ABSOLUTELY CONTINUOUS AND PURE POINT SPECTRA OF DISCRETE OPERATORS WITH SPARSE POTENTIALS

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ABSTRACT. We consider the discrete Schrödinger operator $H = -\Delta + V$ with a sparse potential V and find conditions guaranteeing either existence of wave operators for the pair H and $H_0 = -\Delta$, or presence of dense purely point spectrum of the operator H on some interval $[\lambda_0, 0]$ with $\lambda_0 < 0$.

1. INTRODUCTION

The class of sparse potentials was introduced in the spectral theory of one-dimensional Schrödinger operators by D. Pearson [10]. Such potentials are intermediate between compactly supported (or fast decaying) potentials associated to the scattering theory, and the generic bounded potentials (such as periodic, almost periodic, random ergodic potentials) which are studied in the solid state physics.

Typical sparse potentials have the form

$$V(x) = \sum_{n\in\mathbb{N}} a_n \phi(x - x_n) \tag{1.1}$$

where $\phi \in C_0^\infty$ is a fixed bump function and x_n is a sequence for which the quantity $d(n) = \operatorname{dist}(x_n, \bigcup_{j\neq n}\{x_j\})$ grows as $n \to \infty$. The amplitudes a_n either slowly tend to zero, or are of order $O(1)$ (as in the case where they are independent identically distributed random variables).

The paper by Kiselev, Last and Simon [4] as well as the paper by Molchanov [5] extend the results of [10] in different directions:

Let $H = -\frac{d^2}{dx^2} + V$ be the operator with a potential of the form (1.1) acting in the space $L^2[0,\infty)$. Assume that the boundary condition at the origin is the Dirichlet condition. Suppose also that $\phi \geq 0$, the sequence x_n is monotone and $a_n \to 0$ as $n \to \infty$.

1) If $\sum_n a_n^2 < \infty$ and $x_n/x_{n+1} = o(1)$ as $n \to \infty$, then the spectrum is pure absolutely continuous on $(0,\infty)$.

2) If $\sum_n a_n^2 = \infty$ and $x_n/x_{n+1} = o(1)$ as $n \to \infty$, then the spectrum is purely singular continuous on $(0,\infty)$.

These statements could be compared with the results of the paper by Kotani and Ushiroya [8] where the authors study the transition from the absolutely continuous spectrum to the pure point spectrum for the operator with a random potential.

Technically, the analysis of the absolutely continuous spectrum of the operator H under the condition $\sum_n a_n^2 < \infty$ resembles the theory of lacunary Fourier series and is based on the idea of the stochastization of the phase of the solution of the equation $H\psi = k^2\psi$, $k \in \mathbb{R}\setminus\{0\}$, as one passes from one bump to another.

One should also notice that the spectral properties of the Hamiltonian H depend on the nature of the elementary bump ϕ. For instance, if all $a_n = 1$ and ϕ is a reflectionless potential, then the spectrum of

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H is pure absolutely continuous on $(0,\infty)$ provided the sequence $\{x_n\}$ is exponentially sparse (see [5]). At the same time, if the reflection coefficient $r(k)$ constructed for the bump ϕ is not identically zero, then the spectrum of H is purely singular continuous on $(0,\infty)$ provided some additional mild technical assumptions are fulfilled (see [5]).

One expects the multi-dimensional spectral theory of Schrödinger operators with sparse potentials to be different from the one-dimensional theory, because the waves "reflected" from one individual bump decay at infinity, if $d \geq 2$. More precisely, the solution of the equation $-\Delta\psi + a_n\phi(x-x_n)\psi = k^2\psi$ constructed for one bump decays as $O(|x|^{-(d-1)/2})$ as $|x| \to \infty$. The latter leads to the fact that the absolutely continuous spectrum of the milti-dimensional Schrödinger operator with a sparse potential can cover the positive half-line even in the case where the coefficients a_n do not decay.

Indeed, one of the results of the paper by Safronov [11] says that if V_ξ is a function of the form

$$V_\xi(x) = \sum_{n\in\mathbb{Z}^d} a_n \xi_n \chi(x-n),$$

where ξ_n are independent identically distributed bounded random variables such that $\mathbb{E}(\xi_n) = 0$ and χ is the characteristic function of the cube $[0,1)^d$, then the absolutely continuous spectrum of the operator $-\Delta + tV_\xi$ almost surely covers the positive half-line for almost every $t \in \mathbb{R}$ provided

$$\sum_{n\neq 0} \frac{a_n^2}{|n|^{d-1}} < \infty. \tag{1.2}$$

This statement should be compared with the results of Bourgain [1] and Denisov [3] in which the authors assume that $|a_n| \leq C(1+|n|)^{-1/2-\varepsilon}$ with $\varepsilon > 0$ and prove the corresponding claim for every t. The main difference between [1] and [3] is that the first paper deals with the discrete operator on the lattice $\mathbb{Z}^2$ while the second one handles the continuous operator on $\mathbb{R}^d$. The potentials considered in the paper [7] are also decaying at infinity.

One should mention that the corresponding statement about the absolutely continuous spectrum of the discrete Schrödinger operator on $\mathbb{Z}^d$ under the condition similar to (1.2) has not been proved. Neither has been proved a discrete analogue the Laptev-Naboko-Safronov (see [9]) theorem saying that if $V \geq 0$ and

$$\int_{\mathbb{R}^d} \frac{V(x)}{(1+|x|)^{d-1}} dx < \infty, \tag{1.3}$$

then the absolutely continuous spectrum of the operator $-\Delta + V$ covers the half-line $[0,\infty)$. Note that both conditions (1.2) and (1.3) are fulfilled for some sparse potentials that do not decay at infinity.

The operator H with a sparse potential that does not decay at infinity can have pure point spectrum outside of the spectrum of the free Laplace operator. As a result, one can expect the discrete operator H with a negative random sparse potential to have absolutely continuous spectrum on $[0,4d]$ and dense pure point spectrum on some interval $[-a,0)$ with $a > 0$. This phenomenon was proved in the paper [6] by Molchanov but without the specification of the physical nature of the absolutely continuous component of the spectrum. In the present paper, we develop the scattering theory for a class of discrete operators on the lattice $\mathbb{Z}^d$ with sparse potentials V and prove existence of wave operators.

2. Statement of the main results

Let $H_0 = -\Delta$ be the "free" operator on the lattice $\mathbb{Z}^d$ whose action is defined by the formula

$$\big[H_0 u\big](n) = \sum_{|n-j|=1} \Big(u(n) - u(j)\Big).$$

Let $H = -\Delta + V$, where V is the operator of multiplication by a bounded real-valued function $V : \mathbb{Z}^d \to \mathbb{R}$. We are interested in the question of existence of the wave operators

$$\text{s - }\lim_{t\to\pm\infty} e^{-itH} e^{itH_0} =: W_\pm. \tag{2.1}$$

Theorem 2.1. *Assume that*

$$\sum_{n\in\mathbb{Z}^d\setminus\{0\}} \frac{|V(n)|}{|n|^{(d-1)/2}} < \infty.$$

Then the wave operators (2.1) *exist.*

Note that Theorem 2.1 is applicable to operators with sparse potentials, for which the quantity

$$d(n) = \text{dist}\,\big(n,\ \text{supp}(V) \setminus \{n\}\big) \tag{2.2}$$

tends to infinity sufficiently fast as $|n| \to \infty$. Another theorem that deals with sparse potentials is the followings statement, in which χ_n is the characteristic function of the one point set $\{n\} \subset \mathbb{Z}^d$.

Theorem 2.2. *For an arbitrary number $\lambda_0 < 0$, let*

$$a = \frac{1}{\big((H_0 - \lambda_0)^{-1}\chi_0, \chi_0\big)}.$$

Let Ω be a fixed subset of the lattice $\mathbb{Z}^d$, such that

$$V_\xi(n) = \begin{cases} \xi_n, & \textit{if} \quad n \in \Omega, \\ 0, & \textit{if} \quad n \notin \Omega, \end{cases}$$

where $\{\xi_n\}_{n\in\mathbb{Z}^d}$ are independent random variables uniformly distributed on the interval $[-a, 0]$. Assume that that the quantity $d(n)$ defined by (2.2) *obeys the condition*

$$\lim_{|n|\to\infty,\, n\in\Omega} \frac{d(n)}{|n|^\delta} = \infty$$

for some $\delta > 0$. Then the operator $H_\xi = -\Delta + V_\xi$ almost surely has a dense pure point spectrum in the interval $[\lambda_0, 0)$.

According to Theorems 2.1 and 2.2, there are discrete Schrödinger operators whose absolutely continuous spectrum "fills" the interval $[0, 4d]$ while their pure point spectrum is dense in $[\lambda_0, 0)$.

3. Proof of Theorem 2.1.

Standard notations. For a closed linear operator T, the symbols $\mathcal{D}(T)$, $\sigma(T)$ and $\sigma_p(T)$ denote its domain, its spectrum and the set of eigenvalues (which does not have to be closed). If T is self-adjoint, then $E_T(\cdot)$ denotes the operator-valued spectral measure of the operator T.

Besides the standard wave operators $W_\pm$, we will consider the modified operators $W_\pm(J)$ defined by

$$W_\pm(J) = \text{s - }\lim_{t\to\pm\infty} e^{-itH} J e^{itH_0}, \tag{3.1}$$

where $J : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ is assumed to be bounded. We will employ one of the statements of the scattering theory, using the notion of a trace class operator. One says that an operator T on a separable Hilbert space belongs to the class $\mathfrak{S}_p$, if the sequence $\{s_j(T)\}$ of singular values of this operator is an ℓ^p-sequence. In this case, the norm of T in $\mathfrak{S}_p$ is defined as the norm of the sequence $\{s_j(T)\}$ in ℓ^p. The classes $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are called the trace class and the Hilbert-Schmidt class, correspondingly.

One of theorems by Pearson (see [14]) leads to the following result:

Theorem 3.1. *Let $E_0(\cdot)$ and $E(\cdot)$ be the operator-valued spectral measures of H_0 and H correspondingly. Assume that the operator*

$$E(a,b)\big(HJ - JH_0\big)E_0(a,b) \in \mathfrak{S}_1 \tag{3.2}$$

is a trace class operator for any bounded interval (a,b). Then the limits $W_\pm(J)$ in (3.1) *exist.*

We will apply this theorem in the case where J commutes with H_0. In this case, the condition (3.2) turns into the relation

$$E(a,b)VJE_0(a,b) \in \mathfrak{S}_1, \tag{3.3}$$

while the limit (3.1) coincides with the limit

$$\text{s - }\lim_{t\to\pm\infty} e^{-itH} e^{itH_0} J.$$

The latter observation implies the following lemma.

Lemma 3.1. *Let $\{J_l\}_{l\in\mathbb{N}}$ be a family of bounded operators commuting with H_0 and such that*

$$s\text{-}\lim_{l\to\infty} J_l = I. \tag{3.4}$$

Assume that

$$VJ_lE_0(a,b) \in \mathfrak{S}_1$$

for any bounded interval (a,b) and for any l. Then the limits $W_\pm$ in (2.1) *exist.*

Remark. Replacement of (3.4) by the assumption

$$\text{s-}\lim_{l\to\infty} J_l = J.$$

leads to the existence of the limits (3.1).

In order to construct operators J needed for our purposes, one has to introduce the operator $\Phi : L^2(\mathbb{T}^d) \to \ell^2(\mathbb{Z}^d)$ setting

$$\big[\Phi u\big](n) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} e^{-i\xi n} u(\xi)\,d\xi, \qquad \mathbb{T}^d = [0,2\pi)^d.$$

It is very well known that

$$H_0 = \Phi\big[a\big]\Phi^{-1}, \tag{3.5}$$

where $[a]$ denotes the operator of multiplication by the function

$$a(\xi) = \sum_{j=1}^{d} (2 - 2\cos(\xi_j)), \qquad \xi = (\xi_1, \xi_2, \dots, \xi_d),$$

called the symbol of the operator H_0.

Definition of a regular set. An open set $\Omega \subset \mathbb{T}^d$ is called regular, if
(1) $\nabla a(\xi) \neq 0, \qquad \forall \xi \in \Omega.$
(2) The curvatures of the level surfaces $a(\xi) = const$ are different from zero at all points $\xi \in \Omega$.
(3) Ω is diffeomorphic to a set $[\alpha, \beta] \times U$, where U is an open set in $\mathbb{R}^{d-1}$. Moreover, the corresponding diffeomorphism

$$\tau : \Omega \to [\alpha, \beta] \times U$$

is a function $\tau = (\tau_1, \tau_2, \dots, \tau_d)$ having the property $\tau_1(\xi) = a(\xi), \quad \forall \xi \in \Omega$. Without loss of generality, we will assume that the diffeomorphism τ can be extended into a larger domain $\tilde{\Omega}$ containing Ω, so that the corresponding Jacobian is a bounded separated from zero function on Ω.

Note that for any $\varepsilon > 0$, there is a disjoint collection of sets $\{\Omega_j\}_{j=1}^N$ having the properties (1)-(3) such that the Lebesgue measure of the set

$$\mathbb{T}^d \setminus \bigcup_{j=1}^{N} \Omega_j$$

is smaller than ε. Therefore, the collection of orthogonal projections P_{Ω_j} in $L^2(\mathbb{T}^d)$ onto the space of functions vanishing outside of Ω_j is a family of operators with the property

$$\sum_j P_{\Omega_j} f \to f, \qquad \text{as} \quad \varepsilon \to 0, \quad \forall f \in L^2(\mathbb{T}^d).$$

Consequently, we obtain the following statement:

Corollary 3.1. *Let $J_\Omega : \ell^2(\mathbb{Z}^d) \mapsto \ell^2(\mathbb{Z}^d)$ be the operator*

$$J_\Omega = \Phi \chi_\Omega \Phi^{-1}$$

where χ_Ω is the characteristic function of a set $\Omega \subset \mathbb{T}^d$. If the limits $W_\pm(J_\Omega)$ defined by (3.1) *with $J = J_\Omega$ exist for all sets Ω having the properties (1)-(3), then the wave operators $W_\pm$ defined in* (2.1) *also exist.*

The next step in our arguments will be an approximation of the operators J_Ω by operators $J_{\Omega,n}$ commuting with H_0 and having the property that

$$J_{\Omega,n} f \to J_\Omega f, \qquad \text{as} \quad n \to \infty, \quad \forall f \in \ell^2(\mathbb{Z}^d).$$

Obviously, existence of the limits (3.1) with $J = J_{\Omega,n}$ would imply existence of operators $W_\pm(J_\Omega)$. Therefore, according to Corollary 3.1, the wave operators $W_\pm$ would also exist.

Note that, for any $f \in \ell^2(\mathbb{Z}^d)$, the element $J_\Omega f \in \ell^2(\mathbb{Z}^d)$ is the sequence

$$\big[J_\Omega f\big](j) = \frac{1}{(2\pi)^{d/2}} \int_{[\alpha,\beta]\times U} e^{i\, j\cdot\xi(\tau)} \omega(\tau) \big[\Phi^{-1} f\big](\xi(\tau)) d\tau, \quad j \in \mathbb{Z}^d, \tag{3.6}$$

where $\xi(\cdot)$ is the inverse of the mapping $\tau(\cdot)$ and $\omega(\tau) = \left|\frac{\partial \xi}{\partial \tau}\right|$ is its Jacobian.

Let $\{\phi_k\}_{k=1}^\infty$ be an orthonormal basis in $L^2(U)$. Without loss of generality, one can assume that $\phi_k \in C_0^\infty(U)$. We define the orthogonal projections $P_n : L^2([\alpha,\beta]\times U) \to L^2([\alpha,\beta]\times U)$ setting

$$\big[P_n u\big](\tau) = \sum_{k=1}^n \phi_k(\tau') \int_U u(\tau_1,\eta)\bar{\phi}_k(\eta)d\eta, \qquad \tau = (\tau_1,\tau') \in [\alpha,\beta]\times U.$$

After that, we define the operators $J_{\Omega,n}$ by

$$\big[J_{\Omega,n} f\big](j) = \frac{1}{(2\pi)^{d/2}} \int_{[\alpha,\beta]\times U} e^{i\,j\cdot\xi(\tau)}\omega(\tau)\Big[P_n\big[[\Phi^{-1}f](\xi(\cdot))\big]\Big](\tau)\,d\tau, \quad j \in \mathbb{Z}^d. \tag{3.7}$$

Taking into account the fact that $a(\xi(\tau)) = \tau_1$ for all $\tau \in [\alpha,\beta]\times U$, that is, the symbol of H_0 coincides with the first of the τ-coordinates, we obtain from (3.5) and (3.7) that $\big[P_n([\Phi^{-1}H_0 f]\circ\xi)\big](\tau) = \tau_1\big[P_n([\Phi^{-1}f]\circ\xi)\big](\tau)$. Hence, the operators $J_{\Omega,n}$ commute with H_0. Furthermore, since the sequence P_n converges to I strongly in $L^2([\alpha,\beta]\times U)$, it converges to the same limit in the weighted space $L^2([\alpha,\beta]\times U,\omega)$, because the norms in these two spaces are equivalent. Comparing (3.6) and (3.7), we conclude that $J_{\Omega,n} \to J_\Omega$ strongly as $n \to \infty$.

Thus, it remains to establish existence of limits (3.1) for $J = J_{\Omega,n}$. The latter follows from the statement below.

Proposition 3.1. *Let χ_j be the characteristic function of the one-point set $\{j\} \subset \mathbb{Z}^d$. Let $J_{\Omega,n}$ be the operators defined above. Then there is a positive constant $C = C(d,\Omega,n)$ depending only on d,Ω and the choice of the collection $\{\phi_k\}_{k=1}^n$ such that*

$$\|V\chi_j J_{\Omega,n}\| \le C\frac{|V(j)|}{1+|j|^{(d-1)/2}}, \qquad \forall j \in \mathbb{Z}^d. \tag{3.8}$$

.

Proof. The relation (3.8) follows from the estimate

$$Q_k(\tau_1,j) := \Big|\int_U e^{ij\cdot\xi(\tau_1,\tau')}\phi_k(\tau')\omega(\tau_1,\tau')d\tau'\Big| \le \frac{C_{\tau_1}}{1+|j|^{(d-1)/2}}, \qquad \forall j \in \mathbb{Z}^d, \tag{3.9}$$

which holds for each fixed $\tau_1 \in [\alpha,\beta]$. Such inequalities were studied systematically in the paper [12] by Shaban and Vainberg. In particular, it was shown that C_{τ_1} is a bounded function of τ_1 on $[\alpha,\beta]$. To prove (3.8), one needs to observe that, for any $u \in L^2(\mathbb{T}^d)$,

$$\Big|\big[VJ_{\Omega,n}\Phi u\big](j)\Big| \le (2\pi)^{-d/2}|V(j)|\sum_{k=1}^n \int_\alpha^\beta Q_k(\tau_1,j)g_k(\tau_1)d\tau_1, \tag{3.10}$$

where

$$g_k(\tau_1) = \Big|\int_U u\big(\xi(\tau_1,\tau')\big)\bar{\phi}_k(\tau')d\tau'\Big| \le \Big(\int_U |u\big(\xi(\tau_1,\tau')\big)|^2 d\tau'\Big)^{1/2}. \tag{3.11}$$

The inequality (3.11) implies that $\int_\alpha^\beta g_k(\tau_1)d\tau_1 \le C\|u\|_{L^2(\mathbb{T}^d)}$. Combining this estimate with the relation (3.9), we infer (3.8) from (3.10). □

Since $V\chi_j J_{\Omega,n}$ is a rank one operator, its trace class norm coincides with the usual norm. Consequently, we obtain:

Corollary 3.2. *Assume that*

$$\sum_{j\in\mathbb{Z}^d\setminus\{0\}} \frac{|V(j)|}{|j|^{(d-1)/2}} < \infty.$$

Then the operator $VJ_{\Omega,n}$ is a trace class operator.

Theorem 2.1 follows from Lemma 3.1 and Corollary 3.2. □

4. PROOF OF THEOREM 2.2

Our study of the pure point spectrum is based on the following celebrated result of Simon and Wolff [13]:

Theorem 4.1. *Let A be a self-adjoint operator in a Hilbert space $\mathfrak{H}$ such that ϕ is a cyclic vector for A. Let μ_ϕ be the spectral measure of A corresponding to the vector ϕ. Let P be the orthogonal projection onto the space of scalar multiples of ϕ. The spectrum of $A + tP$ is pure point on a Borel set $\Omega \subset \mathbb{R}$ for almost every $t \in \mathbb{R}$, if and only if*

$$\int_{\mathbb{R}} \frac{d\mu_\phi(t)}{(t-\lambda)^2} < \infty, \qquad \textit{for almost every} \quad \lambda \in \Omega. \tag{4.1}$$

In applications to discrete Schrödinger operators, it is convenient to interpret (4.1) as the condition that $(A-\lambda)^{-1}\phi$ is also an element of the Hilbert space $\mathfrak{H}$. Indeed, let us define $U : \mathfrak{H} \to L^2(\mathbb{R}, \mu_\phi)$ as a bounded operator mapping an element of the form $f(A)\phi$ to the function $f(t)$ (Here, one needs to consider only such functions that $\phi \in \mathcal{D}(f(A))$). This operator U is a unitary operator from $\mathfrak{H}$ onto $L^2(\mathbb{R}, \mu_\phi)$. Moreover, U diagonalizes the operator$A$ in the sense that UAU^{-1} is the operator of multiplication by the independent variable t. Note now that $U\phi$ is the function that is identically equal to 1. Consequently, using the fact that the operator $(A-\lambda)^{-1}$ is well defined for any λ that is not an eigenvalue of A, we obtain:

$$\frac{1}{t-\lambda} \in L^2(\mathbb{R}, \mu_\phi) \iff \lambda \notin \sigma_p(A) \quad \text{and} \quad \phi \in \mathcal{D}\big((A-\lambda)^{-1}\big).$$

Thus, the Simon-Wolff theorem can be reformulated as follows.

Theorem 4.2. *Let A be a self- adjoint operator in a Hilbert space $\mathfrak{H}$ such that ϕ is a cyclic vector for A. Let P be the orthogonal projection onto the space of scalar multiples of ϕ. The spectrum of $A+tP$ is pure point on a Borel set Ω for almost every $t \in \mathbb{R}$, if and only if for almost every $\lambda \in \Omega \setminus \sigma_p(A)$,*

$$\phi \in \mathcal{D}\big((A-\lambda)^{-1}\big). \tag{4.2}$$

One is tempted to say that this theorem could be applied to Schrödinger operators on the lattice $\mathbb{Z}^d$, because one can decompose $H = -\Delta + V$ into the orthogonal sum of operators whose spectra have multiplicities equal to 1. However, the arguments that allow us to use Theorem 4.2 are more complicated.

Proposition 4.1. *Let $\mathfrak{H}_0$ be an invariant subspace for the operator $H = -\Delta + V$ that is orthogonal to vectors χ_j for all $j \in suppV$. Then the spectrum of the restriction of H to the subspace $\mathfrak{H}_0$ is contained in $[0, 4d]$.*

Proof. The statement is obvious and its proof is left as an exercise. □

Proposition 4.2. *Let $j \in \mathbb{Z}^d$ be a fixed point of the lattice. Assume that*

$$\chi_j \in \mathcal{D}\big((H-\lambda)^{-1}\big)$$

for almost every $\lambda \in (a,b)\setminus\sigma_p(H)$. Let $\mathfrak{H}_0(t)$ be an invariant subspace for the operator $H+t\chi_j(\cdot,\chi_j)$ on which the restriction of this operator has coninuous spectrum that is contained in $[a,b]$. Then χ_j is orthogonal to $\mathfrak{H}_0(t)$ for almost every $t \in \mathbb{R}$.

Proof. Let $\mathfrak{H}_1$ be the span of vectors $\{\chi_j, H\chi_j, H^2\chi_j, \dots\}$. The subspace $\mathfrak{H}_1$ is invariant for the operator $H + t\chi_j(\cdot,\chi_j)$ whose restriction to $\mathfrak{H}_1$ is an operator having pure point spectrum in $[a,b]$ for almost every t. Consequently, $\mathfrak{H}_0(t)$ is orthogonal to $\mathfrak{H}_1$. □

Corollary 4.1. *Let $\{\xi_n\}_{n\in\mathbb{Z}^d}$ be independent random variables uniformly distributed on some interval $[-\alpha,0]$ where $\alpha>0$. Let $\Omega\subset\mathbb{Z}^d$ be a fixed subset of the lattice, such that*

$$V_\xi(n) = \begin{cases} \xi_n, & \textit{if} \quad n\in\Omega,\\ 0, & \textit{if} \quad n\notin\Omega. \end{cases}$$

Let $H_\xi = -\Delta + V_\xi$. Let also $a<b<0$. Assume that, for all ξ and all $j\in\Omega$,

$$\chi_j \in \mathcal{D}\big((H_\xi-\lambda)^{-1}\big)$$

for almost every $\lambda \in (a,b)\setminus\sigma_p(H_\xi)$. Then the operator H_ξ almost surely has pure point spectrum in (a,b).

Proof. Let $\mathfrak{H}_c(\xi)$ be the invariant subspace on which the restriction of H_ξ has continuous spectrum contained in $[a,b]$. By Proposition 4.2, χ_j is orthogonal to $\mathfrak{H}_c(\xi)$ for almost every $\xi_j \in [-\alpha,0]$ provided the values of all other random variables are fixed. By Fubini's theorem, χ_j is orthogonal to $\mathfrak{H}_c(\xi)$ almost surely. By Proposition 4.1, the spectrum of the restriction of H_ξ to $\mathfrak{H}_c(\xi)$ does not intersect the interval $[a,b]$, because $b<0$. The obtained contradiction implies that a nontrivial subspace $\mathfrak{H}_c(\xi)$ does not exist. □

Let $H = -\Delta + V$ be a Schrödinger operator on the lattice $\mathbb{Z}^d$. Our goal is to find conditions guaranteeing that the sequence

$$\psi(n) = \big[(H-\lambda)^{-1}\chi_j\big](n)$$

is square summable for any fixed $j\in\mathbb{Z}^d$. This litteraly means that $\chi_j \in \mathcal{D}\big((H-\lambda)^{-1}\big)$. We remind the reader that, for $\lambda\notin[0,4d]$, the sequence ψ is a solution of the equation

$$\psi = \psi_0 - (H_0-\lambda)^{-1}V\psi, \tag{4.3}$$

where ψ_0 is the sequence

$$\psi_0(n) = \big[(H_0-\lambda)^{-1}\chi_j\big](n)$$

One the other hand, once we know the values $\psi(n)$ for $n\in\mathrm{supp}(V)$, we immediately obtain ψ for all n, from (4.3).

$$\psi(n) = \psi_0(n) - \Big((H_0-\lambda)^{-1}\chi_n,\chi_n\Big)V(n)\psi(n) - \sum_{l\neq n}\Big((H_0-\lambda)^{-1}\chi_l,\chi_n\Big)V(l)\psi(l), \tag{4.4}$$

First, observe that, for any $\varepsilon>0$, there are constants $\gamma_\varepsilon>0$ and $C_\varepsilon>0$ depending on ε such that

$$\Big|\Big((H_0-\lambda)^{-1}\chi_l,\chi_n\Big)\Big| \le C_\varepsilon e^{-\gamma_\varepsilon|n-l|}, \qquad \forall n,l\in\mathbb{Z}^d, \qquad \forall\lambda\in\mathbb{R}\setminus[-\varepsilon,4d+\varepsilon]. \tag{4.5}$$

Proposition 4.3. *Let V be a bounded potential and let $\varepsilon > 0$. Then for almost every $\lambda \notin [-\varepsilon, 4d+\varepsilon]$, there is a number $k(\lambda)$ such that*

$$\Big|1 + \big((H_0 - \lambda)^{-1}\chi_n, \chi_n\big)V(n)\Big| \geq \frac{1}{|n|^{-d-\varepsilon}} \tag{4.6}$$

for $|n| > k(\lambda)$.

Proof. Observe first, that if $\big|\big((H_0 - \lambda)^{-1}\chi_n, \chi_n\big)V(n)\big| < 1/2$ and $|n| > 2$, then (4.6) holds. Threfore, we only need to consider the case $\big|\big((H_0 - \lambda)^{-1}\chi_n, \chi_n\big)V(n)\big| \geq 1/2$. The latter implies that we only need to consider the points $\lambda \in [-2\|V\|_\infty, -\varepsilon] \cup [4d+\varepsilon, 4d+2\|V\|_\infty]$ and points $n \in \mathbb{Z}^d$ for which $|V(n)| \geq \varepsilon/2$. On that set of λ-s, the absolute value of the derivative of the monotone function

$$f(\lambda) = 1 + \big((H_0 - \lambda)^{-1}\chi_n, \chi_n\big)V(n)$$

is bounded below by $\varepsilon/\|V\|_\infty^2$. Therefore, the Lebesgue measure of the set

$$S_\varepsilon(n) := \Big\{\lambda \in \mathbb{R} \setminus [-\varepsilon, 4d+\varepsilon] : \qquad \Big|1 + \big((H_0 - \lambda)^{-1}\chi_n, \chi_n\big)V(n)\Big| < \frac{1}{|n|^{d+\varepsilon}}\Big\}$$

is bounded by

$$|S_\varepsilon(n)| \leq \varepsilon^{-1}\|V\|_\infty^2 |n|^{-d-\varepsilon}.$$

Consequently,

$$\sum_{n \in \mathbb{Z}^d} |S_\varepsilon(n)| < \infty.$$

Using the Borel-Cantelli lemma, we conclude that, for almost every $\lambda \in [-\|V\|_\infty, -\varepsilon] \cup [4d+\varepsilon, 4d+\|V\|_\infty]$, there is a number $k(\lambda)$ such that

$$\Big|1 + \big((H_0 - \lambda)^{-1}\chi_n, \chi_n\big)V(n)\Big| \geq \frac{1}{|n|^{-d-\varepsilon}}$$

for $|n| > k(\lambda)$. $\square$

The equation (4.4) can be written in the form

$$\psi(n) = \alpha(n)\psi_0(n) - \alpha(n)\sum_{l \neq n}\Big((H_0 - \lambda)^{-1}\chi_l, \chi_n\Big)V(l)\psi(l), \tag{4.7}$$

where the sequence

$$\alpha(n) = \big(1 + \big((H_0 - \lambda)^{-1}\chi_n, \chi_n\big)V(n)\big)^{-1}$$

obeys the bound

$$|\alpha(n)| \leq |n|^{d+\varepsilon} \quad \text{for} \quad |n| > k(\lambda). \tag{4.8}$$

Consequently, (4.7) can be written in the form

$$\psi = \tilde{\psi}_0 - T\psi, \tag{4.9}$$

where $\tilde{\psi}(n) = \alpha(n)\psi(n)$ and T is the operator defined by

$$[Tu](n) = \alpha(n)\sum_{l \neq n}\Big((H_0 - \lambda)^{-1}\chi_l, \chi_n\Big)V(l)u(l), \tag{4.10}$$

Combining (4.5) with (4.8), we conclude that T is a compact operator on $\ell^2(\mathrm{supp}(V))$ provided the quantity

$$d(n) = \text{dist}\big(n,\ \mathrm{supp}(V)\setminus\{n\}\big) \tag{4.11}$$

obeys the sparseness condition

$$\lim_{|n|\to\infty,\, n\in\mathrm{supp}(V)} \frac{d(n)}{|n|^\delta} = \infty, \qquad \text{for some } \delta > 0. \tag{4.12}$$

Proposition 4.4. *Let V be a bounded potential such that* (4.12) *holds. Let $\chi_{\mathrm{supp}(V)}$ be the characteristic function of the support of V. Let T be defined by* (4.10)*. Then the operator $\chi_{\mathrm{supp}(V)}T$ is compact.*

Proof. For any $R > 0$, let χ_{B_R} be the characteristic function of the set $B_R = \{n \in \mathbb{Z}^d : \ |n| \le R\}$. Consider the decomposition

$$T = \chi_{B_R}T + (1-\chi_{B_R})T.$$

Since $\chi_{B_R}T$ is a finite rank operator, it is sufficient to prove that

$$\|(1-\chi_{B_R})\chi_{\mathrm{supp}(V)}T\| \to 0, \qquad \text{as} \quad R\to\infty.$$

The latter property follows from the estimate

$$\|(1-\chi_{B_R})\chi_{\mathrm{supp}(V)}T\| \le C_\varepsilon\tilde{C} \sup_{n\in\mathrm{supp}(V):\,|n|>R} \Big(|n|^{d+\varepsilon}\big(\sum_{l:\,|l-n|\ge d(n)} e^{-\gamma_\varepsilon|n-l|}\big)^{1/2}\Big), \tag{4.13}$$

which follows from the Schur estimate for the norm of an operator K whose matrix consists of the elements $k(n,m) = \rho_1(n,m)\rho_2(n,m)$. This estimate says that

$$\|K\| \le \sup_n \Big(\sum_m |\rho_1(n,m)|^2\Big)^{1/2} \sup_m \Big(\sum_n |\rho_2(n,m)|^2\Big)^{1/2}.$$

The constant $\tilde{C}$ in (4.13) is equal to the quantity $(\sum_{n\in\mathbb{Z}^d} e^{-\gamma_\varepsilon|n|})^{1/2}$. $\square$

Proposition 4.4 implies that either (4.9) is uniquely solvable, or the equation

$$\psi = -T\psi \tag{4.14}$$

has a non-trivial solution. On the other hand, since (4.14) is equivalent to the equation

$$\psi = -(H_0-\lambda)^{-1}V\psi,$$

which can be written in the form $H\psi = \lambda\psi$, the equation (4.14) has a non-zero solution if and only if λ is an eigenvalue of the operator H. Consequently, (4.9) has a unique square summable solution ψ for almost every $\lambda \notin [0,4d]$. Thus, we obtain the following result.

Theorem 4.3. *Let V be bounded. Assume that the quantity* (4.11) *obeys*

$$\lim_{|n|\to\infty,\, n\in supp(V)} \frac{d(n)}{|n|^\delta} = \infty$$

for some $\delta > 0$. Then the sequence

$$\psi(n) = \big[(H-\lambda)^{-1}\chi_j\big](n)$$

is square summable for almost every $\lambda \notin [0,4d]$.

As a consequence, we obtain the following result.

Theorem 4.4. *Let the conditions of Theorem 2.2 be fulfilled. Then the operator H_ξ almost surely has pure point spectrum on $\mathbb{R} \setminus [0, 4d]$.*

Let us now explain why the spectrum of the operator H_ξ in Theorem 2.2 fills the interval $[\lambda_0, 0]$.

Theorem 4.5. *Let V be a bounded potential for which the quantity* (4.11) *obeys the condition*

$$d(n) \to \infty, \qquad \textit{as} \quad |n| \to \infty.$$

Assume that there is a sequence of ponts $n_j \in \mathbb{Z}^d$ such that

$$V(n_j) \to \beta < 0, \qquad \textit{as} \quad j \to \infty.$$

Let $\lambda < 0$ satisfy the equation

$$1 + \beta\big((H_0 - \lambda)^{-1}\chi_0, \chi_0\big) = 0.$$

Then $\lambda \in \sigma(-\Delta + V)$.

Proof. To prove Theorem 4.5, we use the three statements below.

Proposition 4.5. *Let $H = -\Delta + V$ and $H_n = -\Delta + V_n$ be Schrodinger operators on the lattice $\mathbb{Z}^d$ with bounded potentials V and V_n correspondingly. Assume that the sequence V_n converges to V pointwise so that*

$$\sup_{n \in \mathbb{N}} \|V_n\|_\infty < \infty.$$

Then the sequence of measures

$$\mu_n(\cdot) = \big((E_{H_n}(\cdot)\chi_0, \chi_0)\big)$$

converges weakly to the measure

$$\mu(\cdot) = \big((E_H(\cdot)\chi_0, \chi_0)\big), \qquad \textit{as} \quad |n| \to \infty.$$

Proof. Observe that

$$\int_{\mathbb{R}} \frac{d\mu(t)}{t - z} = \big((H - z)^{-1}\chi_0, \chi_0\big), \qquad z \in \mathbb{C}_+ = \{z \in \mathbb{C} : \quad \operatorname{Im} z > 0\},$$

and a similar representation can be written for the measure μ_n. Since the span of functions of the form

$$f(t) = \operatorname{Im} \frac{1}{t - z}, \qquad z \in \mathbb{C}_+,$$

is dense in the space of continuous decaying at infinity functions $C_0(\mathbb{R})$, it is sufficient to show that

$$\Big((H_n - z)^{-1}\chi_0, \chi_0\Big) \to \big((H - z)^{-1}\chi_0, \chi_0\big), \qquad \text{as} \quad n \to \infty,$$

uniformly on compact subsets of $\mathbb{C}_+$. The latter follows from the Hilbert identity:

$$\big((H_n - z)^{-1}\chi_0, \chi_0\big) - \big((H - z)^{-1}\chi_0, \chi_0\big) = \big((H_n - z)^{-1}(V - V_n)(H - z)^{-1}\chi_0, \chi_0\big).$$

□

Corollary 4.2. *Let $H = -\Delta + V$ be the Schrödinger operator with a bounded potential V for which the quantity* (4.11) *obeys the condition*

$$d(n) \to \infty, \qquad as \quad |n| \to \infty.$$

Let $n_j \in \mathbb{Z}^d$ be a sequence of distinct points such that

$$V(n_j) \to \beta < 0, \qquad as \quad j \to \infty.$$

Then the sequence of measures

$$\mu_j(\cdot) = \big(E_H(\cdot)\chi_{n_j}, \chi_{n_j}\big)$$

converges weakly to the measure

$$\mu(\cdot) = \big(E_{-\Delta+\beta P_0}(\cdot)\chi_0, \chi_0\big),$$

where $P_0 = (\cdot, \chi_0)\chi_0$ is the operator of multiplication by the function χ_0.

Proof. It is sufficient to note that

$$\big(E_H(\cdot)\chi_n, \chi_n\big) = \big(E_{H_n}(\cdot)\chi_0, \chi_0\big)$$

where $H_n = -\Delta + V_n$ with $V_n(x) = V(x - n)$. □

Corollary 4.3. *Let the conditions of Corollary 4.2 be fulfilled. Let $\lambda < 0$ satisfy the equation*

$$1 + \beta\big((H_0 - \lambda)^{-1}\chi_0, \chi_0\big) = 0.$$

Then $\lambda \in \sigma(H)$.

Proof. Indeed, since the spectrum of H contains the support of the measure μ_n for all $n \in \mathbb{Z}^d$, we conclude that

$$\sigma(H) \supset \operatorname{supp}\big(E_{-\Delta+\beta P_0}(\cdot)\chi_0, \chi_0\big) \ni \lambda.$$

□

Now Theorem 4.5 follows from Corollary 4.3. □

Finally, we use the following obvious assertion.

Proposition 4.6. *Let Ω be an unbounded subset of the lattice $\mathbb{Z}^d$. Let $\{\xi_n\}_{n\in\Omega}$ be independent random variables uniformly distributed on $[-a, 0]$ with some $a > 0$. Then, for any $\beta \in [-a, 0]$ and almost surely, there is a sequence of distinct points $n_j \in \Omega$ such that*

$$\xi_{n_j} \to \beta, \qquad as \quad j \to \infty.$$

Proof. Let $B_R = \{n \in \mathbb{Z}^d : \ |n| \le R\}$, where $R > 0$. It is enough to observe that for every $\varepsilon > 0$ and every $R > 0$, the probability of the event

$$\{\xi_n \notin (\beta - \varepsilon, \beta + \varepsilon), \qquad \forall n \in \Omega \setminus B_R\}$$

is zero. Consequently, there is at least one $n_1 \in \Omega \setminus B_R$ for which $\xi_{n_1} \in (\beta - \varepsilon, \beta + \varepsilon)$. Continuing inductively, we construct a sequence n_j such that $\xi_{n_j} \in (\beta - 2^{-j+1}\varepsilon, \beta + 2^{-j+1}\varepsilon)$. □

Theorem 4.4, Theorem 4.5 and Proposition 4.6 imply Theorem 2.2.

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