

Several novel inequalities associated with the Riesz-type fractional integral operator

Lu-Lu Geng^{1,2}, Xiao-Jun Yang^{1,2,3,4†}

1. School of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu, 221116, P. R. China.
2. State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou, Jiangsu, 221116, P. R. China.
3. Department of Mathematics Faculty of Science King Abdulaziz University P.O. Box 80257, Jeddah 21589, Saudi Arabia.
4. Department of Mathematics, College of Science, Kyung Hee University, Seoul, Republic of Korea.

Abstract:

In this paper, we investigate some new inequalities based on the Riesz-type fractional integral operator for synchronous and bounded functions.

Key words: Riesz-type fractional integral; Chebychev functional; synchronous function; bounded function; Young's inequality.

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1. Introduction

Throughout this paper, let \mathbb{N} , \mathbb{R} , and \mathbb{C} be the sets of the natural, real, and complex numbers, respectively.

For $0 < \alpha < 1$, the Riesz-type fractional integral on a bounded domain is defined as (see [1], p.130)

$${}_{Rz}I_{[a,b]}^{\alpha}f(t) = \frac{1}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})} \int_a^b \frac{f(\tau)}{|\tau - t|^{1-\alpha}} d\tau, \quad (1)$$

where

$${}_L I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(\tau - t)^{1-\alpha}} d\tau, \quad (2)$$

and

$${}_L I_{b-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(\tau)}{(t - \tau)^{1-\alpha}} d\tau. \quad (3)$$

[†]Corresponding author E-mail: gengllcumt@163.com

The Chebychev's functional is expressed as [2-7]

$$\begin{aligned}
T(f, g, j, k) = & \int_a^b k(x)dx \int_a^b j(x)f(x)g(x)dx \\
& + \int_a^b j(x)dx \int_a^b k(x)f(x)g(x)dx \\
& - [\int_a^b k(x)f(x)dx][\int_a^b j(x)g(x)dx] \\
& - [\int_a^b j(x)f(x)dx][\int_a^b k(x)g(x)dx],
\end{aligned} \tag{4}$$

where $f, g : [a, b] \rightarrow \mathbb{R}$ are two integrable functions on $[a, b]$ and j, k are positive integrable functions on $[a, b]$.

It is well-known that (4) has wide applicability in numerical quadrature [8], transform theory [9], probability and statistical problems [10], and the bounding of special functions [11].

For $x, y \in [a, b]$, if f and g are synchronous on $[a, b]$, then [12, 13]

$$[f(x) - f(y)][g(x) - g(y)] \geq 0, \tag{5}$$

and

$$T(f, g, j, k) \geq 0. \tag{6}$$

What's more, if f and g are asynchronous on $[a, b]$, then [14]

$$[f(x) - f(y)][g(x) - g(y)] \leq 0. \tag{7}$$

The aim of this paper is to study of some inequalities by means of the Riesz-type fractional integral operator.

The structure of this paper is as follows. In section 2, based on the Riesz-type fractional integral operator, we introduce some inequalities about synchronous functions. In section 3, by the Riesz-type fractional integral operator, we present some inequalities about bounded functions.

2. Inequalities involving Riesz-type fractional integral operator for synchronous functions

In this section, we study some inequalities via the Riesz-type fractional integral operator for synchronous functions.

Let f and g be two continuous and synchronous functions on $[a, b]$, u, v, l, m and n be continuous functions on $[a, b]$. Then we have some theorems as follows:

Theorem 1. If $\alpha \in \mathbb{R}^+$, $a, b \in \mathbb{R}_0^+$, then we have

$$\begin{aligned} & {}_{Rz}I_{[a,b]}^\alpha \{v\}(t) {}_{Rz}I_{[a,b]}^\alpha \{fgu\}(t) + {}_{Rz}I_{[a,b]}^\alpha \{fgv\}(t) {}_{Rz}I_{[a,b]}^\alpha \{u\}(t) \\ & \geq {}_{Rz}I_{[a,b]}^\alpha \{gv\}(t) {}_{Rz}I_{[a,b]}^\alpha \{fu\}(t) + {}_{Rz}I_{[a,b]}^\alpha \{fv\}(t) {}_{Rz}I_{[a,b]}^\alpha \{gu\}(t). \end{aligned} \quad (8)$$

Proof. Based on (5), we have

$$[f(\tau) - f(\rho)][g(\tau) - g(\rho)] \geq 0, \quad (9)$$

where $\tau, \rho \in [a, b]$.

Multiplying (9) by

$$\frac{1}{2\Gamma(\alpha) \cos(\frac{\pi\alpha}{2})} \frac{u(\tau)}{|\tau - t|^{1-\alpha}}, \quad (10)$$

we have

$$\begin{aligned} & \frac{1}{2\Gamma(\alpha) \cos(\frac{\pi\alpha}{2})} \frac{1}{|\tau - t|^{1-\alpha}} f(\tau)g(\tau)u(\tau) + f(\rho)g(\rho) \frac{1}{2\Gamma(\alpha) \cos(\frac{\pi\alpha}{2})} \frac{1}{|\tau - t|^{1-\alpha}} u(\tau) \\ & \geq g(\rho) \frac{1}{2\Gamma(\alpha) \cos(\frac{\pi\alpha}{2})} \frac{1}{|\tau - t|^{1-\alpha}} f(\tau)u(\tau) + f(\rho) \frac{1}{2\Gamma(\alpha) \cos(\frac{\pi\alpha}{2})} \frac{1}{|\tau - t|^{1-\alpha}} g(\tau)u(\tau). \end{aligned} \quad (11)$$

Integrating (11), we have

$$\begin{aligned} & {}_{Rz}I_{[a,b]}^\alpha \{fgu\}(t) + f(\rho)g(\rho) {}_{Rz}I_{[a,b]}^\alpha \{u\}(t) \\ & \geq g(\rho) {}_{Rz}I_{[a,b]}^\alpha \{fu\}(t) + f(\rho) {}_{Rz}I_{[a,b]}^\alpha \{gu\}(t). \end{aligned} \quad (12)$$

Then, multiplying (12) by

$$\frac{1}{2\Gamma(\alpha) \cos(\frac{\pi\alpha}{2})} \frac{v(\tau)}{|\tau - t|^{1-\alpha}}, \quad (13)$$

we can see

$$\begin{aligned} & \frac{1}{2\Gamma(\alpha) \cos(\frac{\pi\alpha}{2})} \frac{1}{|\rho - t|^{1-\alpha}} v(\rho) {}_{Rz}I_{[a,b]}^\alpha \{fgu\}(t) \\ & + \frac{1}{2\Gamma(\alpha) \cos(\frac{\pi\alpha}{2})} \frac{1}{|\rho - t|^{1-\alpha}} f(\rho)g(\rho)v(\rho) {}_{Rz}I_{[a,b]}^\alpha \{u\}(t) \\ & \geq \frac{1}{2\Gamma(\alpha) \cos(\frac{\pi\alpha}{2})} \frac{1}{|\rho - t|^{1-\alpha}} g(\rho)v(\rho) {}_{Rz}I_{[a,b]}^\alpha \{fu\}(t) \\ & + \frac{1}{2\Gamma(\alpha) \cos(\frac{\pi\alpha}{2})} \frac{1}{|\rho - t|^{1-\alpha}} f(\rho)v(\rho) {}_{Rz}I_{[a,b]}^\alpha \{gu\}(t). \end{aligned} \quad (14)$$

Integrating (14), we obtain

$$\begin{aligned} & {}_{Rz}I_{[a,b]}^\alpha \{v\}(t) {}_{Rz}I_{[a,b]}^\alpha \{fgu\}(t) + {}_{Rz}I_{[a,b]}^\alpha \{fgv\}(t) {}_{Rz}I_{[a,b]}^\alpha \{u\}(t) \\ & \geq {}_{Rz}I_{[a,b]}^\alpha \{gv\}(t) {}_{Rz}I_{[a,b]}^\alpha \{fu\}(t) + {}_{Rz}I_{[a,b]}^\alpha \{fv\}(t) {}_{Rz}I_{[a,b]}^\alpha \{gu\}(t). \end{aligned} \quad (15)$$

Therefore, (8) is true. \square

Theorem 2. Let $\alpha, \beta \in \mathbb{R}^+$, $a, b \in \mathbb{R}_0^+$, $\rho \in [a, t]$, and $t \in [a, b]$. Then

$$\begin{aligned} & {}_{Rz}I_{[a,b]}^\beta \{v\}(t) {}_{Rz}I_{[a,b]}^\alpha \{fgu\}(t) + {}_{Rz}I_{[a,b]}^\beta \{fgv\}(t) {}_{Rz}I_{[a,b]}^\alpha \{u\}(t) \\ & \geq {}_{Rz}I_{[a,b]}^\beta \{gv\}(t) {}_{Rz}I_{[a,b]}^\alpha \{fu\}(t) + {}_{Rz}I_{[a,b]}^\beta \{fv\}(t) {}_{Rz}I_{[a,b]}^\alpha \{gu\}(t). \end{aligned} \quad (16)$$

Proof. Multiplying (12) by

$$\frac{1}{2\Gamma(\beta) \cos(\frac{\pi\beta}{2})} \frac{v(\rho)}{|\rho - t|^{1-\beta}}, \quad (17)$$

we have

$$\begin{aligned} & \frac{1}{2\Gamma(\beta) \cos(\frac{\pi\beta}{2})} \frac{1}{|\rho - t|^{1-\beta}} v(\rho) {}_{Rz}I_{[a,b]}^\alpha \{fgu\}(t) \\ & + \frac{1}{2\Gamma(\beta) \cos(\frac{\pi\beta}{2})} \frac{1}{|\rho - t|^{1-\beta}} f(\rho)g(\rho)v(\rho) {}_{Rz}I_{[a,b]}^\alpha \{u\}(t) \\ & \geq \frac{1}{2\Gamma(\beta) \cos(\frac{\pi\beta}{2})} \frac{1}{|\rho - t|^{1-\beta}} g(\rho)v(\rho) {}_{Rz}I_{[a,b]}^\alpha \{fu\}(t) \\ & + \frac{1}{2\Gamma(\beta) \cos(\frac{\pi\beta}{2})} \frac{1}{|\rho - t|^{1-\beta}} f(\rho)v(\rho) {}_{Rz}I_{[a,b]}^\alpha \{gu\}(t). \end{aligned} \quad (18)$$

Integrating (18), we get

$$\begin{aligned} & {}_{Rz}I_{[a,b]}^\beta \{v\}(t) {}_{Rz}I_{[a,b]}^\alpha \{fgu\}(t) + {}_{Rz}I_{[a,b]}^\beta \{fgv\}(t) {}_{Rz}I_{[a,b]}^\alpha \{u\}(t) \\ & \geq {}_{Rz}I_{[a,b]}^\beta \{gv\}(t) {}_{Rz}I_{[a,b]}^\alpha \{fu\}(t) + {}_{Rz}I_{[a,b]}^\beta \{fv\}(t) {}_{Rz}I_{[a,b]}^\alpha \{gu\}(t). \end{aligned} \quad (19)$$

Therefore, the proof of (16) can be completed. \square

Theorem 3. Suppose $\alpha \in \mathbb{R}^+$, $a, b \in \mathbb{R}_0^+$, and $t \in [a, b]$. Then

$$\begin{aligned} & {}_{Rz}I_{[a,b]}^\alpha \{l\}(t) [{}_{Rz}I_{[a,b]}^\alpha \{fgn\}(t) {}_{Rz}I_{[a,b]}^\alpha \{m\}(t) + {}_{Rz}I_{[a,b]}^\alpha \{n\}(t) {}_{Rz}I_{[a,b]}^\alpha \{fgm\}(t)] \\ & + 2 {}_{Rz}I_{[a,b]}^\alpha \{m\}(t) {}_{Rz}I_{[a,b]}^\alpha \{n\}(t) {}_{Rz}I_{[a,b]}^\alpha \{fgl\}(t) \\ & \geq {}_{Rz}I_{[a,b]}^\alpha \{l\}(t) [{}_{Rz}I_{[a,b]}^\alpha \{gn\}(t) {}_{Rz}I_{[a,b]}^\alpha \{fm\}(t) + {}_{Rz}I_{[a,b]}^\alpha \{fn\}(t) {}_{Rz}I_{[a,b]}^\alpha \{gm\}(t)] \\ & + {}_{Rz}I_{[a,b]}^\alpha \{m\}(t) [{}_{Rz}I_{[a,b]}^\alpha \{gn\}(t) {}_{Rz}I_{[a,b]}^\alpha \{fl\}(t) + {}_{Rz}I_{[a,b]}^\alpha \{fn\}(t) {}_{Rz}I_{[a,b]}^\alpha \{gl\}(t)] \\ & + {}_{Rz}I_{[a,b]}^\alpha \{n\}(t) [{}_{Rz}I_{[a,b]}^\alpha \{gm\}(t) {}_{Rz}I_{[a,b]}^\alpha \{fl\}(t) + {}_{Rz}I_{[a,b]}^\alpha \{fm\}(t) {}_{Rz}I_{[a,b]}^\alpha \{gl\}(t)], \end{aligned} \quad (20)$$

is true.

Proof. For (8), consider $u = m$ and $v = n$, we have

$$\begin{aligned} & {}_{Rz}I_{[a,b]}^\alpha \{n\}(t) {}_{Rz}I_{[a,b]}^\alpha \{fgm\}(t) + {}_{Rz}I_{[a,b]}^\alpha \{fgn\}(t) {}_{Rz}I_{[a,b]}^\alpha \{m\}(t) \\ & \geq {}_{Rz}I_{[a,b]}^\alpha \{gn\}(t) {}_{Rz}I_{[a,b]}^\alpha \{fm\}(t) + {}_{Rz}I_{[a,b]}^\alpha \{fn\}(t) {}_{Rz}I_{[a,b]}^\alpha \{gm\}(t). \end{aligned} \quad (21)$$

According to ${}_{Rz}I_R^\alpha\{l\}(t) \geq 0$, and multiplying (21) by ${}_{Rz}I_R^\alpha\{l\}(t) \geq 0$, we have

$$\begin{aligned} & {}_{Rz}I_{[a,b]}^\alpha\{l\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{n\}(t){}_{Rz}I_{[a,b]}^\alpha\{fgm\}(t) + {}_{Rz}I_{[a,b]}^\alpha\{fgn\}(t){}_{Rz}I_{[a,b]}^\alpha\{m\}(t)] \\ & \geq {}_{Rz}I_{[a,b]}^\alpha\{l\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{gn\}(t){}_{Rz}I_{[a,b]}^\alpha\{fm\}(t) + {}_{Rz}I_{[a,b]}^\alpha\{fn\}(t){}_{Rz}I_{[a,b]}^\alpha\{gm\}(t)]. \end{aligned} \quad (22)$$

What's more, for (8), we try to use l, n instead of u, v and l, m instead of u, v , then multiplying by ${}_{Rz}I_{[a,b]}^\alpha\{m\}(t)$ and ${}_{Rz}I_{[a,b]}^\alpha\{n\}(t)$, we have

$$\begin{aligned} & {}_{Rz}I_{[a,b]}^\alpha\{m\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{n\}(t){}_{Rz}I_{[a,b]}^\alpha\{fgl\}(t) + {}_{Rz}I_{[a,b]}^\alpha\{fgn\}(t){}_{Rz}I_{[a,b]}^\alpha\{l\}(t)] \\ & \geq {}_{Rz}I_{[a,b]}^\alpha\{m\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{gn\}(t){}_{Rz}I_{[a,b]}^\alpha\{fl\}(t) + {}_{Rz}I_{[a,b]}^\alpha\{fn\}(t){}_{Rz}I_{[a,b]}^\alpha\{gl\}(t)], \end{aligned} \quad (23)$$

and

$$\begin{aligned} & {}_{Rz}I_{[a,b]}^\alpha\{n\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{m\}(t){}_{Rz}I_{[a,b]}^\alpha\{fgl\}(t) + {}_{Rz}I_{[a,b]}^\alpha\{fgm\}(t){}_{Rz}I_{[a,b]}^\alpha\{l\}(t)] \\ & \geq {}_{Rz}I_{[a,b]}^\alpha\{n\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{gm\}(t){}_{Rz}I_{[a,b]}^\alpha\{fl\}(t) + {}_{Rz}I_{[a,b]}^\alpha\{fm\}(t){}_{Rz}I_{[a,b]}^\alpha\{gl\}(t)]. \end{aligned} \quad (24)$$

Thus, we can obtain (20). \square

Theorem 4. If $\alpha, \beta \in \mathbb{R}^+$, $a, b \in \mathbb{R}_0^+$, and $t \in [a, b]$, then

$$\begin{aligned} & {}_{Rz}I_{[a,b]}^\alpha\{l\}(t)[2{}_{Rz}I_{[a,b]}^\beta\{fgn\}(t){}_{Rz}I_{[a,b]}^\alpha\{m\}(t) + {}_{Rz}I_{[a,b]}^\alpha\{n\}(t){}_{Rz}I_{[a,b]}^\beta\{fgm\}(t)] \\ & + {}_{Rz}I_{[a,b]}^\beta\{n\}(t){}_{Rz}I_{[a,b]}^\alpha\{fgm\}(t) + {}_{Rz}I_{[a,b]}^\alpha\{fgl\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{m\}(t){}_{Rz}I_{[a,b]}^\beta\{n\}(t) \\ & + {}_{Rz}I_{[a,b]}^\alpha\{n\}(t){}_{Rz}I_{[a,b]}^\beta\{m\}(t)] \\ & \geq {}_{Rz}I_{[a,b]}^\alpha\{l\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{mf\}(t){}_{Rz}I_{[a,b]}^\beta\{ng\}(t) + {}_{Rz}I_{[a,b]}^\alpha\{mg\}(t){}_{Rz}I_{[a,b]}^\beta\{nf\}(t)] \\ & + {}_{Rz}I_{[a,b]}^\alpha\{m\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{lf\}(t){}_{Rz}I_{[a,b]}^\beta\{ng\}(t) + {}_{Rz}I_{[a,b]}^\alpha\{lg\}(t){}_{Rz}I_{[a,b]}^\beta\{nf\}(t)] \\ & + {}_{Rz}I_{[a,b]}^\alpha\{n\}(t)[{}_{Rz}I_{[a,b]}^\alpha\{lf\}(t){}_{Rz}I_{[a,b]}^\beta\{mg\}(t) + {}_{Rz}I_{[a,b]}^\alpha\{lg\}(t){}_{Rz}I_{[a,b]}^\beta\{mf\}(t)]. \end{aligned} \quad (25)$$

Proof. For (16), setting $u = m$ and $v = n$, we have

$$\begin{aligned} & {}_{Rz}I_{[a,b]}^\beta\{n\}(t){}_{Rz}I_{[a,b]}^\alpha\{fgm\}(t) + {}_{Rz}I_{[a,b]}^\beta\{fgn\}(t){}_{Rz}I_{[a,b]}^\alpha\{m\}(t) \\ & \geq {}_{Rz}I_{[a,b]}^\beta\{gn\}(t){}_{Rz}I_{[a,b]}^\alpha\{fm\}(t) + {}_{Rz}I_{[a,b]}^\beta\{fn\}(t){}_{Rz}I_{[a,b]}^\alpha\{gm\}(t). \end{aligned} \quad (26)$$

Multiplying (26) by ${}_{Rz}I_R^\alpha\{l\}(t)$, we receive

$$\begin{aligned} & {}_{Rz}I_{[a,b]}^\alpha\{l\}(t)[{}_{Rz}I_{[a,b]}^\beta\{n\}(t){}_{Rz}I_{[a,b]}^\alpha\{fgm\}(t) + {}_{Rz}I_{[a,b]}^\beta\{fgn\}(t){}_{Rz}I_{[a,b]}^\alpha\{m\}(t)] \\ & \geq {}_{Rz}I_{[a,b]}^\alpha\{l\}(t)[{}_{Rz}I_{[a,b]}^\beta\{gn\}(t){}_{Rz}I_{[a,b]}^\alpha\{fm\}(t) + {}_{Rz}I_{[a,b]}^\beta\{fn\}(t){}_{Rz}I_{[a,b]}^\alpha\{gm\}(t)]. \end{aligned} \quad (27)$$

Replacing u, v by l, n and u, v by l, m in (16), we gain

$$\begin{aligned} & {}_{Rz}I_{[a,b]}^\beta\{n\}(t){}_{Rz}I_{[a,b]}^\alpha\{fgl\}(t) + {}_{Rz}I_{[a,b]}^\beta\{fgn\}(t){}_{Rz}I_{[a,b]}^\alpha\{l\}(t) \\ & \geq {}_{Rz}I_{[a,b]}^\beta\{gn\}(t){}_{Rz}I_{[a,b]}^\alpha\{fl\}(t) + {}_{Rz}I_{[a,b]}^\beta\{fn\}(t){}_{Rz}I_{[a,b]}^\alpha\{gl\}(t), \end{aligned} \quad (28)$$

and

$$\begin{aligned} & {}_{Rz}I_{[a,b]}^{\beta}\{m\}(t){}_{Rz}I_{[a,b]}^{\alpha}\{fgl\}(t) + {}_{Rz}I_{[a,b]}^{\beta}\{fgm\}(t){}_{Rz}I_{[a,b]}^{\alpha}\{l\}(t) \\ & \geq {}_{Rz}I_{[a,b]}^{\beta}\{gm\}(t){}_{Rz}I_{[a,b]}^{\alpha}\{fl\}(t) + {}_{Rz}I_{[a,b]}^{\beta}\{fm\}(t){}_{Rz}I_{[a,b]}^{\alpha}\{gl\}(t). \end{aligned} \quad (29)$$

Then multiplying (28) and (29) by ${}_{Rz}I_{[a,b]}^{\alpha}\{m\}(t)$ and ${}_{Rz}I_{[a,b]}^{\alpha}\{n\}(t)$, we can get (25). \square

3. Inequalities involving Riesz-type fractional integral operator for bounded functions

In this section, we investigate some inequalities based on the Riesz-type fractional integral operator for bounded functions.

Theorem 5. *Let $\alpha \in \mathbb{R}^+$, $a, b \in \mathbb{R}_0^+$, $t \in [a, b]$, f be an integrable function on $[a, b]$, u, v be continuous function on $[a, b]$, $\phi_1, \phi_2 \in [a, b]$, and $\phi_1(t) \leq f(t) \leq \phi_2(t)$. Then*

$$\begin{aligned} & {}_{Rz}I_{[a,b]}^{\alpha}\{u\phi_2\}(t){}_{Rz}I_{[a,b]}^{\alpha}\{vf\}(t) + {}_{Rz}I_{[a,b]}^{\alpha}\{uf\}(t){}_{Rz}I_{[a,b]}^{\alpha}\{v\phi_1\}(t) \\ & \geq {}_{Rz}I_{[a,b]}^{\alpha}\{u\phi_2\}(t){}_{Rz}I_{[a,b]}^{\alpha}\{v\phi_1\}(t) + {}_{Rz}I_{[a,b]}^{\alpha}\{uf\}(t){}_{Rz}I_{[a,b]}^{\alpha}\{vf\}(t). \end{aligned} \quad (30)$$

Proof. If $\tau \geq [a, b]$ and $\rho \geq [a, b]$, then we have

$$[\phi_2(\tau) - f(\tau)][f(\rho) - \phi_1(\rho)] \geq 0. \quad (31)$$

Multiplying (31) by

$$\frac{1}{2\Gamma(\alpha) \cos(\frac{\pi\alpha}{2})} \frac{u(\tau)}{|\tau - t|^{1-\alpha}}, \quad (32)$$

and integrating, we have

$$\begin{aligned} & {}_{Rz}I_{[a,b]}^{\beta}\{u\phi_2\}(t)f(\rho) + {}_{Rz}I_{[a,b]}^{\beta}\{uf\}(t)\phi_1(\rho) \\ & \geq {}_{Rz}I_{[a,b]}^{\beta}\{u\phi_2\}(t)\phi_1(\rho) + {}_{Rz}I_{[a,b]}^{\beta}\{uf\}(t)f(\rho). \end{aligned} \quad (33)$$

Multiplying (33) by

$$\frac{1}{2\Gamma(\alpha) \cos(\frac{\pi\alpha}{2})} \frac{v(\rho)}{|\rho - t|^{1-\alpha}}, \quad (34)$$

and integrating, we can see that (30) is true. \square

Theorem 6. *Let $\alpha \in \mathbb{R}^+$, $a, b \in \mathbb{R}_0^+$, $\tau, \rho \geq 0$, f be an integrable function on $[a, b]$, $u, v : [a, b] \rightarrow [a, b]$ be continuous function, $\lambda_1, \lambda_2 > 0$, and $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1$. Then*

$$\begin{aligned} & \frac{1}{\lambda_1} {}_{Rz}I_{[a,b]}^{\alpha}\{v\}(t){}_{Rz}I_{[a,b]}^{\alpha}\{u(\phi_2 - f)^{\lambda_1}\}(t) + \frac{1}{\lambda_2} {}_{Rz}I_{[a,b]}^{\alpha}\{u\}(t){}_{Rz}I_{[a,b]}^{\alpha}\{v(f - \phi_1)^{\lambda_2}\}(t) \\ & + {}_{Rz}I_{[a,b]}^{\alpha}\{u\phi_2\}(t){}_{Rz}I_{[a,b]}^{\alpha}\{v\phi_1\}(t) + {}_{Rz}I_{[a,b]}^{\alpha}\{uf\}(t){}_{Rz}I_{[a,b]}^{\alpha}\{vf\}(t) \\ & \geq {}_{Rz}I_{[a,b]}^{\alpha}\{u\phi_2\}(t){}_{Rz}I_{[a,b]}^{\alpha}\{vf\}(t) + {}_{Rz}I_{[a,b]}^{\alpha}\{uf\}(t){}_{Rz}I_{[a,b]}^{\alpha}\{v\phi_1\}(t). \end{aligned} \quad (35)$$

Proof. Based on the well-known Young's inequality [15]

$$\frac{1}{\lambda_1}x^{\lambda_1} + \frac{1}{\lambda_2}y^{\lambda_2} \geq xy, \quad (x, y \geq 0), \quad (36)$$

Setting $x = \phi_2(\tau) - f(\tau)$ and $y = f(\rho) - \phi_1(\rho)$, we have

$$\begin{aligned} & \frac{1}{\lambda_1}[\phi_2(\tau) - f(\tau)]^{\lambda_1} + \frac{1}{\lambda_2}[f(\rho) - \phi_1(\rho)]^{\lambda_2} \\ & \geq [\phi_2(\tau) - f(\tau)][f(\rho) - \phi_1(\rho)]. \end{aligned} \quad (37)$$

Multiplying (37) by

$$\frac{1}{2\Gamma^2(\alpha) \cos^2(\frac{\pi\alpha}{2})} \frac{u(\tau)v(\rho)}{|\tau - t|^{2(1-\alpha)}}, \quad (38)$$

and integrating, we can acquire (35). \square

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Conflict of interest

The authors declare that they have no conflict of interest.

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