Received 20 December 2021; | Revised XX XXXX 2022; | Accepted XX XXXX 2022

DOI: xxx/xxxx

ARTICLE TYPE

Finite-time boundary stabilization of fractional reaction-diffusion systems

Run-Jie Zhang | Liming Wang* | Kai-Ning Wu

Department of Mathematics, Harbin Insititute of Technology, Weihai 264209, China

Correspondence
*Liming Wang, Department of Mathematics, Harbin Insititute of Technology, Weihai 264209, China. Email: limingwang@hit.edu.cn

Summary

This paper investigates the boundary finite-time stabilization of fractional reaction-diffusion systems (FRDSs). First, a distributed controller is designed, and sufficient conditions are obtained to ensure the finite-time stability (FTS) of FRDSs under the designed controller. Then, a boundary controller is presented to achieve the FTS. By virtue of Lyapunov functional method and inequality techniques, sufficient conditions are presented to ensure the FTS of FRDSs via the designed boundary controller. The effect of diffusion term of FRDSs on the FTS is also investigated. Both Neumann and mixed boundary conditions are considered. Moreover, the robust finite-time stabilization of uncertain FRDSs is studied when there are uncertainties in the system's coefficients. Under the designed boundary controller, sufficient conditions are presented to guarantee the robust FTS of uncertain FRDSs. Finally, numerical examples are presented to verify the effectiveness of our theoretical results.

KEYWORDS:
fractional reaction-diffusion systems, distributed control, boundary control, finite-time stability, robust finite-time stability.

1 | INTRODUCTION

The fractional calculus attracted much concern in recent few decades since it is an effective way to describe many phenomena such as the economy, electromagnetism, bioengineering, fluid mechanics, ecology, and so on, see [1,2,3,4,5,6] and the references therein. Comparing with integer-order differential systems, fractional-order differential systems have several advantages. First, it can elegantly describe the memory and genetic characteristics of various phenomenons. Second, its memory is unlimited. Third, it has more degrees of freedom [7]. The diffusion phenomenon is inevitable when there exists the density non-uniformity for the state of the considered systems, such as, the air pressure, the temperature, and the electron [8].The reaction-diffusion model can excellently explain such systems with the diffusion phenomenon. In recent years, it has been widely used to describe the system dynamics found in many applications, such as the chemical processes, fluid flows, neural networks and biological pattern formation [9,10,11,12,13]. The fractional reaction-diffusion equation was investigated in [14] for a continuous-time random walk model with temporal memory and sources. From then on, a significant number of results on the fractional reaction-diffusion systems (FRDSs) were reported [15,16,17,18,19,20]. Due to measurement errors and external disturbances, usually, the system parameters are uncertain. For uncertain FRDSs, some results are presented [21,22].

Finite-time stability (FTS) arises from practical applications. In the real applications, we usually prefer the states of the system to be stable in a finite time rather than in an infinite time like asymptomatic stability and exponential stability. For example, we need the robot reach the specified position in a finite time and meet the specified speed [23]. The concept of FTS was introduced in[24], and further developed by Bhat and Bernstein[25]. Actually, the FTS analysis for the fractional-order differential systems also has tremendous significance. Many results on FTS of fractional-order differential systems were reported[26,27,28,29,30]. In[26], the authors summarized the processing methods of FTS of fractional differential systems and divided these methods into two groups. One mainly uses the Holder's inequality[29] while the other treats it by the generalized Gronwall's inequality[27]. The stabilization of reaction-diffusion systems (RDSs) has attracted widely attention as well[31,32,33]. For RDSs, there exists a specific control strategy, boundary control, to achieve FTS[33]. Boundary control only places the actuators on the boundary of the spatial region. It is an effective and commonly used control method in practical applications, and has been well researched for FRDSs[34,35]. There was few results for the finite-time stabilization of FRDSs via boundary control, while FTS of RDSs by boundary control and FTS of fractional ordinary differential systems has been sufficiently investigated, respectively.

Motivated by the above analysis, we study the FTS of FRDSs. Both FRDSs and uncertain FRDSs are considered. These are several difficulties during our work. First, the main difficulty is the controller design. Because of the special form of Caputo fractional derivative defined by integral, the finite-time controllers, adopted for the ordinary differential systems, fail for FRDSs. Second, how to deal with diffusion term is also a difficulty in stabilizing the FRDSs. Due to the existence of the diffusion term, common methods to deal with the fractional-order ordinary differential systems, such as Gronwall's inequality, Holder's inequality and other inequality technology, are inaccessible to FRDSs. Third, the uncertainty is also a difficulty for the analysis of the robustness.

In this paper, we focus on the finite-time stabilization of FRDSs. First, the distributed control is investigated to satisfy FTS of FRDSs. Then, boundary controllers are designed. Using the FTS lemma and Wirtinger's inequality, sufficient conditions of FTS are derived for FRDSs based on the designed controllers. The uncertain FRDSs are also investigated under the designed boundary controller. Moreover, the influences of the diffusion term and uncertainty term on the stability are considered. Finally, we present numerical simulations to verify the effectiveness of our boundary controllers. The main contributions of this work are listed as follows

- Boundary controllers are designed for FRDSs with different boundary conditions, and sufficient conditions of FTS for FRDSs are obtained.
- The robust FTS of uncertain FRDSs is also investigated based on the boundary controller, and sufficient conditions are derived.
- The effects of diffusion item and uncertain item on the stability are displayed from our theoretical results.

Notations: $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ represent the n-dimensional Euclidean space and set of $n\times m$ real matrices, respectively. I_n represents the n-dimensional unit matrix. $W^{l,2}([0,L];\mathbb{R}^n)$ is the Sobolev space which contains absolutely n-dimensional continuous functions $y(x):[0,L]\to\mathbb{R}^n$ with l-th square integrable derivatives $\frac{d^l y(x)}{dx^l}$ of the order $l\geq 1$. $L^2([a,b],\mathbb{R}^n)$ is the set of vector-valued functions $u(x):[a,b]\to\mathbb{R}^n$ that is square-integrable. $\|u(\cdot)\| = \left(\int_a^b u^{\mathrm{T}}(x)u(x)\mathrm{d}x\right)^{\frac{1}{2}}$ is norm in $L^2([a,b],\mathbb{R}^n)$. $|y(\cdot)|$ denotes the vector norm, that is, $|y(\cdot)| = \sqrt{y^{\mathrm{T}}(\cdot)y(\cdot)}$. $M\geq 0$ ($M\leq 0$) means that the constant symmetric matrix M is a positive semi-definite (negative semi-definite) matrix. $\lambda_{\min}(M)$ is the minimum eigenvalue of matrix M.

2 | MODEL DESCRIPTION AND PRELIMINARIES

Definition 1 ([36]). The Caputo fractional derivative of order $0<\alpha<1$ of a continuous function $f(t)$ is given by

$$ {}_0^C D_t^\alpha y(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{\partial y(s,t)}{\partial s} \frac{\mathrm{d}s}{(t-s)^\alpha}, \tag{1} $$

in which

$$ \Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1}\mathrm{d}t. $$

Definition 2 ([36]). The Mittag-Leffler function with one parameter is defined as follows:

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \tag{2} $$

where the parameter $\alpha > 0$ and $z \in \mathbb{C}$. For $\alpha = 1$, we have $E_1(z) = e^z$.

Lemma 1 ([37]). Let $v(t) \in \mathbb{R}^n$ be a continuous vector function, for any time instant $t > 0$, $0 < \alpha < 1$, we have

$$ {}_0^C D_t^\alpha v^{\mathrm{T}}(t)V(t) \le 2v^{\mathrm{T}}(t){}_0^C D_t^\alpha v(t). \tag{3} $$

We consider the following fractional reaction-diffusion systems (FRDSs)

$$ {}_0^C D_t^\alpha y(x,t) = f(y(x,t)) + B\frac{\partial^2 y(x,t)}{\partial x^2}, x \in (0,1), t > 0, \tag{4} $$

where x is the space variable, t is the time variable, $y(x,t) \in \mathbb{R}^n$ is the system state, and $B \in \mathbb{R}^{n\times n}$ is a positive definite matrix. ${}_0^C D_t^\alpha$ is the Caputo derivative of order $0 < \alpha < 1$.

The initial value of system (4) is given as

$$ y(x,0) = \phi(x), \tag{5} $$

where $\phi(x)$ is a given continuous function.

Assumption 1. We assume that $f(y)$ satisfies Lipschitz condition. That is, there exists a positive constant L, for any $p, q \in \mathbb{R}^n$, we have

$$ [f(p) - f(q)]^{\mathrm{T}} [f(p) - f(q)] \le L(p-q)^{\mathrm{T}}(p-q). \tag{6} $$

Definition 3 ([38]). If there exists a constant t^* satisfies

$$ \lim_{t\to t^*} \| y(x,t) \| = 0, \tag{7} $$

and

$$ \| y(x,t) \| = 0, \quad t > t^*, \tag{8} $$

then FRDS (4) achieves finite-time stability (FTS). We call t^* is the settling time.

Lemma 2 ([38]). Let $V(t)$ be a positive continuous and differentiable function on $[t_0, +\infty)$. If

$$ {}_0^C D_t^\alpha V(t) \le -kV(t) - \gamma, $$

where $0 < \alpha < 1$, $k > 0$ and $\gamma > 0$, then for any $t \ge t_0$, we obtain

$$ V(t) \le \left(V(t_0) + \frac{\gamma}{k} \right) E_\alpha(-k(t-t_0)^\alpha) - \frac{\gamma}{k}. \tag{9} $$

When $k = 0$, one has

$$ V(t) \le V(t_0) - \frac{\gamma(t-t_0)^\alpha}{\Gamma(\alpha+1)}. \tag{10} $$

Lemma 3 (Wirtinger's Inequality[39]). Let $y \in W^{l,2}([0,L];\mathbb{R}^n)$ be a vector function with $y(0) = 0$ or $y(l) = 0$. Then for positive definite matrix R, the following integral inequality holds

$$ \int_0^L y^{\mathrm{T}}(\eta)Ry(\eta)\mathrm{d}\eta \le \frac{4L^2}{\pi^2} \int_0^L (\frac{\mathrm{d}y}{\mathrm{d}\eta})^{\mathrm{T}} R(\frac{\mathrm{d}y}{\mathrm{d}\eta})\mathrm{d}\eta. $$

3 | MAIN RESULTS

In this section, we consider the FTS of FRDSs. First, we design a distributed controller to achieve the FTS of FRDSs and give sufficient conditions for FTS. In addition, since the boundary control is easier to implement and more economical in practical applications, we investigate it and obtain sufficient conditions to ensure the FTS of FRDSs. Both the Neumann boundary and mixed boundary conditions are considered. The effect of diffusion term on FTS is also investigated. Moreover, for uncertain FRDS, robust FTS is investigated under the designed boundary controller, and the corresponding sufficient condition is presented to ensure the robust FTS.

3.1 | Distributed control

Consider the following FRDSs with the distributed controller

$$ {}_{0}^{C}D_{t}^{\alpha} y(x,t) = f(y(x,t)) + B\frac{\partial^2 y(x,t)}{\partial x^2} + u(x,t), x \in (0,1), t > 0, \tag{11}$$

where $u(x,t)$ is the distributed controller.

We consider the Neumann boundary conditions as follows

$$ \left.\frac{\partial y(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial y(x,t)}{\partial x}\right|_{x=1} = 0. \tag{12}$$

The distributed controller is designed as follows to obtain the FTS,

$$ u(t) = \begin{cases} \left(-\frac{a}{2} - \frac{\gamma}{2|y(x,t)|^2}\right) y(x,t), & y(x,t) \neq 0, \\ 0, & y(x,t) = 0, \end{cases} \tag{13}$$

where a and γ are constants to be determined, and constant γ is positive and $|y(x,t)|^2 = y^{\mathrm{T}}(x,t)\, y(x,t)$.

Theorem 1. If the following inequality holds

$$ 1 + L - a \leq 0, \tag{14}$$

then, system (11) achieves FTS, and if $k = -(1+L-a) > 0$, the settling time t^* satisfies

$$ t^* \leq \left(-\frac{\sigma}{k}\right)^{\frac{1}{\alpha}}, \tag{15}$$

where

$$ \sigma = \max\left\{ z | E_\alpha(z) = \frac{\gamma}{kV(0)+\gamma} \right\} = \max\left\{ z | E_\alpha(z) = \frac{\gamma}{k\int_0^1 \phi^{\mathrm{T}}(x)\phi(x)\mathrm{d}x + \gamma} \right\}. \tag{16}$$

If $k = -(1+L-a) = 0$, the settling time t^* satisfies

$$ t^* \leq \left(\frac{\Gamma(\alpha+1)V(0)}{\gamma}\right)^{\frac{1}{\alpha}}. \tag{17}$$

Proof. Define the Lyapunov functional as follows

$$ V(t) = \int_0^1 y^{\mathrm{T}}(x,t) y(x,t) \mathrm{d}x. \tag{18}$$

Taking Caputo fractional derivative of $V(t)$ along system (11) and using Lemma 1, we have

$$ {}_{0}^{C}D_{t}^{\alpha} V(t) \leq 2\int_0^1 y^{\mathrm{T}}(x,t)\, {}_{0}^{C}D_{t}^{\alpha} y(x,t)\mathrm{d}x = 2\int_0^1 y^{\mathrm{T}} \left[f(y(x,t)) + B\frac{\partial^2 y(x,t)}{\partial x^2} + u(x,t) \right] \mathrm{d}x. \tag{19}$$

With $f(0) = 0$, Assumption 1 and the inequality $2y^{\mathrm{T}} f(y) \leq y^{\mathrm{T}} y + f(y)^{\mathrm{T}} f(y)$, we obtain the following inequality

$$ 2\int_0^1 y^{\mathrm{T}} f(y)\mathrm{d}x \leq \int_0^1 y^{\mathrm{T}} y + f(y)^{\mathrm{T}} f(y)\mathrm{d}x \leq (1+L)\int_0^1 y^{\mathrm{T}} y\mathrm{d}x. \tag{20}$$

Hereafter, for simplicity, the variables (t,x) are omitted. Using the boundary conditions (12), we have

$$ \int_0^1 y^{\mathrm{T}} B\frac{\partial^2 y}{\partial x^2}\mathrm{d}x = y^{\mathrm{T}} B\left.\frac{\partial y}{\partial x}\right|_{x=0}^{x=1} - \int_0^1 \frac{\partial y^{\mathrm{T}}}{\partial x} B\frac{\partial y}{\partial x}\mathrm{d}x = -\int_0^1 \frac{\partial y^{\mathrm{T}}}{\partial x} B\frac{\partial y}{\partial x}\mathrm{d}x. \tag{21}$$

Boundary conditions (12) can not yield either $y(0,t) = 0$ or $y(1,t) = 0$, which means that Lemma 3 cannot work in (21). To deal with this problem, we take the transformation as follows

$$ \bar{y}(x,t) = y(x,t) - y(1,t), \tag{22}$$

then we have $\bar{y}^{\mathrm{T}}(1,t)=0$. From Lemma 3, we have

$$\int_0^1 y^{\mathrm{T}} B \frac{\partial^2 y}{\partial x^2}\mathrm{d}x = -\int_0^1 \frac{\partial y^{\mathrm{T}}}{\partial x} B \frac{\partial y}{\partial x}\mathrm{d}x = -\int_0^1 \frac{\partial \bar{y}^{\mathrm{T}}}{\partial x} B \frac{\partial \bar{y}}{\partial x}\mathrm{d}x \le -\frac{\pi^2}{4}\int_0^1 \bar{y}^{\mathrm{T}} B \bar{y}\mathrm{d}x. \tag{23}$$

Substituting inequalities (20) and (23) into (19) and making use of inequality (14), we obtain

$$\begin{aligned} {}_0^C D_t^{\alpha} V(t) &\le (1+L)\int_0^1 \left(y^{\mathrm{T}} y + 2y^{\mathrm{T}} u\right)\mathrm{d}x - \frac{\pi^2}{2}\int_0^1 \bar{y}^{\mathrm{T}} B\bar{y}\mathrm{d}x \\ &= \int_0^1 \left\{ y^{\mathrm{T}}\left[(1+L-a)I_n\right] y - \frac{\pi^2}{2}\bar{y}^{\mathrm{T}} B\bar{y} - \gamma \right\}\mathrm{d}x \\ &\le -\int_0^1 k y^{\mathrm{T}} y \mathrm{d}x - \gamma \\ &= -kV(t) - \gamma, \end{aligned} \tag{24}$$

where $k=-(1+L-a)$.

If $k=0$, that is

$$ {}_0^C D_t^{\alpha} V(t) \le -\gamma. $$

Making use of Lemma 2 and (18), one has

$$\|y(\cdot,t)\|^2 \le \|y(\cdot,0)\|^2 - \frac{\gamma t^{\alpha}}{\Gamma(\alpha+1)}.$$

To get the t^* satisfying $\|y(\cdot,t^*)\|=0$, let $\|y(\cdot,0)\|^2 - \frac{\gamma t^{\alpha}}{\Gamma(\alpha+1)}=0$, and we obtain the solution of t that

$$t_1 = \left(\frac{\Gamma(\alpha+1)V(0)}{\gamma}\right)^{\frac{1}{\alpha}}.$$

Now we state that $\|y(\cdot,t)\|=0$ for any $t>t_1$. If this is invalid, that is, there is a $t_1^*>t_1$ satisfies $\|y(\cdot,t_1^*)\|>0$. Then, we get the following contradiction

$$0<\|y(\cdot,t_1^*)\|^2 \le \|y(\cdot,0)\|^2 - \frac{\gamma (t_1^*)^{\alpha}}{\Gamma(\alpha+1)} < \|y(\cdot,0)\|^2 - \frac{\gamma t_1^{\alpha}}{\Gamma(\alpha+1)} = 0.$$

Therefore, one has $\|y(\cdot,t)\|=0$ for any $t\ge t_1$ and the settling time t^* is estimated as follows

$$t^* \le \left(\frac{\Gamma(\alpha+1)V(0)}{\gamma}\right)^{\frac{1}{\alpha}}.$$

If $k>0$, that is

$$ {}_0^C D_t^{\alpha} V(t) \le -kV(t) - \gamma. $$

Making use of Lemma 2 and (18), one has

$$\|y(\cdot,t)\|^2 \le \left(\|y(\cdot,0)\|^2 + \frac{\gamma}{k}\right) E_{\alpha}(-kt^{\alpha}) - \frac{\gamma}{k}. \tag{25}$$

Similar to the proof of[38], let

$$g(t) = \left(\|y(\cdot,0)\|^2 + \frac{\gamma}{k}\right) E_{\alpha}(-kt^{\alpha}) - \frac{\gamma}{k},$$

where $t\ge 0$. Note that $E_{\alpha}(-kt^{\alpha})$ is non-increasing for all $k>0$, it is easy to derive that $g(t)$ is a non-increasing function. Note that $E_{\alpha}(0)=1$ and $\lim_{t\to+\infty} E_{\alpha}(-t)=0$[40], one has

$$g(0)=\|y(\cdot,0)\|^2>0,$$

and

$$\lim_{t\to+\infty} g(t) = -\frac{\gamma}{k} < 0.$$

From the continuity of $g(t)$, we obtain that there exists t_2 satisfying $g(t_2) = 0$, which is equivalent to

$$E_\alpha(-kt_2^\alpha) = \frac{\gamma}{k\|y(\cdot,0)\|^2 + \gamma} = \frac{\gamma}{kV(0) + \gamma}.$$

Notice that $E_\alpha(z)$ is a non-negative and non-increasing function, and t_2 satisfies

$$t_2 = \left(-\frac{\sigma}{k}\right)^{\frac{1}{\alpha}},$$

in which

$$\sigma = \max\left\{ z | E_\alpha(z) = \frac{\gamma}{kV(0) + \gamma} \right\}.$$

Then we state that $\|y(\cdot,t)\| = 0$ for any $t > t_2$. If this is invalid, that is, there is a $t_2^* > t_2$ satisfying $\|y(\cdot,t_2^*)\| > 0$. Making use of non-increasing function $g(t)$ and inequality (25), we obtain the following contradiction

$$0 < \|y(\cdot,t_2^*)\|^2 \le g(t_2^*) \le g(t_2) = 0.$$

This means that the settling time t^* satisfies

$$t^* \le \left(-\frac{\sigma}{k}\right)^{\frac{1}{\alpha}}, \tag{26}$$

where

$$\sigma = \max\left\{ z | E_\alpha(z) = \frac{\gamma}{kV(0) + \gamma} \right\}, \tag{27}$$

and

$$V(0) = \int_0^1 \phi^{\mathrm{T}}(x)\phi(x)\mathrm{d}x.$$

This proof is complete.

□

Theorem 1 is given for the FTS of system (11) with the Neumann boundary conditions. Next, we consider system (11) with the following mixed boundary conditions

$$\frac{\partial y(0,t)}{\partial x} = y(1,t) = 0, \qquad \text{or} \qquad \frac{\partial y(1,t)}{\partial x} = y(0,t) = 0. \tag{28}$$

FTS of FRDSs using distributed controller (13) can be proved following the line of the proof of Theorem 1. What we need to pay attention to is that Lemma 3 can be directly applied without the transformation (22), then we have the following proposition.

Proposition 1. System (11) achieves FTS under the distributed controller (13), if the following matrix inequality holds

$$(1 + L - a)I_n - \frac{\pi^2}{2}B \le 0. \tag{29}$$

Moreover, we set $k = -(1 + L - a) + \frac{\pi^2}{2}\lambda_{\min}(B)$. If $k > 0$, the settling time t^* satisfies inequality (15). If $k = 0$, the settling time t^* satisfies inequality (17).

3.2 | Boundary control

We have designed a distributed controller to achieve FTS in above subsection. Distributed control means that every point in the spatial domain is equipped with the controllers. In practical applications, the boundary control is more economical and easier to be implemented, and the boundary control has become a better choice for the FTS of the controlled systems.

In this subsection, we consider system (4) with the following Neumann boundary conditions

$$\frac{\partial y(0,t)}{\partial x} = 0, \qquad \text{and} \qquad \frac{\partial y(1,t)}{\partial x} = u(t), \tag{30}$$

where $u(t)$ is the boundary controller.

We design the following boundary controller to obtain FTS for FRDS (4).

$$u(t)=\begin{cases}\dfrac{y(1,t)}{2\left|y(1,t)\right|^2}\left(-\dfrac{\gamma}{\lambda_1}-\mu\int_0^1 y^{\mathrm{T}}y\mathrm{d}x\right), & y(1,t)\neq 0,\\ 0, & y(1,t)=0,\end{cases} \tag{31}$$

where $\lambda_1=\lambda_{\min}(B)$ and $\left|y(1,t)\right|^2=y^{\mathrm{T}}(1,t)y(1,t)$, and γ and μ are positive constants. Now, we present a finite-time stabilization criterion for system (4) under the boundary controller (31).

Theorem 2. If the constant μ satisfies the following inequality

$$1+L-\lambda_1\mu\leq 0, \tag{32}$$

then, system (4) achieves FTS, and if $k=-(1+L-\lambda_1\mu)>0$, the settling time t^* satisfies

$$t^*\leq\left(-\frac{\sigma}{k}\right)^{\frac{1}{\alpha}}, \tag{33}$$

in which

$$\sigma=\max\left\{z|E_\alpha(z)=\frac{\gamma}{kV(0)+\gamma}\right\}, \tag{34}$$

where $V(0)=\int_0^1\phi^{\mathrm{T}}(x)\phi(x)\mathrm{d}x$. If $k=0$, settling time satisfies

$$t^*\leq\left(\frac{\Gamma(\alpha+1)V(0)}{\gamma}\right)^{\frac{1}{\alpha}}. \tag{35}$$

Proof. Construct the Lyapunov functional

$$V(t)=\int_0^1 y^{\mathrm{T}}(x,t)y(x,t)\mathrm{d}x. \tag{36}$$

Taking Caputo fractional derivative of $V(t)$ along system (4) and using Lemma 1, we have

$${}_0^CD_t^\alpha V(t)\leq 2\int_0^1 y^{\mathrm{T}}\,{}_0^CD_t^\alpha y\mathrm{d}x \quad =2\int_0^1 y^{\mathrm{T}}\left[f(y)+B\frac{\partial^2 y}{\partial x^2}\right]\mathrm{d}x. \tag{37}$$

Integrating by parts and using boundary conditions (30) result in

$$\int_0^1 y^{\mathrm{T}}B\frac{\partial^2 y}{\partial x^2}\mathrm{d}x=y^{\mathrm{T}}B\frac{\partial y}{\partial x}\Big|_{x=0}^{x=1}-\int_0^1\frac{\partial y^{\mathrm{T}}}{\partial x}B\frac{\partial y}{\partial x}\mathrm{d}x=y^{\mathrm{T}}(1,t)Bu(t)-\int_0^1\frac{\partial y^{\mathrm{T}}}{\partial x}B\frac{\partial y}{\partial x}\mathrm{d}x. \tag{38}$$

Adopting transformation (22) and Lemma 3, we obtain

$$\int_0^1 y^{\mathrm{T}}B\frac{\partial^2 y}{\partial x^2}\mathrm{d}x=y^{\mathrm{T}}(1,t)Bu(t)-\int_0^1\frac{\partial y^{\mathrm{T}}}{\partial x}B\frac{\partial y}{\partial x}\mathrm{d}x\leq y^{\mathrm{T}}(1,t)Bu(t)-\frac{\pi^2}{4}\int_0^1\bar{y}^{\mathrm{T}}B\bar{y}\mathrm{d}x. \tag{39}$$

By virtue of Assumption 1, we obtain

$$2\int_0^1 y^{\mathrm{T}}f(y)\mathrm{d}x\leq(1+L)\int_0^1 y^{\mathrm{T}}y\mathrm{d}x. \tag{40}$$

Substituting (39) and (40) into (37) and making use of (31), we obtain

$$
\begin{aligned}
{}_0^C D_t^\alpha V(t) &\leq (1+L)\int_0^1 y^{\mathrm{T}} y \mathrm{d}x + 2y^{\mathrm{T}}(1,t)\, Bu(t) - \frac{\pi^2}{2}\int_0^1 \bar{y}^{\mathrm{T}} B\bar{y}\mathrm{d}x \\
&= \int_0^1 \left[(1+L)y^{\mathrm{T}}y - \frac{\pi^2}{2}\bar{y}^{\mathrm{T}}B\bar{y}\right]\mathrm{d}x + \frac{y^{\mathrm{T}}(1,t)By(1,t)}{|y(1,t)|^2}\left(-\frac{\gamma}{\lambda_1} - \mu\int_0^1 y^{\mathrm{T}}y\mathrm{d}x\right) \\
&\leq \int_0^1 \left[(1+L)y^{\mathrm{T}}y - \frac{\pi^2}{2}\bar{y}^{\mathrm{T}}B\bar{y}\right]\mathrm{d}x - \gamma - \lambda_1\mu\int_0^1 y^{\mathrm{T}}y\mathrm{d}x \\
&= \int_0^1 \left\{(1+L-\lambda_1\mu)y^{\mathrm{T}}y - \frac{\pi^2}{2}\bar{y}^{\mathrm{T}}B\bar{y}\right\}\mathrm{d}x - \gamma \\
&\leq -kV(t) - \gamma,
\end{aligned}
\tag{41}
$$

where $k = -(1 + L - \lambda_1\mu) \geq 0$. Making use of Lemma 2 and techniques used in the proof of Theorem 1 yields that system (4) achieves FTS. Moreover, the settling time t^* satisfies (33) if $k > 0$ or inequality (35) if $k = 0$.

This ends the proof. □

Next, we study the case of system (4) with the following mixed boundary conditions

$$
y(0,t) = 0, \qquad \text{and} \qquad \frac{\partial y(1,t)}{\partial x} = u(t), \tag{42}
$$

in which $u(t)$ is the boundary control.

We present a sufficient condition to achieve the FTS of system (4) with the boundary controller (31) under boundary conditions (42).

Proposition 2. With boundary conditions (42), system (4) achieves FTS under the boundary controller (31), if the following matrix inequality holds

$$
(1 + L - \lambda_1\mu)I_n - \frac{\pi^2}{2}B \leq 0. \tag{43}
$$

Take $k = -\lambda_{\max}\left((1 + L - \lambda_1\mu)I_n - \frac{\pi^2}{2}B\right) = -(1 + L - \lambda_1\mu) + \frac{\pi^2}{2}\lambda_1$, then the settling time t^* satisfies inequality (33) if $k > 0$ or inequality (35) if $k = 0$.

Remark 1. From inequality (33), we see that the larger γ and k in the boundary controller, the smaller the settling time t^* and the faster convergent speed. Moreover, since k depends on control gain μ, then the constants γ and μ in the boundary controller (31) can be chosen to determine the desired settling time t^*.

Remark 2. From inequality (43), one sees that the diffusion coefficient B affects the stabilization of FRDSs. The larger $\lambda_{\min}(B)$, the easier to achieve the stability. Example 1 verifies this result.

3.3 | Robust finite-time stabilization

When there are uncertainties in the system's coefficients, it is necessary to study the robust stability of the uncertain FRDSs. In this subsection, we consider the following uncertain FRDSs

$$
{}_0^C D_t^\alpha y(x,t) = f(y(x,t)) + \Delta f(y(x,t)) + (B + \Delta B(t))\frac{\partial^2 y(x,t)}{\partial x^2}, x \in (0,1), t > 0, \tag{44}
$$

where $B > 0$ is a known matrix and $f(y)$ is a known non-linear function satisfying Assumption 1. $\Delta B(t)$ is an unknown matrix presenting the temporal uncertainty with the following property

$$
-\epsilon B \leq \Delta B(t) \leq \epsilon B, \tag{45}
$$

where $\epsilon \in (0,1)$ is a known constant. $\Delta f(y)$ is an uncertain factor, which presents potential errors such as modeling errors, and determined by the system state. We assumed that $\Delta f(y)$ satisfies the following condition

$$\Delta f^{\mathrm{T}}(y)\Delta f(y) \leq \beta. \tag{46}$$

Definition 4 ([33]). System (44) achieves the robust FTS if the uncertain FRDS is finite-time stable for all admissible $\Delta B(t)$ and $\Delta f(y(x,t))$.

Here we again use boundary controller (31) to achieve FTS for uncertain FRDS (44) with Neumann boundary conditions (30).

Theorem 3. If constants γ and μ satisfy the following inequalities

$$2 + L + (1-\epsilon)\lambda_1\mu \leq 0, \qquad \text{and} \qquad \beta - (1-\epsilon)\gamma \leq 0, \tag{47}$$

then, system (44) achieves FTS, and if $k = -(2 + L + (1-\epsilon)\lambda_1\mu) > 0$, the settling time t^* satisfies

$$t^* \leq \left(-\frac{\sigma}{k}\right)^{\frac{1}{\alpha}}, \tag{48}$$

where

$$\sigma = \max\left\{ z | E_\alpha(z) = \frac{\gamma_1}{kV(0) + \gamma_1} \right\}, \tag{49}$$

and

$$\gamma_1 = (1-\epsilon)\gamma - \beta, \quad V(0) = \int_0^1 \phi^{\mathrm{T}}(x)\phi(x)\mathrm{d}x. \tag{50}$$

If $k = 0$, settling time satisfies

$$t^* \leq \left(\frac{\Gamma(\alpha+1)V(0)}{\gamma}\right)^{\frac{1}{\alpha}}. \tag{51}$$

Proof. Let

$$V(t) = \int_0^1 y^{\mathrm{T}}(x,t)y(x,t)\mathrm{d}x. \tag{52}$$

In light of (45) and (46), we have the following inequality

$$\int_0^1 2y^{\mathrm{T}}\Delta f(y)\mathrm{d}x \leq \int_0^1 \left(y^{\mathrm{T}}y + \Delta f(y)^{\mathrm{T}}\Delta f(y)\right)\mathrm{d}x \leq \int_0^1 \left(y^{\mathrm{T}}y + \beta\right)\mathrm{d}x, \tag{53}$$

and

$$\begin{aligned} 2\int_0^1 y^{\mathrm{T}}(B+\Delta B(t))\frac{\partial^2 y}{\partial x^2}\mathrm{d}x &= 2y^{\mathrm{T}}(1,t)(B+\Delta B(t))u(t) - 2\int_0^1 \frac{\partial \bar{y}^{\mathrm{T}}}{\partial x}(B+\Delta B(t))\frac{\partial \bar{y}}{\partial x}\mathrm{d}x \\ &\leq \frac{y^{\mathrm{T}}(1,t)(B+\Delta B(t))y(1,t)}{\|y(1,t)\|^2}\left(-\frac{\gamma}{\lambda_1} - \mu\int_0^1 y^{\mathrm{T}}y\mathrm{d}x\right) - \frac{\pi^2}{2}\int_0^1 \bar{y}^{\mathrm{T}}(B+\Delta B(t))\bar{y}\mathrm{d}x \\ &\leq -(1-\epsilon)\gamma - (1-\epsilon)\lambda_1\mu\int_0^1 y^{\mathrm{T}}y\mathrm{d}x - \frac{\pi^2}{2}\int_0^1 \bar{y}^{\mathrm{T}}(B+\Delta B(t))\bar{y}\mathrm{d}x. \end{aligned} \tag{54}$$

Following the line of the proof of Theorem 2, substituting equation (39), (53), (54) and (40) into (37) and making use of (47), we obtain

$$
\begin{aligned}
{}_0^C D_t^{\alpha} V(t) &\le 2\int_0^1 y^{\mathrm{T}}\left[f(y)+\Delta f(y)+(B+\Delta B(t))\frac{\partial^2 y}{\partial x^2}\right]\mathrm{d}x \\
&\le \int_0^1 (2+L)y^{\mathrm{T}}y-\frac{\pi^2}{2}\bar{y}^{\mathrm{T}}(B+\Delta B(t))\bar{y}\mathrm{d}x+2y^{\mathrm{T}}(1,t)(B+\Delta B(t))u(t)+\beta \\
&\le -(1-\epsilon)\gamma+(2+L-(1-\epsilon)\lambda_1\mu)\int_0^1 y^{\mathrm{T}}y-\frac{\pi^2}{2}\bar{y}^{\mathrm{T}}(B+\Delta B(t))\bar{y}\mathrm{d}x+\beta \\
&\le \beta-(1-\epsilon)\gamma-k\int_0^1 y^{\mathrm{T}}y\mathrm{d}x=-\gamma_1-kV(t),
\end{aligned}
\tag{55}
$$

where $k=-(2+L-(1-\epsilon)\lambda_1\mu)$ and $\gamma_1=(1-\epsilon)\gamma-\beta$. Making use of Lemma 2 and techniques used in the proof of Theorem 1 yields that system (44) achieves FTS. Moreover, the settling time t^* satisfies (48).

□

For the case of system (44) with mixed boundary conditions (42), we have the following criterion to ensure the FTS.

Theorem 4. If matrix K and constant γ in boundary controller (31) satisfy the following inequalities

$$
(2+L-(1-\epsilon)\lambda_1\mu)I_n-\frac{(1-\epsilon)\pi^2}{2}B\le 0, \qquad \text{and} \qquad \beta-(1-\epsilon)\gamma\le 0. \tag{56}
$$

Then the system (44) achieves FTS, and the settling time t^* satisfies (48) in which $k=-(2+L-(1-\epsilon)\lambda_1\mu)+\frac{(1-\epsilon)\pi^2}{2}\lambda_1$ and $\gamma_1=(1-\epsilon)\gamma-\beta$.

Remark 3. Boundary controller (31) can achieve both FTS of FRDSs and robust FTS of uncertain FRDSs. In fact, by virtue of (50), (47) and the results in Remark 1, one sees that the larger ϵ and β are, and larger t^* is, and the harder to satisfy inequality (47). That is to say, uncertain terms ϵ and β have negative effects on the FTS.

Remark 4. We consider one form of mixed boundary conditions (42) in this paper. In fact, there exists another form of mixed boundary conditions

$$
y(1,t)=0, \qquad \text{and} \qquad \frac{\partial y(0,t)}{\partial x}=u(t). \tag{57}
$$

To solve FTS of FRDSs under the boundary condition (57), we design the following boundary controller

$$
u(t=\begin{cases} \dfrac{y(0,t)}{2\,|y(0,t)|^2}\left(\dfrac{\gamma}{\lambda_1}+\mu\int_0^1 y^{\mathrm{T}}y\mathrm{d}x\right), & y(0,t)\ne 0, \\ 0, & y(0,t)=0, \end{cases} \tag{58}
$$

where $\lambda_1=\lambda_{\min}(B)$ and $|y(0,t)|^2=y^{\mathrm{T}}(0,t)y(0,t)$. Constants γ and μ are positive constants. With the boundary controller (58), the FRDSs (4) and uncertain FRDSs (44) with boundary conditions 57 can achieve FTS , and the corresponding sufficient conditions are easy to present and can be proven by the techniques used in the previous proofs and we omit them.

4 | NUMERICAL SIMULATIONS

In this section, we give numerical examples to show the effectiveness of our theoretical results.

Example 1. Considering the following FRDSs

$$
{}_0^C D_t^{0.95} y(x,t)=0.05y(x,t)+0.23\frac{\partial^2 y(x,t)}{\partial x^2}, x\in(0,1), t>0, \tag{59}
$$

where $\alpha=0.95$, $B=0.23$ and $f(y)=0.05y(x,t)$. System (59) has the following initial function

$$
\phi(x)=1.5+0.25\sin(2\pi x),
$$

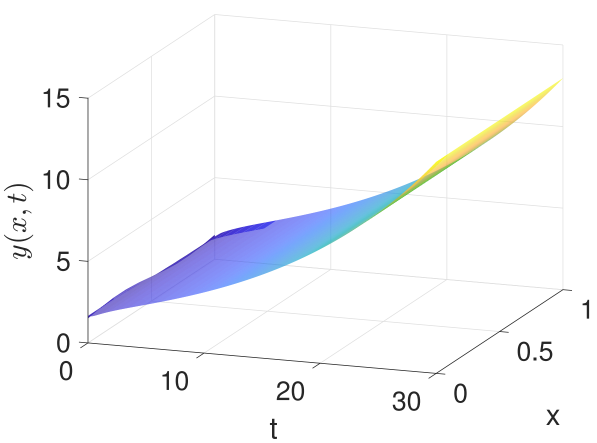
and the boundary conditions (30). By solving inequality (32) in Theorem 2, we have the following boundary controller

$$u(t) = \begin{cases} \dfrac{y(1,t)}{2\|y(1,t)\|^2}\left(-\dfrac{0.1}{0.29} - 7.0\int_0^1 y^{\mathrm{T}} y \mathrm{d}x\right), & y(1,t) \neq 0, \\ 0, & y(1,t) = 0, \end{cases} \tag{60}$$

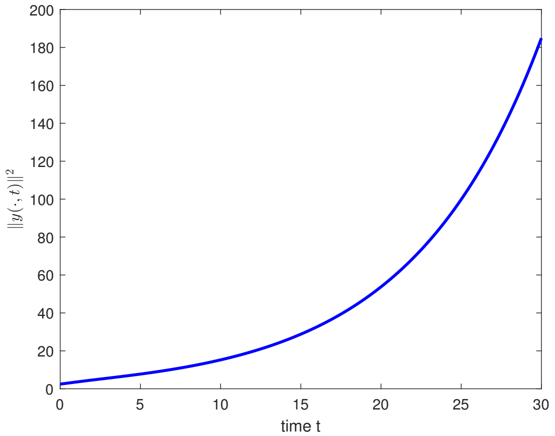
in which $\gamma = 0.1$ and $\mu = 7.0$.

Through the calculations, the settling time $t^* \leq 11.0494$. This means that system (59) achieves FTS theoretically. To show the effectiveness of our designed controller (60), we set $u(t) = 0$ in (31), and give the system state and corresponding norm in Figure 1 . We observe that system (59) is not finite-time stable. Then, we apply boundary controller (60), and the corresponding state $y(x,t)$ and corresponding norm are shown in Figure 2 . We notice that system (59) achieves FTS with the boundary controller (60).

Now, we show the influence of diffusion item B on the FTS which stated in Remark 2. We take $B = 0.41$ in system (59). Figure 3 shows the corresponding state norms of the systems. Note that the larger B is, the smaller t^* is and system is easier to achieve FTS, which verifies our statement in Remark 2.



(a) $y(x,t)$ for system (59)



(b) $\|y(\cdot,t)\|^2$ for system (59)

FIGURE 1 State responses of system (59) without a controller.

Example 2. To verify the uncertain finite-time stabilization of the boundary controller, the following robust FRDS is given

$${}_0^C D_t^{0.90} y(x,t) = 0.29 y(x,t) + 0.1 \sin y + (0.26 + 0.026 \sin(\pi t)) \frac{\partial^2 y(x,t)}{\partial x^2}, x \in (0,1), t > 0, \tag{61}$$

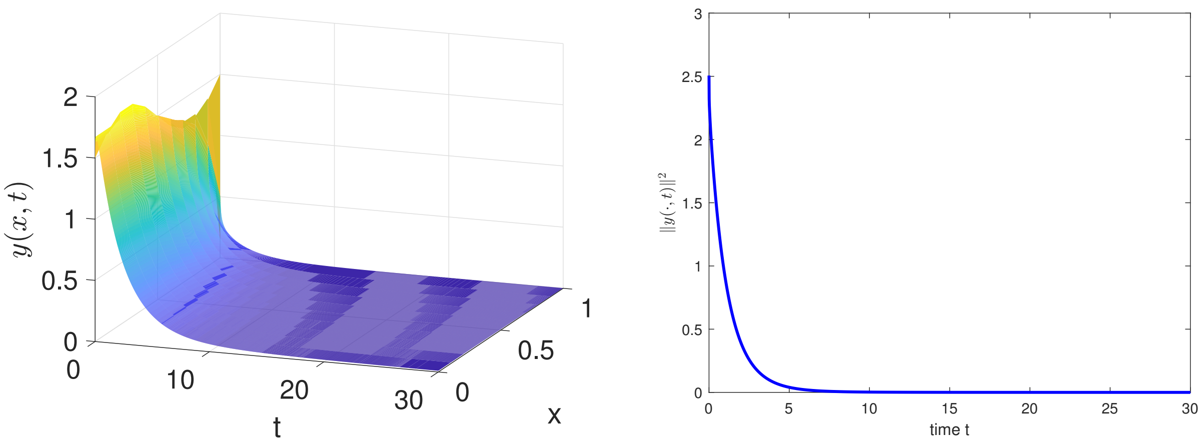
where $\Delta f(y(x,t)) = 0.1 \sin y$, $\Delta B(t) = 0.026 \sin(\pi t)$ satisfy conditions (45) and (46) with $\epsilon = 0.1$ and $\beta = 0.01$. System (61) is subject to the following initial function

$$y(x,0) = 1.5 - \sin(\frac{\pi}{3} x),$$

and Neumann boundary conditions (30). The boundary controller is designed as

$$u(t) = \begin{cases} \frac{y(1,t)}{2\|y(1,t)\|^2}\left(-\dfrac{0.1}{0.26} - 11.5\int_0^1 y^{\mathrm{T}} y \mathrm{d}x\right), & y(1,t) \neq 0, \\ 0, & y(1,t) = 0. \end{cases} \tag{62}$$

According to Theorem 3, we verify that system (61) achieve robust FTS under the boundary controller theoretically, and the settling time $t^* \leq 13.9593$. Figures 4 and 5 present the system states $y(x,t)$ and corresponding norms $\|y(\cdot,t)\|^2$ without and with the boundary controller, respectively. These figures verify our designed boundary controller is effective.



(a) $y(x,t)$ for system (59)

(b) $\|y(\cdot,t)\|^2$ for system (59)

FIGURE 2 State responses of system states with boundary controller (60).

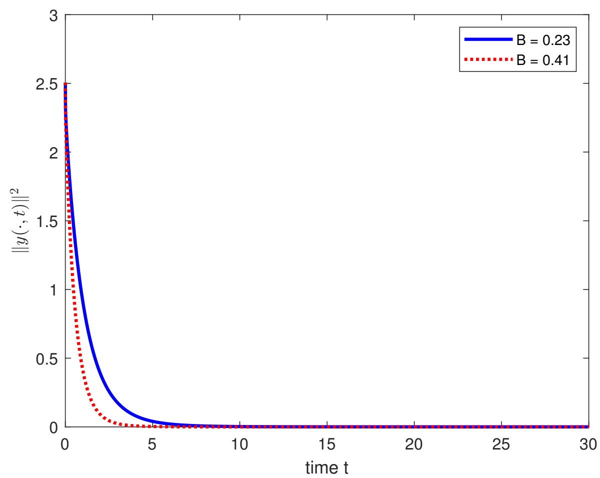


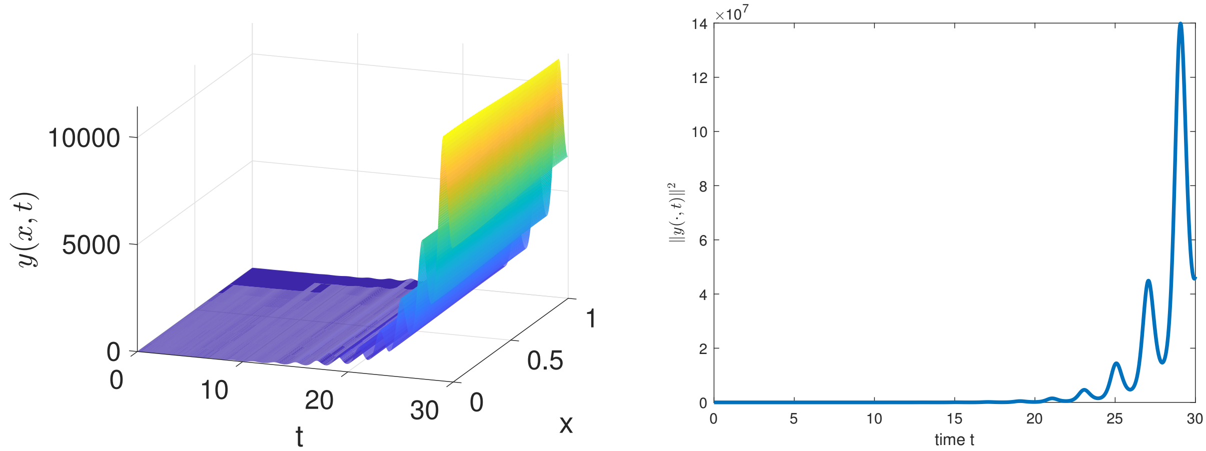
FIGURE 3 Corresponding state norms of the systems.

5 | CONCLUSIONS

This paper addresses the problem of finite-time stability (FTS) of fractional reaction-diffusion systems (FRDSs) under the boundary control. Using Lyapunov functional method and Wirtinger's inequality, sufficient conditions are obtained to ensure the FTS of FRDSs. Then the robust FTS for the uncertain FRDSs is investigated through the boundary controller. We also study the effects of diffusion coefficient and the strength of uncertain items. Finally, numerical examples are given to show the effectiveness of boundary controllers we designed. The design of the boundary controller to achieve the FTS of FRDSs may apply to the FTS of other FRDSs, such as the delay and stochastic FRDSs. Indeed, these are our future works.

ACKNOWLEDGEMENTS

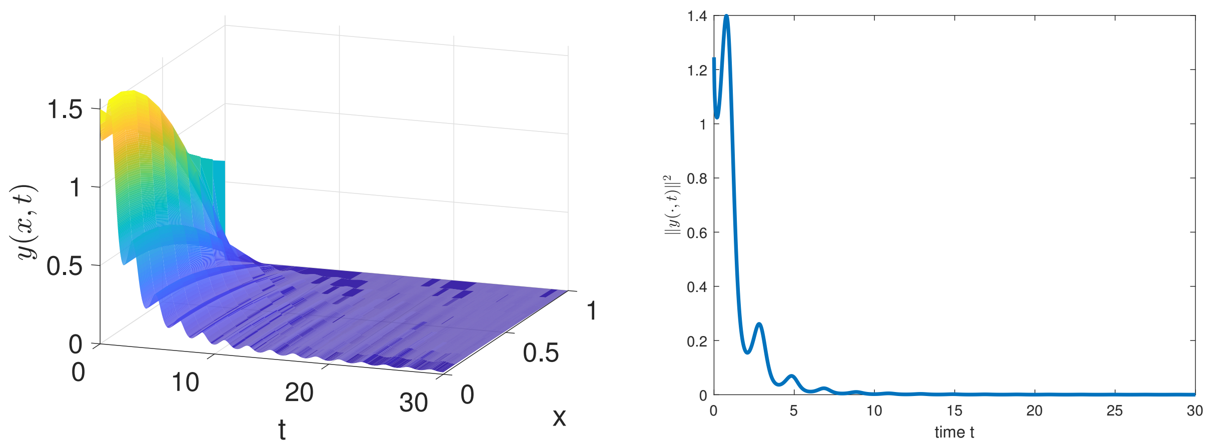
This work is supported by Natural Science Foundations of Shandong Province under Grant ZR2021MF007.



(a) $y(x,t)$ for system (61)

(b) $\|y(\cdot,t)\|^2$ for system (61)

FIGURE 4 State responses of system states without a controller.



(a) $y(x,t)$ for system (61)

(b) $\|y(\cdot,t)\|^2$ for system (61)

FIGURE 5 State responses of system states with boundary controller (62).

Conflict of interest

The authors declare no potential conflict of interests.

References

1. Barshan Billur, Ayrulu Birsel. Fractional Fourier transform pre-processing for neural networks and its application to object recognition. *Neural Networks.* 2002;15(1):131-140.

2. Engheta N.. On fractional calculus and fractional multipoles in electromagnetism. *IEEE Transactions on Antennas and Propagation.* 1996;44(4):554-566.

3. Laskin Nick. Fractional market dynamics. *Physica A: Statistical Mechanics and its Applications.* 2000;287(3):482-492.

4. Kulish Vladimir V., Lage Jose' L.. Application of fractional calculus to fluid mechanics. *Journal of Fluids Engineering.* 2002;124(3):803-806.

5. Kumar Sunil, Kumar Ranbir, Agarwal Ravi P., Samet Bessem. A study of fractional Lotka-Volterra population model using Haar wavelet and Adams-Bashforth-Moulton methods. *Mathematical Methods in the Applied Sciences.* 2020;43(8):5564-5578.

6. Khan Hasib, Gómez-Aguilar J.F., Alkhazzan Abdulwasea, Khan Aziz. A fractional order HIV-TB coinfection model with nonsingular Mittag-Leffler Law. *Mathematical Methods in the Applied Sciences.* 2020;43(6):3786-3806.

7. Velmurugan G., Rakkiyappan R., Cao Jinde. Finite-time synchronization of fractional-order memristor-based neural networks with time delays. *Neural Networks.* 2016;73(1):36-46.

8. Gajewski H., Groger K.. Reaction-diffusion processes of electrically charged species. *Mathematische Nachrichten.* 1996;177(1):109-130.

9. Ma Qian, Feng Gang, Xu Shengyuan. Delay-dependent stability criteria for reaction-diffusion neural networks with time-varying delays. *IEEE Transactions on Cybernetics.* 2013;43(6):1913-1920.

10. Kondo Shigeru, Miura Takashi. Reaction-diffusion model as a framework for understanding biological pattern formation. *Science.* 2010;329(5999):1616-1620.

11. Gatenby RA, Gawlinski ET. A reaction-diffusion model of cancer invasion. *Cancer Research.* 1996;56(24):5745-5753.

12. Gan Qintao. Exponential synchronization of stochastic Cohen-Grossberg neural networks with mixed time-varying delays and reaction-diffusion via periodically intermittent control. *Neural Networks.* 2012;31(1):12-21.

13. Edwards Roderick. Approximation of Neural Network Dynamics by Reaction-Diffusion Equations. *Mathematical Methods in the Applied Sciences.* 1996;19(8):651-677.

14. Henry B.I, Wearne S.L. Fractional reaction–diffusion. *Physica A: Statistical Mechanics and its Applications.* 2000;276(3):448-455.

15. Zeng Fanhai, Liu Fawang, Li Changpin, Burrage Kevin, Turner Ian, Anh V.. A Crank–Nicolson ADI spectral method for a two-dimensional Riesz space fractional nonlinear reaction-diffusion equation. *SIAM Journal on Numerical Analysis.* 2014;52(6):2599-2622.

16. Gafiychuk V., Datsko B., Meleshko V.. Mathematical modeling of time fractional reaction-diffusion systems. *Journal of Computational and Applied Mathematics.* 2008;220(1):215-225.

17. Zhou Yong, Shangerganesh L., Manimaran J., Debbouche Amar. A class of time-fractional reaction-diffusion equation with nonlocal boundary condition. *Mathematical Methods in the Applied Sciences.* 2018;41(8):2987-2999.

18. Atangana Abdon. On the new fractional derivative and application to nonlinear Fisher's reaction–diffusion equation. *Applied Mathematics and Computation.* 2016;273(1):948-956.

19. Seki Kazuhiko, Wojcik Mariusz, Tachiya M.. Fractional reaction-diffusion equation. *The Journal of Chemical Physics.* 2003;119(4):2165-2170.

20. Khan Hasib, Li Yongjin, Khan Aftab, Khan Aziz. Existence of solution for a fractional-order Lotka-Volterra reaction-diffusion model with Mittag-Leffler kernel. *Mathematical Methods in the Applied Sciences.* 2019;42(9):3377-3387.

21. Cao Jinde, Stamov Gani, Stamova Ivanka, Simeonov Stanislav. Almost periodicity in impulsive fractional-order reaction-diffusion neural networks with time-varying delays. *IEEE Transactions on Cybernetics.* 2021;51(1):151-161.

22. Guo Boling, Zhou Guoli. Ergodicity of the stochastic fractional reaction–diffusion equation. *Nonlinear Analysis: Theory, Methods & Applications.* 2014;109(1):1-22.

23. Matoba S., Nakamura N., Nakamura H., Nishitani H.. Robust finite-time control of robot manipulators. *Proceedings of the 18th World Congress the International Federation of Automatic Control Milano.* 2011;.

24. Haimo V. T.. Finite time controllers. *SIAM Journal on Control and Optimization.* 1986;24(4):760-770.

25. Bhat Sanjay P., Bernstein Dennis S.. Finite-time stability of continuous autonomous systems. *SIAM Journal on Control and Optimization.* 2000;38(3):751-766.

26. Naifar Omar, Nagy A. M., Makhlouf Abdellatif Ben, Kharrat Mohamed, Hammami Mohamed Ali. Finite-time stability of linear fractional-order time-delay systems. *International Journal of Robust and Nonlinear Control.* 2019;29(1):180-187.

27. Lazarevic Mihailo P., Spasic Aleksandar M.. Finite-time stability analysis of fractional order time-delay systems: Gronwall's approach. *Mathematical and Computer Modelling.* 2009;49(3):475-481.

28. Velmurugan G., Rakkiyappan R., Cao Jinde. Finite-time synchronization of fractional-order memristor-based neural networks with time delays. *Neural Networks.* 2016;73:36-46.

29. Yang Xujun, Song Qiankun, Liu Yurong, Zhao Zhenjiang. Finite-time stability analysis of fractional-order neural networks with delay. *Neurocomputing.* 2015;152:19-26.

30. Yang Shuai, Hu Cheng, Yu Juan, Jiang Haijun. Finite-time cluster synchronization in complex-variable networks with fractional-order and nonlinear coupling. *Neural Networks.* 2021;135:212-224.

31. Zhou Jianping, Xu Shengyuan, Shen Hao, Zhang Baoyong. Passivity analysis for uncertain BAM neural networks with time delays and reaction–diffusions. *International Journal of Systems Science.* 2013;44(8):1494-1503.

32. Ma Qian, Feng Gang, Xu Shengyuan. Delay-dependent stability criteria for reaction-diffusion neural networks with time-varying delays. *IEEE Transactions on Cybernetics.* 2013;43(6):1913-1920.

33. Wu Kai-Ning, Sun Han-Xiao, Shi Peng, Lim Cheng-Chew. Finite-time boundary stabilization of reaction-diffusion systems. *International Journal of Robust and Nonlinear Control.* 2018;28(5):1641-1652.

34. Chen Juan, Cui Baotong, Chen Yang Quan. Observer-based output feedback control for a boundary controlled fractional reaction diffusion system with spatially-varying diffusivity. *IET Control Theory and Applications.* 2018;12(11):1561-1572.

35. Liu Xiao-Zhen, Li Ze-Tao, Wu Kai-Ning. Boundary Mittag-Leffler stabilization of fractional reaction–diffusion cellular neural networks. *Neural Networks.* 2020;132:269-280.

36. Podlubny Igor. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Academic Press, Cambridge, MA; 1998.

37. Duarte-Mermoud Manuel A., Aguila-Camacho Norelys, Gallegos Javier A., Castro-Linares Rafael. Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems. *Communications in Nonlinear Science and Numerical Simulation.* 2015;22(1):650-659.

38. Liu Peng, Zeng Zhigang, Wang Jun. Asymptotic and finite-time cluster synchronization of coupled fractional-order neural networks with time delay. *IEEE Transactions on Neural Networks and Learning Systems.* 2020;31(11):4956-4967.

39. Wu Huai-Ning, Wang Jun-Wei, Li Han-Xiong. Fuzzy boundary control design for a class of nonlinear parabolic distributed parameter systems. *IEEE Transactions on Fuzzy Systems.* 2014;22(3):642-652.

40. Yang Shuai, Hu Cheng, Yu Juan, Jiang Haijun. Finite-time cluster synchronization in complex-variable networks with fractional-order and nonlinear coupling. *Neural Networks.* 2021;135(1):212-224.

□