Modified gravity over the linearized metric perturbation for two body dynamics

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Abstract:
In this paper Modified gravity is studied over the weak field linearized metric perturbation in post-Minkowskian theory. This is a different aspect for studying the two body dynamics or binary system. Here despite of usual self force originated from the radiative backscattering of gravitational waves we are considering new paradigm of perturbation that is multiplicative approach. The new perturbed metric is determined over the multiplication of one isolated background metric with the others metric deformation in post-Newtonian theory. As one can think for consecutive Lorentz Transformations in Special Theory of Relativity. To verify the model and the theoretical result the binary system of Milky Way central super massive black hole to Sun is considered. The computation shows remarkable result without MOND for galactic flat rotation curve and solar rotational speed 249km/sec, obviously very good agreement with recent observed data.

Keywords: modified gravity, Einstein linearized metric, galactic rotation curve, suns rotational speed.

1. Introduction:

In studying the result for flat rotation curve of distant stars of galaxy after Vera Rubin et al, [**1, 2**] the concept of dark matter arises [**3**]. Also another candidate replacing the dark mater hypothesis is the Modified Newtonian Dynamics (MOND) by Milgrom [**4**] in an empirical level. In post Newtonian concept possibly the best-studied modified gravity without dark matter accounts for Scalar-tensor theories [**5, 6**] and TeVeS, the tensor-vector-scalar theory proposed by Bekenstein [**7**]. But its successes not free from shortcomings as discussed by Sanders [**8**]. So extension of gravitational theory or its modification is necessary in post Newtonian formalism. In this realm wide study has been done over from Einstein Hilbert action by setting suitable Lagrangian in the different post Newtonian (PN) level [**9, 10, 11** and **12**].

Let us go for a completely new paradigm(static orbit without scattering), the motion of the small compact object is not like a point mass moving along geodesic in the fabric of space time described by massive black hole following linearized Einstein's equation[**13**]. We must take into account the gravitational effects of the mass of the compact body which deform the curved space-time. The self gravity of the small body perturbs the space-time created by gravity of large black hole in absences of the compact body. The linearized Einstein's equations can be implied for weak gravitational field with non relativistic velocities of the point particle. In context of strong gravitational fields, large orbital velocities, need fresh approach over PN theory and numerical relativity.

The computations of Self-force [**14**] approach(force for back reaction) must be done in curved space-time, one might begin a treatment of gravitational self-force by considering a metric perturbation, $h_{\mu\nu}$in a background metric, $g_{\mu\nu}$ sourced by the stress-energy tensor of a "point particle" of mass [**15**].

The above clues prompt to think over post Newtonian dynamics for self gravity (not to be confused with Self Force). The perturbation acts on the compact object and alters its motion, which is no longer geodesic in the space-time of the large black hole and in the following that will be introduced in a multiplicative perturbation approach to study two body dynamics. Especially the rotations of galactic stars are computed around super massive black hole (SMBH).

2.Multiplicative perturbation approach:

Here a two body dynamics is considered over the resultant composite metric, unlike the background metric of the spacetime curvature formed only by the massive one. Without ignoring the effect of the small mass which also perturbates the background spacetime fabric, the new metric is formed in an inductive way in a curved space-time for two different massive sources

The line element in a general coordinates system,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu (1)$$

Where the space-time metric in linearized theory is nearly Cartesian that of perturbed over flat space-time [**13, 16, 17**],

$$g_{\mu\nu} = (\eta_{\mu\nu} + h_{\mu\nu}) \ (2)$$

The $h_{\mu\nu}$are assumed to be so small that all expressions can be linearised with respect to the $h_{\mu\nu}$and their derivatives of$h_{\mu\nu}$.

Choosing orthogonal coordinate system, the flat spacetime Minkowski metric [**16**] in matrix form,

$$[\eta_{\mu\nu}] = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ (3)$$

The linear perturbation in matrix form,

$$[h_{\mu\nu}] = \begin{pmatrix} h_{00} & 0 & 0 & 0 \\ 0 & h_{11} & 0 & 0 \\ 0 & 0 & h_{22} & 0 \\ 0 & 0 & 0 & h_{33} \end{pmatrix} \ (4)$$

Implies the perturbed metric in matrix representation,

$$[g_{\mu\nu}] = [\eta_{\mu\nu}] + [h_{\mu\nu}] \ (5)$$

The linearized representation of the metric in a curved space-time can be placed in a little tricky way as the multiplicative factor with initial metric that is the Minkowski metric.

$$[g_{\mu\nu}] = [\eta_{\mu\nu}][\mathbb{1} + [\eta_{\mu\nu}]^{-1}[h_{\mu\nu}]] \ (6)$$

Square bracket tells about the deformation of unit scale in curved spacetime corresponds to the flat space-time or Minkowski space-time, not using Einstein summation convention.

Where identity matrix, $\mathbb{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $[\eta_{\mu\nu}]^{-1}$ is the inverse of matrix, $[\eta_{\mu\nu}]$

$$[\eta_{\mu\nu}]^{-1}[h_{\mu\nu}] = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h_{00} & 0 & 0 & 0 \\ 0 & h_{11} & 0 & 0 \\ 0 & 0 & h_{22} & 0 \\ 0 & 0 & 0 & h_{33} \end{pmatrix} = \begin{pmatrix} -h_{00} & 0 & 0 & 0 \\ 0 & h_{11} & 0 & 0 \\ 0 & 0 & h_{22} & 0 \\ 0 & 0 & 0 & h_{33} \end{pmatrix} (7)$$

Implying equation (6) for two body system, where the fabric of background spacetime for the mass $'M'(source1)$is perturbed by the weak field self gravity of the mass $'m'(source2)$and considering the initial metric get changes as the same fashion in a multiplicative way so that new metric will be

$$[g_{\mu\nu}] = [\eta_{\mu\nu}][\mathbb{1} + [\eta_{\mu\nu}]^{-1}[h^{(1)}_{\mu\nu}]][\mathbb{1} + [\eta_{\mu\nu}]^{-1}[h^{(2)}_{\mu\nu}]] \ (8)$$

The word "multiplicative" refers here that we are not considering only linear perturbation rather it arises as nonlinear fashion. As one can think for consecutive Lorentz Transformations in Special Theory of Relativity.The new perturbed metric for two different massive sources is thus multiplication of one isolated background metric with the others metric deformation. The equation (8) is obvious in regard that offing any one of the sources from the system reproduces the corresponding metric for the existing single source in the linearized perturbation from post-Minkowskian theory.

3. The geodesic motion and perturbation of metric:

The geodesic motion from least action principle [**15, 16**] in a curved spacetime is

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx}{d\tau}^\sigma = 0 \ (9)$$

$$d\tau \rightarrow propertime, (x^\mu; x^0 = ct, x^1, x^2, x^3)$$

Light speed in vacuum, $c = 2.99 \times 10^8 meter.sec^{-1}$

The affine connection,

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}(\frac{\partial g_{\nu\sigma}}{\partial x^\rho} + \frac{\partial g_{\nu\rho}}{\partial x^\sigma} - \frac{\partial g_{\rho\sigma}}{\partial x^\nu}) \ (10)$$

For weak field approximation in nearly Cartesian coordinates system [**18**]

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}c^2\nabla g_{00} \ (11)$$

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}c^2\nabla h_{00} \ (12)$$

Equation of motion using Newtonian mechanics for freely falling body,

$$\frac{d^2x^i}{dt^2} = -\nabla\phi \ (13)$$

Gravitational potential, $\phi = -\frac{GM}{r}$

Gravitational constant, $G = 6.673 \times 10^{-11}\ meter^3 kg^{-1} \sec^{-2}$

$$\nabla\phi = -\frac{1}{2}c^2\nabla g_{00} \ (14)$$

In context of the Einstein field equations [13]

$$R_{\mu\nu} - g_{\mu\nu}R = -8\pi G T_{\mu\nu} \ (15)$$

The static isotropic metric solution of the equations is given by Schwarzschild [13,18]

$$ds^2 = -[1 + \frac{2\phi}{c^2}]c^2dt^2 + [1 + \frac{2\phi}{c^2}]^{-1}dr^2 + r^2d\theta + r^2\sin^2\theta d\varphi^2 \ (16)$$

The perturbed component from equations (14) and (16)

$$h_{00} = -\frac{2\phi}{c^2} \ (17)$$

Then from equation (17) the perturbations for two different masses M and m,

$$h_{00}^{(1)} = \frac{2GM}{c^2r} \ ; \ Inside\ the\ sphere, h_{00}^{(2)} = \frac{Gm}{c^2R^3}(3R^2 - r_0^2)$$

But for these two sources using equation (8) the perturbed metric has the component

$$g_{00} = -1 + (h_{00}^{(1)} + h_{00}^{(2)}) - h_{00}^{(1)}h_{00}^{(2)} \ (18)$$

Where h_{00} is dimensionless and equation (18) contains nonlinear multiplicative perturbation

$$|h_{00}^{(1)}| and |h_{00}^{(2)}| \ll 1 \ (19)$$

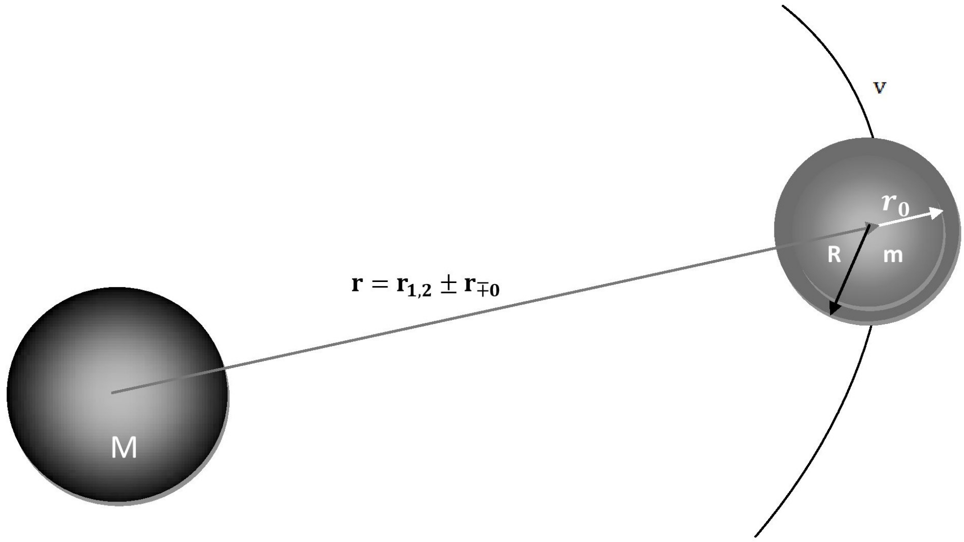

Fig.1

Rotation of star of mass *m* around SMBH of mass *M*

In our system Fig.1, $M \gg m$, $r \gg r_0$and$|h_{00}^{(1)}| \ll |h_{00}^{(2)}|$

Using equations (14) and (18) implies modification of gravity for binary systems as in Fig-1 and (Appendix-I)

$$| - \nabla\phi|_{r_1}| = | - \{\tfrac{GM}{r_1^2} - \tfrac{Gmr_0}{R^3} - \tfrac{G^2Mm}{c^2r_1^2R^3}(3R^2 - r_0^2) + \tfrac{2G^2Mmr_0}{c^2r_1R^3}\}| \quad (20)$$

The equation (18) is quite interesting as it is not giving the resultant gravitational field just as the Newtonian gravity rather it consists additional two terms

$(\frac{G^2Mm}{c^2r_1^2R^3}(3R^2 - r_0^2) - \frac{2G^2Mmr_0}{c^2r_1R^3})$, that gives birth the so-called modification over self gravity, where each individual field is enhanced by the other source.

At r_2 the field becomes

$$| - \nabla\phi|_{r_2}| = | - \{\tfrac{GM}{r_2^2} + \tfrac{Gmr_0}{R^3} - \tfrac{G^2Mm}{c^2r_2^2R^3}(3R^2 - r_0^2) - \tfrac{2G^2Mmr_0}{c^2r_2R^3}\}| \quad (21)$$

Now in case of geodesic motion we consider the path traced by the point mass 'm' i.e. the world line in the background of spacetime created by the mass$M \gg m$. But in real situation the mass has significant physical size and shape hence the geodesic path of the physical body can be described over world tube [**19**].

4. The motion through world tube:

Let us consider a so called static spherical massive body having significant size. The space-time curvature that builds gravitational field at the surface around the whole sphere is same and also the energy density. If there is little imbalance of energy density then there must be a force on the body that will lead the body along world tube.

Here our choice of r_0 is the radius of the spherical body of mass m which is placed on the background of spacetime fabric of the mass ". From equations (19) and (20) between two opposite elementary surfaces there manifest an imbalance in gravitational energy density [**20**] for each and every spherical shell elements embedded in the corresponding gravitational field. The imbalance in energy density for each pair of elementary surfaces is function of r and r_0 **.**

$$\frac{|\nabla\phi|^2|_{r_1}-|\nabla\phi|^2|_{r_2}}{8\pi G}=\frac{|\nabla\Phi|^2(r_0,r)}{8\pi G} \quad (22)$$

Here Φ is the assigned equivalent potential corresponds to the unbalanced energy density with corresponding field $-\nabla\Phi$ that pushes the elementary pair.

The Newtonian mechanics allows us to present the equation of motion for the shell element as

$$|\nabla\Phi|^2(r_0,r)=\frac{v^4}{r^2} \quad (23)$$

v is the rotational speed with radius of curvature $'r'$

5. The galactic rotation curve for Sun:

As an example we can consider a binary system of the two body extreme mass ratio problem we can choose the galactic super massive black hole (SMBH) to Sun

Solar distance from galactic centre, $r_1 \sim r_2 \sim 10^{20} meter$ [**21**]

For SMBH of our Milky Way galactic centre, $M = 4 \times 10^5 M\odot$[**22**]

For Sun, $R = R\odot$=6.957$\times 10^8 meter$[**23, 24**]

$m = M\odot$=1.988$\times 10^{30} kg$[**23, 24**]

$surfacegravity, \frac{Gm}{R^2}= g\odot$=2.74$\times 10^2 meter.sec^{-2}$[**24, 25**]

$\frac{GM}{r_{1,2}^2} \sim 10^{-15} meter.sec^{-2}$

$\frac{Gm}{R^2} = g\odot \sim 10^2 meter.sec^{-2}$

$\frac{G^2Mm}{c^2 r_{1,2}^2 R} \sim 10^{-22} meter.sec^{-2}$

$\frac{G^2Mm}{c^2 r_{1,2} R^2} \sim 10^{-10} meter.sec^{-2}$

Thus for distant star like the Sun we could ignore the first and third terms in the equations (20) and (21), the fourth term always dominating over the first term (the Newtonian term) only when $r > 10^{15} meter$. From equations (22) and (23) only considering the asymmetric part for whole object with rotational velocity v, we have

$$\frac{v^4}{r^2} = \frac{4GM\langle g_0^2 r_0\rangle}{c^2 r^2} \text{ ; Where } g_0 = \frac{Gmr_0}{R^3} \quad (24)$$

For spherical massive body here considering the Sun having uniform density, the average (appendix-II)

$$\langle g_0^2 r_0\rangle \sim \frac{1}{32} g\odot^2 R\odot \quad (25)$$

$$v^4 = \frac{GM}{8c^2} g\odot^2 R\odot \quad (26)$$

The rotational velocity of Sun around the galactic centre is computed as $v = 249 km.\sec^{-1}$, quite relevant as far as recent observation [**21**, **26, 27, 28, 29**], though IAU recommended rotational velocity is $220 km.\sec^{-1}$,[**26, 30**].

Conclusions and remarks:

The modified gravity from self gravity for two gravitating bodies is derived in terms of linearized Einstein metric. Here perturbation is introduced in multiplicative way for each of the masses; the result in this approach is satisfactory for two body dynamics especially when the Newtonian field is vanishingly small (where choosing suitable Lagrangian is out of scope in getting directly orbital equation of motion for the one body spacetime metric). Equation (25) the independence of distance to radial velocity or rotation curve for large distant galactic stars there is no need of theory for modification of Newtonian dynamics MOND by Milgrom [**4**] Further we could skip the dark matter concept in describing the violation of Newtonian gravity to describe the flat rotation curve [**1, 2**]. The widely reported mass of the Milky Way SMBH is $\sim 4 \times 10^6 M\odot$[31], observation is not free from the self gravity of the rotating star near the galactic centre, so preference is given to the lowest order measured value to compute the suns rotational velocity.

Further computing the rotation of the Sun, all the effects for other massive objects like other black holes or stars in the galaxy are discarded, because the equation (18) tells self gravity effect is quadratic in linear perturbations and it is obvious the major contribution comes from SMBH. More over it is considered that the mass distribution in large scale around the Sun is homogeneous and only the anisotropy comes from the galactic nucleus.

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DATA AVAILABILITY

This study used the data from NASA publicly available at https://nssdc.gsfc.nasa.gov/planetary/factsheet/sunfact.html and http://hyperphysics.phy-astr.gsu.edu/hbase/Solar/sun.html.For SMBH of our Milky Way galactic centre, mass is $4{\times}10^5 M\odot (Doeleman, S., Weintroub, J.,$ Rogers, A.*et al,* 2008, **455,**78-80)DOI:10.1038/nature07245.

The Solar distance from galactic centre, $\sim 10^{20} meter$ (Sch¨onrich, R., 2012, MNRAS, 14) https://arxiv.org/abs/1207.3079. and $\sim 4.3{\times}10^6 M\odot$(Eckart, A., et al. 2017).

The rotational velocity of Sun around the galactic centre available at https://www.nsf.gov/discoveries/disc_summ.jsp?cntn_id=114090&org=NSF, (Schönrich, R., 2012, MNRAS, 14) https://arxiv.org/abs/1207.3079; (Reid, M. J. et al. 2009, Astrophys.J.700:137-148, 13) https://arxiv.org/pdf/0902.3913.pdf ; Ghez, A. M., Salim, S., Weinberg, N. N., et al. 2008, ApJ, 689:1044 https://arxiv.org/pdf/0808.2870v1.pdf (Bovy. J., et al, 2012, ApJ 759,131) http://arxiv.org/abs/1209.0759v1. The data used to compute the rotational speed for the Sun and it is the main them for the proposed modified gravity by the corresponding author.

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Appendix-I

From Fig-1, we have

$$\mathbf{r}_1 = \mathbf{r} - \mathbf{r}_{-0}$$

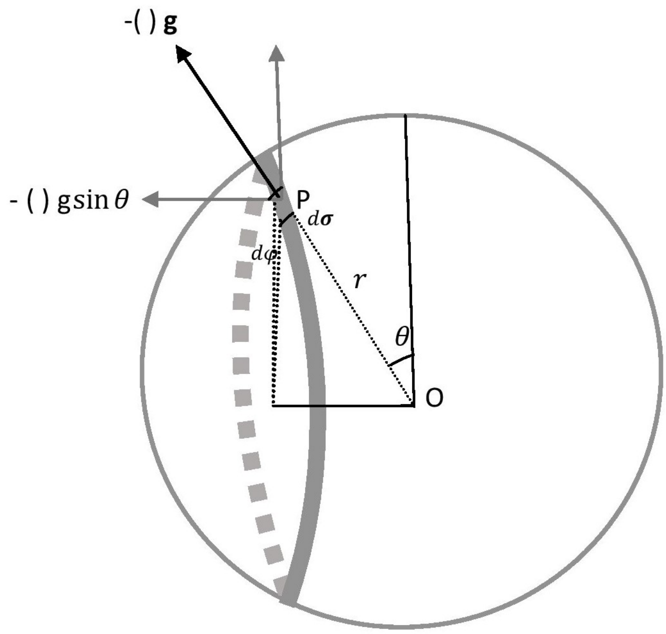
$$\mathbf{r}_2 = \mathbf{r} + \mathbf{r}_{+0}$$

$$\text{At fixed } \mathbf{r},\ \nabla|_{r_1} = \hat{\mathbf{r}}\frac{\partial}{\partial r_1} = -\hat{\mathbf{r}}\frac{\partial}{\partial r_{-0}}\ ,$$

$$\text{and at fixed } \mathbf{r},\ \nabla|_{r_2} = \hat{\mathbf{r}}\frac{\partial}{\partial r_2} = \hat{\mathbf{r}}\frac{\partial}{\partial r_{+0}}\ ,$$

Appendix-II

In arriving to the average value as in equation (25) of the self gravity for a homogeneous spherical body, let us follow the steps using the following Figure.



The effective term as in equations (20) and (21)
$-\frac{2G^2Mm}{c^2 r_{1.2} r^2} = -(\frac{2GM}{c^2 r_{1.2}})g = -f(r_{1.2})$g
Surface element at point, P on the annular ring, $d\sigma = rd\theta . r\cos\theta . d\phi$

Due to asymmetry only non-vanishing term of gravitational field that contributes to unbalanced force on the hemispherical shell

$$\int g \sin\theta.d\sigma{=}gr^2 \int_0^{\frac{\pi}{2}} \sin\theta.cos\theta.d\theta. \int_0^{2\pi} d\phi{=}g\pi r^2$$

Area of hemispherical shell is$2\pi r^2$

The average force field $g_{av}{=}\ \frac{1}{2}g$

Now considering the homogeneous matter density for spherical body 'g' is proportional to radius 'r'.

Taking r=kg_{av}; k is constant, for the whole sphere,

$$\langle g_{av}^2 r\rangle = \langle \tfrac{k}{8}g^3\rangle = \frac{k}{8}\frac{\int_0^{g_0} g^3.dg}{\int_0^{g_0} dg}{=}\ \frac{1}{32}g_0^2 R$$

where g_0 and R, the surface gravity and radius of the spherical body respectively

To get simplistic realisation of the model one can imagine the below-

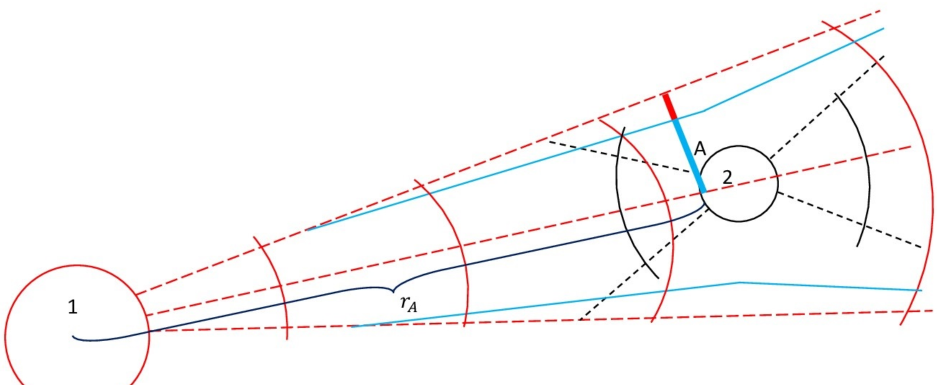


Fig.3

Schematic diagram of 2 dimensional space for two different sources 1 and 2.

The potential/field at r_A is less in absence of source 2, perturbation **(BLUE LINES)** due to presence of source 2 at r_A indicates magnification of local field (contraction of space). As described by the modification of gravity due to self gravity of the 2nd body in a binary system.

The dynamics especially geodesic motion of mass 2 is now governed by the perturbed spacetime!