ARONSON-BÉNILAN ESTIMATES FOR WEIGHTED POROUS MEDIUM EQUATIONS UNDER THE GEOMETRIC FLOW

SHAHROUD AZAMI

Abstract. In this paper, we study Aronson-Bénilan gradient estimates for positive solutions of weighted porous medium equations

$$\partial_t u(x,t) = \Delta_\phi u^p(x,t), \quad (x,t) \in M \times [0,T]$$

coupled with the geometric flow $\frac{\partial g}{\partial t} = 2h(t)$, $\frac{\partial \phi}{\partial t} = \Delta\phi$ on a complete measure space $(M^n, g, e^{-\phi}dv)$. As an application, by integrating the gradient estimates, we derive the corresponding Harnack inequalities.

1. Introduction

An n-dimensional smooth metric measure space (or smooth weighted Riemannian manifold) is denoted by the triple $(M^n, g, e^{-\phi}dv)$ where (M^n, g) is an n-dimensional complete manifold with the Riemannian metric g and a weighted volume element $e^{-\phi}dv$ such that $\phi \in C^2(M)$ and dv is the volume element of g on M. The weighted Riemannian manifolds are naturally endowed with analogue of Ricci tensor, called Bakry-Émery tensor tensor and with analogue of Laplace operator called weighted Laplace operator (or Witten-Laplace operator). The weighted Laplacian is a symmetric diffusion operator given by

$$\Delta_\phi := \Delta - \nabla\phi \cdot \nabla$$

where Δ is the Laplace-Beltrami operator. The Bakry-Émery tensor (see [3]) on metric measure space $(M^n, g, e^{-\phi}dv)$ is defined by

$$Ric_\phi := Ric + \mathrm{Hess}\phi$$

where Ric is the Ricci tensor of the manifold M of the metric g. For any integer $m > n$, an $(m-n)$-Bakry-Émery tensor (see [4]) is defined by

$$Ric_\phi^{m-n} := Ric + \mathrm{Hess}\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n}.$$

Also, the weighted Bochner formula for any smooth function f on an n-dimensional metric measure space $(M, g, e^{-\phi}dv)$ is as follow

$$\frac{1}{2}\Delta_\phi|\nabla f|^2 = |\mathrm{Hess} f|^2 + \langle \nabla\Delta_\phi f, \nabla f\rangle + Ric_\phi(\nabla f, \nabla f). \tag{1.1}$$

In present paper, we will prove local Aronson-Bénilan gradient estimates for positive solution to the weighted porous medium equation

$$u_t(x,t) = \Delta_\phi u^p(x,t), \quad (x,t) \in M \times [0,T], \quad p > 1, \tag{1.2}$$

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on a n-dimensional metric measure space $(M, g, e^{-\phi}dv)$ evolving by the geometric flow system

$$\frac{\partial}{\partial t}g(x,t) = 2h(x,t), \quad \frac{\partial}{\partial t}\phi = \Delta\phi \tag{1.3}$$

where h is a general time-dependent symmetric $(0,2)$-tensor on metric measure space and T is taken to be the maximum time of existence for the geometric flow system. In the geometric flow (1.3),

1) if $h = -Ric$ then it becomes the Ricci flow [16] where Ric is the Ricci tensor,
2) if $h = -\frac{1}{2}Rg$ then it becomes Yamabe flow [9] where R is the scalar curvature,
3) if $h = -Ric + \rho Rg$ then it becomes Ricci-Bourguignon flow [8] where ρ is constant,
4) if $h = -Ric + \alpha\nabla\phi\otimes\nabla\phi$ then it becomes the extended Ricci flow [24] where $\alpha(t)$ is a nonincreasing function and ϕ is a smooth scalar function.

For various values of $p > 1$, the equation (1.2) has arisen in different application to model diffusion phenomena see [12, 14, 25, 30] and the references therein. The weighted porous medium equations are generalization of porous medium equations, the appearance of the weighted e^{ϕ} correspond to spacial nonhomegeneity of the medium, either as concerns was density and as concerns the diffusion coefficient.

In 1979, Aronson and Bénilan [1] established a second order differential inequality for any positive solution to the porous medium equation $u_t = \Delta u^p$ on the Euclidean space $\mathbb{R}^n$ with $p > (1 - \frac{2}{n})^+$. On the other hand, for any complete Riemannian manifold (M^n, g) with a fixed metric and Ricci curvature bounded from below by $-K$, where $K \geq 0$, Li and Yau [21] obtained the celebrated differential gradient estimates for positive solutions u to the heat equation $u_t = \Delta u$, now widely called the Li-Yau estimate,

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \leq \frac{n\alpha^2 K}{2(\alpha-1)} + \frac{n\alpha^2}{2t} \tag{1.4}$$

where $\alpha > 1$ is a constant. Since then, this method plays a powerful role in study of the elliptic and parabolic equations and there have arisen various gradient estimates for these equation. In 1989, Davies [13] improved Li-Yau's estimate (1.4) as follows

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \leq \frac{n\alpha^2 K}{4(\alpha-1)} + \frac{n\alpha^2}{2t}.$$

In 1993, Hamilton [17] generalized the constant α in inequality (1.4) to the function $\alpha(t) = e^{2Kt}$, in fact, he derived the following inequality

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt}\frac{u_t}{u} \leq e^{4Kt}\frac{n}{2t}.$$

In [22], Li and Xu derived another type estimate for this positive solution to $u_t = \Delta u$ as

$$\frac{|\nabla u|^2}{u^2} - \Big(1 + \frac{\sinh(Kt)\cosh(Kt) - Kt}{\sinh^2(Kt)}\Big)\frac{u_t}{u} \leq \frac{nK}{2}(1 + \coth(Kt)),$$

and its linearized version [5, 28],

$$\frac{|\nabla u|^2}{u^2} - (1 + \frac{2}{3}Kt)\frac{u_t}{u} \leq \frac{n}{2}(\frac{1}{t} + K + \frac{1}{3}K^2 t).$$

In [30], Vázquez proved Aronson-Bénilan and Li-Yau type estimate for positive solutions to porous medium equation $u_t = \Delta u^p$ on complete manifold with nonnegative Ricci curvature. In 2009, Lu et al. [25] established more general Aronson-Bénilan and Li-Yau type gradient estimate for the porous medium equations $u_t = \Delta u^p$ on Riemannian manifolds with a fixed metric under the weaker condition that the Ricci curvature of M is bounded from below by $-K$ where $K \geq 0$. Then Huang et al. [18] generalized the results of Lu et al. and obtain the Li-Yau-Hamilton type and Li-Xu type gradient estimates for the porous medium equations. Also, Wang and Chen [31] proved the sharp global Li-Yau type gradient estimates for positive solution to doubly nonlinear diffusion equation $u_t = \Delta_p u^\gamma$ on complete Riemannian manifolds where $\gamma > 0$, $p > 1$.
Recently, many authors used similar techniques to prove gradient estimates and Harnack inequalities for positive solutions of parabolic equations under the geometric flows, see for example [2, 7, 15, 19, 23, 34, 29]. In 2014, B. Ma and J. Li [26] obtained Li-Yau type estimates for porous medium equation $u_t = \Delta u^p$ in $M \times (0, T]$ under the Ricci flow for $p > 1$. In 2015, Cao and Zhu [11] obtained Aronson-Bénilan estimates for solution to the porous medium equation under the Ricci flow. In 2018, Wang et al. [32] derived another local Aronson-Bénolan type gradient estimates for the porous medium type equation under the Ricci flow.

Throughout the paper, we assume u be a positive smooth solution to the general parabolic equation (1.2). We denote by n the dimension of the metric measure space M, and by $d(x, y, t)$ the geodesic distance between $x, y \in M$ under $g(t)$. In addition, for any fixed $x_0 \in M$, $R > 0$ we define the compact set

$$Q_{2R,T} := \{(x, t) : d(x, x_0, t) \leq 2R, \ \ 0 \leq t \leq T\} \subset M^n \times (-\infty, +\infty).$$

If $v = \frac{p}{p-1} u^{p-1}$ then by (1.2) the function v satisfies

$$v_t = (p-1) v \Delta_\phi v + |\nabla v|^2. \tag{1.5}$$

Similar to [32], on Riemannian manifold $(M^n, g(t))$ we introduce three C^1 functions $\alpha(t)$, $\beta(t)$, and $\gamma(t) : [0, +\infty) \to (0, +\infty)$. Let these functions satisfy the following conditions

(A1) $\alpha(t) > 1$ and $\gamma(t) > 0$,
(A2) $\alpha(t) > 1$ and $\gamma(t)$ are non-decreasing functions,
(A3) $\frac{\gamma'}{\gamma} + \frac{\alpha'}{\alpha} - 2\big[\frac{1}{m(p-1)} + 1\big]\frac{\beta}{\alpha} \leq 0$,
(A4) $\alpha\beta' + \big[\frac{1}{m(p-1)} + 1\big]\beta^2 \geq 0$,
(A5) $2\big[\frac{1}{m(p-1)} + 1\big]\frac{\alpha-1}{\alpha}\beta + \frac{\alpha'}{\alpha} \geq 0$,
(A6) $\frac{\gamma}{\alpha-1} \leq c$ for some constant c or $\frac{\gamma}{\alpha-1}$ is a nondecreasing function.

Firstly, we give a local space-time gradient estimate for (1.2)-(1.3) with conditions of Ric_ϕ^{m-n} is lower bounded.

Theorem 1.1. *Let $(M, g(0), e^{-\phi_0} dv)$ be a complete metric measure space, and let $g(t), \phi(t)$ evolve by (1.3) for $t \in [0, T]$. Given x_0 and $R > 0$, let u be a positive solution to (1.2) in $Q_{2R,T} =$ such that $u^{p-1} \leq \frac{p-1}{p} k$ for some positive constant k. Suppose that there exist constants k_1, k_2, k_3, k_4 such that*

$$Ric_\phi^{m-n} \geq -(m-1)k_1 g, \qquad -k_2 g \leq h \leq k_3 g, \qquad |\nabla h| \leq k_4,$$

on $Q_{2R,T}$. Let there exist three functions $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ satisfy conditions (A1)-(A6). Suppose that

$$\theta_1 := \sup_{Q_{2R,T}} |\nabla\phi|, \qquad \theta_2 := \sup_{Q_{2R,T}} |\nabla\Delta\phi|.$$

For any constant $p>1$,

(1) *if $\frac{\gamma}{\alpha-1} \leq c$ then there exist positive constants c_0, c_1, and c_2 such that*

$$\frac{|\nabla v|^2}{v} - \alpha\frac{v_t}{v} \leq \frac{m(p-1)}{1+m(p-1)}\left(\alpha^2 K_1 + \alpha\sqrt{K_2\frac{1+m(p-1)}{m(p-1)}}\right) + \alpha\beta, \tag{1.6}$$

on $Q_{2R,T}$ where

$$\begin{aligned} K_1 &= c_2k_2 + k\Big(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R}) + \frac{c_1}{R^2}\Big) + 2(p-1)k\frac{c_1}{R^2} \\ &\quad + \frac{1}{\gamma}\Big(\frac{mp^2(p-1)\alpha}{2\big(1+m(p-1)\big)}\frac{c_1}{R^2}ck\Big) \\ &\quad + \frac{1}{2\gamma}c\Big[2(p-1)k\big(k_1+k_4\big) + (p-1)\alpha\big(2k_2+1\big) + 2(\alpha-1)k_3\Big], \end{aligned}$$

and

$$K_2 = \alpha^2(p-1)n\max\{k_2^2, k_3^2\} + \frac{9}{8}(p-1)n\alpha^2k_4 + \frac{1}{2}k_2\alpha(p-1)k\theta_1^2 + \frac{1}{4}\alpha(p-1)k\theta_2^2.$$

(2) *If $\frac{\gamma}{\alpha-1}$ be a nondecreasing function then there exist positive constants c_0, c_1, and c_2 such that*

$$\frac{|\nabla v|^2}{v} - \alpha\frac{v_t}{v} \leq \frac{m(p-1)}{1+m(p-1)}\left(K_3 + \alpha\sqrt{K_2\frac{1+m(p-1)}{m(p-1)}}\right) + \alpha\beta \tag{1.7}$$

on $Q_{2R,T}$ where

$$\begin{aligned} K_3 &= \alpha^2\Big[c_2k_2 + k\Big(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R}) + \frac{c_1}{R^2}\Big) + 2(p-1)k\frac{c_1}{R^2}\Big] \\ &\quad + \frac{mp^2(p-1)}{2\big(1+m(p-1)\big)}\frac{c_1}{R^2}\frac{\alpha^3}{\alpha-1}k \\ &\quad + \frac{1}{2}\frac{\alpha^2}{\alpha-1}\Big[2(p-1)k\big(k_1+k_4\big) + (p-1)\alpha\big(2k_2+1\big) + 2(\alpha-1)k_3\Big]. \end{aligned}$$

The three functions $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are the following

1. *Li-Yau type:*

$$\alpha(t) = \text{constant} > 1, \qquad \beta(t) = \frac{m(p-1)}{m(p-1)+1}\frac{\alpha}{t} + \frac{kk_1}{\alpha-1},$$
$$\gamma(t) = t^\theta, \qquad 0<\theta<2.$$

2. *Hamilton type:*

$$\alpha(t) = e^{kk_1t}, \qquad \beta(t) = \frac{m(p-1)}{(m(p-1)+1)t}e^{2kk_1t},$$
$$\gamma(t) = te^{kk_1t}.$$

3. *Li-Xu type:*

$$\alpha(t) = 1 + \frac{\sinh(kk_1t)\cosh(kk_1t) - kk_1t}{\sinh^2(kk_1t)},$$

$$\beta(t) = \frac{kk_1m(p-1)}{m(p-1)+1}(1+\coth(kk_1t)), \qquad \gamma(t) = \tanh(kk_1t).$$

4. *Linear Li-Xu type:*

$$\alpha(t) = 1 + kk_1t, \qquad \beta(t) = \frac{m(p-1)}{m(p-1)+1}\Big(\frac{1}{t} + kk_1\Big)$$

$$\gamma(t) = kk_1t.$$

Corollary 1.2. Let $(M, g(0))$ be a complete noncompact metric measure space without boundary, and $g(t)$, $\phi(t)$ evolve by (1.3) for $t \in (0, T]$. Let u be a positive solution to (1.2) in M such that $u^{p-1} \leq \frac{p-1}{p}k$ for some positive constant k. Suppose that there exist constants k_1, k_2, k_3, k_4 such that

$$Ric_\phi^{m-n} \geq -(m-1)k_1 g, \qquad -k_2 g \leq h \leq k_3 g, \qquad |\nabla h| \leq k_4,$$

on M. Let there exist three functions $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ satisfy conditions (A1)-(A6). Suppose that

$$\Theta_1 := \sup_{M\times[0,T]} |\nabla\phi|, \qquad \Theta_2 := \sup_{M\times[0,T]} |\nabla\Delta\phi|.$$

For any constant $p > 1$,

(1) if $\frac{\gamma}{\alpha-1} \leq c$ then there exist positive constants c_0, c_1, and c_2 such that

$$\frac{|\nabla v|^2}{v} - \alpha\frac{v_t}{v} \leq \frac{m(p-1)}{1+m(p-1)}\left(\alpha^2 K_4 + \alpha\sqrt{K_5\frac{1+m(p-1)}{m(p-1)}}\right) + \alpha\beta, \tag{1.8}$$

on M where

$$K_4 = c_2k_2 + \frac{1}{2\gamma}c\Big[2(p-1)k\big(k_1+k_4\big) + (p-1)\alpha\big(2k_2+1\big) + 2(\alpha-1)k_3\Big],$$

and

$$K_5 = \alpha^2(p-1)n\max\{k_2^2, k_3^2\} + \frac{9}{8}(p-1)n\alpha^2k_4 + \frac{1}{2}k_2\alpha(p-1)k\Theta_1^2 + \frac{1}{4}\alpha(p-1)k\Theta_2^2.$$

(2) If $\frac{\gamma}{\alpha-1}$ be a nondecreasing function then there exist positive constants c_0, c_1, and c_2 such that

$$\frac{|\nabla v|^2}{v} - \alpha\frac{v_t}{v} \leq \frac{m(p-1)}{1+m(p-1)}\left(\alpha^2 K_6 + \alpha\sqrt{K_5\frac{1+m(p-1)}{m(p-1)}}\right) + \alpha\beta \tag{1.9}$$

on M where

$$K_6 = c_2k_2 + \frac{1}{2}\frac{1}{\alpha-1}\Big[2(p-1)k\big(k_1+k_4\big) + (p-1)\alpha\big(2k_2+1\big) + 2(\alpha-1)k_3\Big].$$

As an application of the global gradient estimates obtained in Corollary 1.2, by integrating the gradient estimates in space-time we derive the following Harnack inequality. We first introduce the following notation. Given $(y_1, s_1) \in M \times (0, T]$ and $(y_2, s_2) \in M \times (0, T]$ satisfying $s_1 < s_2$, define

$$\mathcal{J}(y_1, s_1, y_2, s_2) = \inf \int_{s_1}^{s_2} |\zeta'(t)|^2_{g(t)}dt,$$

and the infimum is taken over the all smooth curves $\zeta : [s_1, s_2] \to M$ jointing y_1 and y_2.

Corollary 1.3. With the same assumptions in Corollary 1.3, for $(y_1, s_1) \in M \times (0, T]$ and $(y_2, s_2) \in M \times (0, T]$ such that $s_1 < s_2$, if $\frac{\gamma}{\alpha-1} \leq c$ then we have

$$\begin{aligned} v(y_1, s_1) \quad &\leq \quad v(y_2, s_2) \exp\left\{ \frac{\alpha(s_2)}{4\tilde{k}} \mathcal{J}(y_1, s_1, y_2, s_2) \right. \\ &\left. + \int_{s_1}^{s_2} \left\{ \frac{m(p-1)}{1+m(p-1)} \left(\alpha^2 K_4 + \alpha \sqrt{K_5 \frac{1+m(p-1)}{m(p-1)}} \right) + \alpha\beta \right\} dt \right\}, \end{aligned}$$

and if $\frac{\gamma}{\alpha-1}$ be a nondecreasing function then

$$\begin{aligned} v(y_1, s_1) \quad &\leq \quad v(y_2, s_2) \exp\left\{ \frac{\alpha(s_2)}{4\tilde{k}} \mathcal{J}(y_1, s_1, y_2, s_2) \right. \\ &\left. + \int_{s_1}^{s_2} \left\{ \frac{m(p-1)}{1+m(p-1)} \left(\alpha^2 K_6 + \alpha \sqrt{K_5 \frac{1+m(p-1)}{m(p-1)}} \right) + \alpha\beta \right\} dt \right\}, \end{aligned}$$

where $\tilde{k} = \inf\limits_{M \times [0,T]} v$. In particular, for Li-Yau type estimate, we have

$$\begin{aligned} v(y_1, s_1) \quad &\leq \quad v(y_2, s_2) \Big(\frac{s_2}{s_1} \Big)^{\frac{m(p-1)\alpha^2}{1+m(p-1)}} \exp\left\{ \frac{\alpha}{4\tilde{k}} \mathcal{J}(y_1, s_1, y_2, s_2) \right. \\ &\left. + \left\{ \frac{m(p-1)}{1+m(p-1)} \left(\alpha^2 K_6 + \alpha \sqrt{K_5 \frac{1+m(p-1)}{m(p-1)}} \right) + \frac{\alpha k k_1}{\alpha - 1} \right\} (s_2 - s_1) \right\}, \end{aligned}$$

2. Proofs of results

For prove our results, we need the following lemmas. From [10] we have

Lemma 2.1. Let the metric evolves by (1.3). Then for any smooth function f, we have

$$\frac{\partial}{\partial t} |\nabla f|^2 = -2h(\nabla f, \nabla f) + 2\langle \nabla f, \nabla f_t \rangle$$

and

$$\begin{aligned} (\Delta_\phi f)_t \quad &= \quad \Delta_\phi f_t - 2\langle h, \mathrm{Hess} f \rangle - 2\langle \mathrm{div} h - \frac{1}{2} \nabla(\mathrm{tr}_g h), \nabla f \rangle \\ &\quad +2h(\nabla\phi, \nabla f) - \langle \nabla f, \nabla \Delta\phi \rangle \end{aligned}$$

where $\mathrm{div} h$ is the divergence of h.

Let

$$F = \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} - \alpha\beta$$

where $\alpha = \alpha(t)$ and $\beta = \beta(t)$ are functions dependenig to t let

$$\mathcal{L} = \partial_t - (p-1) v \Delta_\phi.$$

Lemma 2.2. Let $(M^n, g, e^{-\phi}dv)$ be a metric measure space, $g(t)$ evolves by (1.3) for $t \in [0,T]$ satisfies the hypotheses of Theorem 1.1. Then

$$\begin{aligned}\mathcal{L}(F) \leq\ & 2p\langle\nabla v, \nabla F\rangle - \frac{1+m(p-1)}{m(p-1)}\Big((p-1)\Delta_\phi v\Big)^2 - \alpha'\frac{v_t}{v} - \alpha'\beta - \alpha\beta' \\ & +C_1\frac{|\nabla v|^2}{v} + C_2 \end{aligned}\tag{2.1}$$

where

$$C_1 = 2(p-1)k\big(k_1+k_4\big) + (p-1)\alpha\big(2k_2+1\big) + 2(\alpha-1)k_3$$

and

$$C_2 = \alpha^2(p-1)n\max\{k_2^2, k_3^2\} + \frac{9}{8}(p-1)n\alpha^2k_4 + \frac{1}{2}k_2\alpha(p-1)k\theta_1^2 + \frac{1}{4}\alpha(p-1)k\theta_2^2.$$

Proof. Using Lemma 2.1 and simple calculation we obtain

$$\begin{aligned}\mathcal{L}(v_t) &= \big[(p-1)v\Delta_\phi v + |\nabla v|^2\big]_t - (p-1)v\Delta_\phi(v_t) \\ &= (p-1)v_t\Delta_\phi v + (p-1)v(\Delta_\phi v)_t - 2h(\nabla v, \nabla v) + 2\langle\nabla v, \nabla v_t\rangle \\ &\quad -(p-1)v\Delta_\phi(v_t) \\ &= (p-1)v_t\Delta_\phi v - 2(p-1)v\langle h, \mathrm{Hess}v\rangle - 2(p-1)v\langle \mathrm{div}h - \frac{1}{2}\nabla(\mathrm{tr}_g h), \nabla v\rangle \\ &\quad +2(p-1)vh(\nabla\phi, \nabla v) - (p-1)v\langle\nabla v, \nabla\Delta\phi\rangle - 2h(\nabla v, \nabla v) \\ &\quad +2\langle\nabla v, \nabla v_t\rangle.\end{aligned}\tag{2.2}$$

From the weighted Bochner formula (1.1) we obtain

$$\begin{aligned}\mathcal{L}(|\nabla v|^2) &= -2h(\nabla v, \nabla v) + 2\langle\nabla v, \nabla v_t\rangle \\ &\quad -2(p-1)v\Big(|\mathrm{Hess}v|^2 + \langle\nabla\Delta_\phi v, \nabla v\rangle + Ric_\phi(\nabla v, \nabla v)\Big) \\ &= -2h(\nabla v, \nabla v) + 2\langle\nabla v, \nabla\left[(p-1)v\Delta_\phi v + |\nabla v|^2\right]\rangle \\ &\quad -2(p-1)v\Big(|\mathrm{Hess}v|^2 + \langle\nabla\Delta_\phi v, \nabla v\rangle + Ric_\phi(\nabla v, \nabla v)\Big) \\ &= -2\big((p-1)vRic_\phi + h\big)(\nabla v, \nabla v) + 2(p-1)|\nabla v|^2\Delta_\phi v \\ &\quad +2\langle\nabla v, \nabla|\nabla v|^2\rangle - 2(p-1)v|\mathrm{Hess}v|^2.\end{aligned}\tag{2.3}$$

By direct computaion, we have

$$\mathcal{L}\Big(\frac{f}{g}\Big) = \frac{1}{g}\mathcal{L}(f) - \frac{f}{g^2}\mathcal{L}(g) + 2(p-1)v\langle\nabla\Big(\frac{f}{g}\Big), \nabla\log g\rangle, \qquad \forall f, g \in C^\infty(M).\tag{2.4}$$

Using (2.2) and (2.3) into (2.4) yields

$$\begin{aligned}\mathcal{L}\Big(\frac{v_t}{v}\Big) &= \frac{1}{v}\mathcal{L}(v_t) - \frac{v_t}{v^2}\mathcal{L}(v) + 2(p-1)v\langle\nabla\Big(\frac{v_t}{v}\Big), \nabla\log v\rangle \\ &= (p-1)\frac{v_t}{v}\Delta_\phi v - 2(p-1)\langle h, \mathrm{Hess}v\rangle - 2(p-1)\langle\mathrm{div}h - \frac{1}{2}\nabla(\mathrm{tr}_g h), \nabla v\rangle \\ &\quad +2(p-1)h(\nabla\phi, \nabla v) - (p-1)\langle\nabla v, \nabla\Delta\phi\rangle - \frac{2}{v}h(\nabla v, \nabla v) \\ &\quad +\frac{2}{v}\langle\nabla v, \nabla v_t\rangle - \frac{v_t}{v}\frac{|\nabla v|^2}{v} + 2(p-1)v\langle\nabla\Big(\frac{v_t}{v}\Big), \nabla\log v\rangle\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}\Big(\frac{|\nabla v|^2}{v}\Big) &= -2\big((p-1)Ric_\phi+\frac{1}{v}h\big)(\nabla v,\nabla v)+2(p-1)\frac{|\nabla v|^2}{v}\Delta_\phi v\\ &\quad+\frac{2}{v}\langle\nabla v,\nabla|\nabla v|^2\rangle-2(p-1)|\mathrm{Hess}v|^2-\frac{|\nabla v|^4}{v^2}\\ &\quad+2(p-1)v\langle\nabla\Big(\frac{|\nabla v|^2}{v}\Big),\nabla\log v\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{L}(F) &= \mathcal{L}\Big(\frac{|\nabla v|^2}{v}\Big)-\alpha\mathcal{L}\Big(\frac{v_t}{v}\Big)-\alpha'\frac{v_t}{v}-\alpha'\beta-\alpha\beta'\\ &= -2\big((p-1)Ric_\phi+\frac{1}{v}h\big)(\nabla v,\nabla v)+2(p-1)\frac{|\nabla v|^2}{v}\Delta_\phi v-\frac{|\nabla v|^4}{v^2}\\ &\quad+\frac{2}{v}\langle\nabla v,\nabla|\nabla v|^2\rangle-2(p-1)|\mathrm{Hess}v|^2+2(p-1)v\langle\nabla\Big(\frac{|\nabla v|^2}{v}\Big),\nabla\log v\rangle\\ &\quad-\alpha(p-1)\frac{v_t}{v}\Delta_\phi v+2\alpha(p-1)\langle h,\mathrm{Hess}v\rangle\\ &\quad+2\alpha(p-1)\langle\mathrm{div}h-\frac{1}{2}\nabla(\mathrm{tr}_g h),\nabla v\rangle-2\alpha(p-1)h(\nabla\phi,\nabla v)\\ &\quad+\alpha(p-1)\langle\nabla v,\nabla\Delta\phi\rangle+\alpha\frac{2}{v}h(\nabla v,\nabla v)-\alpha\frac{2}{v}\langle\nabla v,\nabla v_t\rangle+\alpha\frac{v_t}{v}\frac{|\nabla v|^2}{v}\\ &\quad-2\alpha(p-1)v\langle\nabla\Big(\frac{v_t}{v}\Big),\nabla\log v\rangle-\alpha'\frac{v_t}{v}-\alpha'\beta-\alpha\beta'.\end{aligned}\tag{2.5}$$

Also,

$$2(p-1)v\langle\nabla\Big(\frac{|\nabla v|^2}{v}\Big),\nabla\log v\rangle-2\alpha(p-1)v\langle\nabla\Big(\frac{v_t}{v}\Big),\nabla\log v\rangle=2(p-1)\langle\nabla v,\nabla F\rangle$$

and

$$\frac{2}{v}\langle\nabla v,\nabla|\nabla v|^2\rangle-\alpha\frac{2}{v}\langle\nabla v,\nabla v_t\rangle=2(F+\beta)\frac{|\nabla v|^2}{v}+2\langle\nabla v,\nabla F\rangle$$

imply that

$$\begin{aligned}&2(p-1)v\langle\nabla\Big(\frac{|\nabla v|^2}{v}\Big),\nabla\log v\rangle-2\alpha(p-1)v\langle\nabla\Big(\frac{v_t}{v}\Big),\nabla\log v\rangle\\ &+\frac{2}{v}\langle\nabla v,\nabla|\nabla v|^2\rangle-\alpha\frac{2}{v}\langle\nabla v,\nabla v_t\rangle\\ &=2p\langle\nabla v,\nabla F\rangle+2(F+\beta)\frac{|\nabla v|^2}{v}\\ &=2p\langle\nabla v,\nabla F\rangle+2\Big(\frac{|\nabla v|^2}{v}-\alpha\frac{v_t}{v}\Big)\frac{|\nabla v|^2}{v}.\end{aligned}\tag{2.6}$$

Using (1.5) we conclude

$$\begin{aligned}&2(p-1)\frac{|\nabla v|^2}{v}\Delta_\phi v-\frac{|\nabla v|^4}{v^2}-\alpha(p-1)\frac{v_t}{v}\Delta_\phi v+\alpha\frac{v_t}{v}\frac{|\nabla v|^2}{v}\\ &=2\frac{|\nabla v|^2}{v}\Big(\frac{v_t}{v}-\frac{|\nabla v|^2}{v}\Big)-\frac{|\nabla v|^4}{v^2}-\alpha\frac{v_t}{v}\Big(\frac{v_t}{v}-\frac{|\nabla v|^2}{v}\Big)+\alpha\frac{v_t}{v}\frac{|\nabla v|^2}{v}\\ &=(2\alpha+2)\frac{v_t}{v}\frac{|\nabla v|^2}{v}-3\frac{|\nabla v|^4}{v^2}-\alpha\Big(\frac{v_t}{v}\Big)^2.\end{aligned}\tag{2.7}$$

Equations (2.6) and (2.7) yield

$$
\begin{aligned}
&2(p-1)v\langle\nabla\Big(\frac{|\nabla v|^2}{v}\Big),\nabla\log v\rangle-2\alpha(p-1)v\langle\nabla\Big(\frac{v_t}{v}\Big),\nabla\log v\rangle+\frac{2}{v}\langle\nabla v,\nabla|\nabla v|^2\rangle\\
&-\alpha\frac{2}{v}\langle\nabla v,\nabla v_t\rangle+2(p-1)\frac{|\nabla v|^2}{v}\Delta_\phi v-\frac{|\nabla v|^4}{v^2}-\alpha(p-1)\frac{v_t}{v}\Delta_\phi v+\alpha\frac{v_t}{v}\frac{|\nabla v|^2}{v}\\
&=2p\langle\nabla v,\nabla F\rangle-\Big(\frac{|\nabla v|^2}{v}-\frac{v_t}{v}\Big)^2-(\alpha-1)\Big(\frac{v_t}{v}\Big)^2\\
&=2p\langle\nabla v,\nabla F\rangle-\Big((p-1)\Delta_\phi v\Big)^2-(\alpha-1)\Big(\frac{v_t}{v}\Big)^2. \qquad (2.8)
\end{aligned}
$$

Plugging (2.8) into (2.5) we deduce

$$
\begin{aligned}
\mathcal{L}(F)\;=\;&2p\langle\nabla v,\nabla F\rangle-\Big((p-1)\Delta_\phi v\Big)^2-(\alpha-1)\Big(\frac{v_t}{v}\Big)^2-\alpha'\frac{v_t}{v}-\alpha'\beta-\alpha\beta'\\
&-2\Big((p-1)Ric_\phi+\frac{1}{v}h\Big)(\nabla v,\nabla v)-2(p-1)|\mathrm{Hess}v|^2\\
&+2\alpha(p-1)\langle h,\mathrm{Hess}v\rangle+2\alpha(p-1)\langle\mathrm{div}h-\frac{1}{2}\nabla(\mathrm{tr}_g h),\nabla v\rangle \qquad (2.9)\\
&-2\alpha(p-1)h(\nabla\phi,\nabla v)+\alpha(p-1)\langle\nabla v,\nabla\Delta\phi\rangle+\alpha\frac{2}{v}h(\nabla v,\nabla v).
\end{aligned}
$$

The assumption $-k_2 g\le h\le k_3 g$ implies that

$$|h|^2\le n\max\{k_2^2,k_3^2\}.$$

By Yaung's inequality

$$
\begin{aligned}
\langle h,\mathrm{Hess}v\rangle\;&\le\;\frac{1}{2\alpha}|\mathrm{Hess}v|^2+\frac{\alpha}{2}|h|^2\\
&\le\;\frac{1}{2\alpha}|\mathrm{Hess}v|^2+\frac{n\alpha}{2}\max\{k_2^2,k_3^2\} \qquad (2.10)
\end{aligned}
$$

. We also have

$$|\mathrm{div}h-\frac{1}{2}\nabla(\mathrm{tr}_g h)|=|g^{ij}\nabla_i h_{jl}-\frac{1}{2}g^{ij}\nabla_l h_{ij}|\le\frac{3}{2}|g||\nabla h|\le\frac{3}{2}\sqrt{n}k_4. \qquad (2.11)$$

Notice also that for any $m>n$ we derive

$$
\begin{aligned}
0\;&\le\;\Big(\sqrt{\frac{m-n}{mn}}\Delta v+\sqrt{\frac{n}{m(m-n)}}\langle\nabla v,\nabla\phi\rangle\Big)^2\\
&=\;(\frac{1}{n}-\frac{1}{m})(\Delta v)^2+\frac{2}{m}\Delta v\langle\nabla v,\nabla\phi\rangle+(\frac{1}{m-n}-\frac{1}{m})\langle\nabla v,\nabla\phi\rangle^2\\
&\le\;|\mathrm{Hess}v|^2-\frac{1}{m}\Big((\Delta v)^2-2\Delta v\langle\nabla v,\nabla\phi\rangle+\langle\nabla v,\nabla\phi\rangle^2\Big)+\frac{1}{m-n}\langle\nabla v,\nabla\phi\rangle^2\\
&=\;|\mathrm{Hess}v|^2-\frac{(\Delta_\phi v)^2}{m}+\frac{1}{m-n}\langle\nabla v,\nabla\phi\rangle^2.
\end{aligned}
$$

Therefore

$$|\mathrm{Hess}v|^2\ge\frac{(\Delta_\phi v)^2}{m}-\frac{1}{m-n}\langle\nabla v,\nabla\phi\rangle^2. \qquad (2.12)$$

For $p>1$, substituting (2.13), (2.11), and (2.12) into (2.9) we arrive at

$$\begin{aligned}\mathcal{L}(F) &\leq 2p\langle\nabla v,\nabla F\rangle-\Big((p-1)\Delta_\phi v\Big)^2-(\alpha-1)\Big(\frac{v_t}{v}\Big)^2-\alpha'\frac{v_t}{v}-\alpha'\beta-\alpha\beta'\\ &\quad -2(p-1)Ric_\phi(\nabla v,\nabla v)+2(\alpha-1)k_3\frac{|\nabla v|^2}{v}-(p-1)|\mathrm{Hess}v|^2\\ &\quad +\alpha^2(p-1)n\max\{k_2^2,k_3^2\}+3\alpha(p-1)\sqrt{n}k_4|\nabla v|\\ &\quad +2k_2\alpha(p-1)\langle\nabla v,\nabla\phi\rangle+\alpha(p-1)\langle\nabla v,\nabla\Delta\phi\rangle\\ &\leq 2p\langle\nabla v,\nabla F\rangle-\frac{1+m(p-1)}{m(p-1)}\Big((p-1)\Delta_\phi v\Big)^2\\ &\quad -\alpha'\frac{v_t}{v}-\alpha'\beta-\alpha\beta'+2(p-1)k_1|\nabla v|^2+2(\alpha-1)k_3\frac{|\nabla v|^2}{v}\\ &\quad +\alpha^2(p-1)n\max\{k_2^2,k_3^2\}+3\alpha(p-1)\sqrt{n}k_4|\nabla v|\\ &\quad +2k_2\alpha(p-1)\langle\nabla v,\nabla\phi\rangle+\alpha(p-1)\langle\nabla v,\nabla\Delta\phi\rangle.\end{aligned}\tag{2.13}$$

Youn's inequality implies

$$3\alpha\sqrt{n}k_4|\nabla v|\leq 2k_4k\frac{|\nabla v|^2}{v}+\frac{9}{8}n\alpha^2k_4.\tag{2.14}$$

Using Cauchy's inequality, we deduce

$$\langle\nabla v,\nabla\phi\rangle\leq\theta_1k^{\frac{1}{2}}\frac{|\nabla v|}{v^{\frac{1}{2}}}\leq\frac{|\nabla v|^2}{v}+\frac{1}{4}k\theta_1^2\tag{2.15}$$

and

$$\langle\nabla v,\nabla\Delta\phi\rangle\leq\theta_2k^{\frac{1}{2}}\frac{|\nabla v|}{v^{\frac{1}{2}}}\leq\frac{|\nabla v|^2}{v}+\frac{1}{4}k\theta_2^2.\tag{2.16}$$

Substituting (2.14)-(2.16) into (2.13), we get

$$\begin{aligned}\mathcal{L}(F) &\leq 2p\langle\nabla v,\nabla F\rangle-\frac{1+m(p-1)}{m(p-1)}\Big((p-1)\Delta_\phi v\Big)^2-\alpha'\frac{v_t}{v}-\alpha'\beta-\alpha\beta'\\ &\quad +\Big[(p-1)\big(2k_1k+2k_4k+2k_2\alpha+\alpha\big)+2(\alpha-1)k_3\Big]\frac{|\nabla v|^2}{v}\\ &\quad +\alpha^2(p-1)n\max\{k_2^2,k_3^2\}+\frac{9}{8}(p-1)n\alpha^2k_4+\frac{1}{2}k_2\alpha(p-1)k\theta_1^2\\ &\quad +\frac{1}{4}\alpha(p-1)k\theta_2^2.\end{aligned}$$

This completes the proof of Lemma. □

Proof of theorem 1.1. Since the Ricci tensor and the evolution of the Riemannian metric are bonded we imply that $g(t)$ is uniformly equivalent to the initial metric $g(0)$ (see [9, Corollary 6.11]),

$$e^{-2k_2T}g(0)\leq g(t)\leq e^{2k_3T}g(0).$$

Therefore the manifold $(M,g(t))$ is also complete for $t\in[0,T]$. Now let $\psi(s)$ be a C^2-function on $[0,+\infty)$ such that

$$\psi(s)=\begin{cases}1, & s\in[0,1],\\ 0, & s\in[2,+\infty),\end{cases}$$

and it satisfies $\psi(s) \in [0,1]$, $-c_0 \le \psi'(s) \le 0$, $\psi''(s) \ge -c_1$, and $\frac{|\psi''(s)|^2}{\psi(s)} \le c_1$, where c_0 and c_1 are absolute constants. Let $R \ge 1$ and define a function

$$\eta(x,t) = \psi(\frac{r(x,t)}{R}),$$

where $r(x,t) = d(x, x_0, r)$. By using the argument of [6, 21], we can apply maximum principle and invoke Calabi's trick to assume everywhere smoothness of $\eta(x,t)$. Also, for obtain inequalities of $\eta(x,t)$, we use generalization Laplacian comparison theorem [3, 20, 27, 33]. Since $Ric_\phi^{m-n} \ge -(m-1)k_1$, the generalization Lpalacian comparison theorem implies that

$$\Delta_\phi r(x) \le (m-1)\sqrt{k_1}\coth(\sqrt{k_1}r(x))$$

and

$$\begin{aligned}\Delta_\phi \eta &= \psi'\frac{\Delta_\phi r}{R} + \psi''\frac{|\nabla r|^2}{R^2} \\ &\ge -\frac{c_0}{R}(m-1)\sqrt{k_1}\coth(\sqrt{k_1}r(x)) - \frac{c_1}{R^2} \\ &\ge -\frac{c_0}{R}(m-1)(\sqrt{k_1} + \frac{2}{R}) - \frac{c_1}{R^2}.\end{aligned} \tag{2.17}$$

Also, we have

$$\frac{|\nabla\eta|^2}{\eta} = \frac{|\psi'|^2|\nabla r|^2}{R^2\psi} \le \frac{c_1}{R^2}. \tag{2.18}$$

Let $G = \gamma(t)\eta G$. Fix arbitrary $T_1 \in (0,T]$ and assume that G achieves its maximum at point $(x_0,t_0) \in Q_{2R,T_1}$. If $G(x_0,t_0) \le 0$, then the result holds trivially and we done. Hence, we may assume that $G(x_0,t_0) > 0$. In this point we have

$$\nabla G = 0, \qquad \mathcal{L}G \ge 0.$$

Therefore, we conclude

$$\nabla F = -\frac{F}{\eta}\nabla\eta \tag{2.19}$$

and

$$0 \le \mathcal{L}G = \gamma'\eta F + \gamma F\mathcal{L}\eta + \gamma\eta\mathcal{L}F - 2\gamma(p-1)v\langle\nabla\eta, \nabla F\rangle. \tag{2.20}$$

By [29, p. 494], there exist a constant c_2 such that

$$-F\eta_t \ge -c_2k_2F. \tag{2.21}$$

Replacing (2.17)-(2.19) and (2.21) into (2.20) we get

$$\begin{aligned}0 \le\ & \frac{\gamma'}{\gamma}G + \Big[c_2k_2 + k\Big(\frac{c_0}{R}(m-1)(\sqrt{k_1} + \frac{2}{R}) + \frac{c_1}{R^2}\Big)\Big]\gamma F \\ & + \gamma\eta\mathcal{L}F - 2\gamma(p-1)v\langle\nabla\eta, \nabla F\rangle.\end{aligned} \tag{2.22}$$

Then at point (x_0,t_0), it follows that

$$\begin{aligned}0 \le\ & \frac{\gamma'}{\gamma}G + \Big[c_2k_2 + k\Big(\frac{c_0}{R}(m-1)(\sqrt{k_1} + \frac{2}{R}) + \frac{c_1}{R^2}\Big)\Big]\gamma F \\ & -2\gamma(p-1)v\langle\nabla\eta, \nabla F\rangle + 2p\gamma\eta\langle\nabla v, \nabla F\rangle - \gamma\eta\alpha'\frac{v_t}{v} - \gamma\eta\alpha'\beta - \gamma\eta\alpha\beta' \\ & -\gamma\eta\Big[\frac{1}{m(p-1)} + 1\Big]\Big((p-1)\Delta_\phi v\Big)^2 + \gamma\eta C_1\frac{|\nabla v|^2}{v} + \gamma\eta C_2\end{aligned} \tag{2.23}$$

By equation (1.5) and definition of F we have

$$\Big((p-1)\Delta_\phi v\Big)^2 = \frac{1}{\alpha^2}F^2 + \frac{2(\alpha-1)}{\alpha^2}F\frac{|\nabla v|^2}{v} + \frac{(\alpha-1)^2}{\alpha^2}\frac{|\nabla v|^4}{v^2} + \beta^2 + \frac{2\beta}{\alpha}F + \frac{2\beta(\alpha-1)}{\alpha}\frac{|\nabla v|^2}{v} \tag{2.24}$$

and

$$\frac{v_t}{v} = -\frac{F}{\alpha} + \frac{1}{\alpha}\frac{|\nabla v|^2}{v} - \beta. \tag{2.25}$$

By Cauchy's inequality, we derive

$$\eta\langle\nabla v, \nabla F\rangle = -F\langle\nabla v, \nabla\eta\rangle \le \frac{\sqrt{c_1}}{R}\eta^{\frac{1}{2}}k^{\frac{1}{2}}F\frac{|\nabla v|}{v^{\frac{1}{2}}}. \tag{2.26}$$

Inequality (2.18) yields

$$-\langle\nabla\eta, \nabla F\rangle = F\frac{|\nabla\eta|^2}{\eta} \le \frac{c_1}{R^2}F. \tag{2.27}$$

Putting (2.24)-(2.27) into (2.23), we get

$$\begin{aligned}
0 \le\ & \frac{\gamma'}{\gamma}G + \Big[c_2k_2 + k\Big(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R}) + \frac{c_1}{R^2}\Big) + 2(p-1)k\frac{c_1}{R^2}\Big]\gamma F\\
&+2p\gamma\frac{\sqrt{c_1}}{R}\eta^{\frac{1}{2}}k^{\frac{1}{2}}F\frac{|\nabla v|}{v^{\frac{1}{2}}} - \gamma\eta\Big[\frac{1}{m(p-1)}+1\Big]\frac{F^2}{\alpha^2}\\
&-\gamma\eta\Big[\frac{1}{m(p-1)}+1\Big]\frac{(\alpha-1)^2}{\alpha^2}\frac{|\nabla v|^4}{v^2} - 2\gamma\eta\Big[\frac{1}{m(p-1)}+1\Big]\frac{(\alpha-1)}{\alpha}F\frac{|\nabla v|^2}{v}\\
&-\gamma\eta\Big[\frac{1}{m(p-1)}+1\Big]\frac{2(\alpha-1)\beta}{\alpha}\frac{|\nabla v|^2}{v} - \gamma\eta\Big[\frac{1}{m(p-1)}+1\Big]\beta^2\\
&-\Big[\frac{1}{m(p-1)}+1\Big]\frac{2\beta}{\alpha}G + \gamma\eta C_1\frac{|\nabla v|^2}{v} + \gamma\eta C_2\\
&-\gamma\eta\Big(-\frac{\alpha'}{\alpha}F + \frac{\alpha'}{\alpha}\frac{|\nabla v|^2}{v} + \alpha\beta'\Big)
\end{aligned}$$

then

$$\begin{aligned}
0 \le\ & \Big(\frac{\gamma'}{\gamma} + \frac{\alpha'}{\alpha} - \Big[\frac{1}{m(p-1)}+1\Big]\frac{2\beta}{\alpha}\Big)G\\
&+\Big[c_2k_2 + k\Big(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R}) + \frac{c_1}{R^2}\Big) + 2(p-1)k\frac{c_1}{R^2}\Big]\gamma F\\
&+2p\frac{\sqrt{c_1}}{R}\eta^{-\frac{1}{2}}k^{\frac{1}{2}}G\frac{|\nabla v|}{v^{\frac{1}{2}}} - \gamma\eta\Big[\frac{1}{m(p-1)}+1\Big]\frac{F^2}{\alpha^2}\\
&-\gamma\eta\Big[\frac{1}{m(p-1)}+1\Big]\frac{(\alpha-1)^2}{\alpha^2}\frac{|\nabla v|^4}{v^2} - 2\Big[\frac{1}{m(p-1)}+1\Big]\frac{(\alpha-1)}{\alpha}G\frac{|\nabla v|^2}{v}\\
&+\gamma\eta\Big(-\Big[\frac{1}{m(p-1)}+1\Big]\frac{2(\alpha-1)\beta}{\alpha} - \frac{\alpha'}{\alpha} + C_1\Big)\frac{|\nabla v|^2}{v}\\
&+\gamma\eta\Big(C_2 - \alpha\beta' - \Big[\frac{1}{m(p-1)}+1\Big]\beta^2\Big).
\end{aligned} \tag{2.28}$$

Using conditions (A3)-A(5) we obtain

$$\begin{aligned}0 \leq\ & \Big[c_2k_2 + k\Big(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R}) + \frac{c_1}{R^2}\Big) + 2(p-1)k\frac{c_1}{R^2}\Big]\gamma F \\ & +2p\frac{\sqrt{c_1}}{R}\eta^{-\frac{1}{2}}k^{\frac{1}{2}}G\frac{|\nabla v|}{v^{\frac{1}{2}}} - \gamma\eta\Big[\frac{1}{m(p-1)}+1\Big]\frac{F^2}{\alpha^2} \\ & -\gamma\eta\Big[\frac{1}{m(p-1)}+1\Big]\frac{(\alpha-1)^2}{\alpha^2}\frac{|\nabla v|^4}{v^2} - 2\Big[\frac{1}{m(p-1)}+1\Big]\frac{(\alpha-1)}{\alpha}G\frac{|\nabla v|^2}{v} \\ & +\gamma\eta C_1\frac{|\nabla v|^2}{v} + \gamma\eta C_2. \end{aligned} \tag{2.29}$$

By virtue of the inequality $-Ax^2 + Bx \leq \frac{B^2}{4A}$ for a positive number A and any number B, we get

$$-\gamma\eta\Big[\frac{1}{m(p-1)}+1\Big]\frac{(\alpha-1)^2}{\alpha^2}\frac{|\nabla v|^4}{v^2} + \gamma\eta C_1\frac{|\nabla v|^2}{v} \leq \gamma\eta C_3 \tag{2.30}$$

where

$$C_3 = \frac{m(p-1)\alpha^2}{4(\alpha-1)^2\big(1+m(p-1)\big)}C_1^2.$$

and

$$\begin{aligned} & -2\Big[\frac{1}{m(p-1)}+1\Big]\frac{(\alpha-1)}{\alpha}G\frac{|\nabla v|^2}{v} + 2p\frac{\sqrt{c_1}}{R}\eta^{-\frac{1}{2}}k^{\frac{1}{2}}G\frac{|\nabla v|}{v^{\frac{1}{2}}} \\ & \leq \frac{mp^2(p-1)\alpha}{2(\alpha-1)\big(1+m(p-1)\big)}\frac{c_1}{R^2}\eta^{-1}kG. \end{aligned} \tag{2.31}$$

Plugging (2.30) and (2.31) into (2.29), we infer

$$\begin{aligned}0 \leq\ & \Big[c_2k_2 + k\Big(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R}) + \frac{c_1}{R^2}\Big) + 2(p-1)k\frac{c_1}{R^2}\Big]\gamma F \\ & +\frac{mp^2(p-1)\alpha}{2(\alpha-1)\big(1+m(p-1)\big)}\frac{c_1}{R^2}\eta^{-1}kG - \gamma\eta\Big[\frac{1}{m(p-1)}+1\Big]\frac{F^2}{\alpha^2} \\ & +\gamma\eta\Big(C_2 + C_3\Big). \end{aligned} \tag{2.32}$$

Multiplying (2.32) by $\gamma\eta$ we conclude

$$\begin{aligned}0 \leq\ & -\Big[\frac{1}{m(p-1)}+1\Big]\frac{G^2}{\alpha^2} + \Big\{\gamma\Big[c_2k_2 + k\Big(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R}) \\ & +\frac{c_1}{R^2}\Big) + 2(p-1)k\frac{c_1}{R^2}\Big] + \frac{mp^2(p-1)\alpha}{2(\alpha-1)\big(1+m(p-1)\big)}\frac{c_1}{R^2}\gamma\eta^{\frac{1}{2}}k\Big\}G \\ & +\gamma^2\eta^2\Big(C_2 + C_3\Big). \end{aligned} \tag{2.33}$$

For a positive number $\tilde{a}$, a number $\tilde{b}$, and a nonnegative number $\tilde{c}$, the quadratic inequality of the form $-\tilde{a}x^2 + \tilde{b}x + \tilde{c} \geq 0$ implies that $x \leq \frac{1}{2\tilde{a}}(\tilde{b} + \sqrt{\tilde{b}^2 + 4\tilde{a}\tilde{c}})$. Set

$$\begin{aligned} C_4 =\ & \gamma\Big[c_2k_2 + k\Big(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R}) + \frac{c_1}{R^2}\Big) + 2(p-1)k\frac{c_1}{R^2}\Big] \\ & +\frac{mp^2(p-1)\alpha}{2(\alpha-1)\big(1+m(p-1)\big)}\frac{c_1}{R^2}\gamma\eta^{\frac{1}{2}}k \end{aligned}$$

and

$$C_5 = C_2 + C_3.$$

Then

$$\begin{aligned}
G &\leq \frac{1}{2}\frac{m(p-1)\alpha^2}{1+m(p-1)}\left(C_4+\sqrt{C_4^2+4\gamma^2\eta^2C_5\frac{1+m(p-1)}{m(p-1)\alpha^2}}\right)\\
&\leq \frac{m(p-1)\alpha^2}{1+m(p-1)}\left(C_4+\gamma\eta\sqrt{C_5\frac{1+m(p-1)}{m(p-1)\alpha^2}}\right) \qquad (2.34)\\
&\leq \frac{m(p-1)}{1+m(p-1)}\left(\alpha^2C_4+\gamma\eta C_1\frac{\alpha^2}{2(\alpha-1)}+\gamma\eta\alpha\sqrt{C_2\frac{1+m(p-1)}{m(p-1)}}\right).
\end{aligned}$$

If $\frac{\gamma}{\alpha-1}\leq c$ then

$$G \leq \frac{m(p-1)}{1+m(p-1)}\left(\alpha^2C_6+\gamma\eta\alpha\sqrt{C_2\frac{1+m(p-1)}{m(p-1)}}\right),$$

where

$$\begin{aligned}
C_6 &= \gamma\Big[c_2k_2+k\Big(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R})+\frac{c_1}{R^2}\Big)+2(p-1)k\frac{c_1}{R^2}\Big]\\
&\quad+\frac{mp^2(p-1)\alpha}{2\big(1+m(p-1)\big)}\frac{c_1}{R^2}c\eta^{\frac{1}{2}}k+\frac{1}{2}\eta cC_1.
\end{aligned}$$

If $\frac{\gamma}{\alpha-1}$ be a nondecreasing function then

$$G \leq \frac{m(p-1)}{1+m(p-1)}\left(C_7+\gamma\eta\alpha\sqrt{C_2\frac{1+m(p-1)}{m(p-1)}}\right),$$

where

$$\begin{aligned}
C_7 &= \gamma\alpha^2\Big[c_2k_2+k\Big(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R})+\frac{c_1}{R^2}\Big)+2(p-1)k\frac{c_1}{R^2}\Big]\\
&\quad+\frac{mp^2(p-1)}{2\big(1+m(p-1)\big)}\frac{c_1}{R^2}\frac{\gamma\alpha^3}{\alpha-1}\eta^{\frac{1}{2}}k+\frac{1}{2}\eta\frac{\gamma\alpha^2}{\alpha-1}C_1.
\end{aligned}$$

To obtain the required result on $F(x,t)$ for an appropriate range of $x\in M$, we get $\eta(x,T_1)=1$ whenever $d(x,x_0,T_1)<2R$ and since (x_0,t_0) is the maximum point of G in $Q_{2R,T}$, we have

$$F(x,T_1)=\frac{G(x,T_1)}{\gamma(T_1)}\leq\frac{G(x_0,t_0)}{\gamma(T_1)}$$

for all $x\in M$, such that $d(x,x_0,T_1)<R$ and $T_1\in(0,T]$ was arbitrary. Now, if $\frac{\gamma}{\alpha-1}\leq c$ then

$$F(x,T_1)\leq\frac{m(p-1)}{1+m(p-1)}\left(\alpha^2C_8+\alpha\sqrt{C_2\frac{1+m(p-1)}{m(p-1)}}\right)\Big|_{t=T_1},$$

where

$$\begin{aligned} C_8 &= c_2k_2 + k\Big(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R})+\frac{c_1}{R^2}\Big)+2(p-1)k\frac{c_1}{R^2} \\ &\quad +\frac{1}{\gamma}\Big(\frac{mp^2(p-1)\alpha}{2\big(1+m(p-1)\big)}\frac{c_1}{R^2}ck+\frac{1}{2}cC_1\Big). \end{aligned}$$

If $\frac{\gamma}{\alpha-1}$ be a nondecreasing function then

$$F(x,T_1) \leq \frac{m(p-1)}{1+m(p-1)}\left(C_9+\alpha\sqrt{C_2\frac{1+m(p-1)}{m(p-1)}}\right)\Big|_{t=T_1},$$

where

$$\begin{aligned} C_9 &= \alpha^2\Big[c_2k_2 + k\Big(\frac{c_0}{R}(m-1)(\sqrt{k_1}+\frac{2}{R})+\frac{c_1}{R^2}\Big)+2(p-1)k\frac{c_1}{R^2}\Big] \\ &\quad +\frac{mp^2(p-1)}{2\big(1+m(p-1)\big)}\frac{c_1}{R^2}\frac{\alpha^3}{\alpha-1}k+\frac{1}{2}\frac{\alpha^3}{\alpha-1}C_1. \end{aligned}$$

Since T_1 is arbitrary, then inequalities (1.6) and (1.7) hold. Note that

(1) Li-Yau type gradient estimate
Let

$$\alpha(t) = \text{constant} > 1, \qquad \beta(t) = \frac{m(p-1)}{m(p-1)+1}\frac{\alpha}{t}+\frac{kk_1}{\alpha-1},$$
$$\gamma(t) = t^{\theta}, \qquad 0<\theta<2.$$

By a direct calculation, we have

$$\frac{\gamma'}{\gamma}+\frac{\alpha'}{\alpha}-2\big[\frac{1}{m(p-1)}+1\big]\frac{\beta}{\alpha} = \frac{\theta}{t}-\frac{2}{t}-2\big[\frac{1}{m(p-1)}+1\big]\frac{kk_1}{\alpha-1}\alpha \leq 0,$$

$$\alpha\beta'+\big[\frac{1}{m(p-1)}+1\big]\beta^2 = \frac{m(p-1)+1}{m(p-1)}\Big[2\frac{m(p-1)}{m(p-1)+1}\frac{\alpha}{t}\frac{kk_1}{\alpha-1}+\big(\frac{kk_1}{\alpha-1}\big)^2\Big] \geq 0,$$

and

$$2\big[\frac{1}{m(p-1)}+1\big]\frac{\alpha-1}{\alpha}\beta+\frac{\alpha'}{\alpha} = 2\frac{\alpha-1}{t}+2\big[\frac{1}{m(p-1)}+1\big]\frac{kk_1}{\alpha} \geq 0.$$

Also, $\alpha(t)>1$, $\gamma(t)$, and $\frac{\gamma}{\alpha-1}$ are nondecreasing functions. Hence α, β, and γ in this case satisfy the conditions (A1)-(A6).

(2) Hamilton type gradient estimate
Let

$$\alpha(t) = e^{kk_1t}, \qquad \beta(t) = \frac{m(p-1)}{(m(p-1)+1)t}e^{2kk_1t},$$
$$\gamma(t) = te^{kk_1t}.$$

Direct calculation shows

$$\frac{\gamma'}{\gamma}+\frac{\alpha'}{\alpha}-2\big[\frac{1}{m(p-1)}+1\big]\frac{\beta}{\alpha} = \frac{1}{t}\Big(1-2kk_1t-2e^{kk_1t}\Big) \leq 0,$$

$$\alpha\beta'+\big[\frac{1}{m(p-1)}+1\big]\beta^2 = \frac{m(p-1)}{m(p-1)+1}\frac{e^{3kk_1t}}{t^2}\Big[e^{kk_1t}-1-2kk_1t\Big] \geq 0,$$

and

$$2\big[\frac{1}{m(p-1)}+1\big]\frac{\alpha-1}{\alpha}\beta+\frac{\alpha'}{\alpha} = \frac{2}{t}(e^{kk_1t}-1)e^{kk_1t}+kk_1 \geq 0.$$

We see that $\alpha(t)>1$, $\gamma(t)$, and $\frac{\gamma}{\alpha-1}$ are nondecreasing functions. So, the functions α, β, and γ in this case satisfy the conditions (A1)-(A6).

(3) Li-Xu type gradient estimate
Let

$$\alpha(t)=1+\frac{\sinh(kk_1t)\cosh(kk_1t)-kk_1t}{\sinh^2(kk_1t)},$$
$$\beta(t)=\frac{kk_1m(p-1)}{m(p-1)+1}(1+\coth(kk_1t)),\qquad \gamma(t)=\tanh(kk_1t).$$

We have

$$\alpha'(t)=2kk_1-2(\alpha-1)kk_1\coth(kk_1t).$$

The conditions (A3), (A4), and (A5) become

$$\begin{aligned}&\frac{\gamma'}{\gamma}+\frac{\alpha'}{\alpha}-2\big[\frac{1}{m(p-1)}+1\big]\frac{\beta}{\alpha}\\&=\frac{1}{\alpha}\Big[\frac{kk_1\alpha}{\sinh(kk_1t)\cosh(kk_1t)}+2kk_1-2(\alpha-1)kk_1\coth(kk_1t)\\&-2kk_1(1+\coth(kk_1t)\Big]\\&=\frac{kk_1}{\sinh(kk_1t)\cosh(kk_1t)}\Big[1-2\cosh^2(kk_1t)\Big]\leq 0,\end{aligned}$$

$$\begin{aligned}&\alpha\beta'+\big[\frac{1}{m(p-1)}+1\big]\beta^2\\&=\frac{(kk_1)^2m(p-1)}{(m(p-1)+1)\sinh^2(kk_1t)}\Big[\Big(1+\coth(kk_1t)\Big)^2\sinh^2(kk_1t)-1\\&-\frac{\sinh(kk_1t)\cosh(kk_1t)-kk_1t}{\sinh^2(kk_1t)}\Big]\\&=\frac{(kk_1)^2m(p-1)}{(m(p-1)+1)\sinh^2(kk_1t)}\frac{e^{2kk_1t}}{(e^{2kk_1t}-1)^2}\Big[e^{4kk_1t}-4e^{2kk_1t}+3+4kk_1t\Big]\geq 0,\end{aligned}$$

and

$$2\big[\frac{1}{m(p-1)}+1\big]\frac{\alpha-1}{\alpha}\beta+\frac{\alpha'}{\alpha}=2kk_1\geq 0.$$

On the other hand, $\lim\limits_{t\to 0}\frac{\gamma}{\alpha-1}=\frac{3}{2}$ and $\lim\limits_{t\to+\infty}\frac{\gamma}{\alpha-1}=1$, then $\frac{\gamma}{\alpha-1}\leq c$ for some constant c. Thus, the functions α, β, and γ in this case satisfy the conditions (A1)-(A6).

(4) Linear Li-Xu type gradient estimate
Let

$$\alpha(t)=1+kk_1t,\qquad \beta(t)=\frac{m(p-1)}{m(p-1)+1}\Big(\frac{1}{t}+kk_1\Big)$$
$$\gamma(t)=kk_1t.$$

Then we get

$$\frac{\gamma'}{\gamma}+\frac{\alpha'}{\alpha}-2\big[\frac{1}{m(p-1)}+1\big]\frac{\beta}{\alpha}=\frac{-1}{t(1+kk_1t)}\leq 0,$$
$$\alpha\beta'+\big[\frac{1}{m(p-1)}+1\big]\beta^2=\frac{m(p-1)}{m(p-1)+1}\frac{kk_1}{t}(1+kk_1t)\geq 0,$$

and

$$2\Big[\frac{1}{m(p-1)}+1\Big]\frac{\alpha-1}{\alpha}\beta+\frac{\alpha'}{\alpha}=2kk_1+\frac{kk_1}{1+kk_1t}\geq 0.$$

Therefore, the functions α, β, and γ in this case satisfy the conditions (A1)-(A5) and $\frac{\gamma}{\alpha-1}=1$.

□

Proof of Corollary 1.2. Since $g(t)$ is uniformly equivalent to the initial metric $g(0)$, the $(M,g(t))$ is complete noncompact for $t\in[0,T]$. If $R\to+\infty$ in (1.6) and (1.7) then we obtain inequality (1.8) and (1.9). □

Proof of Corollary 1.3. Let $\zeta(t)$ be a shortest the geodesic joining y_1 and y_2 with $\zeta(s_1)=y_1$ and $\zeta(s_2)=y_2$. Now consider the path $(\zeta(t),t)$ in space-time. If $\frac{\gamma}{\alpha-1}\leq c$ and $\tilde{k}=\inf\limits_{M\times[0,T]} v$ then from Corollary 1.2, we have the following gradient estimate

(2.35)

$$-\partial_t(\ln v)\leq -\frac{1}{\alpha}v|\nabla(\ln v)|^2+\frac{m(p-1)}{1+m(p-1)}\left(\alpha^2K_4+\alpha\sqrt{K_5\frac{1+m(p-1)}{m(p-1)}}\right)+\alpha\beta,$$

Integrating this inequality along ζ, we get

$$\begin{aligned}
&\log\frac{v(y_1,s_1)}{v(y_2,s_2)}\\
&=-\int_{s_1}^{s_2}\frac{d}{dt}\big(\ln v(\zeta(t),t)\big)dt\\
&=-\int_{s_1}^{s_2}\Big(\partial_t(\ln v)+\langle\nabla(\ln v)(\zeta(t),t),\dot{\zeta}(t)\rangle\Big)dt\\
&\leq\int_{s_1}^{s_2}\left\{-\frac{\tilde{k}}{\alpha}|\nabla(\ln v)|^2+\frac{m(p-1)}{1+m(p-1)}\left(\alpha^2K_4+\alpha\sqrt{K_5\frac{1+m(p-1)}{m(p-1)}}\right)+\alpha\beta\right.\\
&\left.-\langle\nabla(\ln v),\dot{\zeta}(t)\rangle\right\}dt\\
&\leq\int_{s_1}^{s_2}\left\{\frac{\alpha|\dot{\zeta}(t)|^2}{4\tilde{k}}+\left(\frac{m(p-1)}{1+m(p-1)}\left(\alpha^2K_4+\alpha\sqrt{K_5\frac{1+m(p-1)}{m(p-1)}}\right)+\alpha\beta\right)\right\}dt\\
&\leq\frac{1}{4\tilde{k}}\int_{s_1}^{s_2}\alpha|\dot{\zeta}(t)|^2dt+\int_{s_1}^{s_2}\left\{\frac{m(p-1)}{1+m(p-1)}\left(\alpha^2K_4+\alpha\sqrt{K_5\frac{1+m(p-1)}{m(p-1)}}\right)+\alpha\beta\right\}dt
\end{aligned}$$

where in the computation above we have used (2.35) to obtain the inequality in the third line and used inequality $-\tilde{a}x^2-\tilde{b}x\leq\frac{\tilde{b}^2}{4\tilde{a}}$ to arrive at the inequality in the fourth line. Since α is anondecreasing function, by exponentiation we have

$$\begin{aligned}
v(y_1,s_1)\quad\leq\quad & v(y_2,s_2)\exp\left\{\frac{\alpha(s_2)}{4\tilde{k}}\mathcal{J}(y_1,s_1,y_2,s_2)\right.\\
&\left.+\int_{s_1}^{s_2}\left\{\frac{m(p-1)}{1+m(p-1)}\left(\alpha^2K_4+\alpha\sqrt{K_5\frac{1+m(p-1)}{m(p-1)}}\right)+\alpha\beta\right\}dt\right\}.
\end{aligned}$$

Silmilarly, if $\frac{\gamma}{\alpha-1}$ be a nodecreasing function then we obtain

$$\begin{aligned} v(y_1,s_1) &\leq v(y_2,s_2)\exp\left\{\frac{\alpha(s_2)}{4\tilde{k}}\mathcal{J}(y_1,s_1,y_2,s_2)\right. \\ &\left.+\int_{s_1}^{s_2}\left\{\frac{m(p-1)}{1+m(p-1)}\left(\alpha^2K_6+\alpha\sqrt{K_5\frac{1+m(p-1)}{m(p-1)}}\right)+\alpha\beta\right\}dt\right\}. \end{aligned}$$

In Li-Yau type gradient estimate α is a constant, then

$$\frac{m(p-1)}{1+m(p-1)}\left(\alpha^2K_6+\alpha\sqrt{K_5\frac{1+m(p-1)}{m(p-1)}}\right)$$

is a constant. Therefore,

$$\begin{aligned} &\int_{s_1}^{s_2}\left\{\frac{m(p-1)}{1+m(p-1)}\left(\alpha^2K_6+\alpha\sqrt{K_5\frac{1+m(p-1)}{m(p-1)}}\right)+\alpha\beta\right\}dt \\ &=\frac{m(p-1)}{1+m(p-1)}\left(\alpha^2K_6+\alpha\sqrt{K_5\frac{1+m(p-1)}{m(p-1)}}\right)(s_2-s_1)+\frac{kk_1\alpha}{\alpha-1}(s_2-s_1) \\ &+\frac{m(p-1)\alpha^2}{1+m(p-1)}\ln(\frac{s_2}{s_1}). \end{aligned}$$

Hence,

$$\begin{aligned} v(y_1,s_1) &\leq v(y_2,s_2)\Big(\frac{s_2}{s_1}\Big)^{\frac{m(p-1)\alpha^2}{1+m(p-1)}}\exp\left\{\frac{\alpha}{4\tilde{k}}\mathcal{J}(y_1,s_1,y_2,s_2)\right. \\ &\left.+\left\{\frac{m(p-1)}{1+m(p-1)}\left(\alpha^2K_6+\alpha\sqrt{K_5\frac{1+m(p-1)}{m(p-1)}}\right)+\frac{\alpha kk_1}{\alpha-1}\right\}(s_2-s_1)\right\}, \end{aligned}$$

This completes the proof of Corollary. □

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Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran., Tel.: +98-28-33901321, Fax: +98-28-33780083,
Email address: `azami@sci.ikiu.ac.ir`