Existence and nonexistence of nontrivial solutions for the Schrödinger-Poisson system with zero mass potential *

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Abstract: In this paper, we study the existence and nonexistence of nontrivial solutions for the following Schrödinger-Poisson system with zero mass potential

$$\begin{cases} -\triangle u + \phi u = -a|u|^{p-2}u + f(u), & x \in \mathbb{R}^3, \\ -\triangle \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $a > 0$ and $p \in (2, \frac{12}{5})$. We obtain the existence of nontrivial radial solutions and the nonexistence of nontrivial solutions for the above system under weaker assumptions on a and f. In particular, applying our results to the following system:

$$\begin{cases} -\triangle u + \phi u = -|u|^{p-2}u + |u|^{q-2}u, & x \in \mathbb{R}^3, \\ -\triangle \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

a sufficient and necessary condition is obtained on the existence of nontrivial radial solutions.

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1 Introduction

In this paper, we are concerned with the existence and nonexistence of nontrivial solutions

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for the Schrödinger-Poisson system with zero mass potential

$$\begin{cases} -\triangle u + \phi u = -a|u|^{p-2}u + f(u), & x \in \mathbb{R}^3, \\ -\triangle \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $a > 0, p \in (2, 12/5)$ and f satisfies

(F1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, and there exist constants $\mathcal{C}_0 > 0$ and $q \in (p, 6)$ such that

$$|f(t)| \leq \mathcal{C}_0 \left(1 + |t|^{q-1}\right), \quad \forall\, t \in \mathbb{R};$$

(F2) $f(t) = o(|t|^{p-1})$ as $t \to 0$.

System (1.1) is a special form of the following Schrödinger-Poisson system

$$\begin{cases} -\triangle u + \lambda u + \mu \phi u = g(u), & x \in \mathbb{R}^3, \\ -\triangle \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

which also known as the nonlinear Schrödinger-Maxwell system, was first introduced in [4] as a model describing solitary waves for the nonlinear stationary Schrödinger equations interacting with the electrostatic field. It has a strong physical meaning because it appears in quantum mechanics models (see e.g. [6, 7, 20]) and in semiconductor theory [5, 22, 23]. For more details in the physical aspects, we refer the readers to [4, 5]. In recent years, there has been increasing attention to systems like (1.2) on the existence of positive solutions, ground state solutions, multiple solutions and semiclassical states, see e.g. [2, 3, 8, 9, 10, 14, 15, 16, 17, 18, 25, 30, 31].

When $\lambda = 1$ and $g(u) = |u|^{q-2}u$, then (1.2) reduces to the following special form:

$$\begin{cases} -\triangle u + u + \mu \phi u = |u|^{q-2}u, & x \in \mathbb{R}^3, \\ -\triangle \phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.3}$$

For (1.3), there are many results on the existence of solutions. For example, in [12, 13] a radial positive solution of (1.3) is found for $4 < q < 6$, in which it is easy to verify the mountain-pass geometry and the the boundedness of (PS)-sequences for the energy functional associated (1.3). However, these arguments do not work for the case $2 < q \leq 4$. By introducing Nehari-Pohozaev manifold, Ruiz [25] first proved that for all $\mu > 0$, (1.3) admits a positive radial solution for the case when $3 < q \leq 4$; whereas for $2 < q < 3$, (1.3) has two different positive solution for μ small enough; but for $\mu \geq \frac{1}{4}$, (1.2) does not admit any nontrivial solution.

In recent years, systems like (1.2) or more general forms have begun to receive much attention, see, for example, [2, 3, 8, 9, 12, 14, 26, 27, 28, 30]. However for system (1.2) with $\lambda = 0$, to the best of our knowledge, there are no results on the existence or nonexistence for nontrivial solutions. In general, there is wide difference for case when $\lambda > 0$ and $\lambda = 0$. In the present paper, we try studying the existence or nonexistence for nontrivial solutions for system (1.2) with $\lambda = 0$.

First at least formally, the energy functional associated with (1.2) ($\lambda = 0$) is

$$\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u(x) u^2 \mathrm{d}x - \int_{\mathbb{R}^3} G(u) \mathrm{d}x, \tag{1.4}$$

where $G(t) := \int_0^t g(s)\mathrm{d}s$ and

$$\phi_u(x) := \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}\mathrm{d}y = \frac{1}{|x|} * u^2 \tag{1.5}$$

is the distributional solution of the Poisson equation $-\triangle\phi = u^2$ belongs to $D^{1,2}(\mathbb{R}^3)$ (see e.g. [25] for more details). By Hardy-Littlewood-Sobolev inequality, one has

$$\int_{\mathbb{R}^3} \phi_u(x) u^2 \mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \le \frac{8\sqrt[3]{2}}{3\sqrt[3]{\pi}}\|u\|_{12/5}^4, \qquad u \in L^{12/5}(\mathbb{R}^3). \tag{1.6}$$

It is well known that Ψ is well-defined on $H^1(\mathbb{R}^3)$. However, $H^1(\mathbb{R}^3)$ is not the working space for system (1.2) with $\lambda = 0$, because there is not an equivalent term to $\|u\|_2^2$ in the energy functional $\Psi(u)$. So it is necessary to add a negative feedback $a|u|^{p-2}u$ with $2 < p \le \frac{12}{5}$ in the nonlinearity $g(u)$ to guarantee Ψ is well-defined in new working space. In what follows, we are concerned with the existence and nonexistence of nontrivial solutions for (1.1).

Obviously, the energy functional associated with (1.2) is

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{a}{p}\int_{\mathbb{R}^3} |u|^p \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u(x) u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(u) \mathrm{d}x, \tag{1.7}$$

where $F(t) := \int_0^t f(s)\mathrm{d}s$. The natural working space E for the energy functional $\Phi(u)$ is given by

$$E := \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^3) : u(x) = u(|x|),\ \int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x < \infty,\ \int_{\mathbb{R}^2} |u|^p \mathrm{d}x < \infty \right\}.$$

To state our results, we make the following assumptions on the nonlinearity f.

(F3) $\lim_{|t|\to\infty} \frac{F(t)}{|t|^3} = \infty$;

(F4) $F(t) \ge 0,\ \forall\, t \in \mathbb{R}$, and there exists $\theta \in (0,1)$ such that

$$f(t)t - 3F(t) + \frac{(3-p)\theta a}{p}|t|^p \ge 0, \qquad \forall\, t \in \mathbb{R};$$

(F5) $\limsup_{|t|\to\infty} \frac{|f(t)|}{|t|^{1+\frac{2p}{3}}} = 0$ or $\liminf_{|t|\to\infty} \frac{f(t)t}{F(t)} > 3$;

(F6) $f(t)t \le 2|t|^3 + a|t|^p$ for all $t \in \mathbb{R}$ and $t = 0$ is the isolated zero of the function $2t^3 + a|t|^p - f(t)t$;

(F7) $f(t)t - 2F(t) \leq \frac{2}{3}|t|^3 + \frac{a(p-2)}{p}|t|^p$ for all $t \in \mathbb{R}$ and $t = 0$ is the isolated zero of the function $\frac{2}{3}t^3 + \frac{a(p-2)}{p}|t|^p - f(t)t + 2F(t)$.

Now, we state our results of this paper.

Theorem 1.1. *Assume that f satisfies* (F1)-(F5)*. Then system* (1.1) *has a nontrivial solution.*

Theorem 1.2. *Assume that f satisfies* (F1), (F2), (F6) *or* (F7)*. Then then* (1.1) *does not admit any nontrivial solution.*

Applying the above theorems to the special form of (1.1):

$$\begin{cases} -\triangle u + \phi u = -a|u|^{p-2}u + b|u|^{q-2}u, & x \in \mathbb{R}^3, \\ -\triangle \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.8}$$

we have the following corollary.

Corollary 1.3. *The following conclusions hold:*

(i) *If $3 < q < 6$ and $b > 0$, then* (1.8) *has a nontrivial solution.*

(ii) *If $p < q < 3$ and $0 < b \leq b_0$, then* (1.8) *does not admit any nontrivial solution, where*

$$b_0 := (3-p)\left(\frac{2}{q-p}\right)^{\frac{q-p}{3-p}}\left(\frac{a}{3-q}\right)^{\frac{3-q}{3-p}}.$$

Further, we have the following theorem.

Theorem 1.4. *Let $q = 3$. If $b > \frac{9009\pi}{2^{18}}\left(\frac{7}{2}\right)^{\frac{5}{6}}\left(\frac{425\sqrt[3]{2}\pi}{2}\right)^{\frac{1}{2}} = 8.894113027...$, then* (1.8) *has a nontrivial radial solution. If $0 < b \leq 2$, then* (1.8) *does not admit any nontrivial solution.*

Combining Corollary 1.3 with Theorem 1.4, we have the following corollary.

Corollary 1.5. *Assume that $p < q < 6$. Then*

$$\begin{cases} -\triangle u + \phi u = -|u|^{p-2}u + |u|^{q-2}u, & x \in \mathbb{R}^3, \\ -\triangle \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.9}$$

has a nontrivial radial solution if and only if $3 < q < 6$.

The paper is organized as follows. In Section 2, we give some notation and preliminaries. In Section 3, we complete the proof of existence results. Section 4 is devoted to proving the theorems on the nonexistence.

Throughout this paper, we let $u_t(x) := u(tx)$ for $t > 0$, and denote the norm of $L^s(\mathbb{R}^3)$ by $\|u\|_s = \left(\int_{\mathbb{R}^3}|u|^s \mathrm{d}x\right)^{1/s}$ for $s \geq 2$, $B_r(x) = \{y \in \mathbb{R}^3 : |y - x| < r\}$, and positive constants possibly different in different places, by $C_1, C_2, \cdots$.

2 Preliminary results

Define

$$\|u\| := \sqrt{\|\nabla u\|_2^2 + \|u\|_p^2}, \quad \forall\, u \in E.$$

Then E is a separable Banach space with the above norm. Let

$$\mathcal{D}^{1,2}(\mathbb{R}^3) = \left\{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\right\}.$$

$\mathcal{D}^{1,2}(\mathbb{R}^3)$ is a Banach space equipped with the norm defined by

$$\|u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x.$$

By (1.5), $\phi_u(x) > 0$ when $u \neq 0$, moreover, we have

$$\int_{\mathbb{R}^3} \nabla\phi_u \cdot \nabla v \mathrm{d}x = \int_{\mathbb{R}^3} u^2 v \mathrm{d}x, \quad \forall\, u, v \in E. \tag{2.1}$$

In view of the Gagliardo-Nirenberg inequality [1, 24], one has

$$\|u\|_s^s \leq \mathcal{C}_s^s \|u\|_p^{(6-s)p/(6-p)} \|\nabla u\|_2^{6(s-p)/(6-p)} \quad \text{for } u \in E,\ s > p, \tag{2.2}$$

where $\mathcal{C}_s > 0$ is a constant determined by s.

Lemma 2.1. [11] *Assume that $p \geq 2$. Then for any $u \in E$ and $r_0 > 0$,*

$$|u(x)| \leq \left(\frac{p+2}{8\pi^2}\right)^{\frac{2}{p+2}} \|u\|_p^{\frac{p}{p+2}} \|\nabla u\|_2^{\frac{2}{p+2}} |x|^{-\frac{4}{p+2}}, \quad \forall\, |x| \geq r_0. \tag{2.3}$$

Lemma 2.2. *The embeddings $E \hookrightarrow L^s(\mathbb{R}^2)$ are continuous for all $s \in [p, \infty)$ and compact for all $s \in (p, 6)$.*

Proof. We give only the proof of the compactness, because the continuousness can be proved similarly. Let $\{u\} \subset E$ be such that $u_n \rightharpoonup 0$. For any $s \in (p, 6)$, $u_n \to 0$ in $L^s_{\text{loc}}(\mathbb{R}^3)$. Hence, it follows from Lemma 2.1 that

$$\begin{aligned}
\int_{\mathbb{R}^2} |u_n|^s \mathrm{d}x &= \int_{B_R} |u_n|^s \mathrm{d}x + \int_{B_R^c} |u_n|^s \mathrm{d}x \\
&\leq \int_{B_R} |u_n|^s \mathrm{d}x + C_1 \|u_n\|_p^{\frac{p(s-p)}{p+2}} \|\nabla u_n\|_2^{\frac{2(s-p)}{p+2}} \int_{B_R^c} |u_n|^p |x|^{-\frac{4(s-p)}{p+2}} \mathrm{d}x \\
&\leq \int_{B_R} |u_n|^s \mathrm{d}x + C_1 \|u_n\|_p^{\frac{2(s+2)}{p+2}} \|\nabla u_n\|_2^{\frac{2(s-p)}{p+2}} R^{-\frac{4(s-p)}{p+2}} \\
&= o_n(1) + o_R(1), \quad n \to \infty,\ R \to \infty.
\end{aligned}$$

This shows that the embeddings $E \hookrightarrow L^s(\mathbb{R}^3)$ with $s \in (p, 6)$ is compact. □

Lemma 2.3. [21] *There holds*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)v(y)|}{|x-y|}\mathrm{d}x\mathrm{d}y \le \frac{8\sqrt[3]{2}}{3\sqrt[3]{\pi}}\|u\|_{6/5}\|v\|_{6/5}, \qquad u,v\in L^{6/5}(\mathbb{R}^3). \tag{2.4}$$

By Lemma 2.3, we have the following corollary.

Corollary 2.4. *There holds*

$$N(u) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \le \frac{8\sqrt[3]{2}}{3\sqrt[3]{\pi}}\|u\|^4_{12/5}, \qquad \forall\, u\in E. \tag{2.5}$$

Lemma 2.5. *Suppose that $u_n \rightharpoonup \bar{u}$ in E. Then $N(u_n)$ converges up to a subsequence to $N(\bar{u})$ as $n\to\infty$, and $\langle N'(u_n),\varphi\rangle$ converges up to a subsequence to $\langle N'(\bar{u}),\varphi\rangle$ as $n\to\infty$ for every $\varphi\in E$.*

Proof. Since $u_n \rightharpoonup \bar{u}$ in E, then $\|u_n\|\le C_1$ for some constant $C_1>0$. By Lemma 2.2, we can assume that $\lim_{n\to\infty}\|u_n-\bar{u}\|_s = 0$ for every $s\in(p,6)$. Hence, it follows from (2.4), (2.5) and the Hölder inequality that

$$\begin{aligned}
& |N(u_n)-N(\bar{u})| \\
= & \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u_n^2(x)u_n^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\bar{u}^2(x)\bar{u}^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y\right| \\
\le & \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n^2(x)-\bar{u}^2(x)|u_n^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\bar{u}^2(x)|u_n^2(y)-\bar{u}^2(y)|}{|x-y|}\mathrm{d}x\mathrm{d}y \\
\le & C_1\|u_n-\bar{u}\|_{12/5}\|u_n+\bar{u}\|_{12/5}\|u_n\|^2_{12/5} + C_2\|u_n-\bar{u}\|_{12/5}\|u_n+\bar{u}\|_{12/5}\|\bar{u}\|^2_{12/5} \\
= & o(1).
\end{aligned} \tag{2.6}$$

This shows that $N(u_n)\to N(\bar{u})$ as $n\to\infty$.

Next, we prove that $\langle N'(u_n),\varphi\rangle$ converges up to a subsequence to $\langle N'(\bar{u}),\varphi\rangle$ as $n\to\infty$ for every $\varphi\in E$. Since $E\cap\mathcal{C}_0^\infty(\mathbb{R}^2)$ is density in E, so we can assume that $\varphi\in E\cap\mathcal{C}_0^\infty(\mathbb{R}^2)$. Therefore, we can choose $R>0$ such that $\mathrm{supp}\varphi\subset B_R$. Hence, it follows from (2.4), (2.5) and the Hölder inequality that

$$\begin{aligned}
& |\langle N(u_n)-N(\bar{u}),\varphi\rangle| \\
= & 4\left|\int_{\mathbb{R}^3}[\phi_{u_n}(x)u_n(x)-\phi_{\bar{u}}(x)\bar{u}(x)]\varphi(x)\mathrm{d}x\right| \\
= & \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u_n(x)\varphi(x)u_n^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\bar{u}(x)\varphi(x)\bar{u}^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y\right| \\
\le & \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)-\bar{u}(x)||\varphi(x)|u_n^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|\bar{u}(x)||\varphi(x)||u_n^2(y)-\bar{u}^2(y)|}{|x-y|}\mathrm{d}x\mathrm{d}y \\
\le & C_3\|u_n-\bar{u}\|_{12/5}\|\varphi\|_{12/5}\|u_n\|^2_{12/5} + C_4\|u_n-\bar{u}\|_{12/5}\|u_n+\bar{u}\|_{12/5}\|\bar{u}\|_{12/5}\|\varphi\|_{12/5} \\
= & o(1).
\end{aligned} \tag{2.7}$$

This shows that $\langle N'(u_n),\varphi\rangle \to N'(\bar{u}),\varphi\rangle$ as $n\to\infty$ for every $\varphi\in E$.

□

By using Lemmas 2.3 and 2.5, it is easy to verify that Φ is well-defined of class $\mathcal{C}^1$ functional, and that

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^3} \nabla u \cdot \nabla v \mathrm{d}x + a\int_{\mathbb{R}^3} |u|^{p-2}uv\mathrm{d}x + \int_{\mathbb{R}^3} [\phi_u(x)u - f(u)]\, v\mathrm{d}x. \tag{2.8}$$

3 Existence results

In this section, we give the proof of Theorems 1.1, 1.2 and 1.4.

Proposition 3.1. [19] *Let X be a Banach space and let $J \subset \mathbb{R}^+$ be an interval, and*

$$\Phi_\lambda(u) = A(u) - \lambda B(u), \qquad \forall\ \lambda \in J,$$

be a family of $\mathcal{C}^1$-functional on X such that

(i) *either $A(u) \to +\infty$ or $B(u) \to +\infty$, as $\|u\| \to \infty$;*

(ii) *$B(u) \geq 0$ for all $u \in X$;*

(iii) *there are two points v_1, v_2 in X such that*

$$c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_\lambda(\gamma(t)) > \max\{\Phi_\lambda(v_1), \Phi_\lambda(v_2)\},$$

where

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\lambda \in J$, there exists a sequence such that

(i) *$\{u_n(\lambda)\}$ is bounded in X;*

(ii) *$\Phi_\lambda(u_n(\lambda)) \to c_\lambda$;*

(iii) *$\Phi'_\lambda(u_n(\lambda)) \to 0$ in X^*, where X^* is the dual of X;*

(iv) *c_λ is non-increasing on $\lambda \in J$.*

To apply Proposition 3.1, we use the idea employed by Jeanjean [19] which is an approximation procedure. Precisely, for any $\lambda \in [1/2, 1]$ we study the functional $\Phi_\lambda : E \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\mathrm{d}x + \frac{a}{p}\int_{\mathbb{R}^2} |u|^p\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u(x)u^2\mathrm{d}x - \lambda\int_{\mathbb{R}^2} F(u)\mathrm{d}x. \tag{3.1}$$

Obviously, $\Phi_\lambda \in \mathcal{C}^1(E, \mathbb{R})$, and

$$\langle \Phi'_\lambda(u), u\rangle = \int_{\mathbb{R}^2} |\nabla u|^2\mathrm{d}x + a\int_{\mathbb{R}^2} |u|^p\mathrm{d}x + \int_{\mathbb{R}^3} \phi_u(x)u^2\mathrm{d}x - \lambda\int_{\mathbb{R}^2} f(u)u\mathrm{d}x. \tag{3.2}$$

By a similar argument as the one in [25, Theorem 2.2], we can prove the following lemma.

Lemma 3.2. *Assume that* (F1)-(F3) *hold. Let u be a critical point of Φ_λ in E, then we have the following Pohozaev type identity*

$$\mathcal{P}_\lambda(u) \;:=\; \frac{1}{2}\|\nabla u\|_2^2+\frac{3a}{p}\int_{\mathbb{R}^3}|u|^p\mathrm{d}x+\frac{5}{4}\int_{\mathbb{R}^3}\phi_u(x)u^2\mathrm{d}x-3\lambda\int_{\mathbb{R}^3}F(u)\mathrm{d}x=0. \quad (3.3)$$

Lemma 3.3. *Assume that* (F1)-(F3) *hold. Let $\hat{u}\in E\setminus\{0\}$. Then*

(i) *there exists $T_0>0$ independent of λ such that $\Phi_\lambda(T_0\hat{u}_{T_0})<0$ for all $\lambda\in[1/2,1]$;*

(ii) *there exists a positive constant κ_0 independent of λ such that for all $\lambda\in[1/2,1]$,*

$$c_\lambda:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Phi_\lambda(\gamma(t))\geq\kappa_0>\max\left\{\Phi_\lambda(0),\Phi_\lambda(T_0\hat{u}_{T_0})\right\}, \quad (3.4)$$

where

$$\Gamma=\{\gamma\in\mathcal{C}([0,1],E):\gamma(0)=0,\gamma(1)=T_0\hat{u}_{T_0}\};$$

(iii) *c_λ is non-increasing on $\lambda\in[1/2,1]$.*

Proof. (i). It follows from (3.1) that

$$\begin{aligned}\Phi_\lambda(t^2\hat{u}_t) \;&=\; \frac{t^3}{2}\int_{\mathbb{R}^3}|\nabla\hat{u}|^2\mathrm{d}x+\frac{at^{2p-3}}{p}\int_{\mathbb{R}^3}|\hat{u}|^p\mathrm{d}x+\frac{t^3}{4}\int_{\mathbb{R}^3}\phi_{\hat{u}}(x)\hat{u}^2\mathrm{d}x-\frac{\lambda}{t^3}\int_{\mathbb{R}^3}F(t^2\hat{u})\mathrm{d}x\\ &\leq\; \frac{t^3}{2}\int_{\mathbb{R}^3}|\nabla\hat{u}|^2\mathrm{d}x+\frac{at^{2p-3}}{p}\int_{\mathbb{R}^3}|\hat{u}|^p\mathrm{d}x+\frac{t^3}{4}\int_{\mathbb{R}^3}\phi_{\hat{u}}(x)\hat{u}^2\mathrm{d}x\\ &\quad -\frac{1}{2t^3}\int_{\mathbb{R}^3}F(t^2\hat{u})\mathrm{d}x, \qquad \forall\ \lambda\in[1/2,1]. \end{aligned} \quad (3.5)$$

This, together with (F3), implies that there exists $T_0>0$ independent of λ such that $\Phi_\lambda(T_0\hat{u}_{T_0})<0$ for all $\lambda\in[1/2,1]$.

(ii). In view of the Sobolev inequality, one has

$$\|u\|_6^2\leq\mathcal{S}^{-1}\|\nabla u\|_2^2. \quad (3.6)$$

By (F1) and (F2), there exists $C_1>0$ such that

$$F(t)\leq\frac{a}{2p}|t|^p+C_1|t|^6, \qquad \forall\ t\in\mathbb{R}. \quad (3.7)$$

From (3.7), we obtain

$$\int_{\mathbb{R}^3}F(u)\mathrm{d}x \;\leq\; \frac{a}{2p}\|u\|_p^p+C_2\|u\|_6^6, \qquad \forall\ u\in E. \quad (3.8)$$

Hence, it follows from (3.1), (3.6) and (3.8) that

$$\begin{aligned}\Phi_\lambda(u) \;&=\; \frac{1}{2}\|\nabla u\|_2^2+\frac{1}{4}N(u)+\frac{a}{p}\|u\|_p^p-\lambda\int_{\mathbb{R}^3}F(u)\mathrm{d}x\\ &\geq\; \frac{1}{2}\|\nabla u\|_2^2+\frac{a}{p}\|u\|_p^p-\left[\frac{a}{2p}\|u\|_p^p+C_2\mathcal{S}^{-3}\|\nabla u\|_2^6\right]\end{aligned}$$

$$\geq \frac{1}{2}\|\nabla u\|_2^2 + \frac{a}{2p}\|u\|_p^p - C_2\mathcal{S}^{-3}\|\nabla u\|_2^6. \tag{3.9}$$

Therefore, there exist $\kappa_0 > 0$ and $\rho_0 > 0$ such that

$$\Phi_\lambda(u) \geq \kappa_0, \quad \forall\, u \in S := \left\{u \in E : \|\nabla u\|_2^2 + \|u\|_p^2 = \rho_0^2\right\}, \quad \lambda \in [1/2, 1]. \tag{3.10}$$

This shows that (ii) holds.

(iii) is a direct corollary of (iv) in Proposition 3.1. □

If $b > \frac{9009\pi}{2^{18}}\left(\frac{7}{2}\right)^{\frac{5}{6}}\left(\frac{425\sqrt[3]{2}\pi}{2}\right)^{\frac{1}{2}}$, then we can choose $\lambda_1 \in (0,1)$ such that

$$b\lambda_1 > \frac{9009\pi}{2^{18}}\left(\frac{7}{2}\right)^{\frac{5}{6}}\left(\frac{425\sqrt[3]{2}\pi}{2}\right)^{\frac{1}{2}}. \tag{3.11}$$

Let $\kappa := \frac{425}{2\sqrt[3]{2}\pi}\left(\frac{2}{7}\right)^{\frac{5}{3}}$ and $w = \frac{\kappa}{(1+|x|^2)^{5/2}}$. Then $w \in E$, and

$$\begin{aligned} \|\nabla w\|_2^2 &= \int_{\mathbb{R}^N} |\nabla w|^2 \mathrm{d}x = 100\pi\kappa^2 \int_0^{+\infty} \frac{r^4}{(1+r^2)^7}\mathrm{d}r \\ &= 50\pi\kappa^2 \int_0^{+\infty} \frac{s^{3/2}}{(1+s)^7}\mathrm{d}s = \frac{50\pi\kappa^2\ \Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{9}{2}\right)}{6!} := \frac{175\pi^2\kappa^2}{2^9}, \end{aligned} \tag{3.12}$$

$$\|w\|_s^s = \int_{\mathbb{R}^N} |w|^s \mathrm{d}x = 4\pi\kappa^s \int_0^{+\infty} \frac{r^2}{(1+r^2)^{5s/2}}\mathrm{d}r = \frac{2\pi\kappa^s\ \Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5s-3}{2}\right)}{\Gamma\left(\frac{5s}{2}\right)}, \tag{3.13}$$

$$\|w\|_3^3 = \frac{2\pi\kappa^3\ \Gamma\left(\frac{3}{2}\right)\Gamma(6)}{\Gamma\left(\frac{15}{2}\right)} = \frac{2^{10}\pi\kappa^3}{9009} \tag{3.14}$$

and

$$\|w\|_{12/5}^4 = \left(\int_{\mathbb{R}^3} |w|^{12/5}\mathrm{d}x\right)^{\frac{5}{3}} = \left[\frac{2\pi\kappa^{12/5}\ \Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{9}{2}\right)}{\Gamma(6)}\right]^{\frac{5}{3}} = \left(\frac{7}{2}\right)^{\frac{5}{3}} \frac{\pi^3\sqrt[3]{\pi}\kappa^4}{2^{10}}. \tag{3.15}$$

Both (2.4) and (3.15) imply

$$\int_{\mathbb{R}^3} \phi_w(x) w^2 \mathrm{d}x \leq \frac{8\sqrt[3]{2}}{3\sqrt[3]{\pi}}\|w\|_{12/5}^4 = \frac{\sqrt[3]{2}}{3}\left(\frac{7}{2}\right)^{\frac{5}{3}} \frac{\pi^3\kappa^4}{2^7}. \tag{3.16}$$

Lemma 3.4. *Assume that $f(u) = b|u|u$. Then*

(i) *there exists $T_0 > 0$ independent of λ such that $\Phi_\lambda(T_0 w_{T_0}) < 0$ for all $\lambda \in [\lambda_1, 1]$;*

(ii) *there exists a positive constant κ_0 independent of λ such that for all $\lambda \in [\lambda_1, 1]$,*

$$c_\lambda := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \Phi_\mu(\gamma(t)) \geq \kappa_0 > \max\left\{\Phi_\lambda(0), \Phi_\lambda(T_0 w_{T_0})\right\}, \tag{3.17}$$

where

$$\Gamma = \left\{\gamma \in \mathcal{C}([0,1], E) : \gamma(0) = 0, \gamma(1) = T_0 w_{T_0}\right\};$$

(iii) c_λ *is non-increasing on* $\lambda \in [\lambda_1, 1]$.

Proof. We only prove (i), since (ii) and (iii) can be proved by the same arguments as in Lemma 3.3. To show (i), Then from (3.1), (3.12), (3.13), (3.14) and (3.16), we have

$$\begin{aligned}
\Phi_\lambda(t^2 w_t) &= \frac{t^3}{2}\int_{\mathbb{R}^3}|\nabla w|^2 \mathrm{d}x + \frac{at^{2p-3}}{p}\int_{\mathbb{R}^3}|w|^p \mathrm{d}x + \frac{t^3}{4}\int_{\mathbb{R}^3}\phi_w(x)w^2\mathrm{d}x - \frac{\lambda b t^3}{3}\int_{\mathbb{R}^3}|w|^3\mathrm{d}x \\
&\le \frac{t^3}{2}\int_{\mathbb{R}^3}|\nabla w|^2 \mathrm{d}x + \frac{at^{2p-3}}{p}\int_{\mathbb{R}^3}w^p \mathrm{d}x + \frac{t^3}{4}\int_{\mathbb{R}^3}\phi_w(x)w^2\mathrm{d}x - \frac{\lambda_1 b t^3}{3}\int_{\mathbb{R}^3}|w|^3\mathrm{d}x \\
&\le \left[\frac{175\pi}{2^{10}} + \frac{\sqrt[3]{2}}{3}\left(\frac{7}{2}\right)^{\frac{5}{3}}\frac{\pi^2\kappa^2}{2^9} - \frac{2^{10}\kappa b\lambda_1}{27027}\right]\pi\kappa^2 t^3 + \frac{2\pi\kappa^s\ \Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5p-3}{2}\right)at^{2p-3}}{p\Gamma\left(\frac{5p}{2}\right)}, \\
&\qquad\qquad \forall\ t>0,\ \ \lambda\in[\lambda_1,1]. \qquad (3.18)
\end{aligned}$$

By (3.11), we have

$$\frac{175\pi}{2^{10}} + \frac{\sqrt[3]{2}}{3}\left(\frac{7}{2}\right)^{\frac{5}{3}}\frac{\pi^2\kappa^2}{2^9} - \frac{2^{10}\kappa b\lambda_1}{27027} < 0, \quad \forall\ \lambda\in[\lambda_1,1]. \qquad (3.19)$$

This, together with (3.18), implies that there exists $T_0 > 0$ independent of $\lambda \in [\lambda_1, 1]$ such that $\Phi_\lambda(T_0^2 w_{T_0}) < 0$ for all $\lambda \in [\lambda_1, 1]$. □

Lemma 3.5. *Assume that* (F1)-(F4) *hold. Then for almost every* $\lambda \in [1/2, 1]$*, there exists* $u_\lambda \in E \setminus \{0\}$ *such that*

$$\Phi'_\lambda(u_\lambda) = 0, \qquad \Phi_\lambda(u_\lambda) \le c_\lambda. \qquad (3.20)$$

Proof. By Proposition 3.1 and Lemma 3.3, for almost every $\lambda \in [1/2, 1]$, we deduce that there exists a bounded sequence $\{u_n(\lambda)\} \subset E$ (still denoted by $\{u_n\}$ for simplicity) satisfying

$$\Phi_\lambda(u_n) \to c_\lambda \le c_{1/2}, \qquad \Phi'_\lambda(u_n) \to 0. \qquad (3.21)$$

We may thus assume, passing to a subsequence if necessary, that $u_n \rightharpoonup u_\lambda$ in E, $u_n \to u_\lambda$ in $L^s(\mathbb{R}^3)$ for $s \in (p, 6)$ and $u_n \to u_\lambda$ a.e. on $\mathbb{R}^3$. If $u_\lambda = 0$, then $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for $s \in (p, 6)$. Arguing as in [9, Proof of Theorem 1.4]), we can deduce a contradiction by using (F1), (F2), (2.5), (3.17) and (3.21). Thus, $u_\lambda \neq 0$. By a standard argument, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} f(u_n)\phi \mathrm{d}x = \int_{\mathbb{R}^3} f(u_\lambda)\phi \mathrm{d}x, \qquad \forall\ \phi \in \mathcal{C}_0^\infty(\mathbb{R}^3). \qquad (3.22)$$

By (3.2), (3.21), (3.22) and Lemma 2.5, it is easy to deduce that $\Phi'(u_\lambda) = 0$. Hence, Lemma 3.2 yields that $\mathcal{P}_\lambda(u_\lambda) = 0$. Now from (F4), (3.1), (3.2), (3.3), (3.22) and Fatou's lemma, one has

$$c_\lambda = \lim_{n\to\infty}\left[\Phi_\lambda(u_n) - \frac{2}{3}\langle\Phi'_\lambda(u_n), u_n\rangle + \frac{1}{3}\mathcal{P}_\lambda(u_n)\right]$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left\{\frac{2(3-p)a}{3p}\|u_n\|_p^p+\frac{2\lambda}{3}\int_{\mathbb{R}^3}[f(u_n)u_n-3F(u_n)]\,\mathrm{d}x\right\}\\
&= \lim_{n\to\infty}\left\{\frac{2(1-\lambda)(3-p)a}{3p}\|u_n\|_p^p+\frac{2\lambda}{3}\int_{\mathbb{R}^3}\left[f(u_n)u_n-3F(u_n)+\frac{(3-p)a}{p}|u|^p\right]\mathrm{d}x\right\}\\
&\geq \frac{2(3-p)a}{3p}\|u_\lambda\|_p^p+\frac{2\lambda}{3}\int_{\mathbb{R}^3}[f(u_\lambda)u_\lambda-3F(u_\lambda)]\,\mathrm{d}x\\
&= \Phi_\lambda(u_\lambda)-\frac{2}{3}\langle\Phi_\lambda'(u_\lambda),u_\lambda\rangle+\frac{1}{3}\mathcal{P}_\lambda(u_\lambda)\\
&= \Phi_\lambda(u_\lambda).
\end{aligned}
$$

This shows (3.20) holds. □

Proof of Theorem 1.1. In view of Proposition 3.1, Lemmas 3.3 and 3.5, there exist two sequences of $\{\lambda_n\}\subset[1/2,1]$ and $\{u_{\lambda_n}\}\subset H^1(\mathbb{R}^3)$, denoted by $\{u_n\}$, such that

$$
\lambda_n\to 1,\qquad \Phi_{\lambda_n}'(u_n)=0,\qquad \mathcal{P}_{\lambda_n}(u_n)=0,\qquad \delta_n:=\Phi_{\lambda_n}(u_n)\leq c_{\lambda_n}. \tag{3.23}
$$

From (3.1), (3.2) and (3.23), one has

$$
\begin{aligned}
c_{1/2} &\geq \delta_n=\Phi_{\lambda_n}(u_n)-\frac{2}{3}\langle\Phi_{\lambda_n}'(u_n),u_n\rangle+\frac{1}{3}\mathcal{P}_{\lambda_n}(u_n)\\
&= \frac{2(3-p)a}{3p}\|u_n\|_p^p+\frac{2\lambda_n}{3}\int_{\mathbb{R}^3}[f(u_n)u_n-3F(u_n)]\,\mathrm{d}x\\
&\geq \frac{2(1-\theta)(3-p)a}{3p}\|u_n\|_p^p\\
&\quad+\frac{2\lambda_n}{3}\int_{\mathbb{R}^3}\left[f(u_n)u_n-3F(u_n)+\frac{\theta(3-p)a}{p}|u_n|^p\right]\mathrm{d}x.
\end{aligned}\tag{3.24}
$$

This, together with (F4), shows that $\{\|u_n\|_p\}$ is bounded. Thus, there exists $C_1>0$ such that $\|u_n\|_p\leq C_1$. By (F2), (F4) and (F5), there exist $C_2>0$ such that

$$
f(t)t\leq C_2|t|^p+\frac{1}{2}\mathcal{C}_{2+\frac{2p}{3}}^{-2-\frac{2p}{3}}C_1^{-\frac{2p}{3}}|t|^{2+\frac{2p}{3}},\quad \forall\, t\in\mathbb{R}, \tag{3.25}
$$

or there exist $\mu\in(3,6)$ and $R>0$ such that

$$
f(t)t\geq\mu F(t)\geq 0,\qquad \forall\ |t|\geq R. \tag{3.26}
$$

Next, we demonstrate that $\{\|\nabla u_n\|_2\}$ is also bounded. If (3.25) holds, then according to (F1), (F2), (3.2), (3.23) and (3.26), we have

$$
\begin{aligned}
\|\nabla u_n\|_2^2+N(u_n)+a\|u_n\|_p^p &= \lambda_n\int_{\mathbb{R}^3}f(u_n)u_n\mathrm{d}x\\
&\leq C_2\|u_n\|_p^p+\frac{1}{2}\mathcal{C}_{2+\frac{2p}{3}}^{-2-\frac{2p}{3}}C_1^{-\frac{2p}{3}}\|u_n\|_{2+\frac{2p}{3}}^{2+\frac{2p}{3}}\\
&\leq C_2\|u_n\|_p^p+\frac{1}{2}\|\nabla u_n\|_2^2,
\end{aligned}\tag{3.27}
$$

which, together with the boundedness of $\{\|u_n\|_p\}$, implies that $\{\|\nabla u_n\|_2\}$ is bounded, and so $\{u_n\}$ is bounded in E.

If (3.26) holds, then it follows from (3.24) that

$$
\begin{aligned}
c_{1/2} &\geq \frac{2(1-\theta)(3-p)a}{3p}\|u_n\|_p^p + \frac{2\lambda_n}{3}\int_{\mathbb{R}^3}\left[f(u_n)u_n - 3F(u_n) + \frac{\theta(3-p)a}{p}|u_n|^p\right]\mathrm{d}x \\
&\geq \frac{2(1-\theta)(3-p)a}{3p}\|u_n\|_p^p + \frac{2(\mu-3)}{6\mu}\int_{|u_n|\geq R} f(u_n)u_n\mathrm{d}x. \qquad (3.28)
\end{aligned}
$$

according to (F1), (F2), (3.2), (3.23) and (3.28), we have

$$
\begin{aligned}
\|\nabla u_n\|_2^2 + N(u_n) + a\|u_n\|_p^p &= \lambda_n \int_{\mathbb{R}^3} f(u_n)u_n\mathrm{d}x \\
&\leq C_3\|u_n\|_p^p + \int_{|u_n|\geq R} f(u_n)u_n\mathrm{d}x \\
&\leq C_4, \qquad (3.29)
\end{aligned}
$$

which, together with the boundedness of $\{\|u_n\|_p\}$, implies that $\{\|\nabla u_n\|_2\}$ is bounded, and so $\{u_n\}$ is also bounded in E.

By (F1) and (F2), there exists $C_5 > 0$ such that

$$
f(t)t \leq a|t|^p + C_5|t|^6, \qquad \forall\, t \in \mathbb{R}. \qquad (3.30)
$$

From (3.2), (3.6), (3.23) and (3.30), we have

$$
\begin{aligned}
\|\nabla u_n\|_2^2 + a\|u_n\|_p^p &\leq \|\nabla u_n\|_2^2 + N(u_n) + a\|u_n\|_p^p \\
&= \lambda_n \int_{\mathbb{R}^3} f(u_n)u_n\mathrm{d}x \\
&\leq a\|u_n\|_p^p + C_5\|u_n\|_6^6 \\
&\leq a\|u_n\|_p^p + C_5\mathcal{S}^{-3}\|\nabla u_n\|_2^6, \qquad (3.31)
\end{aligned}
$$

which implies that

$$
\|\nabla u_n\|_2^2 \geq \sqrt{\frac{\mathcal{S}^3}{C_5}}. \qquad (3.32)
$$

Since $\{u_n\}$ is bounded in E, we may assume, passing to a subsequence if necessary, that $u_n \rightharpoonup \bar{u}$ in E, $u_n \to \bar{u}$ in $L^s(\mathbb{R}^2)$ for $s \in (p, \infty)$ and $u_n \to \bar{u}$ a.e. on $\mathbb{R}^2$. Choose $C_6 > 0$ such that $\|u_n\|_p^p \leq C_6$. Hence, from (F1), (F2), (3.2), (3.23) and (3.32), we have

$$
\begin{aligned}
\sqrt{\frac{\mathcal{S}^3}{C_5}} &\leq \lim_{n\to\infty}\left[\|\nabla u_n\|_2^2 + N(u_n) + a\|u_n\|_p^p\right] \\
&= \lim_{n\to\infty} \lambda_n \int_{\mathbb{R}^3} f(u_n)u_n\mathrm{d}x \\
&\leq \lim_{n\to\infty}\int_{\mathbb{R}^3}\left(\frac{1}{2C_6}\sqrt{\frac{\mathcal{S}^3}{C_5}}|u_n|^p + C_5|u_n|^q\right)\mathrm{d}x \\
&\leq \frac{1}{2}\sqrt{\frac{\mathcal{S}^3}{C_5}} + C_7\lim_{n\to\infty}\|u_n\|_q^q
\end{aligned}
$$

$$= \frac{1}{2}\sqrt{\frac{\mathcal{S}^3}{C_5}} + C_7\|\bar{u}\|_q^q. \tag{3.33}$$

This shows that $\bar{u} \neq 0$. By a standard argument, we have $\Phi'(\bar{u}) = 0$.

□

Replace Lemma 3.3 with Lemma 3.4, we can prove the first part in Theorem 1.4 by similar arguments.

4 Nonexistence results

In this section, we give the proof of Theorem 1.2.

Lemma 4.1. [29] *Suppose that $u \in H^1(\mathbb{R}^3)$ and $-\Delta\phi = u^2$. Then there holds*

$$\int_{\mathbb{R}^3} \left(b_1|\nabla u|^2 + b_2\phi u^2\right) \mathrm{d}x \geq 2\sqrt{b_1 b_2}\int_{\mathbb{R}^3} |u|^3 \mathrm{d}x, \qquad \forall\ b_1, b_2 > 0;\ u \in E. \tag{4.1}$$

Proof of Theorem 1.2. Suppose that $(\bar{u}, \bar{\phi})$ is a solution of (1.1). Multiply the first equation by $\bar{u}$ and integrate, we obtain

$$\|\nabla\bar{u}\|_2^2 + N(\bar{u}) + a\|\bar{u}\|_p^p - \int_{\mathbb{R}^2} f(\bar{u})\bar{u}\mathrm{d}x = 0. \tag{4.2}$$

From the Pohozaev identity in Lemma 3.2, it follows that

$$\frac{1}{2}\|\nabla\bar{u}\|_2^2 + \frac{3a}{p}\int_{\mathbb{R}^3} |\bar{u}|^p\mathrm{d}x + \frac{5}{4}N(\bar{u}) - 3\int_{\mathbb{R}^3} F(\bar{u})\mathrm{d}x = 0. \tag{4.3}$$

Combining (4.2) with (4.3), we get

$$2\|\nabla\bar{u}\|_2^2 + \frac{1}{2}N(\bar{u}) + \frac{3a(p-2)}{p}\|\bar{u}\|_p^p - 3\int_{\mathbb{R}^2} [f(\bar{u})\bar{u} - 2F(\bar{u})]\mathrm{d}x = 0. \tag{4.4}$$

By (4.2) and Lemma 4.1, we deduce

$$0 \geq \int_{\mathbb{R}^3} \left(2|\bar{u}|^3 + a|\bar{u}|^p - f(\bar{u})\bar{u}\right) \mathrm{d}x, \tag{4.5}$$

which, together with (F6), implies $\bar{u} = 0$. Similarly, by (4.4) and Lemma 4.1, we deduce

$$0 \geq \int_{\mathbb{R}^3} \left[\frac{2}{3}|\bar{u}|^3 + \frac{a(p-2)}{p}|\bar{u}|^p - f(\bar{u})\bar{u} + 2F(\bar{u})\right] \mathrm{d}x, \tag{4.6}$$

which, together with (F7), implies $\bar{u} = 0$.

□

The first part in Theorem 1.4 is a direct corollary of Theorem 1.2.

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