Received -; | Revised -; | Accepted -

DOI: xxx/xxxx

ARTICLE TYPE

Post quantum Ostrowski–type inequalities for coordinated convex functions

Fongchan Wannalookkhee[1] | Kamsing Nonlaopon*[1] | Sortiris K. Ntouyas[2,3] | Hüseyin Budak[4]

[1]Department of Mathematics, Khon Kaen University, Khon Kaen, Thailand
[2]Department of Mathematics, University of Ioannina, Ioannina, Greece
[3]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
[4]Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

Correspondence
*Kamsing Nonlaopon, Department of Mathematics, Khon Kaen University, Khon Kaen, Thailand. Email: nkamsi@kku.ac.th

ABSTRACT

In this article, we give a new notion of (p,q)-derivatives for continuous functions on coordinates. We also derive post quantum Ostrowski–type inequalities for coordinated convex functions. Our significant results are considered as the generalizations of other results that appeared in the literature.
MSC: 05A30; 26D10; 26D15; 26A51; 26B25; 81P68

KEYWORDS:
Ostrowski inequality, convex function, coordinated convex function, (p,q)-derivative, (p,q)-integral, (p,q)-calculus

1 | INTRODUCTION

Quantum calculus (sometimes is called q-calculus) is known as the study of calculus with no limits. It has been firstly studied by the famous mathematician, Euler (1707-1783). In 1910, F. H. Jackson[1] determined the definite q-integral known as the q-Jackson integral. Quantum calculus has applications in several mathematical areas such as combinatorics, number theory, orthogonal polynomials, basic hypergeometric functions, mechanics, quantum theory, and theory of relativity, see for instance[2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22] and the references therein. The book by V. Kac and P. Cheung[23] covers the fundamental knowledge and also the basic theoretical concepts of quantum calculus.

In 2013, J. Tariboon and S. K. Ntouyas[24] defined the q-derivative and q-integral of a continuous function on finite intervals and proved some of its properties. Many well-known integral inequalities such as Hölder, Hermite–Hadamard, trapezoid, Ostrowski, Cauchy–Bunyakovsky–Schwarz, Grüss and Grüss–Čebyšev inequalities have been studied in the concept of q-calculus, see[25] for more details. Based on these results, there are many outcomes concerning q-calculus, see[26,27,28,29,30,31,32,33,34,35,36,37,38] and the references cited therein.

Post quantum calculus (sometimes is called (p,q)-calculus) is the further generalization of quantum calculus which was first considered by R. Chakrabati and R. Jagannathan[39]. In 2016, M. Tunç and E. Göv[40,41] introduced the (p,q)-derivative and (p,q)-integral on finite intervals, proved some of its properties and gave many integral inequalities by using (p,q)-calculus. Recently, according to works of M. Tunç and E. Göv, many researchers started working in this direction, some more results about (p,q)-calculus are in[42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58]. It is worth to note here that (p,q)-calculus cannot be derived directly by replacing q by q/p in q-calculus, but q-calculus can be retaken by setting $p = 1$ in (p,q)-calculus.

A. Ostrowski[59] established the following interesting integral inequality called Ostrowski inequality in 1983.

Theorem 1. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with bounded derivative, that is, $\|f'\|_\infty := \sup_{x\in(a,b)} |f'(x)| < \infty$. Then we have

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})}{(b-a)^2} \right] (b-a)\|f'\|_\infty, \tag{1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Inequality (1) can be rewritten in the equivalent form

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq \left[\frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right] \|f'\|_\infty.$$

Ostrowski inequality has been studied in many fields of mathematics, such as numerical analysis and probability. Many researchers considered generalizations and extensions of the Ostrowski inequality for absolutely continuous, bounded variation, convex, monotonic, Lipschitzian, and n times differentiable functions with error estimates for some numerical quadrature rules and some special means. Some more results relating to Ostrowski inequality can be found in[60,61,62,63,64,65,66,67,68,69,70,71].

In 2010, M. A. Latif, S. Hussian and S. S. Dragomir[71] presented the Ostrowski-type inequalities for convex functions on coordinates as follows:

Theorem 2. Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a twice differentiable function on $(a,b)\times(c,d)$ such that $\frac{\partial^2 f}{\partial s \partial t}$ is continuous and integrable on $[a,b]\times[c,d]$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|$ is coordinated convex on $[a,b]\times[c,d]$ and $\left|\frac{\partial^2 f}{\partial s \partial t}(x,y)\right| \leq M,\ (x,y) \in [a,b]\times[c,d]$, then we have

$$\begin{aligned} &\left| f(x,y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\, ds\, dt - A_1 \right| \\ &\qquad \leq M \left[\frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right] \left[\frac{(y-c)^2 + (d-y)^2}{2(d-c)} \right], \end{aligned}$$

where

$$A_1(x,y) = \frac{1}{d-c} \int_c^d f(x,s)\, ds + \frac{1}{b-a} \int_a^c f(t,y)\, dt. \tag{2}$$

Theorem 3. Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a twice differentiable function on $(a,b)\times(c,d)$ such that $\frac{\partial^2 f}{\partial s \partial t}$ is continuous and integrable on $[a,b]\times[c,d]$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^\alpha$ is coordinated convex on $[a,b]\times[c,d]$, where $\alpha > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, and $\left|\frac{\partial^2 f}{\partial s \partial t}(x,y)\right| \leq M,\ (x,y) \in [a,b]\times[c,d]$, then we have

$$\begin{aligned} &\left| f(x,y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\, ds\, dt - A_1 \right| \\ &\qquad \leq \frac{M}{(1+\beta)^{2/\beta}} \left[\frac{(x-a)^2 + (b-x)^2}{b-a} \right] \left[\frac{(y-c)^2 + (d-y)^2}{d-c} \right], \end{aligned}$$

where $A_1(x,y)$ is defined in (2).

Theorem 4. Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a twice differentiable function on $(a,b)\times(c,d)$ such that $\frac{\partial^2 f}{\partial s \partial t}$ is continuous and integrable on $[a,b]\times[c,d]$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^\alpha$ is coordinated convex on $[a,b]\times[c,d]$, where $\alpha \geq 1$, and $\left|\frac{\partial^2 f}{\partial s \partial t}(x,y)\right| \leq M,\ (x,y) \in [a,b]\times[c,d]$, then we have:

$$\begin{aligned} &\left| f(x,y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\, ds\, dt - A_1 \right| \\ &\qquad \leq \frac{M}{4} \left[\frac{(x-a)^2 + (b-x)^2}{b-a} \right] \left[\frac{(y-c)^2 + (d-y)^2}{d-c} \right], \end{aligned}$$

where $A_1(x,y)$ is defined in (2).

In 2020, H. Budak, M. A. Ali and T. Tunc[70] proved Ostrowski–type inequality for coordinated functions by using q-calculus. Motivated by the above mentioned literatures, we propose to define new ${}^{b}(p,q)$-derivatives for coordinates and then extend the Ostrowski type inequality in q-calculus for coordinated convex functions to (p,q)-calculus.

2 | PRELIMINARIES

Throughout this paper, we let $[a,b],[c,d] \subseteq \mathbb{R}$, $0 < q < p \leq 1$ and $0 < q_i < p_i \leq 1$ for $i = 1,2$. The definitions of (p,q)-calculus, coordinated functions, q-calculus and (p,q)-calculus for coordinates are given in[31,37,38,40,53,55,57,72]. Moreover, we use the following notation:

$$[n]_{p,q} := \frac{p^n - q^n}{p-q} = p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1}, \qquad \text{for} \quad n \in \mathbb{R}.$$

Definition 1. [40] Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then the ${}_{a}(p,q)$-derivative of f at $x \in (a,b]$ is defined by

$${}_{a}D_{p,q}f(x) = \frac{f(px + (1-p)a) - f(qx + (1-q)a)}{(p-q)(x-a)}.$$

The ${}_{a}(p,q)$-integral of f on $[a,x]$ is defined by

$$\int_a^x f(t)\,{}_{a}d_{p,q}t = (p-q)(x-a)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}x + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right).$$

Definition 2. [55] Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then the ${}^{b}(p,q)$-derivative of f at $x \in [a,b)$ is defined by

$${}^{b}D_{p,q}f(x) = \frac{f(qx + (1-q)b) - f(px + (1-p)b)}{(p-q)(b-x)}.$$

The ${}^{b}(p,q)$-integral of f on $[x,b]$ is defined by

$$\int_x^b f(t)\,{}^{b}d_{p,q}t = (p-q)(b-x)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}x + \left(1 - \frac{q^n}{p^{n+1}}\right)b\right).$$

Definition 3. [72] A function $f : [a,b] \times [c,d] \to \mathbb{R}$ is said to be convex on coordinates, if the partial mappings

$$f_x : [c,d] \ni v \mapsto f(x,v) \in \mathbb{R} \quad \text{and} \quad f_y : [a,b] \ni u \mapsto f(u,y) \in \mathbb{R}$$

are convex for all $x \in (a,b)$ and $y \in (c,d)$.

The definition of coordinated convex functions may be stated as follows:

Definition 4. A function $f : [a,b] \times [c,d] \to \mathbb{R}$ is said to be convex on coordinates, if

$$f(tx + (1-t)z, \lambda y + (1-\lambda)w) \leq t\lambda f(x,y) + t(1-\lambda)f(x,w) + (1-t)\lambda f(z,y) + (1-t)\lambda(1-])f(z,w)$$

holds for all $t, \lambda \in [0,1]$ and $(x,y),(z,w) \in [a,b] \times [c,d]$.

Definition 5. [37] Suppose that $f : [a,b] \times [c,d] \to \mathbb{R}$ is a continuous function of two variables. Then the $(q_1 q_2)$-derivatives are given by

$$\frac{{}_{a}\partial_{q_1} f(x,y)}{{}_{a}\partial_{q_1} x} = \frac{f(x,y) - f\left(q_1 x + (1-q_1)a, y\right)}{(1-q_1)(x-a)}, \quad x \neq a,$$
$$\frac{{}_{c}\partial_{q_2} f(x,y)}{{}_{c}\partial_{q_2} y} = \frac{f(x,y) - f\left(x, q_2 y + (1-q_2)c\right)}{(1-q_2)(y-c)}, \quad y \neq c,$$

and

$$\frac{{}_{a,c}\partial^2_{q_1,q_2} f(x,y)}{{}_{a}\partial_{q_1} x\,{}_{c}\partial_{q_2} y} = \frac{1}{(1-q_1)(1-q_2)(x-a)(y-c)}\Big[f\left(q_1 x + (1-q_1)a, q_2 y + (1-q_2)c\right) \\ - f\left(q_1 x + (1-q_1)a, y\right) - f\left(x, q_2 y + (1-q_2)c\right) + f(x,y)\Big],$$

for $x \neq a$ and $y \neq c$.

Definition 6. [38] Suppose that $f : [a,b]\times[c,d] \to \mathbb{R}$ is a continuous function of two variables. Then the (q_1q_2)-derivatives are given by

$$\frac{{}^{b}\partial_{q_1} f(x,y)}{{}^{b}\partial_{q_1} x} = \frac{f\left(q_1x+(1-q_1)b, y\right) - f(x,y)}{(1-q_1)(b-x)}, \quad x \neq b,$$
$$\frac{{}^{d}\partial_{q_2} f(x,y)}{{}^{d}\partial_{q_2} y} = \frac{f\left(x, q_2y+(1-q_2)d\right) - f(x,y)}{(1-q_2)(d-y)}, \quad y \neq d,$$

$$\begin{aligned}\frac{{}_{c}^{b}\partial^2{}_{q_1,q_2} f(x,y)}{{}^{b}\partial_{q_1} x\,{}_{c}\partial_{q_2} y} &= \frac{1}{(1-q_1)(1-q_2)(b-x)(y-c)}\left[f\left(q_1x+(1-q_1)b, q_2y+(1-q_2)c\right)\right.\\ &\quad\left.-f\left(q_1x+(1-q_1)b, y\right) - f\left(x, q_2y+(1-q_2)c\right) + f(x,y)\right], \quad x\neq b, y\neq c,\\ \frac{{}_{a}^{d}\partial^2{}_{q_1,q_2} f(x,y)}{{}_{a}\partial_{q_1} x\,{}^{d}\partial_{q_2} y} &= \frac{1}{(1-q_1)(1-q_2)(x-a)(d-y)}\left[f\left(q_1x+(1-q_1)a, q_2y+(1-q_2)d\right)\right.\\ &\quad\left.-f\left(q_1x+(1-q_1)a, y\right) - f\left(x, q_2y+(1-q_2)d\right) + f(x,y)\right], \quad x\neq a, y\neq d,\end{aligned}$$

and

$$\begin{aligned}\frac{{}^{b,d}\partial^2{}_{q_1,q_2} f(x,y)}{{}^{b}\partial_{q_1} x\,{}^{d}\partial_{q_2} y} &= \frac{1}{(1-q_1)(1-q_2)(b-x)(d-y)}\left[f\left(q_1x+(1-q_1)b, q_2y+(1-q_2)d\right)\right.\\ &\quad\left.-f\left(q_1x+(1-q_1)b, y\right) - f\left(x, q_2y+(1-q_2)d\right) + f(x,y)\right],\end{aligned}$$

for $x \neq b$ and $y \neq d$.

Definition 7. [53] Suppose that $f : [a,b]\times[c,d] \to \mathbb{R}$ is a continuous function of two variables. Then the (p_1p_2, q_1q_2)-derivatives are given by

$$\frac{{}_{a}\partial_{p_1,q_1} f(x,y)}{{}_{a}\partial_{p_1,q_1} x} = \frac{f\left(p_1x+(1-p_1)a, y\right) - f\left(q_1x+(1-q_1)a, y\right)}{(p_1-q_1)(x-a)}, \quad x \neq a,$$

$$\frac{{}_{c}\partial_{p_2,q_2} f(x,y)}{{}_{c}\partial_{p_2,q_2} y} = \frac{f\left(x, p_2y+(1-p_2)c\right) - f\left(x, q_2y+(1-q_2)c\right)}{(p_2-q_2)(y-c)}, \quad y \neq c,$$

and

$$\begin{aligned}\frac{{}_{a,c}\partial^2{}_{p_1,p_2,q_1,q_2} f(x,y)}{{}_{a}\partial_{p_1,q_1} x\,{}_{c}\partial_{p_2,q_2} y} &= \frac{1}{(p_1-q_1)(p_2-q_2)(x-a)(y-c)}\left[f\left(q_1x+(1-q_1)a, q_2y+(1-q_2)c\right)\right.\\ &\quad -f\left(q_1x+(1-q_1)a, p_2y+(1-p_2)c\right)\\ &\quad -f\left(p_1x+(1-p_1)a, q_2y+(1-q_2)c\right)\\ &\quad \left.+f\left(p_1x+(1-p_1)a, p_2y+(1-p_2)c\right)\right],\end{aligned}$$

for $x \neq a$ and $y \neq c$.

Next, we newly define another derivatives for continuous function of two variables.

Definition 8. Suppose that $f : [a,b]\times[c,d] \to \mathbb{R}$ is a continuous function of two variables. Then the (p_1p_2, q_1q_2)-derivatives are given by

$$\frac{{}^{b}\partial_{p_1,q_1} f(x,y)}{{}^{b}\partial_{p_1,q_1} x} = \frac{f\left(q_1x+(1-q_1)b, y\right) - f\left(p_1x+(1-p_1)b, y\right)}{(p_1-q_1)(b-x)}, \quad x \neq b,$$
$$\frac{{}^{d}\partial_{p_2,q_2} f(x,y)}{{}^{d}\partial_{p_2,q_2} y} = \frac{f\left(x, q_2y+(1-q_2)d\right) - f\left(x, p_2y+(1-p_2)d\right)}{(p_2-q_2)(d-y)}, \quad y \neq d,$$
$$\begin{aligned}\frac{{}_{c}^{b}\partial^2{}_{p_1,p_2,q_1,q_2} f(x,y)}{{}^{b}\partial_{p_1,q_1} x\,{}_{c}\partial_{p_2,q_2} y} &= \frac{1}{(p_1-q_1)(p_2-q_2)(b-x)(y-c)}\left[f\left(q_1x+(1-q_1)b, p_2y+(1-p_2)c\right)\right.\\ &\quad -f\left(p_1x+(1-p_1)b, p_2y+(1-p_2)c\right)\end{aligned}$$

$$
\begin{aligned}
&-f\left(q_1x+(1-q_1)b, q_2y+(1-q_2)c\right)\\
&+f\left(p_1x+(1-p_1)b, q_2y+(1-q_2)c\right)\big], \quad x \neq b,\ y \neq c,
\end{aligned}
$$

$$
\begin{aligned}
\frac{{}_{a}^{d}\partial^2{}_{p_1,p_2,q_1,q_2} f(x,y)}{{}_{a}\partial_{p_1,q_1} x\, {}^{d}\partial_{p_2,q_2} y} = {} & \frac{1}{(p_1-q_1)(p_2-q_2)(x-a)(d-y)}\big[f\left(p_1x+(1-p_1)a, q_2y+(1-q_2)d\right)\\
&-f\left(q_1x+(1-q_1)a, q_2y+(1-q_2)d\right)\\
&-f\left(p_1x+(1-p_1)a, p_2y+(1-p_2)d\right)\\
&+f\left(q_1x+(1-q_1)a, p_2y+(1-p_2)d\right)\big], \quad x \neq a,\ y \neq d,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{{}^{b,d}\partial^2{}_{p_1,p_2,q_1,q_2} f(x,y)}{{}^{b}\partial_{p_1,q_1} x\, {}^{d}\partial_{p_2,q_2} y} = {} & \frac{1}{(p_1-q_1)(p_2-q_2)(b-x)(d-y)}\big[f\left(q_1x+(1-q_1)b, q_2y+(1-q_2)d\right)\\
&-f\left(p_1x+(1-p_1)b, q_2y+(1-q_2)d\right)\\
&-f\left(q_1x+(1-q_1)b, p_2y+(1-p_2)d\right)\\
&+f\left(p_1x+(1-p_1)b, p_2y+(1-p_2)d\right)\big], \quad x \neq b,\ y \neq d.
\end{aligned}
$$

Remark 1. If $p_1 = p_2 = 1$, then Definition 8 reduces to Definition 6.

Example 1. Define a function $f : [0,1]\times[0,1] \to \mathbb{R}$ by $f(x,y) = x^2y^2$, which is a continuous function of two variables. Then by the definition of (p_1p_2, q_1q_2)-derivatives, we obtain

$$
\begin{aligned}
\frac{{}_{0}\partial_{p_1,q_1} f(x,y)}{{}_{0}\partial_{p_1,q_1} x} &= \frac{f(p_1x,y)-f(q_1x,y)}{(p_1-q_1)x} = \frac{p_1^2x^2y^2-q_1^2x^2y^2}{(p_1-q_1)x} = (p_1+q_1)xy^2,\\
\frac{{}_{0}\partial_{p_2,q_2} f(x,y)}{{}_{0}\partial_{p_2,q_2} y} &= \frac{f(x,p_2y)-f(x,q_2y)}{(p_2-q_2)y} = \frac{p_2^2x^2y^2-q_2^2x^2y^2}{(p_2-q_2)y} = (p_2+q_2)x^2y,
\end{aligned}
$$

$$
\begin{aligned}
\frac{{}^{1}\partial_{p_1,q_1} f(x,y)}{{}^{1}\partial_{p_1,q_1} x} &= \frac{f(q_1x+1-q_1,y)-f(p_1x+1-p_1,y)}{(p_1-q_1)(1-x)}\\
&= \frac{(q_1x+1-q_1)^2y^2-(p_1x+1-p_1)^2y^2}{(p_1-q_1)(1-x)}\\
&= ((p_1+q_1)x-(p_1+q_1)+2)y^2,\\
\frac{{}^{1}\partial_{p_2,q_2} f(x,y)}{{}^{1}\partial_{p_2,q_2} y} &= \frac{f(x,q_2y+1-q_2)-f(x,q_1y+1-q_1)}{(p_2-q_2)(1-y)}\\
&= \frac{x^2(q_2y+1-q_2)^2-x^2(q_1y+1-q_1)^2}{(p_2-q_2)(1-y)}\\
&= x^2((p_2+q_2)y-(p_2+q_2)+2),\\
\frac{{}_{a,c}\partial^2{}_{p_1,p_2,q_1,q_2} f(x,y)}{{}_{a}\partial_{p_1,q_1} x_c \partial_{p_2,q_2} y} &= \frac{1}{(p_1-q_1)(p_2-q_2)xy}\big[f(q_1x,q_2y)-f(q_1x,p_2y)\\
&\quad -f(p_1x,q_2y)+f(p_1x,p_2y)\big]\\
&= \frac{q_1^2q_2^2x^2y^2-q_1^2p_2^2x^2y^2-p_1^2q_2^2x^2y^2+p_1^2p_2^2x^2y^2}{(p_1-q_1)(p_2-q_2)xy}\\
&= (p_1+q_1)(p_2+q_2)xy,\\
\frac{{}_{c}^{b}\partial^2{}_{p_1,p_2,q_1,q_2} f(x,y)}{{}^{b}\partial_{p_1,q_1} x_c \partial_{p_2,q_2} y} &= \frac{1}{(p_1-q_1)(p_2-q_2)(1-x)y}\big[f(q_1x+1-q_1,p_2y)\\
&\quad -f(p_1x+1-p_1,p_2y)-f(q_1x+1-q_1,q_2y)\\
&\quad +f(p_1x+1-p_1,q_2y)\big]\\
&= \frac{1}{(p_1-q_1)(p_2-q_2)(1-x)y}\big[(q_1x+1-q_1)^2p_2^2y^2
\end{aligned}
$$

$$
\begin{aligned}
&-(p_1x+1-p_1)^2p_2^2y^2-(q_1x+1-q_1)^2q_2^2y^2\\
&+(p_1x+1-p_1)^2q_2^2y^2\big]\\
&=\big[(p_1+q_1)x-(p_1+q_1)+2\big](p_2+q_2)y,
\end{aligned}
$$

$$
\begin{aligned}
\frac{{}_a^d\partial^2_{p_1,p_2,q_1,q_2}f(x,y)}{{}_a\partial_{p_1,q_1}x\,{}^d\partial_{p_2,q_2}y} &= \frac{1}{(p_1-q_1)(p_2-q_2)x(1-y)}\big[f(p_1x,q_2y+1-q_2)\\
&\quad -f(q_1x,q_2y+1-q_2)-f(p_1x,p_2y+1-p_2)\\
&\quad +f(q_1x,p_2y+1-p_2)\big]\\
&= \frac{1}{(p_1-q_1)(p_2-q_2)x(1-y)}\big[p_1^2x^2(q_2y+1-q_2)^2\\
&\quad -q_1^2x^2(q_2y+1-q_2)^2-p_1^2x^2(p_2y+1-p_2)^2\\
&\quad +q_1^2x^2(p_2y+1-p_2)^2\big]\\
&=(p_1+q_1)x\big[(p_2+q_2)y-(p_2+q_2)+2\big],\quad\text{and}
\end{aligned}
$$

$$
\begin{aligned}
\frac{{}^{b,d}\partial^2_{p_1,p_2,q_1,q_2}f(x,y)}{{}^b\partial_{p_1,q_1}x\,{}^d\partial_{p_2,q_2}y} &= \frac{1}{(p_1-q_1)(p_2-q_2)(1-x)(1-y)}\big[f(q_1x+1-q_1,q_2y+1-q_2)\\
&\quad -f(p_1x+1-p_1,q_2y+1-q_2)-f(q_1x+1-q_1,p_2y+1-p_2)\\
&\quad +f(p_1x+1-p_1,p_2y+1-p_2)\big]\\
&= \frac{1}{(p_1-q_1)(p_2-q_2)(1-x)(1-y)}\big[(q_1x+1-q_1)^2(q_2y+1-q_2)^2\\
&\quad -(p_1x+1-p_1)^2(q_2y+1-q_2)^2-(q_1x+1-q_1)^2(p_2y+1-p_2)^2\\
&\quad +(p_1x+1-p_1)^2(p_2y+1-p_2)^2\big]\\
&=\big[(p_1+q_1)x-(p_1+q_1)+2\big]\big[(p_2+q_2)y-(p_2+q_2)+2\big].
\end{aligned}
$$

Definition 9. [53] Suppose that $f : [a,b]\times[c,d]\to\mathbb{R}$ is a continuous function of two variables. Then the definite (p_1p_2,q_1q_2)-integral is given by

$$
\begin{aligned}
\int_a^x\int_c^y f(t,s)\,{}_cd_{p_2,q_2}s\,{}_ad_{p_1,q_1}t &= (p_1-q_1)(p_2-q_2)(x-a)(y-c)\\
&\quad\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)a,\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)c\right),
\end{aligned}
$$

for $(x,y)\in[a,p_1b+(1-p_1)a]\times[c,p_2d+(1-p_2)c]$.

Definition 10. [57] Suppose that $f : [a,b]\times[c,d]\to\mathbb{R}$ is a continuous function of two variables. Then the definite (p_1p_2,q_1q_2)-integrals are given by

$$
\begin{aligned}
\int_a^x\int_y^d f(t,s)\,{}^dd_{p_2,q_2}s\,{}_ad_{p_1,q_1}t &= (p_1-q_1)(p_2-q_2)(x-a)(d-y)\\
&\quad\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)a,\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)d\right),
\end{aligned}
$$

for $(x,y)\in[a,p_1b+(1-p_1)a]\times[c,p_2d+(1-p_2)c]$,

$$
\begin{aligned}
\int_x^b\int_c^y f(t,s)\,{}_cd_{p_2,q_2}s\,{}^bd_{p_1,q_1}t &= (p_1-q_1)(p_2-q_2)(b-x)(y-c)\\
&\quad\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(\frac{q_1^n}{p_1^{n+1}}x+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)b,\frac{q_2^m}{p_2^{m+1}}y+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)c\right),
\end{aligned}
$$

for $(x, y) \in [a, p_1 b + (1 - p_1)a] \times [c, p_2 d + (1 - p_2)c]$ and

$$\int_x^b \int_y^d f(t,s)\ {}^d d_{p_2,q_2} s\ {}^b d_{p_1,q_1} t = (p_1 - q_1)(p_2 - q_2)(b - x)(d - y)$$
$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} f\left(\frac{q_1^n}{p_1^{n+1}} x + \left(1 - \frac{q_1^n}{p_1^{n+1}}\right) b, \frac{q_2^m}{p_2^{m+1}} y + \left(1 - \frac{q_2^m}{p_2^{m+1}}\right) d \right),$$

for $(x, y) \in [a, p_1 b + (1 - p_1)a] \times [c, p_2 d + (1 - p_2)c]$.

Theorem 5. [31] (Hölder's inequality for double sums). Let $(x_{nm})_{n,m\in\mathbb{N}}$ and $(y_{nm})_{n,m\in\mathbb{N}}$ be sequences of real (or complex) numbers and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ with $\alpha, \beta > 1$. Then the following inequality for double sums holds:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |x_{nm} y_{nm}| \leq \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |x_{nm}|^{\alpha} \right)^{1/\alpha} \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |y_{nm}|^{\beta} \right)^{1/\beta},$$

where all the sums are assumed to be finite.

3 | MAIN RESULTS

In this section, we first introduce (p_1p_2, q_1q_2)-Hölder's inequality and (p_1p_2, q_1q_2)-power mean inequality for functions of two variables.

Lemma 1. ((p_1p_2, q_1q_2)-Hölder's inequality for functions of two variables). Let f, g be (p_1p_2, q_1q_2)-integrable functions on $[a, b] \times [c, d]$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ with $\alpha, \beta > 1$. Then we have

$$\int_a^x \int_c^y |f(t,s) g(t,s)|\ {}_c d_{p_2,q_2} s\ {}_a d_{p_1,q_1} t$$
$$\leq \left(\int_a^x \int_c^y |f(t,s)|^{\alpha}\ {}_c d_{p_2,q_2} s\ {}_a d_{p_1,q_1} t \right)^{1/\alpha} \left(\int_a^x \int_c^y |g(t,s)|^{\beta}\ {}_c d_{p_2,q_2} s\ {}_a d_{p_1,q_1} t \right)^{1/\beta},$$

for all $(x, y) \in [a, b] \times [c, d]$.

Proof. By Theorem 5 and the definition of (p_1p_2, q_1q_2)-integral for functions of two variables, we directly derive that

$$\int_a^x \int_c^y |f(t,s) g(t,s)|\ {}_c d_{p_2,q_2} s\ {}_a d_{p_1,q_1} t$$
$$= (p_1 - q_1)(p_2 - q_2)(x - a)(y - c)$$
$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} \left| fg\left(\frac{q_1^n}{p_1^{n+1}} x + \left(1 - \frac{q_1^n}{p_1^{n+1}}\right) a, \frac{q_2^m}{p_2^{m+1}} y + \left(1 - \frac{q_2^m}{p_2^{m+1}}\right) c \right) \right|$$
$$\leq (p_1 - q_1)(p_2 - q_2)(x - a)(y - c)$$
$$\times \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} \left| f\left(\frac{q_1^n}{p_1^{n+1}} x + \left(1 - \frac{q_1^n}{p_1^{n+1}}\right) a, \frac{q_2^m}{p_2^{m+1}} y + \left(1 - \frac{q_2^m}{p_2^{m+1}}\right) c \right) \right|^{\alpha} \right)^{\frac{1}{\alpha}}$$
$$\times \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} \left| g\left(\frac{q_1^n}{p_1^{n+1}} x + \left(1 - \frac{q_1^n}{p_1^{n+1}}\right) a, \frac{q_2^m}{p_2^{m+1}} y + \left(1 - \frac{q_2^m}{p_2^{m+1}}\right) c \right) \right|^{\beta} \right)^{\frac{1}{\beta}}$$
$$\leq \left(\int_a^x \int_c^y |f(t,s)|^{\alpha}\ {}_c d_{p_2,q_2} s\ {}_a d_{p_1,q_1} t \right)^{1/\alpha} \left(\int_a^x \int_c^y |g(t,s)|^{\beta}\ {}_c d_{p_2,q_2} s\ {}_a d_{p_1,q_1} t \right)^{1/\beta}.$$

□

Another form of of (p_1p_2, q_1q_2)-Hölder's inequality for functions of two variables was given as follows:

Lemma 2. ((p_1p_2, q_1q_2)-power mean inequality for functions of two variables). Let f, g be (p_1p_2, q_1q_2)-integrable functions on $[a,b]\times[c,d]$ and $\alpha \geq 1$. Then we have the following inequality for functions of two variables:

$$\int_a^x\int_c^y |f(t,s)g(t,s)|\ {}_cd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t$$
$$\leq \left(\int_a^x\int_c^y |f(t,s)|\ {}_cd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t\right)^{1-1/\alpha}\left(\int_a^x\int_c^y |f(t,s)||g(t,s)|^{\alpha}\ {}_cd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t\right)^{1/\alpha},$$

for all $(x,y)\in[a,b]\times[c,d]$.

Next, we prove the (p,q)-Ostrowski type inequalities for coordinated convex functions. We may begin with Lemma 3, which are useful in further considerations.

Lemma 3. Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a twice partially (p_1p_2, q_1q_2)-differentiable function on $(a,b)\times(c,d)$. If $\frac{{}_{a,c}\partial^2_{p_1,p_2,q_1,q_2}f(t,s)}{{}_a\partial_{p_1,q_1}t\,{}_c\partial_{p_2,q_2}s}$, $\frac{{}^b_c\partial^2_{p_1,p_2,q_1,q_2}f(t,s)}{{}^b\partial_{p_1,q_1}t\,{}_c\partial_{p_2,q_2}s}$, $\frac{{}^d_a\partial^2_{p_1,p_2,q_1,q_2}f(t,s)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}$ and $\frac{{}^{b,d}\partial^2_{p_1,p_2,q_1,q_2}f(t,s)}{{}^b\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}$ are continuous and (p_1p_2, q_1q_2)-integrable on $[a,b]\times[c,d]$, then we have

$$\frac{1}{(b-a)(d-c)}\left[\int_a^x\int_c^y f(p_1t+(1-p_1)a, p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t\right.$$
$$+\int_x^b\int_c^y f(p_1t+(1-p_1)b, p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s\ {}^bd_{p_1,q_1}t$$
$$+\int_a^x\int_y^d f(p_1t+(1-p_1)a, p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t$$
$$\left.+\int_x^b\int_y^d f(p_1t+(1-p_1)b, p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\ {}^bd_{p_1,q_1}t\right]$$
$$-\frac{1}{d-c}\left[\int_c^y f(x, p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s+\int_y^d f(x, p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\right]$$
$$-\frac{1}{b-a}\left[\int_a^x f(p_1t+(1-p_1)a, y)\ {}_ad_{p_1,q_1}t+\int_x^b f(p_1t+(1-p_1)b, y)\ {}^bd_{p_1,q_1}t\right]+f(x,y)$$
$$=\frac{q_1q_2}{(b-a)(d-c)}\left[(x-a)^2(y-c)^2\int_0^1\int_0^1 ts\frac{{}_{a,c}\partial^2_{p_1,p_2,q_1,q_2}f(tx+(1-t)a, sy+(1-s)c)}{{}_a\partial_{p_1,q_1}t\,{}_c\partial_{p_2,q_2}s}\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right.$$
$$+(b-x)^2(y-c)^2\int_0^1\int_0^1 ts\frac{{}^b_c\partial^2_{p_1,p_2,q_1,q_2}f(tx+(1-t)b, sy+(1-s)c)}{{}^b\partial_{p_1,q_1}t\,{}_c\partial_{p_2,q_2}s}\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t$$
$$+(x-a)^2(d-y)^2\int_0^1\int_0^1 ts\frac{{}^d_a\partial^2_{p_1,p_2,q_1,q_2}f(tx+(1-t)a, sy+(1-s)d)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t$$

$$+(b-x)^2(d-y)^2\int_0^1\int_0^1 ts\frac{{}^{b,d}\partial^2{}_{p_1,p_2,q_1,q_2}f(tx+(1-t)b,sy+(1-s)d)}{{}^b\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\Bigg],\tag{3}$$

for all $(x,y)\in[a,b]\times[c,d]$.

Proof. Let $(x,y)\in[a,b]\times[c,d]$. From Definition 7, we have

$$\begin{aligned}&\frac{{}_{a,c}\partial^2{}_{p_1,p_2,q_1,q_2}f(tx+(1-t)a,sy+(1-s)c)}{{}_a\partial_{p_1,q_1}x\,{}_c\partial_{p_2,q_2}y}\\&=\frac{1}{(p_1-q_1)(p_2-q_2)(x-a)(y-c)ts}\left[f\left(p_1tx+(1-p_1t)a,p_2sy+(1-p_2s)c\right)\right.\\&\quad-f\left(q_1tx+(1-q_1t)a,p_2sy+(1-p_2s)c\right)-f\left(p_1tx+(1-p_1t)a,q_2sy+(1-q_2s)c\right)\\&\quad\left.+f\left(q_1tx+(1-q_1t)a,q_2sy+(1-q_2s)c\right)\right].\end{aligned}\tag{4}$$

Moreover, by Definition 8, we get

$$\begin{aligned}&\frac{{}_c^b\partial^2{}_{p_1,p_2,q_1,q_2}f(tx+(1-t)b,sy+(1-s)c)}{{}^b\partial_{p_1,q_1}x\,{}_c\partial_{p_2,q_2}y}\\&=\frac{1}{(p_1-q_1)(p_2-q_2)(b-x)(y-c)ts}\left[f\left(q_1tx+(1-q_1t)b,p_2sy+(1-p_2s)c\right)\right.\\&\quad-f\left(p_1tx+(1-p_1t)b,p_2sy+(1-p_2s)c\right)-f\left(q_1tx+(1-q_1t)b,q_2sy+(1-q_2s)c\right)\\&\quad\left.+f\left(p_1tx+(1-p_1t)b,q_2sy+(1-q_2s)c\right)\right],\end{aligned}\tag{5}$$

$$\begin{aligned}&\frac{{}_a^d\partial^2{}_{p_1,p_2,q_1,q_2}f(tx+(1-t)a,sy+(1-s)d)}{{}_a\partial_{p_1,q_1}x\,{}^d\partial_{p_2,q_2}y}\\&=\frac{1}{(p_1-q_1)(p_2-q_2)(x-a)(d-y)ts}\left[f\left(p_1tx+(1-p_1t)a,q_2sy+(1-q_2s)d\right)\right.\\&\quad-f\left(q_1tx+(1-q_1t)a,q_2sy+(1-q_2s)d\right)-f\left(p_1tx+(1-p_1t)a,p_2sy+(1-p_2s)d\right)\\&\quad\left.+f\left(q_1tx+(1-q_1t)a,p_2sy+(1-p_2s)d\right)\right],\end{aligned}\tag{6}$$

$$\begin{aligned}&\frac{{}^{b,d}\partial^2{}_{p_1,p_2,q_1,q_2}f(tx+(1-t)b,sy+(1-s)d)}{{}^b\partial_{p_1,q_1}x\,{}^d\partial_{p_2,q_2}y}\\&=\frac{1}{(p_1-q_1)(p_2-q_2)(b-x)(d-y)ts}\left[f\left(q_1tx+(1-q_1t)b,q_2sy+(1-q_2s)d\right)\right.\\&\quad-f\left(p_1tx+(1-p_1t)b,q_2sy+(1-q_2s)d\right)-f\left(q_1tx+(1-q_1t)b,p_2sy+(1-p_2s)d\right)\\&\quad\left.+f\left(p_1tx+(1-p_1t)b,p_2sy+(1-p_2s)d\right)\right].\end{aligned}\tag{7}$$

By (4), we have

$$\begin{aligned}I_1:&=\int_0^1\int_0^1 ts\frac{{}_{a,c}\partial^2{}_{p_1,p_2,q_1,q_2}f(tx+(1-t)a,sy+(1-s)c)}{{}_a\partial_{p_1,q_1}t\,{}_c\partial_{p_2,q_2}s}\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\&=\frac{1}{(p_1-q_1)(p_2-q_2)(x-a)(y-c)}\int_0^1\int_0^1 f\left(p_1tx+(1-p_1t)a,p_2sy+(1-p_2s)c\right)\\&\quad-f\left(q_1tx+(1-q_1t)a,p_2sy+(1-p_2s)c\right)-\left(p_1tx+(1-p_1t)a,q_2sy+(1-q_2s)c\right)\\&\quad+f\left(q_1tx+(1-q_1t)a,q_2sy+(1-q_2s)c\right)\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\&=\frac{1}{(x-a)(y-c)}\left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(\frac{q_1^n}{p_1^n}x+\left(1-\frac{q_1^n}{p_1^n}\right)a,\frac{q_2^m}{p_2^m}y+\left(1-\frac{q_2^m}{p_2^m}\right)c\right)\right.\\&\quad-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(\frac{q_1^{n+1}}{p_1^{n+1}}x+\left(1-\frac{q_1^{n+1}}{p_1^{n+1}}\right)a,\frac{q_2^m}{p_2^m}y+\left(1-\frac{q_2^m}{p_2^m}\right)c\right)\end{aligned}$$

$$-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(\frac{q_1^n}{p_1^n}x+\left(1-\frac{q_1^n}{p_1^n}\right)a,\frac{q_2^{m+1}}{p_2^{m+1}}y+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)c\right)$$
$$\left.+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}f\left(\frac{q_1^{n+1}}{p_1^{n+1}}x+\left(1-\frac{q_1^{n+1}}{p_1^{n+1}}\right)a,\frac{q_2^{m+1}}{p_2^{m+1}}y+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)c\right)\right]$$
$$=\frac{1}{q_1q_2}\left[\frac{1}{(x-a)^2(y-c)^2}\int_a^x\int_c^y f(p_1t+(1-p_1)a,p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t\right.$$
$$-\frac{1}{(x-a)(y-c)^2}\int_c^y f(x,p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s$$
$$\left.-\frac{1}{(x-a)^2(y-c)}\int_a^x f(p_1t+(1-p_1)a,y)\ {}_ad_{p_1,q_1}t+\frac{1}{(x-a)(y-c)}f(x,y)\right].$$

Similarly, by (5), (6) and (7), we get

$$I_2:=\int_0^1\int_0^1 ts\frac{{}^b_c\partial^2_{p_1,p_2,q_1,q_2}f(tx+(1-t)b,sy+(1-s)c)}{{}^b\partial_{p_1,q_1}t\,{}_c\partial_{p_2,q_2}s}\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t$$
$$=\frac{1}{q_1q_2}\left[\frac{1}{(b-x)^2(y-c)^2}\int_x^b\int_c^y f(p_1t+(1-p_1)b,p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s\ {}^bd_{p_1,q_1}t\right.$$
$$-\frac{1}{(b-x)(y-c)^2}\int_c^y f(x,p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s$$
$$\left.-\frac{1}{(b-x)^2(y-c)}\int_x^b f(p_1t+(1-p_1)b,y)\ {}^bd_{p_1,q_1}t+\frac{1}{(b-x)(y-c)}f(x,y)\right],$$

$$I_3:=\int_0^1\int_0^1 ts\frac{{}^d_a\partial^2_{p_1,p_2,q_1,q_2}f(tx+(1-t)a,sy+(1-s)d)}{{}_a\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t$$
$$=\frac{1}{q_1q_2}\left[\frac{1}{(x-a)^2(d-y)^2}\int_a^x\int_y^d f(p_1t+(1-p_1)a,p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t\right.$$
$$-\frac{1}{(x-a)(d-y)^2}\int_y^d f(x,p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s$$
$$\left.-\frac{1}{(x-a)^2(d-y)}\int_a^x f(p_1t+(1-p_1)a,y)\ {}_ad_{p_1,q_1}t+\frac{1}{(x-a)(d-y)}f(x,y)\right]$$

and

$$I_4:=\int_0^1\int_0^1 ts\frac{{}^{b,d}\partial^2_{p_1,p_2,q_1,q_2}f(tx+(1-t)b,sy+(1-s)d)}{{}^b\partial_{p_1,q_1}t\,{}^d\partial_{p_2,q_2}s}\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t$$
$$=\frac{1}{q_1q_2}\left[\frac{1}{(b-x)^2(d-y)^2}\int_x^b\int_y^d f(p_1t+(1-p_1)b,p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\ {}^bd_{p_1,q_1}t\right.$$

$$-\frac{1}{(b-x)(d-y)^2}\int_y^d f(x,p_2s+(1-p_2)d)\ {}^d d_{p_2,q_2}s$$
$$\left.-\frac{1}{(b-x)^2(d-y)}\int_x^b f(p_1t+(1-p_1)b,y)\ {}^b d_{p_1,q_1}t+\frac{1}{(b-x)(d-y)}f(x,y)\right].$$

Finally, we obtain

$$\frac{q_1q_2}{(b-a)(d-c)}\left[(x-a)^2(y-c)^2I_1+(b-x)^2(y-c)^2I_2\right.$$
$$\left.+(x-a)^2(d-y)^2I_3+(b-x)^2(d-y)^2I_4\right]$$
$$=\frac{1}{(b-a)(d-c)}\left[\int_a^x\int_c^y f(p_1t+(1-p_1)a,p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t\right.$$
$$+\int_x^b\int_c^y f(p_1t+(1-p_1)b,p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s\ {}^bd_{p_1,q_1}t$$
$$+\int_a^x\int_y^d f(p_1t+(1-p_1)a,p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t$$
$$\left.+\int_x^b\int_y^d f(p_1t+(1-p_1)b,p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\ {}^bd_{p_1,q_1}t\right]$$
$$-\frac{1}{d-c}\left[\int_c^y f(x,p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s+\int_y^d f(x,p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\right]$$
$$-\frac{1}{b-a}\left[\int_a^x f(p_1t+(1-p_1)a,y)\ {}_ad_{p_1,q_1}t+\int_x^b f(p_1t+(1-p_1)b,y)\ {}^bd_{p_1,q_1}t\right]+f(x,y),$$

which completes the proof. □

Remark 2. If $p_1=p_2=1$, then (3) reduces to

$$\frac{1}{(b-a)(d-c)}\left[\int_a^x\int_c^y f(t,s)\ {}_cd_{q_2}s\ {}_ad_{q_1}t+\int_x^b\int_c^y f(t,s)\ {}_cd_{q_2}s\ {}^bd_{q_1}t\right.$$
$$\left.+\int_a^x\int_y^d f(t,s)\ {}^dd_{q_2}s\ {}_ad_{q_1}t+\int_x^b\int_y^d f(t,s)\ {}^dd_{q_2}s\ {}^bd_{q_1}t\right]$$
$$-\frac{1}{d-c}\left[\int_c^y f(x,s)\ {}_cd_{q_2}s+\int_y^d f(x,s)\ {}^dd_{q_2}s\right]$$
$$-\frac{1}{b-a}\left[\int_a^x f(t,y)\ {}_ad_{q_1}t+\int_x^b f(t,y)\ {}^bd_{q_1}t\right]+f(x,y)$$
$$=\frac{q_1q_2}{(b-a)(d-c)}\left[(x-a)^2(y-c)^2\int_0^1\int_0^1 ts\frac{{}_{a,c}\partial^2{}_{q_1,q_2}f(tx+(1-t)a,sy+(1-s)c)}{{}_a\partial_{q_1}t\,{}_c\partial_{q_2}s}\ {}_0d_{q_2}s\ {}_0d_{q_1}t\right.$$

$$+ (b-x)^2(y-c)^2 \int_0^1\int_0^1 ts \frac{{}^{b}_{c}\partial^2_{q_1,q_2} f(tx+(1-t)b, sy+(1-s)c)}{{}^{b}\partial_{q_1} t\, {}_{c}\partial_{q_2} s}\, {}_0d_{q_2} s\, {}_0d_{q_1} t$$
$$+(x-a)^2(d-y)^2 \int_0^1\int_0^1 ts \frac{{}^{d}_{a}\partial^2_{q_1,q_2} f(tx+(1-t)a, sy+(1-s)d)}{{}_{a}\partial_{q_1} t\, {}^{d}\partial_{q_2} s}\, {}_0d_{q_2} s\, {}_0d_{q_1} t$$
$$+(b-x)^2(d-y)^2 \int_0^1\int_0^1 ts \frac{{}^{b,d}\partial^2_{q_1,q_2} f(tx+(1-t)b, sy+(1-s)d)}{{}^{b}\partial_{q_1} t\, {}^{d}\partial_{q_2} s}\, {}_0d_{q_2} s\, {}_0d_{q_1} t\Bigg],$$

which appears in [70].

For convenience, we will use the following notations:

$$\Phi(t,s) := \frac{{}_{a,c}\partial^2_{p_1,p_2,q_1,q_2} f(t,s)}{{}_{a}\partial_{p_1,q_1} t\, {}_{c}\partial_{p_2,q_2} s}, \quad \Psi(t,s) := \frac{{}^{b}_{c}\partial^2_{p_1,p_2,q_1,q_2} f(t,s)}{{}^{b}\partial_{p_1,q_1} t\, {}_{c}\partial_{p_2,q_2} s},$$
$$\Theta(t,s) := \frac{{}^{d}_{a}\partial^2_{p_1,p_2,q_1,q_2} f(t,s)}{{}_{a}\partial_{p_1,q_1} t\, {}^{d}\partial_{p_2,q_2} s} \quad \text{and} \quad \Omega(t,s) := \frac{{}^{b,d}\partial^2_{p_1,p_2,q_1,q_2} f(t,s)}{{}^{b}\partial_{p_1,q_1} t\, {}^{d}\partial_{p_2,q_2} s}.$$

Theorem 6. Under the assumptions of Lemma 3, in addition, if $|\Phi(t,s)|$, $|\Psi(t,s)|$, $|\Phi(t,s)|$ and $|\Omega(t,s)|$ are coordinated convex on $[a,b]\times[c,d]$, then we have the inequality

$$\Bigg|\frac{1}{(b-a)(d-c)}\Bigg[\int_a^x\int_c^y f(p_1t+(1-p_1)a, p_2s+(1-p_2)c)\, {}_cd_{p_2,q_2} s\, {}_ad_{p_1,q_1} t$$
$$+\int_x^b\int_c^y f(p_1t+(1-p_1)b, p_2s+(1-p_2)c)\, {}_cd_{p_2,q_2} s\, {}^bd_{p_1,q_1} t$$
$$+\int_a^x\int_y^d f(p_1t+(1-p_1)a, p_2s+(1-p_2)d)\, {}^dd_{p_2,q_2} s\, {}_ad_{p_1,q_1} t$$
$$+\int_x^b\int_y^d f(p_1t+(1-p_1)b, p_2s+(1-p_2)d)\, {}^dd_{p_2,q_2} s\, {}^bd_{p_1,q_1} t\Bigg]$$
$$-\frac{1}{d-c}\Bigg[\int_c^y f(x, p_2s+(1-p_2)c)\, {}_cd_{p_2,q_2} s + \int_y^d f(x, p_2s+(1-p_2)d)\, {}^dd_{p_2,q_2} s\Bigg]$$
$$-\frac{1}{b-a}\Bigg[\int_a^x f(p_1t+(1-p_1)a, y)\, {}_ad_{p_1,q_1} t + \int_x^b f(p_1t+(1-p_1)b, y)\, {}^bd_{p_1,q_1} t\Bigg] + f(x,y)\Bigg|$$
$$\le \frac{q_1q_2}{(b-a)(d-c)(p_1+q_1)(p_2+q_2)(p_1^2+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)}$$
$$\times\Big\{(x-a)^2(y-c)^2\Big[(p_1+q_1)(p_2+q_2)|\Phi(x,y)|$$
$$+(p_1+q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Phi(x,c)|$$
$$+(p_2+q_2)(p_1^2+p_1q_1+q_1^2-p_1-q_1)|\Phi(a,y)|$$
$$+(p_1^2+p_1q_1+q_1^2-p_1-q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Phi(a,c)|\Big]$$
$$+(b-x)^2(y-c)^2\Big[(p_1+q_1)(p_2+q_2)|\Psi(x,y)|$$
$$+(p_1+q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Psi(x,d)|$$
$$+(p_2+q_2)(p_1^2+p_1q_1+q_1^2-p_1-q_1)|\Psi(a,y)|$$
$$+(p_1^2+p_1q_1+q_1^2-p_1-q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Psi(a,d)|\Big]$$

$$
\begin{aligned}
&+(x-a)^2(d-y)^2\left[(p_1+q_1)(p_2+q_2)|\Theta(x,y)|\right.\\
&+(p_1+q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Theta(x,d)|\\
&+(p_2+q_2)(p_1^2+p_1q_1+q_1^2-p_1-q_1)|\Theta(a,y)|\\
&\left.+(p_1^2+p_1q_1+q_1^2-p_1-q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Theta(a,d)|\right]\\
&+(b-x)^2(d-y)^2\left[(p_1+q_1)(p_2+q_2)|\Omega(x,y)|\right.\\
&+(p_1+q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Omega(x,d)|\\
&+(p_2+q_2)(p_1^2+p_1q_1+q_1^2-p_1-q_1)|\Omega(b,y)|\\
&\left.\left.+(p_1^2+p_1q_1+q_1^2-p_1-q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Omega(b,d)|\right]\right\},
\end{aligned}
$$

for all $(x,y)\in[a,b]\times[c,d]$.

Proof. Taking modulus in Lemma 3, we have

$$
\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\left[\int_a^x\int_c^y f(p_1t+(1-p_1)a,p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t\right.\right.\\
&+\int_x^b\int_c^y f(p_1t+(1-p_1)b,p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s\ {}^bd_{p_1,q_1}t\\
&+\int_a^x\int_y^d f(p_1t+(1-p_1)a,p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t\\
&\left.+\int_x^b\int_y^d f(p_1t+(1-p_1)b,p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\ {}^bd_{p_1,q_1}t\right]\\
&-\frac{1}{d-c}\left[\int_c^y f(x,p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s+\int_y^d f(x,p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\right]\\
&\left.-\frac{1}{b-a}\left[\int_a^x f(p_1t+(1-p_1)a,y)\ {}_ad_{p_1,q_1}t+\int_x^b f(p_1t+(1-p_1)b,y)\ {}^bd_{p_1,q_1}t\right]+f(x,y)\right|\\
&\le\frac{q_1q_2}{(b-a)(d-c)}\left[(x-a)^2(y-c)^2\int_0^1\int_0^1 ts\ |\Phi(tx+(1-t)a,sy+(1-s)c)|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right.\\
&+\ (b-x)^2(y-c)^2\int_0^1\int_0^1 ts\ |\Psi(tx+(1-t)b,sy+(1-s)c)|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&+(x-a)^2(d-y)^2\int_0^1\int_0^1 ts\ |\Theta(tx+(1-t)a,sy+(1-s)d)|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\\
&\left.+(b-x)^2(d-y)^2\int_0^1\int_0^1 ts\ |\Omega(tx+(1-t)b,sy+(1-s)d)|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right].
\end{aligned}
\tag{8}
$$

Since $|\Phi(t,s)|$ is coordinated convex, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1 ts\,|\Phi(tx+(1-t)a, sy+(1-s)c)|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t \\
&\quad\le \int_0^1\int_0^1 ts\,[\,ts|\Phi(x,y)|+t(1-s)|\Phi(x,c)| \\
&\qquad +(1-t)s|\Phi(a,y)|+(1-t)(1-s)|\Phi(a,c)|]\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t \\
&\quad= \frac{1}{(p_1+q_1)(p_2+q_2)(p_1^2+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)} \\
&\qquad\times\Big[(p_1+q_1)(p_2+q_2)|\Phi(x,y)|+(p_1+q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Phi(x,c)| \\
&\qquad+(p_2+q_2)(p_1^2+p_1q_1+q_1^2-p_1-q_1)|\Phi(a,y)| \\
&\qquad+(p_1^2+p_1q_1+q_1^2-p_1-q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Phi(a,c)|\Big]. \qquad (9)
\end{aligned}$$

By the similar way, as $|\Psi(t,s)|$, $|\Theta(t,s)|$ and $|\Omega(t,s)|$ are coordinated convex, we get

$$\begin{aligned}
&\int_0^1\int_0^1 ts\,|\Psi(tx+(1-t)b, sy+(1-s)c)|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t \\
&\quad\le \frac{1}{(p_1+q_1)(p_2+q_2)(p_1^2+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)} \\
&\qquad\times\Big[(p_1+q_1)(p_2+q_2)|\Psi(x,y)|+(p_1+q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Psi(x,c)| \\
&\qquad+(p_2+q_2)(p_1^2+p_1q_1+q_1^2-p_1-q_1)|\Psi(b,y)| \\
&\qquad+(p_1^2+p_1q_1+q_1^2-p_1-q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Psi(b,c)|\Big], \qquad (10)
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\int_0^1 ts\,|\Theta(tx+(1-t)a, sy+(1-s)d)|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t \\
&\quad\le \frac{1}{(p_1+q_1)(p_2+q_2)(p_1^2+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)} \\
&\qquad\times\Big[(p_1+q_1)(p_2+q_2)|\Theta(x,y)|+(p_1+q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Theta(x,d)| \\
&\qquad+(p_2+q_2)(p_1^2+p_1q_1+q_1^2-p_1-q_1)|\Theta(a,y)| \\
&\qquad+(p_1^2+p_1q_1+q_1^2-p_1-q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Theta(a,d)|\Big] \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\int_0^1 ts\,|\Omega(tx+(1-t)b, sy+(1-s)d)|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t \\
&\quad\le \frac{1}{(p_1+q_1)(p_2+q_2)(p_1^2+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)} \\
&\qquad\times\Big[(p_1+q_1)(p_2+q_2)|\Omega(x,y)|+(p_1+q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Omega(x,d)| \\
&\qquad+(p_2+q_2)(p_1^2+p_1q_1+q_1^2-p_1-q_1)|\Omega(b,y)| \\
&\qquad+(p_1^2+p_1q_1+q_1^2-p_1-q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Omega(b,d)|\Big]. \qquad (12)
\end{aligned}$$

Substituting the inequalities (9)-(12) in (8), the proof is completed. □

Corollary 1. Under conditions of Theorem 6, if $|\Phi(t,s)|, |\Psi(t,s)|, |\Theta(t,s)|, |\Omega(t,s)| \leq M$ for all $(t,s) \in [a,b]\times[c,d]$, then we obtain the following (p,q)-Ostrowski-type inequality for coordinates:

$$\begin{aligned}
&\Bigg|\frac{1}{(b-a)(d-c)}\Bigg[\int_a^x\int_c^y f(p_1t+(1-p_1)a, p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t\\
&+\int_x^b\int_c^y f(p_1b+(1-p_1)t, p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s\ {}^bd_{p_1,q_1}t\\
&+\int_a^x\int_y^d f(p_1t+(1-p_1)a, p_2d+(1-p_2)s)\ {}^dd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t\\
&+\int_x^b\int_y^d f(p_1b+(1-p_1)t, p_2d+(1-p_2)t)\ {}^dd_{p_2,q_2}s\ {}^bd_{p_1,q_1}t\Bigg]\\
&-\frac{1}{d-c}\Bigg[\int_c^y f(x, p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s+\int_y^d f(x, p_2d+(1-p_2)s)\ {}^dd_{p_2,q_2}s\Bigg]\\
&-\frac{1}{b-a}\Bigg[\int_a^x f(p_1t+(1-p_1)a, y)\ {}_ad_{p_1,q_1}t+\int_x^b f(p_1b+(1-p_1)t, y)\ {}^bd_{p_1,q_1}t\Bigg]+f(x,y)\Bigg|\\
&\leq \frac{Mq_1q_2}{(b-a)(d-c)(p_1+q_1)(p_2+q_2)}\left[(x-a)^2+(b-x)^2\right]\left[(y-c)^2+(d-y)^2\right],
\end{aligned}$$

for all $(x,y) \in [a,b]\times[c,d]$.

Remark 3. If $p_1 = p_2 = 1$ and q_1, q_2 tend to 1, then Corollary 1 reduces to Theorem 2, which appears in [71].

Theorem 7. Under the assumptions of Lemma 3, in addition, if $|\Phi(t,s)|^\alpha$, $|\Psi(t,s)|^\alpha$, $|\Phi(t,s)|^\alpha$ and $|\Omega(t,s)|^\alpha$ are coordinated convex on $[a,b]\times[c,d]$, where $\frac{1}{\alpha}+\frac{1}{\beta}=1$ with $\alpha > 1$, then we have the inequality

$$\begin{aligned}
&\Bigg|\frac{1}{(b-a)(d-c)}\Bigg[\int_a^x\int_c^y f(p_1t+(1-p_1)a, p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t\\
&+\int_x^b\int_c^y f(p_1t+(1-p_1)b, p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s\ {}^bd_{p_1,q_1}t\\
&+\int_a^x\int_y^d f(p_1t+(1-p_1)a, p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t\\
&+\int_x^b\int_y^d f(p_1t+(1-p_1)b, p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\ {}^bd_{p_1,q_1}t\Bigg]\\
&-\frac{1}{d-c}\Bigg[\int_c^y f(x, p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s+\int_y^d f(x, p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\Bigg]\\
&-\frac{1}{b-a}\Bigg[\int_a^x f(p_1t+(1-p_1)a, y)\ {}_ad_{p_1,q_1}t+\int_x^b f(p_1t+(1-p_1)b, y)\ {}^bd_{p_1,q_1}t\Bigg]+f(x,y)\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{q_1 q_2}{(b-a)(d-c)} \left(\frac{1}{[\beta+1]_{p_1,q_1} [\beta+1]_{p_2,q_2}} \right)^{1/\beta} \left(\frac{1}{(p_1+q_1)(p_2+q_2)} \right)^{1/\alpha} \\
&\times \Big((x-a)^2 (y-c)^2 \big[|\Phi(x,y)|^\alpha + (p_1+q_1-1)|\Phi(x,c)|^\alpha \\
&+(p_2+q_2-1)|\Phi(a,y)|^\alpha + (p_1+q_1-1)(p_2+q_2-1)|\Phi(a,c)|^\alpha \big]^{1/\alpha} \\
&+(b-x)^2 (y-c)^2 \big[|\Psi(x,y)|^\alpha + (p_1+q_1-1)|\Psi(x,c)|^\alpha \\
&+(p_2+q_2-1)|\Psi(b,y)|^\alpha + (p_1+q_1-1)(p_2+q_2-1)|\Psi(b,c)|^\alpha \big]^{1/\alpha} \\
&+(x-a)^2 (d-y)^2 \big[|\Theta(x,y)|^\alpha + (p_1+q_1-1)|\Theta(x,d)|^\alpha \\
&+(p_2+q_2-1)|\Theta(a,y)|^\alpha + (p_1+q_1-1)(p_2+q_2-1)|\Theta(a,d)|^\alpha \big]^{1/\alpha} \\
&+(b-x)^2 (d-y)^2 \big[|\Omega(x,y)|^\alpha + (p_1+q_1-1)|\Omega(x,d)|^\alpha \\
&+(p_2+q_2-1)|\Omega(b,y)|^\alpha + (p_1+q_1-1)(p_2+q_2-1)|\Omega(b,d)|^\alpha \big]^{1/\alpha} \Big),
\end{aligned}
$$

for all $(x, y) \in [a, b] \times [c, d]$.

Proof. From Lemma 3, we have

$$
\begin{aligned}
&\Bigg| \frac{1}{(b-a)(d-c)} \Bigg[\int_a^x \int_c^y f(p_1 t + (1-p_1)a, p_2 s + (1-p_2)c) \ {}_c d_{p_2,q_2} s \ {}_a d_{p_1,q_1} t \\
&+ \int_x^b \int_c^y f(p_1 t + (1-p_1)b, p_2 s + (1-p_2)c) \ {}_c d_{p_2,q_2} s \ {}^b d_{p_1,q_1} t \\
&+ \int_a^x \int_y^d f(p_1 t + (1-p_1)a, p_2 s + (1-p_2)d) \ {}^d d_{p_2,q_2} s \ {}_a d_{p_1,q_1} t \\
&+ \int_x^b \int_y^d f(p_1 t + (1-p_1)b, p_2 s + (1-p_2)d) \ {}^d d_{p_2,q_2} s \ {}^b d_{p_1,q_1} t \Bigg] \\
&- \frac{1}{d-c} \Bigg[\int_c^y f(x, p_2 s + (1-p_2)c) \ {}_c d_{p_2,q_2} s + \int_y^d f(x, p_2 s + (1-p_2)d) \ {}^d d_{p_2,q_2} s \Bigg] \\
&- \frac{1}{b-a} \Bigg[\int_a^x f(p_1 t + (1-p_1)a, y) \ {}_a d_{p_1,q_1} t + \int_x^b f(p_1 t + (1-p_1)b, y) \ {}^b d_{p_1,q_1} t \Bigg] + f(x,y) \Bigg| \\
&\leq \frac{q_1 q_2}{(b-a)(d-c)} \Bigg[(x-a)^2 (y-c)^2 \int_0^1 \int_0^1 ts \ |\Phi(tx + (1-t)a, sy + (1-s)c)| \ {}_0 d_{p_2,q_2} s \ {}_0 d_{p_1,q_1} t \\
&+ (b-x)^2 (y-c)^2 \int_0^1 \int_0^1 ts \ |\Psi(tx + (1-t)b, sy + (1-s)c)| \ {}_0 d_{p_2,q_2} s \ {}_0 d_{p_1,q_1} t \\
&+ (x-a)^2 (d-y)^2 \int_0^1 \int_0^1 ts \ |\Theta(tx + (1-t)a, sy + (1-s)d)| \ {}_0 d_{p_2,q_2} s \ {}_0 d_{p_1,q_1} t \\
&+ (b-x)^2 (d-y)^2 \int_0^1 \int_0^1 ts \ |\Omega(tx + (1-t)b, sy + (1-s)d)| \ {}_0 d_{p_2,q_2} s \ {}_0 d_{p_1,q_1} t \Bigg]. \qquad (13)
\end{aligned}
$$

Using Lemma 1 and the coordinated convexity of $|\Phi(t,s)|^{\alpha}$, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1 ts\,|\Phi(tx+(1-t)a, sy+(1-s)c)|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t \\
&\leq \left(\int_0^1\int_0^1 t^{\beta}s^{\beta}\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right)^{1/\beta}\left(\int_0^1\int_0^1 |\Phi(tx+(1-t)a, sy+(1-s)c)|^{\alpha}\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right)^{1/\alpha} \\
&\leq \left(\frac{1}{[\beta+1]_{p_1,q_1}[\beta+1]_{p_2,q_2}}\right)^{1/\beta}\left(\int_0^1\int_0^1 \left[\ ts|\Phi(x,y)|^{\alpha}+t(1-s)|\Phi(x,c)|^{\alpha}\right.\right. \\
&\quad \left.\left.+(1-t)s|\Phi(a,y)|^{\alpha}+(1-t)(1-s)|\Phi(a,c)|^{\alpha}\ \right]\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t\right)^{1/\alpha} \\
&= \left(\frac{1}{[\beta+1]_{p_1,q_1}[\beta+1]_{p_2,q_2}}\right)^{1/\beta}\left(\frac{|\Phi(x,y)|^{\alpha}+(p_1+q_1-1)|\Phi(x,c)|^{\alpha}}{(p_1+q_1)(p_2+q_2)}\right. \\
&\quad \left.+\frac{(p_2+q_2-1)|\Phi(a,y)|^{\alpha}+(p_1+q_1-1)(p_2+q_2-1)|\Phi(a,c)|^{\alpha}}{(p_1+q_1)(p_2+q_2)}\right)^{1/\alpha}. \qquad (14)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\int_0^1\int_0^1 ts\,|\Psi(tx+(1-t)b, sy+(1-s)c)|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t \\
&\leq \left(\frac{1}{[\beta+1]_{p_1,q_1}[\beta+1]_{p_2,q_2}}\right)^{1/\beta}\left(\frac{|\Psi(x,y)|^{\alpha}+(p_1+q_1-1)|\Psi(x,c)|^{\alpha}}{(p_1+q_1)(p_2+q_2)}\right. \\
&\quad \left.+\frac{(p_2+q_2-1)|\Psi(b,y)|^{\alpha}+(p_1+q_1-1)(p_2+q_2-1)|\Psi(b,c)|^{\alpha}}{(p_1+q_1)(p_2+q_2)}\right)^{1/\alpha}, \qquad (15)
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\int_0^1 ts\,|\Theta(tx+(1-t)a, sy+(1-s)d)|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t \\
&\leq \left(\frac{1}{[\beta+1]_{p_1,q_1}[\beta+1]_{p_2,q_2}}\right)^{1/\beta}\left(\frac{|\Theta(x,y)|^{\alpha}+(p_1+q_1-1)|\Theta(x,d)|^{\alpha}}{(p_1+q_1)(p_2+q_2)}\right. \\
&\quad \left.+\frac{(p_2+q_2-1)|\Theta(a,y)|^{\alpha}+(p_1+q_1-1)(p_2+q_2-1)|\Theta(a,d)|^{\alpha}}{(p_1+q_1)(p_2+q_2)}\right)^{1/\alpha} \qquad (16)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\int_0^1 ts\,|\Omega(tx+(1-t)b, sy+(1-s)d)|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t \\
&\leq \left(\frac{1}{[\beta+1]_{p_1,q_1}[\beta+1]_{p_2,q_2}}\right)^{1/\beta}\left(\frac{|\Omega(x,y)|^{\alpha}+(p_1+q_1-1)|\Omega(x,d)|^{\alpha}}{(p_1+q_1)(p_2+q_2)}\right. \\
&\quad \left.+\frac{(p_2+q_2-1)|\Omega(b,y)|^{\alpha}+(p_1+q_1-1)(p_2+q_2-1)|\Omega(b,d)|^{\alpha}}{(p_1+q_1)(p_2+q_2)}\right)^{1/\alpha}. \qquad (17)
\end{aligned}$$

Substituting the inequalities (14)-(17) in (13), the proof is completed. □

Corollary 2. Under conditions of Theorem 7, if $|\Phi(t,s)|, |\Psi(t,s)|, |\Theta(t,s)|, |\Omega(t,s)| \leq M$ for all $(t,s) \in [a,b] \times [c,d]$, then we obtain the following (p,q)-Ostrowski-type inequality for coordinates:

$$
\begin{aligned}
&\Bigg| \frac{1}{(b-a)(d-c)} \Bigg[\int_a^x \int_c^y f(p_1 t + (1-p_1)a, p_2 s + (1-p_2)c)\ {}_c d_{p_2,q_2} s\ {}_a d_{p_1,q_1} t \\
&+ \int_x^b \int_c^y f(p_1 t + (1-p_1)b, p_2 s + (1-p_2)c)\ {}_c d_{p_2,q_2} s\ {}^b d_{p_1,q_1} t \\
&+ \int_a^x \int_y^d f(p_1 t + (1-p_1)a, p_2 s + (1-p_2)d)\ {}^d d_{p_2,q_2} s\ {}_a d_{p_1,q_1} t \\
&+ \int_x^b \int_y^d f(p_1 t + (1-p_1)b, p_2 s + (1-p_2)d)\ {}^d d_{p_2,q_2} s\ {}^b d_{p_1,q_1} t \Bigg] \\
&- \frac{1}{d-c} \left[\int_c^y f(x, p_2 s + (1-p_2)c)\ {}_c d_{p_2,q_2} s + \int_y^d f(x, p_2 s + (1-p_2)d)\ {}^d d_{p_2,q_2} s \right] \\
&- \frac{1}{b-a} \left[\int_a^x f(p_1 t + (1-p_1)a, y)\ {}_a d_{p_1,q_1} t + \int_x^b f(p_1 t + (1-p_1)b, y)\ {}^b d_{p_1,q_1} t \right] + f(x,y) \Bigg| \\
&\leq \frac{M q_1 q_2}{(b-a)(d-c)} \left(\frac{1}{[\beta+1]_{p_1,q_1} [\beta+1]_{p_2,q_2}} \right)^{1/\beta} \left[(x-a)^2 + (b-x)^2 \right] \left[(y-c)^2 + (d-y)^2 \right],
\end{aligned} \tag{18}
$$

for all $(x,y) \in [a,b] \times [c,d]$.

Remark 4. If $p_1 = p_2 = 1$, then (18) reduces to

$$
\begin{aligned}
&\Bigg| \frac{1}{(b-a)(d-c)} \Bigg[\int_a^x \int_c^y f(t,s)\ {}_c d_{q_2} s\ {}_a d_{q_1} t + \int_x^b \int_c^y f(t,s)\ {}_c d_{q_2} s\ {}^b d_{q_1} t \\
&+ \int_a^x \int_y^d f(t,s)d)\ {}^d d_{q_2} s\ {}_a d_{q_1} t + \int_x^b \int_y^d f(t,s)\ {}^d d_{q_2} s\ {}^b d_{q_1} t \Bigg] \\
&- \frac{1}{d-c} \left[\int_c^y f(x,s)\ {}_c d_{q_2} s + \int_y^d f(x,s)\ {}^d d_{q_2} s \right] \\
&- \frac{1}{b-a} \left[\int_a^x f(t,y)\ {}_a d_{q_1} t + \int_x^b f(t,y)\ {}^b d_{q_1} t \right] + f(x,y) \Bigg| \\
&\leq \frac{M q_1 q_2}{(b-a)(d-c)} \left(\frac{1}{[\beta+1]_{q_1} [\beta+1]_{q_2}} \right)^{1/\beta} \left[(x-a)^2 + (b-x)^2 \right] \left[(y-c)^2 + (d-y)^2 \right],
\end{aligned}
$$

which appears in [70].

Remark 5. If $p_1 = p_2 = 1$ and q_1, q_2 tend to 1, then Corollary 2 reduces to Theorem 3, which appears in [71].

Theorem 8. Under the assumptions of Lemma 3, in addition, if $|\Phi(t,s)|^{\alpha}$, $|\Psi(t,s)|^{\alpha}$, $|\Phi(t,s)|^{\alpha}$ and $|\Omega(t,s)|^{\alpha}$ are coordinated convex on $[a,b]\times[c,d]$, where $\alpha \geq 1$, then we have the inequality

$$
\begin{aligned}
&\Bigg|\frac{1}{(b-a)(d-c)}\Bigg[\int_a^x\int_c^y f(p_1t+(1-p_1)a, p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t\\
&+\int_x^b\int_c^y f(p_1t+(1-p_1)b, p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s\ {}^bd_{p_1,q_1}t\\
&+\int_a^x\int_y^d f(p_1t+(1-p_1)a, p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t\\
&+\int_x^b\int_y^d f(p_1t+(1-p_1)b, p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\ {}^bd_{p_1,q_1}t\Bigg]\\
&-\frac{1}{d-c}\Bigg[\int_c^y f(x, p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s+\int_y^d f(x, p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\Bigg]\\
&-\frac{1}{b-a}\Bigg[\int_a^x f(p_1t+(1-p_1)a, y)\ {}_ad_{p_1,q_1}t+\int_x^b f(p_1t+(1-p_1)b, y)\ {}^bd_{p_1,q_1}t\Bigg]+f(x,y)\Bigg|\\
&\leq \frac{q_1q_2}{(b-a)(d-c)}\left(\frac{1}{(p_1+q_1)(p_2+q_2)}\right)^{1-1/\alpha}\\
&\times\left(\frac{1}{(p_1+q_1)(p_2+q_2)(p_1^1+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)}\right)^{1/\alpha}\\
&\times\Big((x-a)^2(y-c)^2\big[(p_1+q_1)(p_2+q_2)|\Phi(x,y)|^{\alpha}\\
&+(p_1+q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Phi(x,c)|^{\alpha}\\
&+(p_2+q_2)(p_1^2+p_1q_1+q_1^2-p_1-q_1)|\Phi(a,y)|^{\alpha}\\
&+(p_1^2+p_1q_1+q_1^2-p_1-q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Phi(a,c)|^{\alpha}\big]^{1/\alpha}\\
&+(b-x)^2(y-c)^2\big[(p_1+q_1)(p_2+q_2)|\Psi(x,y)|^{\alpha}\\
&+(p_1+q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Psi(x,c)|^{\alpha}\\
&+(p_2+q_2)(p_1^2+p_1q_1+q_1^2-p_1-q_1)|\Psi(b,y)|^{\alpha}\\
&+(p_1^2+p_1q_1+q_1^2-p_1-q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Psi(b,c)|^{\alpha}\big]^{1/\alpha}\\
&+(x-a)^2(d-y)^2\big[(p_1+q_1)(p_2+q_2)|\Theta(x,y)|^{\alpha}\\
&+(p_1+q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Theta(x,d)|^{\alpha}\\
&+(p_2+q_2)(p_1^2+p_1q_1+q_1^2-p_1-q_1)|\Theta(a,y)|^{\alpha}\\
&+(p_1^2+p_1q_1+q_1^2-p_1-q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Theta(a,d)|^{\alpha}\big]^{1/\alpha}\\
&+(b-x)^2(d-y)^2\big[(p_1+q_1)(p_2+q_2)|\Omega(x,y)|^{\alpha}\\
&+(p_1+q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Omega(x,d)|^{\alpha}\\
&+(p_2+q_2)(p_1^2+p_1q_1+q_1^2-p_1-q_1)|\Omega(b,y)|^{\alpha}\\
&+(p_1^2+p_1q_1+q_1^2-p_1-q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Omega(b,d)|^{\alpha}\big]^{1/\alpha}\Big),
\end{aligned}
$$

for all $(x,y)\in[a,b]\times[c,d]$.

Proof. Using Lemma 2 and the coordinated convexity of $|\Phi(t,s)|^{\alpha}$, we have

$$
\begin{aligned}
&\int_0^1\int_0^1 ts\,\left|\Phi(tx+(1-t)a, sy+(1-s)c)\right| \,{}_0d_{p_2,q_2}s\,{}_0d_{p_1,q_1}t \\
&\le \left(\int_0^1\int_0^1 ts\,{}_0d_{p_2,q_2}s\,{}_0d_{p_1,q_1}t\right)^{1-1/\alpha}\left(\int_0^1\int_0^1 ts\,\left|\Phi(tx+(1-t)a, sy+(1-s)c)\right|^{\alpha}\,{}_0d_{p_2,q_2}s\,{}_0d_{p_1,q_1}t\right)^{1/\alpha} \\
&\le \left(\frac{1}{(p_1+q_1)(p_2+q_2)}\right)^{1-1/\alpha}\left(\int_0^1\int_0^1 ts\,\Big[\,ts|\Phi(x,y)|^{\alpha}+t(1-s)|\Phi(x,c)|^{\alpha}\right. \\
&\quad \left.+(1-t)s|\Phi(a,y)|^{\alpha}+(1-t)(1-s)|\Phi(a,c)|^{\alpha}\,\Big]\,{}_0d_{p_2,q_2}s\,{}_0d_{p_1,q_1}t\right)^{1/\alpha} \\
&= \left(\frac{1}{(p_1+q_1)(p_2+q_2)}\right)^{1-1/\alpha}\left(\frac{(p_1+q_1)(p_2+q_2)|\Phi(x,y)|^{\alpha}}{(p_1+q_1)(p_2+q_2)(p_1^1+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)}\right. \\
&\quad +\frac{(p_1+q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Phi(x,c)|^{\alpha}}{(p_1+q_1)(p_2+q_2)(p_1^1+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)} \\
&\quad +\frac{(p_2+q_2)(p_1^2+p_1q_1+q_1^2-p_1-q_1)|\Phi(a,y)|^{\alpha}}{(p_1+q_1)(p_2+q_2)(p_1^1+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)} \\
&\quad \left.+\frac{(p_1^2+p_1q_1+q_1^2-p_1-q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Phi(a,c)|^{\alpha}}{(p_1+q_1)(p_2+q_2)(p_1^1+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)}\Bigg]\right)^{1/\alpha}.
\end{aligned}
\tag{19}
$$

Similarly, we get

$$
\begin{aligned}
&\int_0^1\int_0^1 ts\,\left|\Psi(tx+(1-t)b, sy+(1-s)c)\right| \,{}_0d_{p_2,q_2}s\,{}_0d_{p_1,q_1}t \\
&\le \left(\frac{1}{(p_1+q_1)(p_2+q_2)}\right)^{1-1/\alpha}\left(\frac{(p_1+q_1)(p_2+q_2)|\Psi(x,y)|^{\alpha}}{(p_1+q_1)(p_2+q_2)(p_1^1+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)}\right. \\
&\quad +\frac{(p_1+q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Psi(x,c)|^{\alpha}}{(p_1+q_1)(p_2+q_2)(p_1^1+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)} \\
&\quad +\frac{(p_2+q_2)(p_1^2+p_1q_1+q_1^2-p_1-q_1)|\Psi(b,y)|^{\alpha}}{(p_1+q_1)(p_2+q_2)(p_1^1+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)} \\
&\quad \left.+\frac{(p_1^2+p_1q_1+q_1^2-p_1-q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Psi(b,c)|^{\alpha}}{(p_1+q_1)(p_2+q_2)(p_1^1+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)}\right)^{1/\alpha},
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&\int_0^1\int_0^1 ts\,\left|\Theta(tx+(1-t)a, sy+(1-s)d)\right| \,{}_0d_{p_2,q_2}s\,{}_0d_{p_1,q_1}t \\
&\le \left(\frac{1}{(p_1+q_1)(p_2+q_2)}\right)^{1-1/\alpha}\left(\frac{(p_1+q_1)(p_2+q_2)|\Theta(x,y)|^{\alpha}}{(p_1+q_1)(p_2+q_2)(p_1^1+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)}\right. \\
&\quad +\frac{(p_1+q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Theta(x,d)|^{\alpha}}{(p_1+q_1)(p_2+q_2)(p_1^1+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)} \\
&\quad +\frac{(p_2+q_2)(p_1^2+p_1q_1+q_1^2-p_1-q_1)|\Theta(a,y)|^{\alpha}}{(p_1+q_1)(p_2+q_2)(p_1^1+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)} \\
&\quad \left.+\frac{(p_1^2+p_1q_1+q_1^2-p_1-q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Theta(a,d)|^{\alpha}}{(p_1+q_1)(p_2+q_2)(p_1^1+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)}\right)^{1/\alpha}
\end{aligned}
\tag{21}
$$

and

$$\int_0^1\int_0^1 ts\,|\Omega(tx+(1-t)b, sy+(1-s)d)|\ {}_0d_{p_2,q_2}s\ {}_0d_{p_1,q_1}t$$
$$\leq \left(\frac{1}{(p_1+q_1)(p_2+q_2)}\right)^{1-1/\alpha}\left(\frac{(p_1+q_1)(p_2+q_2)|\Omega(x,y)|^\alpha}{(p_1+q_1)(p_2+q_2)(p_1^1+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)}\right.$$
$$+\frac{(p_1+q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Omega(x,d)|^\alpha}{(p_1+q_1)(p_2+q_2)(p_1^1+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)}$$
$$+\frac{(p_2+q_2)(p_1^2+p_1q_1+q_1^2-p_1-q_1)|\Omega(b,y)|^\alpha}{(p_1+q_1)(p_2+q_2)(p_1^1+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)}$$
$$\left.+\frac{(p_1^2+p_1q_1+q_1^2-p_1-q_1)(p_2^2+p_2q_2+q_2^2-p_2-q_2)|\Omega(b,d)|^\alpha}{(p_1+q_1)(p_2+q_2)(p_1^1+p_1q_1+q_1^2)(p_2^2+p_2q_2+q_2^2)}\right)^{1/\alpha}. \tag{22}$$

Substituting the inequalities (19)-(22) in (13), the proof is completed. □

Corollary 3. Under conditions of Theorem 8, if $|\Phi(t,s)|, |\Psi(t,s)|, |\Theta(t,s)|, |\Omega(t,s)| \leq M$ for all $(t,s) \in [a,b]\times[c,d]$, then we obtain the following (p,q)-Ostrowski-type inequality for coordinates:

$$\left|\frac{1}{(b-a)(d-c)}\left[\int_a^x\int_c^y f(p_1t+(1-p_1)a, p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t\right.\right.$$
$$+\int_x^b\int_c^y f(p_1t+(1-p_1)b, p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s\ {}^bd_{p_1,q_1}t$$
$$+\int_a^x\int_y^d f(p_1t+(1-p_1)a, p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\ {}_ad_{p_1,q_1}t$$
$$\left.+\int_x^b\int_y^d f(p_1t+(1-p_1)b, p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\ {}^bd_{p_1,q_1}t\right]$$
$$-\frac{1}{d-c}\left[\int_c^y f(x, p_2s+(1-p_2)c)\ {}_cd_{p_2,q_2}s+\int_y^d f(x, p_2s+(1-p_2)d)\ {}^dd_{p_2,q_2}s\right]$$
$$\left.-\frac{1}{b-a}\left[\int_a^x f(p_1t+(1-p_1)a, y)\ {}_ad_{p_1,q_1}t+\int_x^b f(p_1t+(1-p_1)b, y)\ {}^bd_{p_1,q_1}t\right]+f(x,y)\right|$$
$$\leq \frac{Mq_1q_2}{(b-a)(d-c)(p_1+q_1)(p_2+q_2)}\left[(x-a)^2+(b-x)^2\right]\left[(y-c)^2+(d-y)^2\right],$$

for all $(x,y) \in [a,b]\times[c,d]$.

Remark 6. If $p_1 = p_2 = 1$ and q_1, q_2 tend to 1, then Corollary 3 reduces to Theorem 4, which appears in[71].

4 | CONCLUSIONS

We defined new definition of (p,q)-derivatives for continuous functions of two variables. Moreover, we established post quantum Ostrowski–type inequalities for coordinated convex functions. Some previously published results of other researchers are some special cases of our results for p_1, p_2 are unity and q_1, q_2 tend to one.

ACKNOWLEDGMENTS

We would like to thank anonymous referees for comments which are helpful for improvement in this paper. The first author is supported by Development and Promotion of Science and Technology talents project (DPST), Thailand.

Author contributions

All authors contributed equally to this article. They read and approved the final manuscript.

Financial disclosure

Not applicable.

Conflict of interest

The authors declare that they have no competing interests.

SUPPORTING INFORMATION

Not applicable.

References

1. Jackson FH. On a q-definite integrals. *Quart. J. Pure. Appl. Math.* 1910;41:193–203.
2. Jackson FH. q-difference equations. *Amer. J. Math.* 1910;32:305–314.
3. Bangerezaka G. Variational q-calculus. *J. Math. Anal. Appl.* 2004;289:650–665.
4. Asawasamrit S, Sudprasert C, Ntouyas S, Tariboon J. Some results on quantum Hanh integral inequalities. *J. Inequal. Appl.* 2019;2019:154.
5. Bangerezako G. Variational calculus on q-nonuniform. *J. Math. Anal. Appl.* 2005;306:161–179.
6. Exton H. *q-Hypergeometric functions and applications.* Hastead Press: New York, USA 1983.
7. Annyby HM, Mansour SK. *q-fractional calculus and equations.* Springer: Helidelberg, Germany 2012.
8. Ernst T. *A comprehensive treatment of q-calculus.* Springer: Basel, Switzerland 2012.
9. Ernst T. *A history of q-calculus and a new method.* UUDM Report; Uppsala University: Uppsala, Sweden 2000.
10. Rui ACF. Nontrivial solutions for fractional q-difference boundary value problems. *Electron. J. Qual. Theory Differ. Equ.* 2010;2010:1–10.
11. Aslam M, Awan MU, Noor KI. Quantum Ostrowski inequalities for q-differentiable convex function. *J. Math. Inequal.* 2016;10:1013–1018.
12. Aral A, Gupta V, Agarwal RP. *Applications of q-calculus in operator theory.* Springer Science + Business Media: New York, USA 2013.
13. Gauchman H. Integral inequalities in q-calculus. *J. Comput. Appl. Math.* 2002;47:281–300.
14. Dobrogowska A, Odzijewicz A. A second order q-difference equation solvable by factorization method. *J. Comput. Appl. Math.* 2006;193:319–346.

15. Gasper G, Rahman M. Some systems of multivariable orthogonal q-Racah polynomials. *Ramanujan J.* 2007;13:389–405.

16. Ismail MEH, Simeonov P. q-difference operators for orthogonal polynomials. *J. Comput. Appl. Math.* 2009; 233:749–761.

17. Bohner M, Guseinov GS. The h-Laplace and q-Laplace transforms. *J. Comput. Appl. Math.* 2010;365:75–92.

18. El-Shahed M, Hassan HA. Positive solutions of q-difference equation. *Proc. Amer. Math. Soc.* 2010;138:1733–1738.

19. Ahmad B. Boundary-value problems for nonlinear third-order q-difference equations. *Electron. J. Differ. Equ.* 2011;94:1–7.

20. Ahmad B, Alsaedi A, Ntouyas SK. A study of second-order q-difference equations with boundary conditions. *Adv. Differ. Equ.* 2012;2012:35.

21. Ahmad B, Ntouyas SK, Purnaras IK. Existence results for nonlinear q-difference equations with nonlocal boundary conditions. *Comm. Appl. Nonlinear Anal.* 2012;19:59–72.

22. Ahmad B, Nieto JJ. On nonlocal boundary value problems of nonlinear q-difference equation. *Adv. Differ. Equ.* 2012;2012:81.

23. Kac V, Cheung P. *Quantum Calculus.* Springer: New York, USA 2002.

24. Tariboon J, Ntouyas SK. Quantum calculus on finite intervals and applications to impulsive difference equations. *Adv. Differ. Equ.* 2013;2013:282.

25. Tariboon J, Ntouyas SK. Quantum integral inequalities on finite intervals. *J. Inequal. Appl.* 2014;2014:121.

26. Kunt M, Latif MA, Iscan I, Dragomir SS. Quantum Hermite-Hadamard type inequality and some estimates of quantum midpoint type inequalities for double integrals. *Sigma J. Eng. Nat. Sci.* 2019;37:207–223.

27. Bermudo S, Korus P, Valdes JEN. On q-Hermite-Hadamard inequalities for general convex functions. *Acta Math. Hungar.* 2020;162:364–374.

28. Budak H, Ali MA, Tarhanaci M. Some new quantum Hermite-Hadamard-like inequalities for coordinated convex functions. *J. Optim. Theory Appl.* 2020;186:899–910.

29. Latif MA, Dragomir SS, Momoniat E. Some q-analogues of Hermite-Hadamard inequality of functions of two variables on finite rectangles in the plane. *J. King Saud Univ. Sci.* 2017;29:263–273.

30. Jhanthanam S, Tariboon J, Ntouyas SK, Nonlaopon K. On q-Hermite-Hadamard inequalities for differentiable convex functions. *Mathematics* 2019;7:632.

31. Kalsoom H, Wu JD, Hussain S, Latif MA. Simpson's type inequalities for co-ordinated convex functions on quantum calculus. *Symmetry* 2019;11:768.

32. Noor MA, Noor KI, Awan MU. Some quantum estimates for Hermite-Hadamard inequalities. *Appl. Math. Comput.* 2015;251:675–679.

33. Prabseang J, Nonlaopon K, Tariboon J. Quantum Hermite-Hadamard inequalities for double integral and q-differentiable convex functions. *J. Math. Inequal.* 2019;13:675–686.

34. Sudsutad W, Ntouyas SK, Tariboon J. Quantum integral inequalities for convex functions. *J. Math. Inequal.* 2015;9:781–793.

35. Yang W. Some new Fejér type inequalities via quantum calculus on finite intervals. *Sci. Asia.* 2017;43:123–134.

36. Prabseang J, Nonlaopon K, Ntouyas SK. On refinement of quantum Hermite-Hadamard inequalities for convex functions. *J. Math. Inequal.* 2020;14:875–885.

37. Kalsoom H, Idrees M, Baleanu D, Chu YM. New estimates of q_1q_2-Ostrowski-type inequalities within a class of n-polynomial prevexity of functions. *J. Funct. Spaces* 2020;2020: Article ID 3720798.

38. Ali MA, Budak H, Sarikaya MZ. On some new trapezoidal type inequalities for the functions of two variables via quantum calculus. https://www.researchgate.net/ publication/344469986.

39. Chakrabarti R, Jagannathan R. A (p,q)-oscillator realization of two-parameter quantum algebras. *J. Phys. A Math. Gen.* 1991;24:L711–L718.

40. Tunç M, Göv E. Some integral inequalities via (p,q)-calculus on finite intervals. *RGMIA Res. Rep. Coll.* 2016;19:1–12.

41. Tunç M, Göv E. (p,q)-integral inequalities. *RGMIA Res. Rep. Coll.* 2016;19:1–13.

42. Prabseang J, Nonlaopon K, Tariboon J. (p,q)-Hermite-Hadamard inequalities for double integral and (p,q)-differentiable convex functions. *Axioms* 2019;8:68.

43. Kalsoom H, Amer M, Junjua MD, Hassain S, Shahxadi G. (p,q)-estimates of Hermite-Hadamard-type inequalities for coordinated convex and quasi convex function. *Mathematics* 2019;7:683.

44. Burban I. Two-parameter deformation of the oscillator algebra and (p,q)-analog of two-dimensional conformal field theory. *J. Nonlinear Math. Phys.* 1995;2:384–391.

45. Burban IM, Klimyk AU. (p,q)-differentiation, (p,q)-integration and (p,q)-hypergeometric functions related to quantum groups. *Integral Transforms Spec. Funct.* 1994;2:15–36.

46. Hounkonnou MN, Désiré J, Kyemb BR. (p,q)-calculus: differentiation and integration. *SUT J. Math.* 2013;49:145–167.

47. Aral A, Gupta V. Applications of (p,q)-gamma function to Szász durrmeyer operators. *Publ. Inst. Math.* 2017;102:211–220.

48. Sahai V, Yadav S. Representations of two parameter quantum algebras and (p,q)-special functions. *J. Math. Anal. Appl.* 2007;335:268–279.

49. Sadjang PN. On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas. *Results Math.* 2018;73:39.

50. Sadjang PN. On the (p,q)-gamma and the (p,q)-beta functions. *arXiv* 2015. arXiv:1506.07394.

51. Sadjang PN. On two (p,q)-analogues of the laplace transform. *J. Differ. Equ. Appl.* 2017;23:1562–1583.

52. Chu YM, Awan MU, Talib S, Noor MA, Noor KI. New post quantum analogues of Ostrowski-type inequalities using new definitions of left-right (p,q)-derivatives and definite integrals. *Adv. Differ. Equ.* 2020;2020:634.

53. Kalsoom H, Rashid S, Tdrees M, Safdar F, Akram S, Baleanu D, Chu YM. Post quantum inequalities of Hermite-Hadamard-type associated with co-ordinated higher-order generalized strongly pre-index and quasi-pre-index mappings. *Symmetry* 2020;12:443.

54. Kunt M, Iscan I, Alp N, Sarikaya MZ. (p,q)-Hermite-Hadamard and (p,q)-estimates for midpoint type inequalities via convex and quasi-convex functions. *RACSAM* 2018;112:969–992.

55. Ali MA, Budak H, Kalsoom H, Chu YM. Post-quantum Hermite-Hadamard inequalities involving newly defined (p,q)-integral. *Authorea* 2020. doi:10.22541/au.160465507.75463188/v1.

56. Thongjob S, Nonlaopon K, Ntouyas SK. Some (p,q)-Hardy type inequalities for (p,q)-integrable functions. *AMIS Math.* 2020;6:77–89.

57. Wannalookkhee F, Nonlaopon K, Tariboon J, Ntouyas SK. On Hermite-Hadamard type inequalities for coordinated convex functions via (p,q)-calculus. *Mathematics* 2021;9:698.

58. Arunrat N, Nakprasit KM, Nonlaopon K, Tariboon J, Ntouyas SK. On Fejér type inequalities via (p,q)-calculus. *Symmetry* 2021;13:953.

59. Ostrowski A. Uber die absolutabweichung einer differentiebaren funcktion von iherem integralmittelwert. *Comment. Math. Helv.* 1938;10:226–227.

60. Alomari M, Darus M. Some Ostrowski's type inequalities for convex functions with applications. *Rep. Coll.* 2010;13:2.

61. Barnett NS, Dragomir SS. An Ostrowski type inequality for double integrals and applications for cubature formulae. *RGMIA research report collection* 2001;1:1.

62. Cerone P, Dragomir SS. Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions. *Demonstr. Math.* 2004;37:299–308.

63. Dragomir SS, Sofo A. Ostrowski type inequalities for functions whose derivatives are convex. *In: Proceeding of the* 4*th International Conference on Modelling and Simulation* 2002.

64. Dragomir SS, Barnett NS, Cerone P. An n-dimensional version of Ostrowski's inequality for mappings of the Hölder type. *RGMIA research report collection* 1999;2:2.

65. Latif MA, Dragomir SS, Matouk A. New inequalities of Ostrowski type for co-ordinated convex functions via fractional integrals. *J. Fract. Calc. Appl.* 2012;2:1–15.

66. Pachpatte B. On an inequality of Ostrowski type in three independent variables. *J. Math. Anal. Appl.* 2000;249:583–591.

67. Pachpatte B. On a new Ostrowski type inequality in two independent variables. *Tamkang J. Math.* 2001;32:45–49.

68. Pachpatte B. A new Ostrowski type inequality for double integrals. *J. Math. Anal. Appl.* 2006;32:317.

69. Sarikaya M. On the Ostrowski type integral inequality. *Acta Math. Univ. Comen.* 2010;79:129–134.

70. Budak H, Ali MA, Tunc T. Quantum Ostrowski–type inequalities for functions of two variables. *Math. Meth. Appl. Sci.* 2021;2021:1–16.

71. Latif MA, Hussain S, Dragomir SS. New Ostrowski type inequalities for co-ordinated convex functions. *TJMM* 2012;4:125–136.

72. Dragomir SS. On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane. *Taiwanese J. Math.* 2001;5:775–788.

How to cite this article: Wannalookkhee F., Nonlaopon K., Ntouyas S.K., and Budak H. (2021), Ostrowski inequality, convex function, coordinated convex function, (p, q)-derivative, (p, q)-integral, (p, q)-calculus, -, -.