SOME RESULTS OF WATSON, PLANCHEREL TYPE INTEGRAL TRANSFORMS RELATED TO THE HARTLEY, FOURIER CONVOLUTIONS AND APPLICATIONS

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ABSTRACT. In this work, we study the Watson-type integral transforms for the convolutions related to the Hartley and Fourier transformations. We establish necessary and sufficient conditions for these operators to be unitary in the $L_2(\mathbb{R})$ space and get their inverse represented in the conjugate symmetric form. Furthermore, we also formulated the Plancherel-type theorem for the aforementioned operators and prove a sequence of functions that converge to the original function in the defined $L_2(\mathbb{R})$ norm. Next, we study the boundedness of the operators (T_k). Besides, showing the obtained results, we demonstrate how to use it to solve the class of integro-differential equations of Barbashin type, the differential equations and the system of differential equations. And there are numerical examples given to illustrate these.

CONTENTS

1. INTRODUCTION

One of the integral transforms studied by many mathematicians is the Watson integral transform. This transform is built for the Mellin convolution (see [25, 27]) with the kernel $k(xy)$ in the following form:

$$f(x) \longmapsto g = \left(f \underset{M}{*} k\right), \text{ where } g(x) = \int\limits_0^{+\infty} k(xy)f(y)dy,\ k(xy) \text{ is the kernel of transformation.}$$

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Furthermore, we can study the Watson-type integral transform as follows

$$f(x) \longmapsto g = D\,(f * k)\,(x), \tag{1.1}$$

where D is an arbitrary operator and k is a known function. In recent years, there have been some results of integral transformations for convolutions involving Fourier sine, Fourier cosine and Kontorovich-Lebedev transformations in case D is the differential operator of order 2 or $2n$. From there, the authors use these results to solve in a closed form some classes of differential equations, Parabolic equations (see [6, 8, 14, 18, 20, 22, 26]) in the space $L_1(\mathbb{R}_+)$. These above convolutions are also used to solve in a closed form some classes of integral equations of Fredholm and Toeplitz-Hankel type and study the boundedness of the solution (see [11, 19, 21, 23, 28]). As we know, the integro-differential equation of Barbashin type was first introduced by E. A. Barbashin in 1957 (see [4]) and studied in [2, 3], which has the following form

$$\frac{\partial f(t,s)}{\partial t} = \mathcal{C}(t,s)f(t,s) + \int\limits_a^b \mathcal{K}(t,s,\rho)f(t,\rho)d\rho + g(t,s), \tag{1.2}$$

where $\mathcal{C}(t,s), g(t,s)$ are the given functions, $\mathcal{K}(t,s,\rho)$ is the kernel of the equation and f is the unknown function. The equation (1.2) has been applied in many fields such as mathematical physics, radiation propagation, mathematical biology and transport problems, e.g more detail in (see [2], Chapter 4. §19. p421−447). One of the characteristics of the equation (1.2) is that studying solvability of the equation is heavily dependent on the kerner $\mathcal{K}(t,s,\rho)$ of the equation. For example, we can use the Cauchy integral operator to study the equation in case the kerner does not depend on t. In some other cases, we need to use the partial integral operator to study the equation (see [2, 3]). Until now, the solution of the equation (1.2) is still an open problem in case $\mathcal{K}(t,s,\rho)$ is an arbitrary general kerner. On the other hand, if we call $\mathcal{A}$ as the operator defined by $\mathcal{A} := \dfrac{\partial}{\partial t} - \mathcal{C}(t,s)\mathcal{I}$, where $\mathcal{I}$ is the identity operator, then the equation (1.2) is written in the following form

$$\mathcal{A}f(t,s) = \int\limits_a^b \mathcal{K}(t,s,\rho)f(t,\rho)d\rho + \varphi(t,s). \tag{1.3}$$

The main contents of the paper are presented in sections 3, 4 and 5. In section 3, by choosing the operator $D = \left(\mathcal{I} - \dfrac{d^2}{dx^2}\right)$ and using the scheme (1.1), we build Watson-type integral transforms for the convolutions defined by the formula (2.2), (2.4). These convolutions are related to the Hartley, Fourier integral transformations studied in [9, 24]. We establish necessary and sufficient conditions for these operators to be unitary in the space $L_2(\mathbb{R})$ and the inverse symmetric transform, which are the contents of the Theorems 3.1 and 3.2. In [13], the authors proved Plancherel's Theorem in general form for the Fourier transform on $L_2(\mathbb{R})$ and confirmed that this transformation is unitary and this result is also still true for Hartley transforms. The main contents of section 4 are studying the Plancherel-type theorems of the operators presented in Section 3, by using the technique in [8] specifically the study of approximation in the norm of space $L_2(\mathbb{R})$, for the operator (T_k) constructed in the formula (3.13). Specifically, if suppose that the image function $\Psi(x) = (T_k f)\,(x)$ and the original function $f(x)$, then it can approximate to sequences of functions in the space $L_2(\mathbb{R})$ that converge normally in $L_2(\mathbb{R})$ to an arbitrary function $f(x)$, which also belongs to $L_2(\mathbb{R})$. By using Risez's interpolation Theorem [17], we study the boundedness of operator (T_k) from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$ with $\frac{1}{p}+\frac{1}{q}=1$ and $1\le p\le 2$, which are the contents of the Theorems 4.1 and 4.2.

At the end of this paper, in section 5, we use some integral transformation techniques presented in section 3 and some other results in [15, 9, 24, 12] to solve in a closed form some classes of integro−differential equation of Barbashin type (5.1), differential equations (5.9) and system of differential equations (5.12). Results obtained here are to establish the explicit formulas and a space of solutions for the above problems (see Theorems 5.1, 5.2, 5.3 and 5.4). After theorems, we construct numerical examples to illustrate the obtained results. It should be further clarified that the use of convolution and polyconvolution techniques related to Hartley and Fourier integral transforms to solve in a closed form some classes of integro−differential equation of Barbashin type is a new idea in the theory. We have not found any previous results using this method.

2. Recalling some results of the Hartley, Fourier convolutions

In this section, we briefly recall some useful results of the Hartley, Fourier convolutions and the Hartley convolution, which will be used in the next sections of the article.

The generalized convolution for the Hartley−Fourier integral transforms is of the form (see [24]). Let $f,g\in L_1(\mathbb{R})$ then $\left(f\underset{1}{*}g\right)\in L_1(\mathbb{R})$ and

$$H_1\left(f\underset{1}{*}g\right)(y):=(H_1f)(y)(Fg)(y),\ y\in\mathbb{R},\tag{2.1}$$

where

$$\left(f\underset{1}{*}g\right)(x):=\frac{1}{2\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}g(y)\big[f(x+y)+f(x-y)+if(-x-y)-if(-x+y)\big]\,dy,\ x\in\mathbb{R}.\tag{2.2}$$

In (p. 364, [9]), the authors used integral transformations

$$(T_1f)(x)=\frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{+\infty}\cos\left(xy+\frac{\pi}{4}\right)f(y)dy$$

and

$$(T_2f)(x)=\frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{+\infty}\sin\left(xy+\frac{\pi}{4}\right)f(y)dy$$

to study the convolutions $\left(f\underset{T_1}{*}g\right)$ and $\left(f\underset{T_2}{*}g\right)$. It's obvious that, we get: $\cos\left(xy+\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\,\mathrm{cas}(-xy)$ and $\sin\left(xy+\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\,\mathrm{cas}(xy)$. Therefore, $(T_1f)\equiv(H_2f)$ and $(T_2f)\equiv(H_1f)$, where $H_{\left\{\substack{1\\2}\right\}}$ is the Hartley transform defined by (2.5). Consequently, the Theorem 3.5 (p 337 in [9]) and the Theorem 3.14 (p 378 in [9]) can be rewritten in the following form.

Assume that f,g belong to $L_1(\mathbb{R})$ then $\left(f\underset{2}{*}g\right)\in L_1(\mathbb{R})$ and

$$H_{\left\{\substack{1\\2}\right\}}\left(f\underset{2}{*}g\right)(y)=\left(H_{\left\{\substack{1\\2}\right\}}f\right)(y)\ \left(H_{\left\{\substack{1\\2}\right\}}g\right)(y),\ y\in\mathbb{R},\tag{2.3}$$

where

$$\left(f \underset{2}{*} g\right)(x) := \frac{1}{2\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} f(y)\big[g(x+y)+g(x-y)+g(-x+y)-g(-x-y)\big]dy,\ x \in \mathbb{R}. \tag{2.4}$$

Notably, the results for the Hartley convolutions was studied in $\mathbb{R}^n$ by (see [1]). The Hartley transform was proposed as an alternative to the Fourier transform by R. V. L. Hartley in 1942. Compared to the Fourier transform, the Hartley transform has the advantages of transforming real functions to real functions (as opposed to requiring complex numbers) and has following form (see [5, 16, 25]).

$$\left(H_{\left\{\substack{1\\2}\right\}} f\right)(y) := \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} f(x)\ \operatorname{cas}(\pm xy)dx,\ y \in \mathbb{R}. \tag{2.5}$$

And the inverse transform is of the form

$$f(x) := \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \left(H_{\left\{\substack{1\\2}\right\}} f\right)(y)\ \operatorname{cas}(\pm xy)dy,\ x \in R. \tag{2.6}$$

The Fourier (F) integral transforms were studied in (see [5, 16, 25]).

$$(Ff)(y) := \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-ixy} f(x)dx,\ y \in R, \tag{2.7}$$

where: $\operatorname{cas}(xy) = \cos(xy) + \sin(xy) = \dfrac{1+i}{2} e^{-ixy} + \dfrac{1-i}{2} e^{ixy}$ and

$$e^{-ixy} = \frac{1+i}{2} \operatorname{cas}(xy) + \frac{1-i}{2} \operatorname{cas}(-xy). \tag{2.8}$$

$$(Ff)(y) = H_{\left\{\substack{1\\2}\right\}} \left(\frac{1-i}{2} f(\pm x) + \frac{1+i}{2} f(\mp x)\right)(y). \tag{2.9}$$

$$\left(H_{\left\{\substack{1\\2}\right\}} f\right)(y) = F\left(\frac{1 \pm i}{2} f(x) + \frac{1 \mp i}{2} f(-x)\right)(y). \tag{2.10}$$

3. The Watson-type Theorems

In this section, we study the Watson-type integral transform for the convolutions were defined in the formulas (2.2) and (2.4) in the $L_2(\mathbb{R})$ space. We establish the necessary and sufficient conditions for these operators to be unitary. Moreover, the inverse transformations can be represented in the conjugate symmetric form.

Lemma 3.1. *Suppose that, $f, g \in L_2(\mathbb{R})$ then $\left(f \underset{1}{*} g\right), \left(f \underset{2}{*} g\right)$ belong to the space $L_2(\mathbb{R})$ and were determined respectively by* (2.2),(2.4). *We get the factorization equalities* (2.1), (2.3) *and the Parseval's equality is as follows*

$$\left(f \underset{1}{*} g\right)(x) = H_1\Big((H_1 f)(y)(Fg)(y)\Big)(x), \tag{3.1}$$

$$\left(f \underset{2}{*} g\right)(x) = H_{\{\frac{1}{2}\}}\Big(\left(H_{\{\frac{1}{2}\}}f\right)(y)\left(H_{\{\frac{1}{2}\}}g\right)(y)\Big)(x). \tag{3.2}$$

Proof.

$$\begin{aligned}
\int_{-\infty}^{+\infty}\left|\left(f \underset{1}{*} g\right)(x)\right|^2 dx &\leq \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\left|\, g(y)\Big[f(x+y)+f(x-y)+if(-x-y)-if(-x+y)\Big]\right|^2 dy\right)dx \\
&\leq \frac{4}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|g(y)|^2\Big[|f(x+y)|^2+|f(x-y)|^2+|f(-x-y)|^2+|f(-x+y)|^2\Big]\,dydx \\
&= 2\sqrt{\frac{2}{\pi}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|g(y)|^2|f(x-y)|^2\,dxdy + 2\sqrt{\frac{2}{\pi}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|g(y)|^2|f(x+y)|^2\,dxdy
\end{aligned}$$

Since f, g belong to the $L_2(\mathbb{R})$ space, using Fubini's Theorem and the variable transformation formula for $u = x-y;\ u = x+y$, we obtain

$$\int_{-\infty}^{+\infty}\left|\left(f \underset{1}{*} g\right)(x)\right|^2 dx \leq 4\sqrt{\frac{2}{\pi}}\left(\int_{-\infty}^{+\infty}|g(y)|^2dy\right)\left(\int_{-\infty}^{+\infty}|f(u)|^2du\right) < +\infty,$$

On the other hand, the proof of factorization equality of convolution $\left(\cdot \underset{1}{*} \cdot\right)$ in $L_2(\mathbb{R})$ space is similar to in the $L_1(\mathbb{R})$ and combining the unitary property of the Hartley transform, we obtain equality (3.1) (refer Lemma 4.2 in [24]).

Similarly, using the same above method to prove the convolution (2.4), we will get the equalities (2.3), (3.2). The proof of lemma is completed. □

Lemma 3.2. *Suppose that, $h \in L_2(\mathbb{R})$ is a given function a satisfying the following condition $(A_1):\ (1+y^2)|(Fh)(y)| < +\infty,\ y \in \mathbb{R}$ and $f \in L_2(\mathbb{R})$, then*

$$H_1\bigg(\left(1-\frac{d^2}{dx^2}\right)\left(f \underset{1}{*} h\right)(x)\bigg)(y) = (1+y^2)(Fh)(y)(H_1 f)(y),\ y \in \mathbb{R}. \tag{3.3}$$

Proof. It is well-known that $\xi(y),\ y\xi(y),\ \ldots,\ y^n\xi(y) \in L_2(\mathbb{R})$ if and only if $F(\xi)(x),\ \frac{d}{dx}(F\xi)(x),\ \ldots,\ \frac{d^n}{dx^n}(F\xi)(x)$ belong to $L_2(\mathbb{R})$ following (Theorem 68, p.92, [25]). Combined with formulas (2.9), (2.10) the above confirmation is still true for the Hartley $H_{\{\frac{1}{2}\}}$ transform, which means $f(y), yf(y), y^2f(y) \in L_2(\mathbb{R})$ if and only if $\left(H_{\{\frac{1}{2}\}}f\right)(x),\ \frac{d}{dx}\left(H_{\{\frac{1}{2}\}}f\right)(x),\ \frac{d^2}{dx^2}\left(H_{\{\frac{1}{2}\}}f\right)(x)$ belong to $L_2(\mathbb{R})$. Moreover, we have $\frac{d}{dx}\operatorname{cas}(\pm xy) = -y\operatorname{cas}(\mp xy)$. Furthermore, since $H_1(f(\pm t)) = H_2(f(\mp t))$, then

$$\begin{aligned}
\frac{d^2}{dx^2}\left(H_{\{\frac{1}{2}\}}f\right)(x) &= \frac{1}{\sqrt{2\pi}}\frac{d^2}{dx^2}\bigg(\int_{-\infty}^{+\infty}f(y)\operatorname{cas}(\pm t)dy\bigg) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(y)(-y)^2\operatorname{cas}(\pm xy)\,dy = H_{\{\frac{1}{2}\}}(-y^2f(y))(x). \qquad (3.4)
\end{aligned}$$

On the other hand $f(y), yf(y), y^2f(y) \in L_2(\mathbb{R})$, using the formula (3.4), we have

$$\left(1-\frac{d^2}{dx^2}\right)\left(H_{\{\frac{1}{2}\}}f(y)\right)(x) = H_{\{\frac{1}{2}\}}\Big((1+y^2)f(y)\Big)(x) \in L_2(\mathbb{R}). \tag{3.5}$$

Thanks to the formula (3.1) in Lemma 3.1 and (3.5), we obtain

$$\begin{aligned}\left(1-\frac{d^2}{dx^2}\right)\left(f \underset{1}{*} h\right)(x) &= \left(1-\frac{d^2}{dx^2}\right)H_1\Big((H_1f)(y)(Fh)(y)\Big)(x)\\ &= H_1\Big((1+y^2)(H_1f)(y)(Fh)(y)\Big)(x)\end{aligned}$$

According to the condition (A_1): $(1+y^2)(Fh)(y) < +\infty$ and $(H_1f) \in L_2(\mathbb{R})$, which implies that $(1+y^2)(Fh)(y)(H_1f)(y) \in L_2(\mathbb{R})$. And

$$\left(1-\frac{d^2}{dx^2}\right)\left(f \underset{1}{*} h\right)(x) = H_1\Big((1+y^2)(H_1f)(y)(Fh)(y)\Big)(x) \in L_2(\mathbb{R}). \tag{3.6}$$

In addition, the Plancherel's Theorem for the Hartley transform is unitary in the space $L_2(\mathbb{R})$, which means $H_{\{\frac{1}{2}\}} \times H_{\{\frac{1}{2}\}} = \mathcal{I}$. From the formula (3.6), we obtain
$H_1\left(\left(1-\frac{d^2}{dx^2}\right)\left(f \underset{1}{*} h\right)\right)(y) = (1+y^2)(H_1f)(y)(Fh)(y),\ y \in \mathbb{R}.$

The proof of lemma is completed. □

Next, we present the Watson-type integral transform that allows the convolution defined by the formula (2.2). If we fix a function, say $g \equiv h(x)$, which h is a known function, and let the remaining function f vary in certain function spaces, then we get an integral operator of the convolution type. We consider the following operator

$$\begin{aligned}&T_h : L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R})\\ &\qquad f \longmapsto \varphi = T_h\left(f \underset{1}{*} h\right) = D\left(f \underset{1}{*} h\right),\end{aligned}$$

where D is the second order differential operator defined as $D := \left(\mathcal{I} - \frac{d^2}{dx^2}\right)$ and image

$$\begin{aligned}\varphi(x) = (T_hf)(x) &= \left(1-\frac{d^2}{dx^2}\right)\left(f \underset{1}{*} h\right)(x)\\ &= \frac{1}{2\sqrt{2\pi}}\left(1-\frac{d^2}{dx^2}\right)\left\{\int_{-\infty}^{+\infty} h(y)\Big[f(x+y)+f(x-y)+if(-x-y)-if(-x+y)\Big]dy\right\},\ x \in \mathbb{R}.\end{aligned} \tag{3.7}$$

Theorem 3.1. *Suppose that $h \in L_2(\mathbb{R})$ is a given function. Then, the condition (A_2) : $|(Fh)(y)| = \frac{1}{1+y^2}$ with $y \in \mathbb{R}$ is the necessary and sufficient one for operator (T_h) given by formula (3.7) to be unitary in the $L_2(\mathbb{R})$ space. Moreover, the inverse operator of*

(T_h) has a symmetric form and is represented by the formula

(3.8)

$$f(x) = (T_{\bar{h}}\varphi)(x) = \left(1 - \frac{d^2}{dx^2}\right)\left(\varphi \underset{1}{*} \bar{h}\right)(x)$$
$$= \frac{1}{2\sqrt{2\pi}}\left(1 - \frac{d^2}{dx^2}\right)\left\{\int\limits_{-\infty}^{+\infty} \bar{h}(y)\Big[\varphi(x+y) + \varphi(x-y) + i\varphi(-x-y) - i\varphi(-x+y)\Big]dy\right\},\ x \in \mathbb{R},$$

where $\bar{h}$ is a complex conjugate of h function.

Proof. By the (A_2) condition, $(1+y^2)(Fh)(y)$ is a bounded function, and $f \in L_2(\mathbb{R})$ implies that $(H_1 f)(y) \in L_2(\mathbb{R})$. Thus, $(1+y^2)(Fh)(y)(H_1 f)(y)$ is a function that belongs to the $L_2(\mathbb{R})$ space. Using the formula (3.6) in Lemma 3.2, we obtain

(3.9)

$$\varphi(x) = (T_h f)(x) = \left(1 - \frac{d^2}{dx^2}\right)\left(f \underset{1}{*} h\right)(x) = H_1\Big((1+y^2)(Fh)(y)(H_1 f)(y)\Big)(x) \in L_2(\mathbb{R}).$$

According to the formula (3.9) and in the $L_2(\mathbb{R})$ space then $\|H_{\{\frac{1}{2}\}} f\|_{L_2(\mathbb{R})} = \|f\|_{L_2(\mathbb{R})}$, combined with (A_2) condition and we have the following evaluation in the $L_2(\mathbb{R})$ space

(3.10)

$$\|\varphi\|_{L_2(\mathbb{R})} = \|T_h\ f\|_{L_2(\mathbb{R})} = (1+y^2)|(Fh)(y)|\ \|(H_1 f)(y)\|_{L_2(\mathbb{R})} = \|(H_1 f)(y)\|_{L_2(\mathbb{R})} = \|f\|_{L_2(\mathbb{R})}.$$

This means that (T_h) is an isometric transformation or unitary in the $L_2(\mathbb{R})$ space. Using the Plancherel's Theorem for the (H_1) transformation and from the expression (3.9), we obtain

$$(H_1\varphi)(y) = H_1(T_h f)(y) = (1+y^2)(Fh)(y)(H_1 f)(y) \in L_2(\mathbb{R}). \tag{3.11}$$

From the (A_2) condition and the equality (3.11), we obtain

$$(H_1 f)(y) = (1+y^2)(F\bar{h})(y)(H_1\varphi)(y) \in L_2(\mathbb{R}), \tag{3.12}$$

where $\bar{h}$ is a complex conjugate of h function. By the same argument as in the above, we obtain

$$f(x) = H_1\Big((1+y^2)(F\bar{h})(y)(H_1\varphi)(y)\Big)(x) \in L_2(\mathbb{R}).$$

Using the formulas (3.6) and (3.1) consecutively, we have

$$f(x) = \left(1 - \frac{d^2}{dx^2}\right) H_1\Big((F\bar{h})(y)(H_1\varphi)(y)\Big)(x)$$
$$= \left(1 - \frac{d^2}{dx^2}\right)\left(\varphi \underset{1}{*} \bar{h}\right)(x)$$
$$= \frac{1}{2\sqrt{2\pi}}\left(1 - \frac{d^2}{dx^2}\right)\left\{\int\limits_{-\infty}^{+\infty} \bar{h}(y)\Big[\varphi(x+y) + \varphi(x-y) + i\varphi(-x-y) - i\varphi(-x+y)\Big]\ dy\right\}$$
$$\equiv (T_{\bar{h}}\varphi)(x).$$

Sufficient condition: Suppose that the operator (T_h) mapping $f(x) \mapsto (T_h f)(x) \equiv \varphi(x)$ determined by (3.7) with the inverse operator (3.8). We need to show that h satisfies condition (A_2), which means

$$|(Fh)(y)| = \frac{1}{1+y^2},\ y \in \mathbb{R}.$$

Indeed, since $(T_h f) \equiv \varphi(x)$ is unitary in the $L_2(\mathbb{R})$ space and $\varphi(x)$ can be rewritten in the form (3.9), we obtain

$$||H_1\varphi||_{L_2(\mathbb{R})} = ||H_1\left(T_h f\right)||_{L_2(\mathbb{R})} = ||\varphi||_{L_2(\mathbb{R})},$$

equivalent to

$$\begin{aligned}\|H_1\left((1+y^2)(Fh)(y)(H_1f)(y)\right)\|_{L_2(\mathbb{R})} &= (1+y^2)|(Fh)(y)|\ \|(H_1f)(y)\|_{L_2(\mathbb{R})}\\ &= \|H_1f\|_{L_2(\mathbb{R})} = ||f||_{L_2(\mathbb{R})},\ \forall f\in L_2(\mathbb{R}).\end{aligned}$$

This shows that there exists a multiplication operator of the form $\mathcal{M}_\Theta(.)$ defined by $\mathcal{M}_\Theta(f)(y) := \Theta(y).f(y)$, where $\Theta(y) = (1+y^2)|(Fh)(y)|$. The above expression can be rewritten

$$\|H_1f\|_{L_2(\mathbb{R})} = \|\mathcal{M}_\Theta(H_1f)\|_{L_2(\mathbb{R})},\ \forall f\in L_2(\mathbb{R}).$$

It means $\mathcal{M}_\Theta(.)$ is unitary in the $L_2(\mathbb{R})$ space, and this happens if and only if $(1+y^2)|(Fh)(y)| \equiv 1,\ \forall y\in\mathbb{R}$, implies that the function h must satisfy the condition (A_2).

The proof of theorem is completed. □

Lemma 3.3. *(Follow* [10]*) Let $\omega\in L_1(\mathbb{R}_+), \varphi\in L_1(\mathbb{R})$ then $\left(\omega \underset{4}{*} \varphi\right)(x)\in L_1(\mathbb{R})$ and get the factorization equality*

$$H_1\left(\omega \underset{4}{*} \varphi\right)(y) = (F_c\omega)(|y|)(H_1\varphi)(y), y\in\mathbb{R},$$

with $\left(\omega \underset{4}{*} \varphi\right)(x) := \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_0^{+\infty}\left[\omega(x+u)+\omega(x-u)\right]\varphi(u)du,\ x\in\mathbb{R}$. *This result is also true for in case $\omega\in L_2(\mathbb{R}_+), \varphi\in L_2(\mathbb{R})$, where F_c determined by the formula* (5.5).

Remark 1. *We can replace the (A_2) condition in the Theorem* (3.1) *with a weaker condition denoted by $(A_2^{'})$ and state the following:*

Suppose that h is a given function belonging to (L_2) space and satisfying the following condition $(A_2^{'})$: $0 < c_1 \leqslant (1+y^2)|(Fh)(y)| \leqslant c_2 < +\infty$. We have the following assertions
(i) $c_1\|f\|_{L_2(\mathbb{R})} \leqslant \|T_hf\|_{L_2(\mathbb{R})} \leqslant c_2\|f\|_{L_2(\mathbb{R})}$
(ii) *The inverse operator of (T_k) have the form*

$$f(x) = \left(T_\omega\varphi\right)(x) = \left(1-\frac{d^2}{dx^2}\right)\left(\omega \underset{4}{*} \varphi\right)(x).$$

Indeed, from (3.10) *and the $(A_2^{'})$ condition, we obtain*

$$\begin{aligned}c_1\|f\|_{L_2(\mathbb{R})} &= c_1\|H_1f\|_{L_2(\mathbb{R})} \leqslant \|(1+y^2)(Fh)(y)(H_1f)(y)\|_{L_2(\mathbb{R})}\\ &= (1+y^2)|(Fh)(y)|\ \|(H_1f)(y)\|_{L_2(\mathbb{R})} = \|(T_hf)\|_{L_2(\mathbb{R})} \leqslant c_2\|H_1f\|_{L_2(\mathbb{R})} = c_2\|f\|_{L_2(\mathbb{R})}.\end{aligned}$$

Furthermore, from the $(A_2^{'})$ condition, we also get

$$0 < \frac{1}{c_2(1+y^2)} \leqslant \frac{1}{(1+y^2)^2|(Fh)(y)|} \leqslant \frac{1}{c_1(1+y^2)}.$$

Therefore $\dfrac{1}{(1+y^2)^2(Fh)(y)}$ *belongs to the $L_2(\mathbb{R}_+)$ space, and this shows that there exists a function $\omega\in L_2(\mathbb{R}_+)$ such that* $(F_c\omega)(|y|) = \dfrac{1}{(1+y^2)^2(Fh)(y)} \in L_2(\mathbb{R}_+)$. *From the* *formula* (3.12) *and combining with the* (A_2') *condition and Lemma 3.3, we obtain*

$$\begin{aligned}(H_1 f)(y) &= \frac{1}{(1+y^2)(Fh)(y)}(H_1\varphi)(y) = (1+y^2)\frac{1}{(1+y^2)^2(Fh)(y)}(H_1\varphi)(y)\\ &= (1+y^2)(F_c\omega)(|y|)(H_1\varphi)(y) = (1+y^2)H_1\left(\omega \underset{4}{*} \varphi\right)(y).\end{aligned}$$

Continuing to use Plancherel's Theorem for the (H_1) *transformation and combining with the formula* (3.3), *we get the following inverse operator*

$$\begin{aligned}f(x) = (T_\omega\varphi)(x) &= H_1\Big((1+y^2)\big(H_1\left(\omega \underset{4}{*} \varphi\right)(y)\big)\Big)(x) = \left(1-\frac{d^2}{dx^2}\right)H_1\Big(H_1\left(\omega \underset{4}{*} \varphi\right)(y)\Big)(x)\\ &= \left(1-\frac{d^2}{dx^2}\right)\left(\omega \underset{4}{*} \varphi\right)(x).\end{aligned}$$

Another remarkable point is that the inverse operator obtained here does not preserve the symmetry.

An example is given below to illustrate the above result.

Example 3.1. *Let we choose* $h(x) = \sqrt{\frac{\pi}{2}}e^{-|x|} \in L_2(\mathbb{R})$, *then*

$$\begin{aligned}(Fh)(y) &= \sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ixy}e^{-|x|}dx = \frac{1}{2}\int_{-\infty}^{+\infty} e^{(-|x|-ixy)}dx\\ &= \frac{1}{2}\left(\int_{-\infty}^{0} e^{(1-iy)x}dx + \int_{0}^{+\infty} e^{-(1+iy)x}dx\right)\\ &= \frac{1}{2}\left(\frac{1}{1-iy}+\frac{1}{1+iy}\right) = \frac{1}{1+y^2}.\end{aligned}$$

Thus, $|(Fh)(y)|$ *satisfies the condition* (A_2). *Then*

$$(F_c\omega)(|y|) = \frac{1}{(1+y^2)^2(Fh)(y)} = \frac{1}{1+y^2} \in L_2(\mathbb{R}_+).$$

So, there exists a function $\omega(x) = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{+\infty}\frac{1}{1+y^2}\cos(xy)dy = \sqrt{\frac{\pi}{2}}e^{-x} \in L_2(\mathbb{R}_+)$.

Next, we study the Watson-type integral transform for the Hartley convolution defined by the formula (2.4).

Let the operators (T_k) be defined by mapping

$$\begin{aligned}T_k : L_2(\mathbb{R}) &\longrightarrow L_2(\mathbb{R})\\ f &\longmapsto \Psi \equiv T_k\left(f \underset{2}{*} k\right) = D\left(f \underset{2}{*} k\right),\end{aligned}$$

where $D = \left(1 - \frac{d^2}{dx^2}\right)$ and function Ψ is determined by the formula

$$\begin{aligned} \Psi(x) = (T_k f)(x) &= \left(1 - \frac{d^2}{dx^2}\right) \left(f \underset{2}{*} k\right)(x) \\ &= \frac{1}{2\sqrt{2\pi}} \left(1 - \frac{d^2}{dx^2}\right) \left\{ \int_{-\infty}^{+\infty} f(y) \Big[k(x+y) + k(x-y) + k(-x+y) - k(-x-y) \Big] dy \right\}, \ x \in \mathbb{R}. \end{aligned} \tag{3.13}$$

Theorem 3.2. *Suppose that k is a given function belonging to $L_2(\mathbb{R})$ space and satisfying the condition (A_3) : $(1+y^2)\left|\left(H_{\{\frac{1}{2}\}}k\right)(y)\right| = 1$ with $y \in \mathbb{R}$. Then, the (A_3) condition is the necessary and sufficient one for (T_k) to be an unitary operator in the $L_2(\mathbb{R})$ space. Moreover, the inverse operator of (T_k) can be represented in the conjugate symmetric form.*

$$\begin{aligned} f(x) = \left(T_{\bar{k}} \Psi\right)(x) &= \left(1 - \frac{d^2}{dx^2}\right) \left(\Psi \underset{2}{*} \bar{k}\right)(x) \\ &= \frac{1}{2\sqrt{2\pi}} \left(1 - \frac{d^2}{dx^2}\right) \left\{ \int_{-\infty}^{+\infty} \Psi(y) \Big[\bar{k}(x+y) + \bar{k}(x-y) + \bar{k}(-x+y) - \bar{k}(-x-y) \Big] dy \right\}, \ x \in \mathbb{R}. \end{aligned} \tag{3.14}$$

where $\bar{k}$ is a complex conjugate of k function.

Proof. By the same argument as in Theorem 3.1 and consecutively using the formulas (3.5), (3.2), we obtain

$$\begin{aligned} \Psi(x) = (T_k f)(x) &= \left(1 - \frac{d^2}{dx^2}\right) \left(H_{\{\frac{1}{2}\}} \left(\left(H_{\{\frac{1}{2}\}} f\right)(y) \left(H_{\{\frac{1}{2}\}} k\right)\right)(y) \right)(x) \\ &= H_{\{\frac{1}{2}\}} \left((1+y^2) \left(H_{\{\frac{1}{2}\}} k\right)(y) \left(H_{\{\frac{1}{2}\}} f\right)(y) \right)(x) \in L_2(\mathbb{R}). \end{aligned} \tag{3.15}$$

Thus, f belongs to $L_2(\mathbb{R})$ space. Besides, in $L_2(\mathbb{R})$ we obtain $\|H_{\{\frac{1}{2}\}} f\|_{L_2(\mathbb{R})} = \|f\|_{L_2(\mathbb{R})}$. So we have the evaluation of the equality (3.15) and the (A_3) condition as follows

$$\begin{aligned} \|\Psi\|_{L_2(\mathbb{R})} = \|T_k f\|_{L_2(\mathbb{R})} &= \left\| H_{\{\frac{1}{2}\}} \left((1+y^2) \left(H_{\{\frac{1}{2}\}} k\right)(y) \left(H_{\{\frac{1}{2}\}} f\right)(y) \right) \right\|_{L_2(\mathbb{R})} \\ &= (1+y^2) \left| \left(H_{\{\frac{1}{2}\}} k\right)(y) \right| \left\| \left(H_{\{\frac{1}{2}\}} f\right)(y) \right\|_{L_2(\mathbb{R})} = \left\| \left(H_{\{\frac{1}{2}\}} f\right)(y) \right\|_{L_2(\mathbb{R})} \\ &= \|f\|_{L_2(\mathbb{R})}. \end{aligned}$$

Therefore, the (T_k) operator is an isometric transformation or an unitary operator in $L_2(\mathbb{R})$ space, using the formula (3.15), we obtain

$$\left(H_{\{\frac{1}{2}\}} \Psi\right)(y) = H_{\{\frac{1}{2}\}} (T_k f)(y) = (1+y^2) \left(H_{\{\frac{1}{2}\}} k\right)(y) \left(H_{\{\frac{1}{2}\}} f\right)(y), \tag{3.16}$$

From (3.16) together with the (A_3) condition and using Plancherel's theorem for the Hartley transform is unitary $\left(H_{\{\frac{1}{2}\}} \times H_{\{\frac{1}{2}\}} = \mathcal{I}\right)$ in the $L_2(\mathbb{R})$ space, we obtain

$$\left(H_{\{\frac{1}{2}\}} f\right)(y) = (1+y^2) \left(H_{\{\frac{1}{2}\}} \bar{k}\right)(y) \left(H_{\{\frac{1}{2}\}} \Psi\right)(y)$$

$$f(x) = H_{\{\frac{1}{2}\}}\Big((1+y^2)\left(H_{\{\frac{1}{2}\}}\bar{k}\right)(y)\left(H_{\{\frac{1}{2}\}}\Psi\right)(y)\Big)(x) \in L_2(\mathbb{R}).$$

We continue to use formulas (3.5) and by using the Parseval's identity (3.2) in Lemma 3.1, then the above expression can be rewritten

$$\begin{aligned} f(x) = \left(T_{\bar{k}}\Psi\right)(x) &= \left(1-\frac{d^2}{dx^2}\right) H_{\{\frac{1}{2}\}}\left(\left(H_{\{\frac{1}{2}\}}\Psi\right)(y)\left(H_{\{\frac{1}{2}\}}\bar{k}\right)(y)\right)(x) \\ &= \left(1-\frac{d^2}{dx^2}\right)\left(\Psi \underset{2}{*} \bar{k}\right)(x). \end{aligned}$$

To prove the sufficient condition, suppose that To prove the sufficient condition, suppose that the operator (T_k) in (3.13) is unitary in $L_2(\mathbb{R}_+)$ and that the inversion operator $(T_{\bar{k}})$ is given by (3.14), we obtain

$$\begin{aligned} \|\Psi\|_{L_2(\mathbb{R})} = \|T_k f\|_{L_2(\mathbb{R})} &= (1+y^2)\left|\left(H_{\{\frac{1}{2}\}}k\right)(y)\right| \left\|\left(H_{\{\frac{1}{2}\}}f\right)(y)\right\|_{L_2(\mathbb{R})} \\ &= \left\|\left(H_{\{\frac{1}{2}\}}f\right)(y)\right\|_{L_2(\mathbb{R})} = \|f\|_{L_2(\mathbb{R})}. \end{aligned}$$

This equality holds for all $f \in L_2(\mathbb{R})$ if and only if k function satisfies condition (A_3).
The proof of theorem is completed. □

The following example is an illustration of the above result.

Example 3.2. *Now, we choose* $k(x) = \sqrt{\frac{\pi}{2}} e^{-|x|} \in L_2(\mathbb{R})$, *then*

$$\begin{aligned} \left(H_{\{\frac{1}{2}\}}k\right)(y) &= \sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-|x|}\operatorname{cas}(\pm xy)dx \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x|}\big[\cos(xy) \pm \sin(xy)\big]dx \\ &= \int_0^{+\infty} e^{-|x|}\cos(xy)dx = \frac{1}{2}\int_0^{+\infty}\big[e^{-(1-iy)x} + e^{-(1+iy)x}\big]dx \\ &= \frac{1}{2}\left(\frac{1}{1-iy} + \frac{1}{1+iy}\right) = \frac{1}{1+y^2}. \end{aligned}$$

Thus, k is satisfies the condition (A_3), which implies that

$$(1+y^2)\left|\left(H_{\{\frac{1}{2}\}}\left(\sqrt{\frac{\pi}{2}}e^{-|x|}\right)\right)(y)\right| = 1.$$

Remark 2. *We extend the study for the operators (T_h) and (T_k) in the above theorems by replacing the second-order differential operator $D = \left(1-\frac{d^2}{dx^2}\right)$ by the differential operator of order $2m$ in the following form $D^{2m} := \sum_{m=0}^{n}(-1)^m a_m \frac{d^{2m}}{dx^{2m}}$, $n \in \mathbb{N}$ (see [6]). Then, the* *condition (A_2) in Theorem 3.1 becomes*

$$(A_2^*):\ |(Fh)(y)| = \frac{1}{\left(\sum\limits_{m=0}^{n} a_m y^{2m}\right)},\ y \in \mathbb{R},$$

and the condition (A_3) in Theorem 3.2 becomes

$$(A_3^*):\ \left(\sum_{m=0}^{n} a_m y^{2m}\right)\left|\left(H_{\{\frac{1}{2}\}}k\right)(y)\right| = 1,\ y \in \mathbb{R},$$

where $P(y) = \sum_{m=0}^{n} a_m y^{2m}$ is a polynomial with real coefficients without real zero points which implies that, the conditions (A_2^*) and (A_3^*) are well-defined. Then, the operator $D = \left(1 - \dfrac{d^2}{dx^2}\right)$ becomes a special case with $n = 1, a_0 = a_1 = 1$.

4. The Plancherel-type Theorems

In this section we indicate the norms approximation of the operators (T_k) defined by the formula (3.13) in the space $L_2(\mathbb{R})$ and get the Planchernel-type theorems. The approximation here can be understood in the sense of to the norm convergence in $L_2(\mathbb{R})$ space, we study the boundedness of operator (T_k) from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $1 \le p \le 2$.

Theorem 4.1. *Suppose that k belongs to the $L_2(\mathbb{R})$ and satisies the condition (A_3) of Theorem (3.2) such that $K(x) = \left(k(x) - \dfrac{d^2k}{dx^2}\right)$ is a locally bounded function on $\mathbb{R}$. Let $f \in L_2(\mathbb{R})$ for any positive number N, we set*

$$\Psi_N(x) := \frac{1}{2\sqrt{2\pi}} \int\limits_{-N}^{N} f(y)\Big[K(x+y)+K(x-y)+K(-x+y)-K(-x-y)\Big]dy,\ x \in \mathbb{R}. \tag{4.1}$$

Then the following assertions hold
(i) *$\Psi_N(x)$ belongs to the $L_2(\mathbb{R})$.*
(ii) *When $N \longrightarrow +\infty$, the sequence of functions $\{\Psi_N(x)\}$ converge in the sense of the norm to a function $\Psi(x)$ in $L_2(\mathbb{R})$ and satisfy $\|\Psi\|_{L_2(\mathbb{R})} = \|f\|_{L_2(\mathbb{R})}$.*
(iii) *Let $\Psi^N(x) = \Psi(x).\mathcal{X}_{[-N,N]}$ and set*

$$f_N(x) = \left(T_k\Psi^N\right)(x) = \left(1 - \frac{d^2}{dx^2}\right)\left(\Psi^N \underset{2}{*} k\right)(x),$$

then the sequence of functions $\{f_N(x)\}$ is converge in the sense of the norm to the function $f(x)$ in $L_2(\mathbb{R})$ space.

Proof. (i) According to the assumptions of the Theorem 4.1 and f belonging to $L_2(\mathbb{R})$ space, the formula (4.1) is rewritten as

$$\Psi_N(x) = \frac{1}{2\sqrt{2\pi}} \int\limits_{-N}^{N} f(y)\left(1 - \frac{d^2}{dx^2}\right)\big[k(x+y)+k(x-y)+k(-x+y)-k(-x-y)\big]dy,\ x \in \mathbb{R}. \tag{4.2}$$

The integration (4.2) is convergent. We continue to change the order of the integration and the differentiation. Set $f^N = f.\mathcal{X}_{[-N,N]}$, it follows $f^N \in L_2(\mathbb{R})$, which $\mathcal{X}_{[-N,N]}$ is the characteristic function of f on finite interval $[-N, N]$. We obtain

(4.3)

$$\begin{aligned}\Psi_N(x) &= \frac{1}{2\sqrt{2\pi}}\left(1-\frac{d^2}{dx^2}\right)\int\limits_{-\infty}^{+\infty} f^N(y)\big[k(x+y)+k(x-y)+k(-x+y)-k(-x-y)\big]dy\\ &= \left(1-\frac{d^2}{dx^2}\right)\left(f^N \underset{2}{*} k\right)(y) \equiv \left(T_k f^N\right)(y).\end{aligned}$$

By theorem 3.2, we get $\Psi_N(x) \in L_2(\mathbb{R})$. So, the first assertion is completed.

(ii) It is clear that $\|f^N - f\|_{L_2(\mathbb{R})} \longrightarrow 0$ when $N \longrightarrow +\infty$. Following to the formula (3.13) then $\Psi(x) = \left(1-\frac{d^2}{dx^2}\right)\left(f \underset{2}{*} k\right)(x) \in L_2(\mathbb{R})$. Then, subtracting from both sides of the expression (4.3) and (3.13), we obtain

$$\begin{aligned}\left(\Psi_N - \Psi\right)(x) &= \left(1-\frac{d^2}{dx^2}\right)\left(f^N \underset{2}{*} k\right)(x) - \left(1-\frac{d^2}{dx^2}\right)\left(f \underset{2}{*} k\right)(x)\\ &= \left(1-\frac{d^2}{dx^2}\right)\left(f^N - f \underset{2}{*} k\right)(x) \equiv \left(T_k(f^N - f)\right)(x).\end{aligned}$$

Since $f^N - f \in L_2(\mathbb{R})$ and the condition (A_3) in Theorem 3.2 is satisfied, we have $\Psi_N - \Psi \in L_2(\mathbb{R})$ and $\|\Psi_N - \Psi\|_{L_2(\mathbb{R})} = \|f^N - f\|_{L_2(\mathbb{R})}$. However $\|f_N - f\|_{L_2(\mathbb{R})} \longrightarrow 0$ as $N \longrightarrow +\infty$, it follows $\Psi_N \longrightarrow \Psi$ as $N \longrightarrow +\infty$ in $L_2(\mathbb{R})$ space. Furthermore, seeing Ψ as the image of f under the operator (T_k) in $L_2(\mathbb{R})$ space, by Theorem 3.2, we obtain $\|f\|_{L_2(\mathbb{R})} = \|T_k f\|_{L_2(\mathbb{R})} = \|\Psi\|_{L_2(\mathbb{R})}$, which is unitary. Now, the second assertion is completed.

(iii) Let $\Psi^N(x) = \Psi(x).\mathcal{X}_{[-N,N]}$ and

$$f_N(x) := \left(1-\frac{d^2}{dx^2}\right)\left(\Psi^N \underset{2}{*} \bar{k}\right)(x) \equiv \left(T_{\bar{k}}\Psi^N\right)(x). \tag{4.4}$$

Thus, $\Psi \in L_2(\mathbb{R})$ then $\Psi^N \in L_2(\mathbb{R})$. Besides, $\bar{k}$ is a complex conjugate of k and satisfies the conditions (A_3). So, by Theorem 3.2 then $f_N \in L_2(\mathbb{R})$, by using (3.14) and (4.4), we obtain

$$(f_N - f)(x) = \left(1-\frac{d^2}{dx^2}\right)\left(\Psi^N - \Psi \underset{2}{*} \bar{k}\right)(x) \equiv \left(T_{\bar{k}}(\Psi^N - \Psi)\right)(x).$$

Since $\Psi^N - \Psi \in L_2(\mathbb{R})$ it follows $(f_N - f) \in L_2(\mathbb{R})$ and $\|f_N - f\|_{L_2(\mathbb{R})} = \|\Psi^N - \Psi\|_{L_2(\mathbb{R})}$. Moreover, $\Psi^N \longrightarrow \Psi$ when $N \longrightarrow +\infty$, which implies that$f_N \longrightarrow f$ as $N \longrightarrow +\infty$.

The proof of theorem is completed. □

A note that, the functions k were shown in examples 3.2 also satisfying the conditions of Theorem 3.2 respectively and $K(x) = \left(1-\frac{d^2}{dx^2}\right)\left(\sqrt{\frac{\pi}{2}}e^{-|x|}\right) = \sqrt{\frac{\pi}{2}}\frac{2-x^2}{e^x} < +\infty$ with $x > 0$ are the functions locally bounded on $\mathbb{R}$.

In the next theorem, we consider the boundedness of the operators (T_k) from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$, with $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 4.2. *Suppose that $k \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ has second order continuous derivatives on $\mathbb{R}$ with $K(x) = \left(k(x) - \dfrac{d^2k}{dx^2}\right)$ is the locally bounded functions on $\mathbb{R}$ and h satisfy the condition (A_3) of Theorem 3.2. Then, the operator (T_k) is a bounded operator from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$, with $1 \leq p \leq 2$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.*

Proof. If $p_0 = 1$ then $q_0 = \infty$, according to the formula (3.13) then

$$(T_k f)(x) = \frac{1}{2\sqrt{2\pi}}\left(1 - \frac{d^2}{dx^2}\right)\int\limits_{-\infty}^{+\infty} f(y)\Big[k(x+y) + k(x-y) + k(-x+y) - k(-x-y)\Big]dy$$

is convergent. We can change order of taking differentiation and integration, and obtain

$$\begin{aligned}(T_k f)(x) &= \frac{1}{2\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(y)\left(1 - \frac{d^2}{dx^2}\right)\Big[k(x+y) + k(x-y) + k(-x+y) - k(-x-y)\Big]dy \\ &= \frac{1}{2\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(y)\Big[K(x+y) + K(x-y) + K(-x+y) - K(-x-y)\Big]dy.\end{aligned}$$

According to the assumption of the theorem, $K(x) = \left(k(x) - \dfrac{d^2k}{dx^2}\right)$ is the locally bounded functions on $\mathbb{R}$, which means $\exists M > 0$ such that $|K(x)| \leq M, \forall x \in \Omega \subset \mathbb{R}$. Moreover,

$$\begin{aligned}|(T_k f)(x)| &\leq \frac{1}{2\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} |f(y)|.\ |K(x+y) + K(x-y) + K(-x+y) - K(-x-y)|\ dy \\ &\leq \frac{4M}{2\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} |f(t)|dt = M\sqrt{\frac{2}{\pi}}\|f\|_{L_1(\mathbb{R})} < +\infty,\end{aligned}$$

which implies that (T_k) is a bounded operator from $L_1(\mathbb{R}) \longrightarrow L_\infty(\mathbb{R})$. On the other hand, if $p_1 = 2$ then $q_1 = 2$, by theorem 4.1 it follows (T_k) is a bounded operator from $L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R})$. Thus, using Riesz's interpolation Theorem (refer Theorem 1.3, p. 179, Chapter 5 in [17]), we obtain (T_k) is a bounded operator from $L_p(\mathbb{R}) \longrightarrow L_q(R)$, where p, q is a pair of conjugate exponents and p is determined by formula $\dfrac{1}{p} = \dfrac{1-\alpha}{1} + \dfrac{\alpha}{2} = 1 - \dfrac{\alpha}{2}$. The condition $0 < \alpha < 1$ implies $1 < p < 2$. Adding the cases $p = 1, p = 2$ that already hold, we conclude that the operator (T_k) is bounded from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$ for all $1 \leq p \leq 2$.

The proof of theorem is completed. □

Remark 3. *By setting $\Psi(x) = (T_k f)(x) = \lim\limits_{N \longrightarrow \infty}\left(1 - \dfrac{d^2}{dx^2}\right)\left(f^N \underset{2}{*} k\right)(x)$ and $f(x) = \lim\limits_{N \longrightarrow \infty}\left(1 - \dfrac{d^2}{dx^2}\right)\left(\Psi^N \underset{2}{*} \bar{k}\right)(x)$, then Ψ, f are the locally bounded functions from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$ and with $\frac{1}{p} + \frac{1}{q} = 1$, where the limits are understood in the norm sense in the space $L_q(\mathbb{R})$ and $f^N = f.\mathcal{X}_{[-N,N]}$, $\Psi^N = \Psi.\mathcal{X}_{[-N,N]}$.*

We have $\Psi(x) = \lim\limits_{N \to +\infty}\left(T_k f^N\right)(x)$ and $f(x) = \lim\limits_{N \to +\infty}\left(T_{\bar{k}}\Psi^N\right)(x)$ with $\bar{k}$ is conjugates function of k. Following Theorem 4.2, then the (T_k) and $(T_{\bar{k}})$ are bounded operator from

$L_p(\mathbb{R})$ to $L_q(\mathbb{R})$. Implies that Ψ and f are the locally bounded functions from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$.

5. Some Applications

In this section, we will use the obtained results in section 3, combined with the techniques of convolution and polyconvolution in [9, 15, 24] and [12] to study the solving of some classes of the integro-differential equations of Barbashin type (5.1), differential equations (5.9) and system of differential equations (5.12). We will give explicit formulas of solutions and determine the function spaces of solutions. Following each result are illustrative examples.

5.1. **Solving the integro−differential equations of Barbashin type.** The main idea of this section is that we replace the expression $\mathcal{A}f(t,s)$ of the equation (1.3) by the formula $\mathcal{A}f(t,s) := \left(1 - \frac{d^2}{dx^2}\right)(f * g)(x)$. Then the equation (1.3) can be rewritten as follows

$$\left(1 - \frac{d^2}{dx^2}\right)(f * g)(x) = \int\limits_a^b \mathcal{K}(x,u,v) f(u)du + \Phi_i(x), \tag{5.1}$$

where $g, \Phi_i(x),\ i = 1,2$ are the given functions, $\mathcal{K}(x,u,v)$ are some given kernels and f is an unknown function. To solve the equation (5.1), we choose $(f * g)$ defined by the formula (2.2), (2.4) and $\mathcal{K}(x,u,v)$ as the kernel of the convolution, polyconvolution defined in [15, 9].

Lemma 5.1. *(See* [15]*), Suppose that $f, \varphi_2 \in L_2(\mathbb{R}), \varphi_1 \in L_2(\mathbb{R}_+)$. Then, polyconvolution $\left(\underset{3}{*}(f,\varphi_1,\varphi_2)\right)$ belongs to $L_2(\mathbb{R})$ and*

$$H_{\left\{\substack{1\\2}\right\}}\left(\underset{3}{*}(f,\varphi_1,\varphi_2)\right)(y) = \left(H_{\left\{\substack{1\\2}\right\}} f\right)(y)(F_c\varphi_1)(|y|)\left((H_{\left\{\substack{2\\1}\right\}}\varphi_2\right)(y),\ y \in \mathbb{R}, \tag{5.2}$$

where

$$\left(\underset{3}{*}(f,\varphi_1,\varphi_2)\right)(x) := \frac{1}{4\pi}\int\limits_{-\infty}^{+\infty}\int\limits_0^{+\infty} f(u)\varphi_1(v)\Phi(x,u,v)\ dudv,\ x \in \mathbb{R}. \tag{5.3}$$

And

$$\begin{aligned}\Phi(x,u,v) &= \varphi_2(-x+u+v) + \varphi_2(x-u+v) + \varphi_2(-x+u-v) + \varphi_2(x-u+v)\\ &\quad + \varphi_2(-x-u+v) - \varphi_2(x+u-v) + \varphi_2(-x-u-v) - \varphi_2(x+u+v).\end{aligned} \tag{5.4}$$

And

$$(F_c\varphi_1)(y) = \sqrt{\frac{2}{\pi}}\int\limits_0^{+\infty}\varphi_1(x)\cos(xy)dx,\ y > 0. \tag{5.5}$$

- Firstly, we choose $(a,b) = (-\infty,+\infty)$, and $(f * h) \equiv (f \underset{2}{*} k)$ which is defined by the formula (2.4), and the selected kernel of equation $\mathcal{K}(x,u,v) = -\frac{1}{4\pi}\int\limits_0^{+\infty}\Phi(x,u,v)dv$, where

$\Phi(x,u,v)$ is defined by (5.4). Applying the formulas (3.7) and (5.3), then the equation (5.1) can be rewritten as follows

$$(T_k f)(x) + \left(\underset{3}{*}(f,\varphi_1,\varphi_2)\right)(x) = \Phi_1(x), \tag{5.6}$$

where the operator $(T_k f)$ defined by the formula (3.7), the polyconvolution $\left(\underset{3}{*}(f,\varphi_1,\varphi_2)\right)$ is defined by the formula (5.3), $\varphi_1, \varphi_2, k, \Phi_1$ are given functions and f is a unknown function.

Theorem 5.1. *Let φ_2, k, Φ_1 be functions that belong to $L_2(\mathbb{R})$ space and $\varphi_1 \in L_2(\mathbb{R}_+)$ are given functions satisfying the following condition*

$$(A_4): \ \frac{\left(H_{\{\frac{1}{2}\}}\Phi_1\right)(y)}{(1+y^2)\left(H_{\{\frac{1}{2}\}}k\right)(y) + (F_c\varphi_1)(|y|)\left(H_{\{\frac{1}{2}\}}\varphi_2\right)(y)} \in L_2(\mathbb{R}),\ y \in \mathbb{R}.$$

Then, the equation (5.6) *has a unique solution in* $L_2(\mathbb{R})$ *which can be presented in the form*

$$f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} \frac{\left(H_{\{\frac{1}{2}\}}\Phi_1\right)(y)}{(1+y^2)\left(H_{\{\frac{1}{2}\}}k\right)(y) + (F_c\varphi_1)(|y|)\left(H_{\{\frac{1}{2}\}}\varphi_2\right)(y)} \operatorname{cas}(\pm xy)dy,\ x \in \mathbb{R},$$

where $H_{\{\frac{1}{2}\}}, F_c$ are respectively determined by (2.5) *and* (5.5).

Proof. Applying the Hartley $H_{\{\frac{1}{2}\}}$ transformation on both sides of the equation (5.6) and using consecutively the formulas (3.16), (5.2), we obtain

$$H_{\{\frac{1}{2}\}}(T_k f)(y) + H_{\{\frac{1}{2}\}}\left(\underset{3}{*}(f,\varphi_1,\varphi_2)\right)(y) = \left(H_{\{\frac{1}{2}\}}\Phi_1\right)(y),$$

or equivalently,

$$(1+y^2)\left(H_{\{\frac{1}{2}\}}k\right)(y)\left(H_{\{\frac{1}{2}\}}f\right)(y) + \left(H_{\{\frac{1}{2}\}}f\right)(y)(F_c\varphi_1)(|y|)\left(H_{\{\frac{2}{1}\}}\varphi_2\right)(y) = \left(H_{\{\frac{1}{2}\}}\Phi_1\right)(y).$$

Under the condition (A_4), we have

$$\left(H_{\{\frac{1}{2}\}}f\right)(y)\left[(1+y^2)\left(H_{\{\frac{1}{2}\}}k\right)(y) + (F_c\varphi_1)(|y|)\left(H_{\{\frac{2}{1}\}}\varphi_2\right)(y)\right] = \left(H_{\{\frac{1}{2}\}}\Phi_1\right)(y),$$

which means

$$\left(H_{\{\frac{1}{2}\}}f\right)(y) = \frac{\left(H_{\{\frac{1}{2}\}}\Phi_1\right)(y)}{(1+y^2)\left(H_{\{\frac{1}{2}\}}k\right)(y) + (F_c\varphi_1)(|y|)\left(H_{\{\frac{2}{1}\}}\varphi_2\right)(y)} \in L_2(\mathbb{R}),\ y \in \mathbb{R}.$$

By using the inverse transform of Hartley defined in the formula (2.6), we obtain

$$f(x) = \int\limits_{-\infty}^{+\infty} \frac{\left(H_{\{\frac{1}{2}\}}\Phi_1\right)(y)}{(1+y^2)\left(H_{\{\frac{1}{2}\}}k\right)(y) + (F_c\varphi_1)(|y|)\left(H_{\{\frac{2}{1}\}}\varphi_2\right)(y)} \operatorname{cas}(\pm xy)dy,\ x \in \mathbb{R},$$

and $f(x)$ belongs to $L_2(\mathbb{R})$. The proof of theorem is completed. □

An example is given below to illustrate the above result.

Example 5.1. *We choose* $\Phi_1(x) = \varphi_2(x) = k(x) = \sqrt{\frac{\pi}{2}}e^{-|x|}$ *as functions belonging to the* $L_2(\mathbb{R})$ *space and* $\varphi_1(x) = \sqrt{\frac{\pi}{2}}e^{-x} \in L_2(\mathbb{R}_+)$. *We have* $H_{\{\frac{1}{2}\}}\left(\sqrt{\frac{\pi}{2}}e^{-|x|}\right) = F_c\left(\sqrt{\frac{\pi}{2}}e^{-x}\right) = \frac{1}{1+y^2}$ *are satisfying the condition* (A_4) *and*

$$\frac{\left(H_{\{\frac{1}{2}\}}\Phi_1\right)(y)}{(1+y^2)\left(H_{\{\frac{1}{2}\}}k\right)(y) + (F_c\varphi_1)(|y|)\left(H_{\{\frac{2}{1}\}}\varphi_2\right)(y)} = \frac{1+y^2}{(1+y^2)^2+1} \in L_2(\mathbb{R}).$$

With the help of Mathematica 5.1 tool, we have the solution for this case as follows

$$f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\frac{1+y^2}{(1+y^2)^2+1}\operatorname{cas}(\pm xy)\, dy = \sqrt{\frac{2}{\pi}}\int\limits_{0}^{+\infty}\frac{1+y^2}{(1+y^2)^2+1}\cos(xy)\, dy$$

$$= \sqrt{\frac{\pi}{32}}e^{-\sqrt{1+\sqrt{2}x}}\left(\sqrt{2(1-i)}e^{\sqrt{1-ix}} + \sqrt{2(i+1)}e^{\sqrt{1+ix}}\right).$$

• To further study the solvability of the equation (5.1), we choose $(a,b) = (-\infty,+\infty)$ and the convolution $(f*g)(x) = \left(f \underset{1}{*} g\right)(x)$, which is determined by the formula (2.2). Now, the kernel of the equation is chosen by the expression

$$\mathcal{K}(x,y,t) \equiv \mathcal{K}(x,y) = -\frac{1}{2\sqrt{2\pi}}\big[\varphi_3(x+y) + \varphi_3(x-y) + \varphi_3(-x+y) - \varphi_3(-x-y)\big].$$

Applying the formulas (3.7) and (2.4), then the equation (5.1) can be rewritten as follows

$$(T_h f)(x) + \left(f \underset{2}{*} \varphi_3\right)(x) = \Phi_2(x), \tag{5.7}$$

where $(T_h f)(x) = \left(1 - \frac{d^2}{dx^2}\right)\left(f \underset{1}{*} h\right)(x)$ is defined by formula (3.7) and h, φ_3, Φ_2 are the given functions, f is an unknown function.

Theorem 5.2. *Suppose that* h, φ_3, Φ_2 *belong to* $L_2(\mathbb{R})$ *space and satisfy the following condition*

$$(A_5): \ \frac{(H_1\Phi_2)(y)}{(1+y^2)(Fh)(y) + (H_1\varphi_3)(y)} \in L_2(\mathbb{R}).$$

Then the equation (5.7) *has a unique solution in* $L_2(\mathbb{R})$ *which is of the form*

$$f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\frac{(H_1\Phi_2)(y)}{(1+y^2)(Fh)(y) + (H_1\varphi_3)(y)}\operatorname{cas}(xy)\, dy, \ x \in \mathbb{R},$$

where H_1, F *are respectively defined by the formulas* (2.5) *and* (2.7).

Proof. Applying the H_1 transformation on both sides of the equation (5.7) and using the formulas (3.11), (2.3), we obtain

$$H_1(T_h f)(y) + H_1\left(f \underset{2}{*} \varphi_3\right)(y) = (H_1\Phi_2)(y).$$

or equivalently

$$(1+y^2)(H_1 f)(y)(Fh)(y) + (H_1 f)(y)(H_1\varphi_3)(y) = (H_1\Phi_2)(y).$$

Under the condition (A_5), we have $(H_1 f)(y) = \dfrac{(H_1\Phi_2)(y)}{(1+y^2)(Fh)(y)+(H_1\varphi_3)(y)} \in L_2(\mathbb{R})$ and

$$f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \frac{(H_1\Phi_2)(y)}{(1+y^2)(Fh)(y)+(H_1\varphi_3)(y)} \operatorname{cas}(xy)dy,\ x\in\mathbb{R},$$

and $f(x)$ belongs to $L_2(\mathbb{R})$ space. The proof of theorem is completed. □

The following example is an illustration of the above result.

Example 5.2. *Now, choose* $h(x)=\varphi_3(x)=\Phi_2(x)=\sqrt{\dfrac{\pi}{2}}e^{-|x|}\in L_2(\mathbb{R})$. *Then* $F\left(\sqrt{\dfrac{\pi}{2}}e^{-|x|}\right) = H_1\left(\sqrt{\dfrac{\pi}{2}}e^{-|x|}\right) = \dfrac{1}{1+y^2}$ *satisfies the condition* (A_5) *and*

$$\frac{(H_1\Phi_2)(y)}{(1+y^2)(Fh)(y)+(H_1\varphi_3)(y)} = \frac{1}{2+y^2}\in L_2(\mathbb{R}).$$

Following the formula (21, p.616, [7]*), we have the solution for this case as follows*

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\frac{1}{2+y^2}\operatorname{cas}(xy)dy = \sqrt{\frac{2}{\pi}}\int\limits_{0}^{+\infty}\frac{1}{2+y^2}\cos(xy)dy \\ &= \frac{\sqrt{\pi}}{2}e^{-\sqrt{2}x}\in L_2(\mathbb{R}_+)\subset L_2(\mathbb{R}). \end{aligned}$$

5.2. **Solving the differential equations.** According to Theorem 4.1 in (Chapter 4, p. 224, see [12]) gives a closed-form solution of the Cauchy-type problem.

$$\text{(5.8)}\qquad \begin{cases} \left(D^{\alpha}_{a+}f\right)(x) - \lambda f(x) = \Phi(x), & (a< x\leqq b; \alpha>0; \lambda\in\mathbb{R}) \\ \left(D^{\alpha-k}_{a+}f\right)(a+) = b_k, & (b_k\in\mathbb{R}; k=1,\cdots,n=-[-\alpha]), \end{cases}$$

where $\Phi(x)\in C_\gamma[a,b](0\leqq\gamma<1)$ with the Riemann-Liouville fractional derivative $\left(D^{\alpha}_{a+}f\right)(x)$ of order $\alpha>0$ given by

$$\left(D^{\alpha}_{a+}f\right)(x) := \frac{1}{\Gamma(n-\alpha)}\left(\frac{d^n}{dx^n}\right)\left\{\int\limits_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}dt\right\}, (n=[\alpha]+1; x>a).$$

In this part of the paper, we study closed-form solutions of a narrow class of differential equations than equation in the problem (5.8) by choosing $\lambda=-1$ and substituting the operator $\left(D^{\alpha}_{a+}f\right)$ by operator $D(f*g)$ where $D=\left(\mathcal{I}-\dfrac{d^2}{dx^2}\right)$. Then the equation in Cauchy-type problem (5.8) can be rewritten as follow

$$\text{(5.9)}\qquad f(x) + \left(1-\frac{d^2}{dx^2}\right)(f*g)(x) = \Phi_j(x),\ x\in\mathbb{R}.$$

where $\Phi_j,\ j=3,4$ and g are given function, f is an unknown function. The idea to solve the equation (5.9) is that, we use the operators (T_h) or (T_k) studied in section 3 and choose $(f*g)$ as one of the two convolutions studied in results (see [24] or [9]).

• In this case if we choose $(f * g)(x) \equiv \left(f \underset{1}{*} h\right)(x)$, which is defined by the formula (2.2), then the equation (5.9) can be rewritten in the form

$$f(x) + (T_h f)(x) = \Phi_3(x), \tag{5.10}$$

where Φ_3, h are given functions and $(T_h f) = \left(1 - \frac{d^2}{dx^2}\right)\left(f \underset{1}{*} h\right)(x)$ is determined by (3.7).

Theorem 5.3. *Suppose that h, Φ_3 are given functions in $L_2(\mathbb{R})$ satisfying the following condition (A_6) : $\frac{(H_1\Phi_3)(y)}{1 + (1+y^2)(Fh)(y)} \in L_2(\mathbb{R})$. Then the equation (5.10) has a unique solution in $L_2(\mathbb{R})$ which can be presented in the form*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{(H_1\Phi_3)(y)}{1 + (1+y^2)(Fh)(y)} \operatorname{cas}(xy)dx \in L_2(\mathbb{R}),\ x \in \mathbb{R},$$

where H_1, F the respectively defined by the formulas (2.5),(2.7).

Proof. Applying the H_1 transform on both sides of equation (5.10) and using the formula (3.11), we obtain

$$(H_1 f)(y) + H_1(T_h f)(y) = (H_1\Phi_3)(y),$$

or equivalently

$$(H_1 f)(y) + (1+y^2)(H_1 f)(y)(Fh)(y) = (H_1\Phi_3)(y),$$

under the condition (A_6) then $(H_1 f)(y) = \frac{(H_1\Phi_3)(y)}{1 + (1+y^2)(Fh)(y)} \in L_2(\mathbb{R})$. And we have a solution in $L_2(\mathbb{R})$ as follows

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{(H_1\Phi_3)(y)}{1 + (1+y^2)(Fh)(y)} \operatorname{cas}(xy)dy,\ x \in \mathbb{R}.$$

The proof of theorem is completed. $\square$

An example is given below to illustrate the above result.

Example 5.3. *We choose $\Phi_3 = \sqrt{\frac{\pi}{2}} e^{-|x|}$ and $h = \sqrt{\frac{\pi}{2}} |x| e^{-|x|}$ as functions belonging to the $L_2(\mathbb{R})$. Following the formula (5, p611, [7]), we have*

$$F\left(\sqrt{\frac{\pi}{2}} |x| e^{-|x|}\right) = \frac{1-y^2}{(1+y^2)^2}.$$

$$H_1\left(\sqrt{\frac{\pi}{2}} e^{-|x|}\right) = \frac{1}{1+y^2}.$$

Then $\frac{(H_1\Phi_3)(y)}{1 + (1+y^2)(Fh)(y)} = \frac{1}{2(1+y^2)} \in L_2(\mathbb{R})$. And we have the solution for this case as follows

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{2(1+y^2)} \operatorname{cas}(xy)dy = \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} \frac{1}{1+y^2} \cos(xy)dy = \frac{1}{2}\sqrt{\frac{\pi}{2}} e^{-x} \in L_2(\mathbb{R}_+) \subset L_2(\mathbb{R}).$$

• On the other hand, if we choose the convolutions $(f * g)(x) \equiv \left(f \underset{2}{*} k\right)(x)$, it means $\left(1 - \dfrac{d^2}{dx^2}\right)(f * g)(x) = (T_k f)(x)$, which is determined by the formula (3.13). Then the equation (5.9) is rewritten as

$$f(x) + (T_k f)(x) = \Phi_4(x), \tag{5.11}$$

Applying the Hartley $H_{\left\{\frac{1}{2}\right\}}$ transformation on both sides of the equation (5.11) and using the formulas (3.16), we get the following corollary.

Corollary 5.1. *Let $k, \Phi_4 \in L_2(\mathbb{R})$ are functions satisfying the following condition (A_7) :* $\dfrac{\left(H_{\left\{\frac{1}{2}\right\}}\Phi_4\right)(y)}{1 + (1+y^2)\left(H_{\left\{\frac{1}{2}\right\}}k\right)(y)} \in L_2(\mathbb{R})$. *Then the equation* (5.11) *has a solution in* $L_2(\mathbb{R})$ *which can be presented in the form*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\left(H_{\left\{\frac{1}{2}\right\}}\Phi_4\right)(y)}{1 + (1+y^2)\left(H_{\left\{\frac{1}{2}\right\}}k\right)(y)} \operatorname{cas}(\pm xy)dy, \;\; x \in \mathbb{R}.$$

Note that if we choose $\Phi_4 = k = \sqrt{\dfrac{\pi}{2}} e^{-|x|}$, *then* $\dfrac{\left(H_{\left\{\frac{1}{2}\right\}}\Phi_4\right)(y)}{1 + \left(H_{\left\{\frac{1}{2}\right\}}k\right)(y)} = \dfrac{1}{2(1+y^2)} \in L_2(\mathbb{R})$, *and the solution of the equation* (5.11) *in this case has the form*

$$f(x) = \frac{\sqrt{\pi}}{2\sqrt{2}} e^{-x} \in L_2(\mathbb{R}_+) \subset L_2(\mathbb{R}).$$

5.3. **Solving the system of differential equations.** Consider the following system of differential equations

$$\begin{cases} f(x) + \left(1 - \dfrac{d^2}{dx^2}\right)\left(g \underset{1}{*} h\right)(x) & = \Phi_5(x), \;\; x \in \mathbb{R} \\ \left(1 - \dfrac{d^2}{dx^2}\right)\left(f \underset{2}{*} k\right)(x) + g(x) & = \Phi_6(x), \;\; x \in \mathbb{R}, \end{cases} \tag{5.12}$$

where Φ_5, Φ_6 are given functions, f, g are unknown functions. The convolutions $(. \underset{1}{*} .), (. \underset{2}{*} .)$ are respectively determined by the formulas (2.2) and (2.4).

Theorem 5.4. *Suppose that h, k, Φ_5, Φ_6 are given functions belonging to the $L_2(\mathbb{R})$ space and simultaneously satisfy the following conditions*

$$\begin{cases} (A_8 :) & 1 - (1+y^2)^2 H_1\left(k \underset{1}{*} h\right)(y) \neq 0, \forall y \in \mathbb{R} \\ (A_9 :) & \dfrac{(H_1\Phi_5)(y) - (1+y^2)H_1\left(\Phi_6 \underset{1}{*} h\right)(y)}{1 - (1+y^2)^2 H_1\left(k \underset{1}{*} h\right)(y)} \in L_2(\mathbb{R}) \\ (A_{10} :) & \dfrac{(H_1\Phi_6)(y) - (1+y^2)H_1\left(\Phi_5 \underset{2}{*} h\right)(y)}{1 - (1+y^2)^2 H_1\left(k \underset{1}{*} h\right)(y)} \in L_2(\mathbb{R}). \end{cases}$$

Then the problem (5.12) *has solution* $(f(x), g(x)) \in L_2(\mathbb{R}) \times L_2(\mathbb{R})$ *which can be presented in the form*

$$f(x) = H_1\left(\frac{(H_1\Phi_5)(y) - (1+y^2)H_1\left(\Phi_6 \underset{1}{*} h\right)(y)}{1-(1+y^2)^2 H_1\left(k \underset{1}{*} h\right)(y)}\right)(y),\ x \in \mathbb{R},$$

$$g(x) = H_1\left(\frac{(H_1\Phi_6)(y) - (1+y^2)H_1\left(\Phi_5 \underset{2}{*} h\right)(y)}{1-(1+y^2)^2 H_1\left(k \underset{1}{*} h\right)(y)}\right)(y),\ x \in \mathbb{R}.$$

where the (H_1) *transform is defined by* (2.5) *and* $(.\underset{1}{*}.), (.\underset{2}{*}.)$ *are respectively determined by the formulas* (2.2) *and* (2.4) .

Proof. From the formulas (3.7), (3.13), we rewrite the system of differential equations (5.12) as follows

$$\begin{cases} f(x) + (T_h g)(x) = \Phi_5(x),\ x \in \mathbb{R}, \\ (T_k f)(x) + g(x) = \Phi_6(x),\ x \in \mathbb{R}. \end{cases} \tag{5.13}$$

Applying the H_1 transformation respectively on both sides of the first and the second in the system of equations (5.13) and using the formulas (3.11),(3.16), we obtain

$$\begin{cases} (H_1 f)(y) + (1+y^2)(H_1 g)(y)(Fh)(y) = (H_1\Phi_5)(y),\ y \in \mathbb{R}, \\ (1+y^2)(H_1 f)(y)(H_1 k)(y) + (H_1 g)(y) = (H_1\Phi_6)(y),\ y \in \mathbb{R}. \end{cases}$$

Combining the factorization equalities (2.1),(2.3), we have

$$\Delta = \begin{vmatrix} 1 & (1+y^2)(Fh)(y) \\ (1+y^2)(H_1 k)(y) & 1 \end{vmatrix} = 1-(1+y^2)^2 H_1\left(k \underset{1}{*} h\right)(y)$$

$$\Delta_1 = \begin{vmatrix} (H_1\Phi_5)(y) & (1+y^2)(Fh)(y) \\ (H_1\Phi_6)(y) & 1 \end{vmatrix} = (H_1\Phi_6)(y) - (1+y^2)H_1\left(\Phi_6 \underset{1}{*} k\right)(y)$$

$$\Delta_2 = \begin{vmatrix} 1 & (H_1\Phi_5)(y) \\ (1+y^2)(H_1 k)(y) & (H_1\Phi_6)(y) \end{vmatrix} = (H_1\Phi_6)(y) - (1+y^2)H_1\left(\Phi_5 \underset{2}{*} k\right)(y).$$

Under the conditions$(A_8 - A_{10})$, we get

$$(H_1 f)(y) = \frac{\Delta_1}{\Delta} = \frac{(H_1\Phi_5)(y) - (1+y^2)H_1\left(\Phi_6 \underset{1}{*} h\right)(y)}{1-(1+y^2)^2 H_1\left(k \underset{1}{*} h\right)(y)} \in L_2(\mathbb{R}),\ y \in \mathbb{R}$$

$$(H_1 g)(y) = \frac{\Delta_2}{\Delta} = \frac{(H_1\Phi_6)(y) - (1+y^2)H_1\left(\Phi_5 \underset{2}{*} h\right)(y)}{1-(1+y^2)^2 H_1\left(k \underset{1}{*} h\right)(y)} \in L_2(\mathbb{R}),\ y \in \mathbb{R}.$$

Using the inverse transform formula (2.6) then this implies that

$$f(x) = H_1\left(\frac{(H_1\Phi_5)(y) - (1+y^2)H_1\left(\Phi_6 \underset{1}{*} h\right)(y)}{1-(1+y^2)^2H_1\left(k \underset{1}{*} h\right)(y)}\right)(y),\ \ x \in \mathbb{R}$$

$$g(x) = H_1\left(\frac{(H_1\Phi_6)(y) - (1+y^2)H_1\left(\Phi_5 \underset{2}{*} h\right)(y)}{1-(1+y^2)^2H_1\left(k \underset{1}{*} h\right)(y)}\right)(y),\ \ x \in \mathbb{R},$$

and $(f,g) \in L_2(\mathbb{R}) \times L_2(\mathbb{R})$. The proof of theorem is completed. □

We will end the article with an example illustrating the above result

Example 5.4. *Now, we choose* $h = k = e^{-|x|}$ *and* $\Phi_5 = \Phi_6 = \sqrt{\frac{\pi}{2}}e^{-|x|}$ *where* h, k, Φ_5, Φ_6 *are given functions belonging to the* $L_2(\mathbb{R})$ *space.*
We have $F\left(e^{-|x|}\right) = H_1\left(e^{-|x|}\right) = \sqrt{\frac{2}{\pi}}\frac{1}{1+y^2}$ *and* $H_1\Phi_5 = H_1\Phi_6 = \frac{1}{1+y^2}$, *which satisfy the condition* (A_8), *which implies that*

$$1-(1+y^2)^2H_1\left(k \underset{1}{*} h\right)(y) = 1 - \frac{2}{\pi} \neq 0.$$

Furthermore, we obtain $H_1\left(\Phi_6 \underset{1}{*} h\right)(y) = H_1\left(\Phi_5 \underset{2}{*} h\right)(y) = \sqrt{\frac{2}{\pi}}\frac{1}{(1+y^2)^2}$, *which also are functions satisfying the condition* (A_9), *which implies that*

$$\frac{(H_1\Phi_5)(y) - (1+y^2)H_1\left(\Phi_6 \underset{1}{*} h\right)(y)}{1-(1+y^2)^2H_1\left(k \underset{1}{*} h\right)(y)} = \frac{1}{1+\sqrt{\frac{2}{\pi}}}\frac{1}{1+y^2} \in L_2(\mathbb{R}).$$

And under the condition (A_{10}), *we get*

$$\frac{(H_1\Phi_6)(y) - (1+y^2)H_1\left(\Phi_5 \underset{2}{*} h\right)(y)}{1-(1+y^2)^2H_1\left(k \underset{1}{*} h\right)(y)} = \frac{1}{1+\sqrt{\frac{2}{\pi}}}\frac{1}{1+y^2} \in L_2(\mathbb{R}).$$

In conclusion, the solution of the system equations (5.12) *in this case has the form*

$$(f,g) = \left(\frac{\sqrt{\frac{\pi}{2}}}{1+\sqrt{\frac{2}{\pi}}}e^{-|x|},\ \frac{\sqrt{\frac{\pi}{2}}}{1+\sqrt{\frac{2}{\pi}}}e^{-|x|}\right) \in L_2(\mathbb{R}) \times L_2(\mathbb{R}).$$

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References

[1] Al-Omari. S. K. Q and Baleanu. D. Convolution theorems associated with some integral operators and convolutions. *Math. Methods Appl. Sci.*, **42**(2):541–552, (2019).

[2] Appell. J , Kalitvin. A, and Zabrejko. P. *Partial Integral Operators and Integro-Differential Equations: pure and applied mathematics.* Marcel Dekker, New York, 2000.

[3] Appell. J , Diallo. O. W, and Zabrejko. P. P. On linear integro-differential equations of Barbashin type in spaces of continuous and measurable functions. *J. Integral Equations Appl.*, **1**(2):227–247, (1988).

[4] Barbashin. E. A . Conditions for invariance of stability of solutions of integro-differential equations. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (in Russian), 1:25–34, (1957).

[5] Bateman. H . *Tables of Integral Transforms*, volume 1. New York (NY), London, McGraw-Hill Book Company, Inc, 1954.

[6] Britvina. L. E . A class of integral transforms related to the Fourier cosine convolution. *Integral Transforms Spec. Funct.*, **16**(5-6):379–389, (2005).

[7] Debnath. L and Bhatta. D. *Integral Transforms and Their Applications.* 2nd edn. Chapman & Hall/CRC Press, New York, 2007.

[8] F. Al-Musallam and Vu Kim Tuan. Integral transforms related to a generalized convolution. *Result. Math.*, **38**(3):197–208, (2000).

[9] Giang. B. T. , N. V. Mau, and N. M. Tuan. Operational properties of two integral transforms of Fourier type and their convolutions. *Integr. Equ. Oper. Theory.*, **65**(3):363–386, (2009).

[10] H. T. V. Anh. and N. X. Thao. Hartley–Fourier cosine generalized convolution inequalities. *Math. Inequal. Appl.*, **18**(4):1393–1408, (2015).

[11] Hoang. P. V , Tuan. T, Thao. N. X, and Tuan. V. K. Boundedness in weighted L_p spaces for the Kontorovich–Lebedev–Fourier generalized convolutions and applications. *Integral Transforms Spec. Funct.*, **28**(8):590–604, (2017).

[12] Kilbas. A. A , Srivastava. H. M, and Trujillo. J. J. *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam, 2006.

[13] Michel Plancherel . Contribution à l'étude de la représentation d'une fonction arbitraire par des intégrales définies. *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, **30**(1):289–335, (1910).

[14] N. T. Hong , T. Tuan, and N. X. Thao. On the Fourier cosine-Kontorovich-Lebedev generalized convolution transforms. *Applications of Mathematics*, **58**(4):473–486, (2013).

[15] P. T. V. Anh. and N. X. Thao. Integral transforms of Hartley–Fourier cosine polyconvolution type. *Appl. Anal.*, **94**(9):1749–1765, (2015).

[16] Poularikas. A. D . *The Transform and Applications Handbook.* CRC Press, Florida, 227, 1st edition, 1996.

[17] Stein. E. M and Weiss. G. Introduction to Fourier Analysis on Euclidean spaces. *Princeton Mathematical Series*, vol 32, Princeton University Press, Princeton, NJ, 1971.

[18] T. Tuan . On the Fourier-sine and Kontorovich–Lebedev Generalized Convolution Transforms and Their Applications. *Ukrainian Mathematical Journal*, **72**(2):302–316, (2020). https://doi.org/10.1007/s11253-020-01782-1.

[19] T. Tuan . Some classes of integral equations of convolutions-pair generated bythe Kontorovich-Lebedev, Laplace and Fourier transforms. *Commun. Korean Math. Soc.*, **36**(3):485–494, (2021). https://doi.org/10.4134/CKMS.c200183.

[20] T. Tuan , P. V. Hoang, and N. T. Hong. Integral equation of Toeplitz plus Hankel's type and parabolic equation related to the Kontorovich-Lebedev-Fourier generalized convolutions. *Math. Methods Appl. Sci.*, **41**(17):8171–8181, (2018).

[21] T. Tuan and N. T. Hong. A class of Fredholm equations and systems of equations related to the Kontorovich-Lebedev and the Fourier integral transforms. *Turk. J. Math.*, **44**(3):643–655, (2020). https://doi.org/10.3906/mat-2001-24.

[22] T. Tuan , N. T. Hong, and P. V. Hoang. Generalized convolution for the Kontorovich-Lebedev, Fourier transforms and applications to acoustic fields. *Acta. Math. Vietnam.*, **41**(2):355–367, (2016).

[23] T. Tuan , N. X. Thao, and N. V. Mau. On the generalized convolution for the Fourier sine and the Kontorovich-Lebedev transforms. *Acta. Math. Vietnam.*, **35**(2):303–317, (2010).

[24] Thao. N. X. and H. T. V. Anh. On the Hartley–Fourier sine generalized convolution. *Math. Methods Appl. Sci.*, **37**(15):2308–2319, (2014).

[25] Titchmarsh. E. C . *Introduction to the Theory of Fourier Integrals.* Chelsea Publishing Co. New York, 3rd edition, 1986.

[26] Vu Kim Tuan . Integral transforms of Fourier cosine convolution type. *J. Math. Anal. Appl.*, **229**(2):519–529, (1999).

[27] Watson. G. N . General Transforms. *Proc. London Math. Soc.*, **35**(1):156–199, (1933).

[28] Yakubovich. S. B and Britvina. L. E. Convolution operators related to the Fourier cosine and Kontorovich–Lebedev transformations. *Result. Math.*, **55**(1):175–197, (2009).

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