years. The idea of the weak Galerkin method was initially derived from the hybrid mixed finite element method and it was first introduced By Wang and Ye in [3] for the second order elliptic equations. Later, its convergence theory was developed [7, 14] and the method was applied to many other model problems, such as the Helmholtz equations [4], the Stokes problem [6] and Brinkman model problems [8] and so on. The weak Galerkin method has two main features, one is that it uses totally discontinuous finite element functions with the trace of the finite element function on element edge may be independent with its value in the interior of the element, and the other is the common partial derivatives are regarded as distributions or approximations of distributions. Furthermore, by defining weak differential operator instead of traditional differential operator in the weak Galerkin method, the difficulty of selecting approximate function in traditional finite element method can be overcome.

In this paper, we will investigate the weak Galerkin method with two-grid discretization technique for the semilinear elliptic partial differential equations. The two-grid discretization method, first proposed by Xu [9, 10, 11] to study nonsymmetric and nonlinear elliptic equations, is an effective method for nonlinear problems. It has been widely used for solving numerous different model problems in the past thirty years, for instance, the eigenvalue problems by Xu and Zhou [12], nonlinear parabolic problems by Dawson, Wheeler and Woodward [1], reaction-diffusion equations by Wu and Allen [13] and Navier-Stokes equations by Utnes [2], and so on. The two-grid method is based on the fact that the asymmetry and nonlinearity

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of low frequency are controlled by coarse grid, while the related high frequency is controlled by some linear or symmetric positive definite (SPD) operators.

By combining the features of weak Galerkin method and the efficiency of two-grid technique, in this paper, we study a two-grid weak Galerkin method for semilinear elliptic model problem (2.1). The main procedure of the two-grid weak Galerkin method maintains two steps, i.e., solving a nonlinear equations on the coarse mesh with mesh size H and linearized system on the fine mesh with mesh size h. Theoretical analysis shows that when the exact solution of (2.1) maintains sufficient regularity and mesh sizes h and H satisfy $h = H^2$, the numerical approximate solution by two-grid weak Galerkin method achieves the same convergence result as the weak Galerkin method. In order to verify the theoretical results, two numerical examples are given on both triangular and rectangular meshes. The discrete weak Galerkin element $(P_0(T^0), P_0(\partial T), RT_0(T))$ with T as mesh element are used in the numerical implementation. Numerical results in Section 5 proves the theoretical results and also show the efficiency of the two-grid weak Galerkin method.

This paper is organized as follows. In Section 2, we will introduce the model problem and the weak Galerkin finite element method. In Section 3, some auxiliary projection operators and the approximate results of the weak Galerkin method are presented. The two-grid weak Galerkin method as well as the error estimation of this method are given in section 4. Section 5 is devoted to verify the theoretical analysis by two numerical examples on both triangular and rectangular meshes.

2. Model problems and weak formulation

In this paper, we will consider the following homogeneous semilinear elliptic differential equations

$$\begin{cases} -\nabla\cdot(\mathcal{A}\nabla u) + f(u) = 0, x \in \Omega, \\ u = 0, x \in \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is the polygon or polyhedron in $R^d (d = 2, 3)$, $\mathcal{A} = (a_{ij}(x))_{d\times d} \in [L^\infty(\Omega)]^{d^2}$ is a symmetric positive definite matrix. In order to make a theoretical analysis of the two-grid method, we also need to introduce the following assumption for semilinear term $f(u)$.

Assumption 2.1. The semilinear term $f(u)$ has continuously derivative up to the second order and satisfies

$$f_u(u) \le 0,$$

and $|f_u(u)| + |f_{uu}(u)| \le M$ with M being some positive constant.

We introduce some standard notations which will be used later. Denote $W^{m,p}(\Omega)$ as Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|^p_{m,p} = \sum_{|\alpha|\le m} \|D^\alpha v\|^p_{L^p(\Omega)}$ and a semi-norm $|\cdot|_{m,p}$ given by $|v|^p_{m,p} = \sum_{|\alpha|=m} \|D^\alpha v\|^p_{L^p(\Omega)}$. We set $W^{m,p}_0(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H^m_0(\Omega) = W^{m,2}_0(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$.

In order to present the weak form, we first introduce the weak Galerkin space and the definition of weak derivative.

Let K be any polygonal domain with interior K^0 and boundary ∂K. Denote $W(K)$ as the weak functions of K by

$$W(K) = \left\{ v = \{v^0, v^b\} : v^0 \in L^2(K), v^b \in H^{\frac{1}{2}}(\partial K) \right\}.$$

Definition 2.1. For any $v \in W(K)$, the weak gradient of v, denoted as $\nabla_w v$, is defined as linear functional in the dual space of $H(\mathrm{div}, K)$ whose action on each $q \in H(\mathrm{div}, K)$ is given by

$$(\nabla_w v, q) = -\int_K v_0 \nabla \cdot q dK + \int_{\partial K} v_b q \cdot n ds,$$

where n is the unit outward normal direction of ∂K.

In order to define the discrete weak derivative, let $\mathcal{T}_h$ be the triangular or rectangular decompostion of polygon domain K and $T \in \mathcal{T}_h$ be the any element. h_T is used as the diameter of T and $h = \max_{T \in \mathcal{T}_h} h_T$. Associate with the partition, we introduce the discrete L^2 inner product and norm by

$$(u, v)_h = \sum_{T \in \mathcal{T}_h} (u, v)_T = \sum_{T \in \mathcal{T}_h} \int_T uv dx, \quad \|v\|_h^2 = (v, v)_h.$$

For any elment $T \in \mathcal{T}_h$, let $P_j(T^0)$ denote j-th polynomial on T^0 where T^0 is used as the interior of element T. Denote $P_j(\partial T)$ as the polynomial on ∂T with polynomial order less or equal to j. Then, we can define the discrete weak Galerkin space.

Definition 2.2. Given j, ℓ, denote

$$W(T, j, \ell) := \{v = \{v^0, v^b\} : v^0 \in P_j(T^0), v^b \in P_\ell(\partial T)\},$$

and the discrete weak Galerkin space S_h is defined as:

$$S_h(j, \ell) = \{v = \{v^0, v^b\} : \{v^0, v^b\}|_T \in W(T, j, \ell), \forall T \in \mathcal{T}_h\},$$

and the subspace of S_h is defined as

$$S_h^0(j, \ell) = \{v = \{v^0, v^b\} : \{v^0, v^b\}|_T \in S_h(j, \ell), v^b|_{\partial T \cap \partial \Omega} = 0, \forall T \in \mathcal{T}_h\}.$$

Definition 2.3. Let $V(T, r)$ be a subspace of the set of vector-valued polynomials of degree no more than r on T. For any $v \in S_h$, denote $\nabla_{w,r} v \in V(T, r)$ as the discrete weak derivative of v which is defined as: $\forall \mathbf{q} \in V(T, r)$,

$$(\nabla_{w,r} v, \mathbf{q}) := -\int_T v^0 \nabla \cdot \mathbf{q} d\Omega + \int_{\partial T} v^b \mathbf{q} \cdot \mathbf{n} ds. \tag{2.2}$$

Now, we can introduce the weak form of (2.1). The standard weak form of problem (2.1) is to find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \mathcal{A} \nabla u \nabla v dx + \int_\Omega f(u) v^0 dx = 0, \quad \forall v \in H_0^1(\Omega), \tag{2.3}$$

and the linearized weak form with guess μ is: $\forall v \in H_0^1(\Omega)$,

$$\int_\Omega \mathcal{A} \nabla \bar{u} \nabla v dx + \int_\Omega f(\mu) + f_u(\mu)(\bar{u} - \mu) v^0 dx = 0. \tag{2.4}$$

Then, the discrete weak form of weak Galerkin method for the Dirichlet boundary problem (2.1) is to find $u_h = \{u_h^0, u_h^b\} \in S_h^0(j,\ell)$ such that

$$\int_K \mathcal{A}\nabla_{w,r} u_h \nabla_{w,r} v dx + \int_K f(u_h) v^0 dx = 0, \quad \forall v = \{v^0, v^b\} \in S_h^0(j,\ell) \tag{2.5}$$

and the linearized discrete weak form is to find $\bar{u}_h \in S_h^0(j,\ell)$ such that $\forall v \in S_h^0(j,\ell)$,

$$\int_K \mathcal{A}\nabla_{w,r} \bar{u}_h \nabla_{w,r} v dx + \int_K f(\mu) + f_u(\mu)(\bar{u}_h - \mu) v^0 dx = 0. \tag{2.6}$$

Denote

$$a(u_h, v_h) = (\mathcal{A}\nabla_{w,r} u_h, \nabla_{w,r} v_h),$$

and also the linearized bilinear form as

$$a_\mu(\bar{u}_h, v_h) = (\mathcal{A}\nabla_{w,r} \bar{u}_h, \nabla_{w,r} v_h) + (f_u(\mu)\bar{u}_h, v_h^0).$$

3. Projection and approximation

In this section, we will give some projections and the approximation properties of the weak Galerkin solution defined in (2.5).

We first need to introduce some auxiliary projection operators.

Definition 3.1. Let $Q_h : H^1(\Omega) \to S_h$ be the L^2 projection such that $\forall T \in \mathcal{T}_h$, $Q_h u|_T = \{Q_h^0 u, Q_h^b u\}$.

Then, we have the following properties for the L^2 projection.

Lemma 3.1. [15] Let $u \in H^{1+s}(\Omega)$ and $2 \le p \le \infty$, then

$$\left\|Q_h^0 u - u\right\|_{0,p} \le Ch^s \left\|u\right\|_{s,p}, \ 0 \le s \le j+1,$$
$$\left\|\nabla_{w,r} Q_h u - \nabla u\right\|_0 \le Ch^s \left\|u\right\|_{1+s}, \ 0 \le s \le j+2.$$

Definition 3.2. Let Π_h be the projection from $H(\mathrm{div};\Omega)$ to $H(\mathrm{div};\Omega)$. For any $T \in \mathcal{T}_h$, $\Pi_h \mathbf{q} \in V(T,r)$ satisfies

$$(\nabla \cdot \mathbf{q}, v^0)_T = (\nabla \cdot \Pi_h \mathbf{q}, v^0)_T, \ \forall v^0 \in P_j(T^0).$$

We have the following results for Π_h.

Lemma 3.2. [3, 15] For any $\mathbf{q} \in H(\mathrm{div},\Omega)$ and if $\mathbf{q} \in [H^{1+s}]^d, s \ge 0$, then

$$\sum_{T\in\mathcal{T}_h} (-\nabla \cdot \mathbf{q}, v^0)_T = \sum_{T\in\mathcal{T}_h} (\Pi_h \mathbf{q}, \nabla_{w,r} v)_T, \ \forall v = \{v^0, v^b\} \in S_h^0.$$
$$\|\mathbf{q} - \Pi_h \mathbf{q}\|_{0,T} \le Ch^{1+s} \|\mathbf{q}\|_{1+s,T}, \ 0 \le s \le j+1.$$

For both projection operators Q_h, Π_h, we have the following properties.

Lemma 3.3. [3, 15] For any $\mathbf{q} \in H(\mathrm{div},\Omega), \Pi_h \mathbf{q} \in H(\mathrm{div},\Omega)$ and $u \in H^{1+s}(\Omega)$ with $s > 0$,

$$\|\Pi_h(\mathcal{A}\nabla u) - \mathcal{A}\nabla_w(Q_h u)\| \le Ch^s \left\|u\right\|_{1+s},$$
$$\|\nabla u - \nabla_w(Q_h u)\| \le Ch^s \left\|u\right\|_{1+s}.$$

We also need the following results in the estimation.

Lemma 3.4. Let $g(v)$ be a piecewise smooth function on the partition $\mathcal{T}_h$, if $\bar{g}$ is the average value of $g(v)$ on each element T and $\|\nabla g\|_{0,\infty} \le \kappa$, then

$$|(g(v)\phi, \psi) - (\bar{g}\phi, \psi)| \le C\kappa h \|\phi\| \|\psi\|.$$

Proof. By using interpolation estimate, we have

$$|(g(v)\phi,\psi)-(\bar{g}\phi,\psi)| = |\int_\Omega (g(v)-\bar{g})\phi\psi dx| \le \sum_{T\in\mathcal{T}_h}\int_T |g(v)-\bar{g}||\phi\psi|dx$$
$$\le C\kappa h\sum_{T\in\mathcal{T}_h}\int_T |\phi\psi|dx \le C\kappa h\|\phi\|\|\psi\|.$$

□

Lemma 3.5. [15] Let Ω be a polygon domain, then for any $v\in S_h^0$, there exists positive constant C_0 that is independent of h such that

$$\left\|v^0\right\| \le C_0\left\|\nabla_{w,r}v\right\|_h.$$

Lemma 3.6. Let $u\in H^2(\Omega)$ be the solution of (2.3), then:

$$(\Pi_h(\mathcal{A}\nabla u),\nabla_{w,r}v)+(f(u),v^0)=0,\forall v\in S_h^0.$$

Proof. Let $\mathbf{w}\in[H^1(\Omega)]^d$, then

$$-(\mathrm{div}\mathbf{w},v^0) = -(\mathrm{div}\Pi_h\mathbf{w},v^0) = (\Pi_h\mathbf{w},\nabla_{w,r}v),\forall v\in S_h^0,$$

choose $\mathbf{w}=\mathcal{A}\nabla u$ and

$$-(\mathrm{div}(\mathcal{A}\nabla u),v^0)+(f(u),v^0)=0,\forall v\in S_h^0,$$

we have:

$$-(\mathrm{div}\Pi_h\mathbf{w},v^0)=(\Pi_h(\mathcal{A}\nabla u),\nabla_{w,r}v),$$

then

$$(\Pi_h(\mathcal{A}\nabla u),\nabla_{w,r}v)+(f(u),v^0)=0,\forall v\in S_h^0.$$

□

Theorem 3.1. Let u,u_h be the solutions of problem (2.3) and (2.5) respectively, $u\in H^{2+s}(\Omega),(s\ge 0)$, then, we have the following optimal error estimates

$$\|\nabla_{w,r}u_h-\nabla u\| \le Ch^{1+s}\|u\|_{2+s},\ 0\le s\le j.$$

Proof. From the definition of L^2 projection and the Lemma 3.6, we have the following error equation, i.e., $\forall T\in\mathcal{T}_h, v\in S_h^0$,

$$\begin{aligned}
a(Q_hu-u_h,v) &= (\mathcal{A}\nabla_{w,r}Q_hu-\Pi_h\mathcal{A}\nabla u,\nabla_{w,r}v)+(f(u_h)-f(u),v^0)\\
&= (\mathcal{A}\nabla_{w,r}Q_hu-\Pi_h(\mathcal{A}\nabla u),\nabla_{w,r}v)+(f_u(\epsilon)(u_h-u),v^0)\\
&= (\mathcal{A}\nabla_{w,r}Q_hu-\Pi_h(\mathcal{A}\nabla u),\nabla_{w,r}v)+((f_u(\epsilon)-\bar{f}_u)(u_h-u),v^0)\\
&\quad +(\bar{f}_u(u_h-Q_h^0u),v^0),
\end{aligned}$$

where $\bar{f}_u$ is the average value of $f_u(\epsilon)$ on each element of $\mathcal{T}_h$. Then, choose $v^0 = Q_h^0u-u_h^0$, we have

$$\begin{aligned}
\|\nabla_{w,r}Q_hu-\nabla_{w,r}u_h\|_h^2 \le\ & \|\mathcal{A}(\nabla_{w,r}Q_hu-\nabla u)\|_h\|\nabla_{w,r}(Q_hu-u_h)\|_h\\
&+\|\mathcal{A}\nabla u-\Pi_h(\mathcal{A}\nabla u)\|_h\|\nabla_{w,r}(Q_hu-u_h)\|_h\\
&+((f_u-\bar{f}_u)(u_h-u),Q_h^0u-u_h^0)\\
&-(\bar{f}_u(Q_h^0u-u_h^0),Q_h^0u-u_h^0).
\end{aligned}$$

By using Lemma 3.4 and the Assumption 2.1, we get

$$\begin{aligned}\|\nabla_{w,r}Q_hu-\nabla_{w,r}u_h\|_h^2 \le (&\|\mathcal{A}\|_\infty\|\nabla_{w,r}Q_hu-\nabla u\|_h\|\nabla_{w,r}(Q_hu-u_h)\|_h\\ &+\|\mathcal{A}\nabla u-\Pi_h(\mathcal{A}\nabla u)\|_h\|\nabla_{w,r}(Q_hu-u_h)\|_h\\ &+C\kappa h(\|u^0-Q_h^0u\|_h+\|Q_h^0u-u_h^0\|_h)\|Q_h^0u-u_h^0\|_h.\end{aligned}$$

Then, from Lemma 3.1, 3.3 and 3.5, we get the following estimation for $0\le s\le j$,

$$\begin{aligned}\|\nabla_{w,r}Q_hu-\nabla_{w,r}u_h\|_h &\le \|\mathcal{A}\|_\infty\|\nabla_{w,r}Q_hu-\nabla u\|_h+\|\mathcal{A}\nabla u-\Pi_h(\mathcal{A}\nabla u)\|_h\\ &\quad +C\kappa h\|u-Q_hu\|_h\\ &\le Ch^{1+s}\|u\|_{2+s}.\end{aligned}$$

Using the triangle inequality

$$\|\nabla u-\nabla_{w,r}u_h\|\le\|\nabla u-\nabla_{w,r}Q_hu\|+\|\nabla_{w,r}Q_hu-\nabla_{w,r}u_h\|,$$

we finally get the results. □

In the following, we will get the error estimation of u_h in the L^2 norm by dual argument. Consider the dual problem, find $w\in H_0^1(\Omega)\bigcap H^2(\Omega)$ with $w=0$ on $\partial\Omega$ satisfying

$$-\nabla\cdot(\mathcal{A}\nabla w)+f_uw=Q_h^0u-u_h^0,\ \text{on }\Omega. \tag{3.1}$$

Suppose the solution of the dual problem maintain H^2 regularity, i.e., there exists constant C such that

$$\|w\|_2\le C\left\|Q_h^0u-u_h^0\right\|.$$

Theorem 3.2. Let u and u_h are the solution of (2.1) and (2.5) respectively and $u\in H^{2+s}(\Omega)$, $s\ge0$, then, we have the following estimation:

$$\left\|Q_h^0u-u_h^0\right\|\le Ch^{2+s}\cdot\|u\|_{2+s},\quad 0\le s\le j,$$

and the optimal L^2 error estimation:

$$\left\|u_h^0-u\right\|\le Ch^{1+j}\cdot\|u\|_{1+j},\quad j\ge1. \tag{3.2}$$

Proof. From the weak form of the dual problem (3.1) and Lemma 3.6, we have:

$$\begin{aligned}\left\|Q_h^0u-u_h^0\right\|^2 &= (\nabla_{w,r}(Q_hu-u_h),\Pi_h(\mathcal{A}\nabla w))+(f_uw,Q_h^0u-u_h^0)\\ &=(\nabla_{w,r}(Q_hu-u_h),\Pi_h(\mathcal{A}\nabla w)-\mathcal{A}\nabla_{w,r}Q_hw)\\ &\quad +a(Q_hu-u_h,Q_hw)+(f_uw,Q_h^0u-u_h^0)\\ &\le Ch\cdot\|w\|_2\cdot\|\nabla_{w,r}(Q_hu-u_h)\|+a(Q_hu-u_h,Q_hw)\\ &\quad +(f_uw,Q_h^0u-u_h^0)\\ &\le Ch^{2+s}\|u\|_{2+s}\|w\|_2+E,\ 0\le s\le j,\end{aligned} \tag{3.3}$$

where $E=a(Q_hu-u_h,Q_hw)+(f_uw,Q_h^0u-u_h^0)$. In the following, we will give the estimation for E. From the error equation in the proof of Theorem 3.1, we have

$$E=(\mathcal{A}\nabla_{w,r}Q_hu-\Pi_h(\mathcal{A}\nabla u),\nabla_{w,r}Q_hw)+(f_uw,Q_h^0u-u_h^0)+(f(u_h)-f(u),Q_h^0w),$$

by using the Taylor expansion

$$f(u_h)=f(u)+f_u(u_h-u)+\frac{f_{uu}(\epsilon)}{2}(u_h-u)^2,$$

we get

$$
\begin{aligned}
E &= (\mathcal{A}\nabla_{w,r}Q_h u - \Pi_h(\mathcal{A}\nabla u), \nabla_{w,r}Q_h w) + (f_u(u_h - u), Q_h^0 w) \\
&\quad + (\frac{f_{uu}}{2}(u_h - u)^2, Q_h^0 w) + (f_u w, Q_h^0 u - u_h^0) \\
&= (\mathcal{A}\nabla_{w,r}Q_h u - \Pi_h(\mathcal{A}\nabla u), \nabla_{w,r}Q_h w - \nabla w) + (\mathcal{A}\nabla_{w,r}Q_h u - \Pi_h(\mathcal{A}\nabla u), \nabla w) \\
&\quad + (f_u(Q_h^0 u - u^0), Q_h^0 w) + (f_u(Q_h^0 u - u_h^0), w - Q_h^0 w) + (\frac{f_{uu}}{2}(u_h - u)^2, Q_h^0 w) \\
&= E_1 + E_2 + E_3 + E_4 + E_5.
\end{aligned}
$$

For E_1, E_2, we have

$$
\begin{aligned}
E_1 + E_2 &\le Ch^{2+s} \cdot \|u\|_{2+s} \|w\|_2 + (\mathcal{A}\nabla_{w,r}Q_h u - \mathcal{A}\nabla u, \nabla w) \\
&\quad + (\mathcal{A}\nabla u - \Pi_h(\mathcal{A}\nabla u), \nabla w) \\
&= Ch^{2+s} \cdot \|u\|_{2+s} \cdot \|w\|_2 + (\nabla_{w,r}Q_h u - \nabla u, \mathcal{A}^T\nabla w - Q_h(\mathcal{A}^T\nabla w)) \\
&\quad - (\mathrm{div}(\mathcal{A}\nabla u - \Pi_h(\mathcal{A}\nabla u)), w - Q_h w) \\
&\le Ch^{2+s} \cdot \|u\|_{2+s} \|w\|_2\,,\ 0 \le s \le j.
\end{aligned}
$$

For E_3, we have

$$
\begin{aligned}
E_3 &= ((f_u - \bar{f}_u)(Q_h^0 u - u^0), Q_h w) + (\bar{f}_u(Q_h^0 u - u^0), Q_h^0 w) \\
&\le C\kappa h \|Q_h^0 u - u_h^0\| \|Q_h^0 w\| \\
&\le C\kappa h^{s+2} \|u\|_{s+1} \|w\|_2.
\end{aligned}
$$

For E_4, we have

$$
\begin{aligned}
E_4 &= (f_u(Q_h^0 u - u_h^0), w - Q_h^0 w) \\
&\le C\kappa h^2 \|Q_h^0 u - u_h^0\| \|w\|_2.
\end{aligned}
$$

For E_5, by using the inverse inequality and proof in Theorem 3.1, we have

$$
\begin{aligned}
E_5 &= (\frac{f_{uu}}{2}(u_h - u)^2, Q_h^0 w) \\
&\le \|f\|_{2,\infty} \|u - u_h\|_{0,4}^2 \|Q_h^0 w\| \\
&\le \|f\|_{2,\infty} (\|u - Q_h u\|_{0,4}^2 + \|Q_h u - u_h\|_{0,4}^2) \|Q_h^0 w\| \\
&\le C(h^{2s+2} + h^{-\frac{1}{2}} \|Q_h^0 u - u_h^0\|^2) \|w\|_2 \\
&\le C(h^{2s+2} + h^{-\frac{1}{2}} \|\nabla_{w,r}Q_h^0 u - \nabla_{w,r}u_h^0\| \|Q_h^0 u - u_h^0\|) \|w\|_2 \\
&\le C(h^{2s+2} + h^{s+\frac{1}{2}} \|Q_h^0 u - u_h^0\|) \|w\|_2.
\end{aligned}
$$

By using estimations of $E, E_1, E_2, E_3, E_4, E_5$ and $\|w\|_2 \le \left\|Q_h^0 u - u_h^0\right\|$, when h is sufficient small, we have

$$
\left\|Q_h^0 u - u_h^0\right\| \le Ch^{2+s} \cdot \|u\|_{2+s}\,.
$$

Now, by using the triangle inequality, we have

$$
\begin{aligned}
\left\|u_h^0 - u\right\| &\le \left\|u_h^0 - Q_h^0 u\right\| + \left\|Q_h^0 u - u\right\| \\
&\le Ch^{2+s} \cdot \|u\|_{2+s} + Ch^{2+s} \cdot \|u\|_{2+s} \\
&\le Ch^{j+1} \cdot \|u\|_{j+1}\,, (j \ge 0). \qquad (3.4)
\end{aligned}
$$

□

Using the inverse inequality, we can also get the error estimation in L^∞ norm.

Lemma 3.7. [15] Suppose $\mathcal{T}_h$ are uniform divided meshes, u and u_h are the solutions of (2.1) and (2.5). Suppose $u \in W^{1+s,\infty}(\Omega) \bigcap H^{2+s}(\Omega)$, $s \geq 0$, then:

$$\left\|u - u_h^0\right\|_{0,\infty} \leq Ch^{1+s}(\|u\|_{1+s,\infty} + \|u\|_{2+s}).$$

Proof. Using the inverse inequality, we have

$$\left\|Q_h^0 u - u_h^0\right\|_{0,\infty} \leq Ch^{-1}\left\|Q_h^0 u - u_h^0\right\|_{L^2} \leq Ch^{1+s}\|u\|_{2+s},$$

from Lemma 3.1, we have

$$\begin{aligned}
\left\|u - u_h^0\right\|_{0,\infty} &\leq \left\|u - Q_h^0 u\right\|_{0,\infty} + \left\|Q_h^0 u - u_h^0\right\|_{0,\infty} \\
&\leq Ch^{1+s}\|u\|_{1+s,\infty} + Ch^{1+s}\|u\|_{2+s} \\
&\leq Ch^{1+s}\cdot(\|u\|_{1+s,\infty} + \|u\|_{2+s}).
\end{aligned}$$

□

4. The two-grid discretization method and error analysis

In this section, we will present the two-grid weak Galerkin discretizaiton method for the semilinear elliptic equations on the two nested spaces $S_H^0(j,\ell)$ and $S_h^0(j,\ell)$. The main idea of the two-grid method is to reduce a seminlinear elliptic equation into a linear problem by solving a nonlinear elliptic equation on a much smaller space, more specifically, the method is described in detail as follows:

Step 1: On the coarse mesh $\mathcal{T}_H$: solve $u_H \in S_H^0$ such that

$$a(u_H, v_H) + (f(u_H), v_H^0) = 0, \ \forall v_H \in S_H^0. \tag{4.1}$$

Step 2: On the fine mesh $\mathcal{T}_h$, solve the following linearized system for $\tilde{u}_h \in S_h^0$,

$$a(\tilde{u}_h, v_h) + (f(u_H) + f_u(u_H)(\tilde{u}_h - u_H), v_h^0) = 0, \ \forall v_h \in S_h^0. \tag{4.2}$$

For the nonlinear equation (4.1), we use the Newton iteration method which is presented as follows, i.e., on the coarse mesh $\mathcal{T}_H$, given a initial guess $u_H^{(0)}$, for $k = 1, 2, \cdots$, solve the following equation

$$(\mathcal{A}\nabla_{w,r}u_H^{(k)}, \nabla_{w,r}v) + (f_u(u_H^{(k-1)})\cdot u_H^{(k)}, v^0) = (f_u(u_H^{(k-1)})\cdot u_H^{(k-1)} - f(u_H^{(k-1)}), v^0).$$

The matrix form for solving $u_H^{(k)}$ is:

$$(A + J(u_H^{(k-1)}))\cdot u_H^{(k)} = b(u_H^{(k-1)}),$$

where A is the stiffness matrix, $b(u_H^{(k)}) = (-f(u_H^{(k-1)} + f_u(u_H^{(k-1)})\cdot u_H^{(k-1)}, v^0)$ and the Jacobi matrix J is computed from term

$$J(u_H^{(k-1)})u_H^k = (f_u(u_H^{(k-1)})u_H^{(k)}, v^0).$$

In order to make theoretical analysis of the two-grid weak Galerkin method, we need to introduce the following results for the linearized operator $L_\mu = -\nabla\cdot(\mathcal{A}\nabla) + f_u(\mu)$. By using Assumption 2.1, we have the following property.

Lemma 4.1. There exists a constant $\delta > 0$ such that for a given $\mu \in H_0^1(\Omega)$ with $\|u - \mu\| \leq \delta$,

- $L_\mu : H_0^1(\Omega) \cap H_0^1(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$ is bijective and there exists a constant $C = C(\delta)$ such that

$$\|\omega\|_{H^2(\Omega)} \le C\|L_\mu \omega\|, \quad \forall \omega \in H_0^1(\Omega) \cap H^2(\Omega),$$

- if h is sufficient small, there exists a constant $c(\delta)$ such that

$$\sup_{v \in S_h} \frac{a_\mu(\omega_h, v)}{\|\nabla_{w,r} v\|_h} \ge c(\delta)\|\nabla_{w,r}\omega_h\|_h.$$

Next, we will conclude the error estimation of the two-grid discretization method.

Theorem 4.1. Suppose $\tilde{u}_h \in S_h^0$ is the two-grid weak Galerkin solution of (4.2), u is the exact solution of (2.3), $h < H$, then,

$$\|u - \tilde{u}_h^0\| \le C(H^{2(s+1)} + h^{s+1})\|u\|_{2+s}, \ 0 \le s \le j.$$

Proof. Subtract equation (4.2) from (2.5), we get the following error equation

$$(\mathcal{A}\nabla_{w,r}(u_h - \tilde{u}_h), \nabla_{w,r} v_h) + (f(u_h) - f(u_H) - f_u(u_H)(\tilde{u}_h - u_H), v_h^0) = 0,$$

then, use the Taylor expansion for $f(u_h)$ at u_H, i.e.,

$$f(u_h) = f(u_H) + f_u(u_H)(u_h - u_H) + \frac{1}{2} f_{uu}(\xi)(u_h - u_H)^2,$$

replace the last relation into the error equation to get

$$a_{u_H}(u_h - \tilde{u}_h, v_h) + \frac{1}{2}(f_{uu}(\xi)(u_h - u_H)^2, v_h^0) = 0. \tag{4.3}$$

Then, using Lemma 3.5 and 4.1, we have

$$\begin{aligned} c(H^2)\|u_h^0 - \tilde{u}_h^0\| &\le c(H^2)\|\nabla_{w,r}(u_h - \tilde{u}_h)\|_h \le \sup_{v_h \ne 0} \frac{a_{u_H}(u_h - \tilde{u}_h, v_h)}{\|\nabla_{w,r} v_h\|_h} \\ &\le \sup_{v_h \ne 0} \frac{a_{u_H}(u_h - \tilde{u}_h, v_h)}{\|v_h^0\|}. \end{aligned}$$

From error equation (4.3), we have

$$\begin{aligned} c(H^2)\|u_h^0 - \tilde{u}_h^0\| &\le \sup_{v_h \ne 0} \frac{a_{u_H}(u_h - \tilde{u}_h, v_h)}{\|v_h^0\|} = \sup_{v_h \ne 0} \frac{-\frac{1}{2}(f_{uu}(\xi)(u_h - u_H)^2, v_h^0)}{\|v_h^0\|} \\ &\le \frac{1}{2}\|f\|_{2,\infty}\|u_h^0 - u_H^0\|_{L^4}^2 \le C\|u - u_H^0\|_{L^4}^2, \end{aligned}$$

then, we have for any $0 \le s \le j$,

$$\|u_h^0 - \tilde{u}_h^0\| \le \|u^0 - u_H^0\|_{L^4}^2 \le \|u^0 - u_H^0\|_{0,\infty}^2 \le CH^{2s+2}.$$

By using triangle inequality, we have

$$\begin{aligned} \|u - \tilde{u}_h^0\| &\le \|u - u_h^0\| + \|u_h^0 - \tilde{u}_h^0\| \\ &\le C(h^{s+1} + H^{2s+2}), \end{aligned}$$

which finishes the proof. $\square$

5. Numerical experiments

In this section, we present some numerical experiments to verify the theoretical results given in Section 4. We consider the following model problem

$$\begin{aligned} -\nabla \cdot (\mathcal{A}\nabla u) + u^3 &= g, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

with $\Omega = [0,1]^2$.

Four types of norms will be used in the numerical experiments:

$$\|\nabla_{w,r}(u_h - Q_h u)\| = \left(\sum_{T \in \mathcal{T}_h} \int_T |\nabla_{w,r}(u_h - Q_h u)|^2 dx \right)^{1/2}, \tag{5.1}$$

$$\|\nabla_{w,r} u_h - \nabla u\| = \left(\sum_{T \in \mathcal{T}_h} \int_T |\nabla_{w,r} u_h - \nabla u|^2 dx \right)^{1/2}, \tag{5.2}$$

$$\|u_h - Q_h u\| = \left(\sum_{T \in \mathcal{T}_h} \int_T |u_h - Q_h u|^2 dx \right)^{1/2}, \tag{5.3}$$

$$\|u_h - u\| = \left(\sum_{T \in \mathcal{T}_h} \int_T |u_h - u|^2 dx \right)^{1/2}. \tag{5.4}$$

The stopping criterion for Newton iteration is chosen to be the relative error between two adjacent iterates less than a prescribed tolerance, i.e.,

$$\frac{\|U_H^{l+1} - U_H^l\|}{\|U_H^{l+1}\|} \le \epsilon,$$

where $\epsilon = 10^{-3}$ is used in our numerical tests. For the linear system of equations, we use the algebraic multigrid method with tolerance 10^{-9}.

The numerical tests are conducted on a computer with 2.24 GHz 4-core Intel Celeron N3160 CPU and 4 GB RAM memory. The MATLAB finite element package iFEM is used for the implementation [16].

5.1. **weak Galerkin finite element.** In this subsection, we present the weak Galerkin element we used in our numerical experiments on both triangular and rectangular meshes. We mainly consider using $(P_0(T^0), P_0(\partial T), RT_0(T))$ in our numerical implementation.

5.1.1. *weak Galerkin element on triangular mesh.* Let $\mathcal{T}_h$ be triangulation of Ω and $T \in \mathcal{T}_h$ be any element, let $S_h(0,0) = \{v = \{v^0, v^b\} : v^0 \in P_0(T^0), v^b \in P_0(\partial T)\}$, i.e., v is constant on both the interior and boundary of T.

Given three points $P_i = (x_i, y_i), i = 1,2,3$ of triangle T and denote (x_c, y_c) as the barycentric point of T. Let $\chi_1 = \binom{1}{0}, \chi_2 = \binom{0}{1}, \chi_3 = \binom{x-x_c}{y-y_c}$ be the local basis functions of Raviart-Thomas, i.e., $RT_0(T) = Span(\chi_1, \chi_2, \chi_3)$. Since the local basis functions form the orthogonal basis of RT_0 on element T, then, the mass matrix of RT_0 element on triangle T is represented as

$$M_R = \begin{pmatrix} |T| & 0 & 0 \\ 0 & |T| & 0 \\ 0 & 0 & S \end{pmatrix},$$

where $|T|$ is the area of the triangle T and $S = \frac{1}{12}(|e_1|^2+|e_2|^2+|e_3|^2)$ with e_1, e_2, e_3 are three edges of triangle T and $e_i (i = 1, 2, 3)$ be the opposite edge to node (x_i, y_i).

Let ϕ_0 be the basis of $P_0(T^0)$ and $\phi_{b_1}, \phi_{b_2}, \phi_{b_3}$ be the basis functions of weak Galerkin element on $P_0(\partial T)$ and suppose

- $\phi_0 = 1$ in the interior of T but 0 on the boundary of ∂T,
- $\phi_{b_i}, i = 1, 2, 3$ equal 1 on the $i - th$ boundary and equals 0 on the other edges and interior of T.

Then, the weak derivative of these basis functions of space $S_h(T)$ belongs to $RT_0(T)$. Using the basis functions, we can compute the local stiffness matrix as well as the right hand side.

5.1.2. *Weak Galerkin element on rectangular mesh.* Similar as the case on triangular mesh, we can also give the definition of basis functions on rectangular element. For any $T \in \mathcal{T}_h$ be rectangle of Ω, let $S_h(0,0) = \{v = \{v^0, v^b\} : v^0 \in P_0(T^0), v^b \in P_0(\partial T)\}$.

Let ϕ_0 be the basis of $P_0(T^0)$ and $\phi_{b_1}, \phi_{b_2}, \phi_{b_3}, \phi_{b_4}$ be the basis functions of weak Galerkin element on $P_0(\partial T)$ and suppose

- $\phi_0 = 1$ in the interior of T but 0 on the boundary of ∂T,
- $\phi_{b_i}, i = 1, 2, 3, 4$ equal 1 on the $i - th$ boundary and equals 0 on the other edges and interior of T.

Let $T = [0, a] \times [0, b]$ be a rectangle where a, b are positive real numbers. Denote four edges of rectangle element as $e_i, (i = 1, 2, 3, 4)$ by $e_1 : x = 0,\ e_2 : y = b,\ e_3 : x = a,\ e_4 : y = 0$. The unit outward normal direction to each edge is given by

$$n_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, n_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, n_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, n_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

And the local basis function of Raviart-Thomas on T are defined as follows:

$$\chi_1 = \begin{pmatrix} \frac{x}{a} - 1 \\ 0 \end{pmatrix}, \chi_2 = \begin{pmatrix} 0 \\ \frac{y}{b} \end{pmatrix}, \chi_3 = \begin{pmatrix} \frac{x}{a} \\ 0 \end{pmatrix}, \chi_4 = \begin{pmatrix} 0 \\ \frac{y}{b} - 1 \end{pmatrix}.$$

Clearly, each χ_i satisfies

$$\chi_i \cdot n_j|_{e_j} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Accordingly, $RT_0(T) = Span(\chi_1, \chi_2, \chi_3, \chi_4)$. Denote $|e_i|$ as the length of edge $e_i (i = 1, 2, 3, 4)$ and $|T|$ as the area of the rectangle T, the mass matrix $\bar{M}_R$ of RT_0 element on rectangle T is represented as

$$\bar{M}_R = \frac{|T|}{6} \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}.$$

Similarly, we can use the basis functions to compute the local stiffness matrix and right hand side on rectangular mesh.

5.2. **Numerical examples.** In this subsection, we will implement the two-grid discretization method with two concrete examples.

Example 1: We choose $\mathcal{A}$ as the identity matrix. Suppose the exact solution is

$$u = x(1-x)y(1-y),$$

then, the explicit expression of g is

$$g = -2x(x-1) - 2y(y-1) + [x(1-x)y(1-y)]^3.$$

In order to see the convergence order of the two-grid method, we choose mesh size pairs $(1/2, 1/4), (1/4, 1/16)$ and $(1/8, 1/64)$. In Tables 1, 2, 3 and 4, we present the errors of the **Example** 1 by the two-grid weak Galerkin method and the weak Galerkin method on triangular grid and rectangular grid. $u, Q_h u, u_h, \tilde{u}_h$ are used to represent exact solutions, L^2 projection, weak Galerkin solution and two-grid weak Galerkin solution respectively. As we can see from the Tables 1, 2, 3 and 4, the two-grid weak Galerkin solution has the same convergence order as the weak Galerkin method, which verify the theoretical results in Section 4.

We also present the convergence order of the two methods in Figures 1 and 2. Figure 1 shows the convergence order of two-grid weak Galerkin method on triangular mesh and rectangular mesh in norm (5.3) and (5.4). We observe that the convergence order of $\|u_h - Q_h u\|$ is $\mathcal{O}(h^2)$ (equivalently, $\mathcal{O}(N^{-1})$ where N is the total number of degrees of freedom) and the convergence order of $\|u_h - u\|$ is $\mathcal{O}(h)$ (equivalently, $\mathcal{O}(N^{-0.5})$) , which are consistent with the theory. Figure 2 shows the convergence order in norm (5.1) and (5.2) on rectangular mesh and triangular meshes. As we can see from Figures, $\|\nabla_{w,r}(u_h - Q_h u)\|$ and $\|\nabla_{w,r} u_h - \nabla u\|$ have the same convergence order $\mathcal{O}(h)$ (equivalently, $\mathcal{O}(N^{-0.5})$), which is consistent with the theoretical results in Section 4 as well as the numerical results in [5].

Table 1. Example 1: Errors in norms (5.1) and (5.2) on rectangular mesh

(H, h)	$(1/2, 1/4)$	$(1/4, 1/16)$	$(1/8, 1/64)$
$\|\nabla_{w,r}(u_h - Q_h u)\|$	2.0094e-02	4.6812e-03	1.1650e-03
$\|\nabla_{w,r}(\tilde{u}_h - Q_h u)\|$	2.0092e-02	4.6810e-03	1.1650e-03
$\|\nabla_{w,r} u_h - \nabla u\|$	1.0035e-02	2.6832e-03	6.7729e-04
$\|\nabla_{w,r} \tilde{u}_h - \nabla u\|$	1.0035e-02	2.6832e-03	6.7729e-04

Table 2. Example 1: Errors in norms (5.3) and (5.4) on rectangular mesh

(H, h)	$(1/2, 1/4)$	$(1/4, 1/16)$	$(1/8, 1/64)$
$\|u_h - Q_h u\|$	1.8220e-03	1.1378e-04	7.1114e-06
$\|\tilde{u}_h - Q_h u\|$	1.8207e-03	1.1347e-04	7.0289e-06
$\|u_h - u\|$	3.3159e-02	8.4919e-03	2.1261e-03
$\|\tilde{u}_h - u\|$	3.3159e-02	8.4919e-03	2.1261e-03

Example 2: In this example, we also choose $\mathcal{A}$ as the identity and the exact solution of the equation as

$$u = \sin(\pi x_1)\sin(\pi x_2),$$

Table 3. Example 1:Errors in norms (5.1) and (5.2) on triangular mesh

(H,h)	$(1/2,1/4)$	$(1/4,1/16)$	$(1/8,1/64)$
$\|\nabla_{w,r}(u_h-Q_hu)\|$	1.8805e-02	4.8920e-03	1.2273e-03
$\|\nabla_{w,r}(\tilde{u}_h-Q_hu)\|$	1.8804e-02	4.8920e-03	1.2273e-03
$\|\nabla_{w,r}u_h-\nabla u\|$	1.4396e-02	3.1466e-03	7.7707e-04
$\|\nabla_{w,r}\tilde{u}_h-\nabla u\|$	1.4396e-02	3.1466e-03	7.7707e-04

Table 4. Example 1:Errors in norms (5.3) and (5.4) on triangular mesh

(H,h)	$(1/2,1/4)$	$(1/4,1/16)$	$(1/8,1/64)$
$\|u_h-Q_hu\|$	9.1294e-04	6.3215e-05	3.9877e-06
$\|\tilde{u}_h-Q_hu\|$	9.1235e-04	6.3114e-05	3.9609e-06
$\|u_h-u\|$	7.2954e-05	4.8075e-06	3.0135e-07
$\|\tilde{u}_h-u\|$	7.2954e-05	4.8075e-06	3.0135e-07

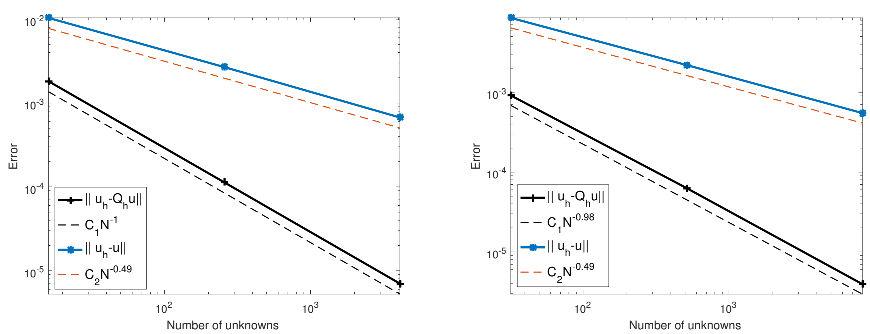

Figure 1. Example 1: the convergence order in norms (5.3) and (5.4) on rectangular (left) and triangular mesh(right).

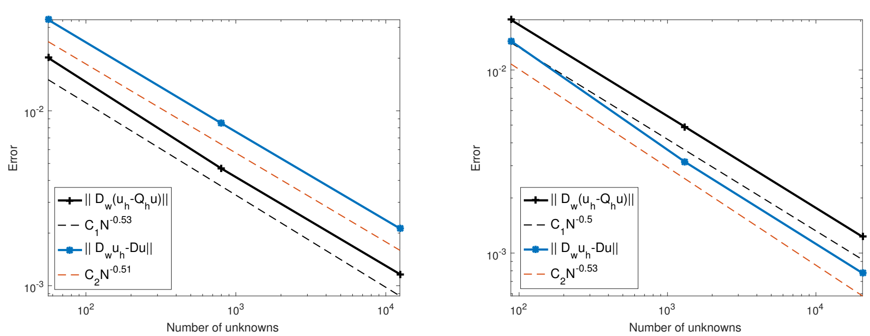

Figure 2. Example 1: the convergence order in norms (5.1) and (5.2) on rectangular (left) and triangular mesh(right).

then, we have

$$g=2\pi^2\sin(\pi x_1)\sin(\pi x_2)+(\sin(\pi x_1)\sin(\pi x_2))^3.$$

Similar as for Example 1, in Tables 5, 6, 7 and 8, we present the error of weak Galerkin method and two-grid weak Galerkin method for Example 2. Also, we show the convergence orders of the solutions in Figures 3 and 4 with norms defined in (5.1),(5.2),(5.3) and (5.4) respectively. As we can see from the data in the Tables and Figures, we get the same conclusion as in Example 1.

TABLE 5. Example 2: Error in norm (5.1) and (5.2) on rectangular mesh

(H,h)	$(1/2,1/4)$	$(1/4,1/16)$	$(1/8,1/64)$
$\|\nabla_{w,r}(u_h - Q_h u)\|$	3.0553e-01	6.3906e-02	1.5754e-02
$\|\nabla_{w,r}(\tilde{u}_h - Q_h u)\|$	2.8834e-01	6.2935e-02	1.5751e-02
$\|\nabla_{w,r} u_h - \nabla u\|$	1.5806e-01	4.0046e-02	1.0019e-02
$\|\nabla_{w,r} \tilde{u}_h - \nabla u\|$	1.5741e-01	4.0022e-02	1.0022e-02

TABLE 6. Example 2: Errors in norms (5.3) and (5.4) on rectangular element

(H,h)	$(1/2,1/4)$	$(1/4,1/16)$	$(1/8,1/64)$
$\|u_h - Q_h u\|$	4.6088e-02	2.9945e-03	1.8759e-04
$\|\tilde{u}_h - Q_h u\|$	4.0391e-02	1.6416e-03	1.5698e-04
$\|u_h - u\|$	8.4035e-01	2.1758e-01	5.4514e-02
$\|\tilde{u}_h - u\|$	8.4116e-01	2.1769e-01	5.4538e-02

TABLE 7. Example 2: Error in norm (5.1) and (5.2) on triangular element

(H,h)	$(1/2,1/4)$	$(1/4,1/16)$	$(1/8,1/64)$
$\|\nabla_{w,r}(u_h - Q_h u)\|$	2.7501e-01	6.3348e-02	1.5745e-02
$\|\nabla_{w,r}(\tilde{u}_h - Q_h u)\|$	2.6931e-01	6.3108e-02	1.5755e-02
$\|\nabla_{w,r} u_h - \nabla u\|$	6.3432e-01	1.6230e-01	4.0635e-02
$\|\nabla_{w,r} \tilde{u}_h - \nabla u\|$	6.3480e-01	1.6236e-01	4.0648e-02

TABLE 8. Example 2: Errors in norms (5.3) and (5.4) on triangular element

(H,h)	$(1/2,1/4)$	$(1/4,1/16)$	$(1/8,1/64)$
$\|u_h - Q_h u\|$	2.5778e-02	1.6642e-03	1.0419e-04
$\|\tilde{u}_h - Q_h u\|$	2.2270e-02	1.0079e-03	1.4166e-04
$\|u_h - u\|$	1.6542e-02	1.0686e-03	6.6923e-05
$\|\tilde{u}_h - u\|$	1.6508e-02	1.0686e-03	6.6961e-05

We also show the computational time of the weak Galerkin method and the two-grid weak Galerkin method in Table 9, as we can see from the result, for small scaled problem, the two-grid weak Galerkin method cost almost the same time as the weak Galerkin method. But for large scaled problems, we can see the two-grid weak Galerkin method cost almost half of that of the weak Galerkin method, which shows the efficiency of the two-grid weak Galerkin method.

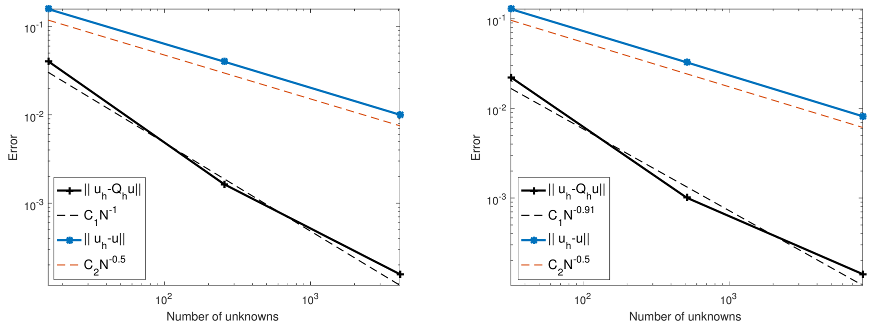

Figure 3. Example 2: the convergence order in norms (5.3) and (5.4) on rectangular (left) and triangular mesh(right).

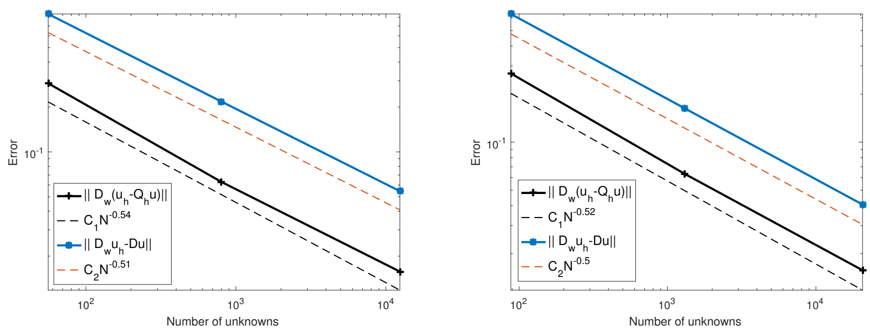

Figure 4. Example 2: the convergence order in norms (5.1) and (5.2) on rectangular (left) and triangular mesh(right).

Table 9. CPU time between two-grid method and weak Galerkin method

(H, h)	Two-grid(s)	WGFEM(s)
$(1/2, 1/4)$	0.0595	0.0321
$(1/4, 1/16)$	0.0793	0.0734
$(1/8, 1/64)$	0.4112	0.7264
$(1/16, 1/256)$	8.9317	20.7955

6. Conclusion

In this paper, we proposed a two-grid weak Galerkin method for the semilinear elliptic equation on both triangular and rectangular meshes. We show that when the mesh sizes H, h satisfy the relation $h = H^2$, the approximate solution by two-grid weak Galerkin method achieves the same convergence order as the weak Galerkin method. Two numerical examples are given to verify the theoretical results. In our future work, we will apply the proposed efficient two-grid weak Galerkin method for more complicated quasilinear or nonlinear partial differential equations.

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