On the planar Schrödinger-Poisson system with zero mass potential *

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Abstract

In this paper, we prove that the following planar Schrödinger-Poisson system with zero mass

$$\begin{cases} -\Delta u + \phi u = f(u), & x \in \mathbb{R}^2, \\ \Delta \phi = 2\pi u^2, & x \in \mathbb{R}^2, \end{cases}$$

admits a nontrivial radially symmetric solution under weaker assumptions on f by using some new analytical approaches.

Keywords: Planar Schrödinger-Poisson system; Logarithmic convolution potential; Ground state solution; Zero mass.

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1 Introduction

Considered the following planar Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(u), & x \in \mathbb{R}^2, \\ \Delta \phi = 2\pi u^2, & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $V \in \mathcal{C}(\mathbb{R}^2, [0, \infty))$ and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$. In the past several years, Alves and Figueiredo [3], Cingolani and Weth [17], Du and Weth [18],Chen and Tang [11, 12, 14, 15] obtained some interesting results on the existence of nontrivial solutions, ground state solutions and infinitely many solutions for (1.1).

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System (1.1) is the special form for the Schrödinger-Poisson system of the type

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(u), & x \in \mathbb{R}^N, \\ \Delta \phi = 2\pi u^2, & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $N \geq 2$. It is well-known that the solutions of (1.2) are related to the solitary wave solutions to the following nonlinear Schrödinger-Poisson system

$$\begin{cases} -i\psi_t - \Delta \psi + E(x)\psi + \mu\phi\psi = f(\psi), & x \in \mathbb{R}^N,\ t > 0, \\ \Delta \phi = |\psi|^2, & x \in \mathbb{R}^N,\ t > 0, \end{cases} \tag{1.3}$$

where $\psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$ is the wave function, $E(x) = V(x) - a$ with $a \in \mathbb{R}$ is a real-valued external potential, ϕ represents an internal potential for a nonlocal self-interaction of the wave function and the nonlinear term f describes the interaction effect among particles. System (1.3) arises from quantum mechanics (see e.g. [8, 9, 24]) and in semiconductor theory [7, 25, 26]. In the last decades, system (1.2) has attracted considerable attention, see [4, 5, 10, 13, 19, 20, 27, 29, 30, 31, 33, 34, 35].

From the second equation in (1.1), we can obtain that $\phi(x) = 2\pi(\Gamma_2 * u^2)(x)$, i.e., the convolution of $2\pi u^2$ with the fundamental solution $\Gamma_2(x) = \frac{1}{2\pi}\ln|x|$ of the Laplacian. With this formal inversion, system (1.1) is converted into an equivalent integro-differential equation

$$-\Delta u + V(x)u + 2\pi(\Gamma_2 * u^2)u = f(u), \quad x \in \mathbb{R}^2. \tag{1.4}$$

Denote by $\phi_{2,u}(x) = 2\pi(\Gamma_2 * u^2)(x)$. Then at least formally, the energy functional associated with (1.4) is

$$\mathcal{I}(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^2} \phi_{2,u}u^2 \mathrm{d}x - \int_{\mathbb{R}^2} F(u)\mathrm{d}x,$$

here and in the sequel $F(t) = \int_0^t f(s)\mathrm{d}s$.

In contrast with the case $N = 3$, the applicability of variational methods is not straightforward for $N = 2$, since the corresponding energy functional $\mathcal{I}$ is not well-defined on the natural Sobolev space $H^1(\mathbb{R}^2)$ due to the appearance of the sign-changing and unbounded logarithmic integral kernel $\ln|x|$. This also exhibits some serious mathematical differences to the case $N = 3$ (see [14, 15, 17, 18]). To overcome this obstacle, Cingolani and Weth [17], inspired by Stubbe [28], developed a variational framework for the following equation

$$-\Delta u + V(x)u + \left(\int_{\mathbb{R}^2} \ln|x-y|\ u^2(y)\mathrm{d}y\right) u = |u|^{q-2}u, \quad x \in \mathbb{R}^2 \tag{1.5}$$

within the smaller Hilbert space

$$X = \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} [V(x) + \ln(1+|x|)]u^2 \mathrm{d}x < \infty \right\}. \tag{1.6}$$

By using the Nehari manifold argument and a new strong compactness condition (Φ satisfies the Cerami condition at arbitrary energy levels after suitable translation), they proved that (1.5) has ground state solutions when $q > 4$ and $V(x)$ is a positive 1-periodic function. In the case where $V(x) \equiv V_0 > 0$, they also obtained the existence of nonradial solutions which have arbitrarily many nodal domains. Based on the strong compactness condition in [17] and the Pohozaev type argument, Du and Weth [18] provided a counterpart of the results of [17] in the case where $V(x) \equiv V_0 > 0$ and $2 < q \leq 4$. In the papers [14] and [15], Chen and Tang developed a new variational framework for system (1.4) in the space which consists of axially symmetric functions. They established a new inequality and proposed a new approach to recover the compactness for the (PS)-sequence. By using this approach, they proved some existence results on nontrivial solutions and ground state solutions under the following assumptions on V and f, which are much weaken than the ones in [17, 18].

(V1) $V \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$, $V(x) = V(|x_1|, |x_2|)$, $\liminf_{|x|\to\infty} V(x) > 0$, and there exists a constant $\Theta_0 > 0$ such that $|V(x)| + |\nabla V(x) \cdot x| \leq \Theta_0$ for all $x \in \mathbb{R}^2$;

(F1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, and there exist constants $\mathcal{C}_0 > 0$ and $p \in (3, \infty)$ such that

$$|f(t)| \leq \mathcal{C}_0 \left(1 + |t|^{p-1}\right), \quad \forall\ (x,t) \in \mathbb{R}^2 \times \mathbb{R};$$

(F2) $f(t) = o(t)$ as $t \to 0$;

(F3) $F(t) \geq 0$ for all $t \in \mathbb{R}$, and there exist constants $\alpha_0, c_0, R_0 > 0$ and $\kappa > 1$ such that

$$f(t)t - 3F(t) + \alpha_0 t^2 \geq 0, \quad \forall\ t \in \mathbb{R},$$

and

$$\left|\frac{f(t)}{t}\right|^\kappa \leq c_0 \left[f(t)t - 3F(t) + \alpha_0 t^2\right], \quad \forall\ |t| \geq R_0.$$

We would like to point out that the condition $\liminf_{|x|\to\infty} V(x) > 0$ plays a crucial role in verifying the Mountain Pass geometry for the energy functional associated with (1.1) in the arguments of [14] and [15].

When $V \equiv 0$, (1.1) reduces to the following planar Schrödinger-Poisson system with zero mass

$$\begin{cases} -\Delta u + \phi u = f(u), & x \in \mathbb{R}^2, \\ \Delta \phi = 2\pi u^2, & x \in \mathbb{R}^2. \end{cases} \tag{1.7}$$

In the recent paper [16], Chen and Tang proved that system (1.7) has a nontrivial axially symmetric solution provided f satisfies (F1) and the following two assumptions:

(F2′) $f(t) = o(t^2)$ as $t \to 0$;

(F3′) $F(t) \geq 0,\ \forall\, t \in \mathbb{R}$, and there exist $\mu > 4$ and $\nu > 0$ such that

$$f(t)t \geq \mu F(t) - \nu t^2, \qquad \forall\, t \in \mathbb{R}.$$

It is easy to see that (F3) is much weaker than (F3′). In the present paper, inspired by [14] and [16], we will establish the existence of nontrivial radially symmetric solutions for (1.7) under (F1), (F3) and the following condition:

(F2″) $\lim_{t\to 0} \frac{f(t)}{|t|t} = l \in [0, +\infty)$.

Obviously, (F2″) is weaker than (F2′). In particular, $f(t) = a|t|^{p-2}t$ with $a > 0$ and $p \geq 3$ satisfies (F1), (F2″) and (F3), but it does not satisfy (F2′) and (F3′) when $p = 3$ and $p \in [3, 4]$, respectively. Moreover, there are many functions satisfying (F1), (F2″) and (F3). For example,

i) $f(t) = \left(|t|^{p-2} + b|t|^{q-2}\right) t$ with $b \geq 0$ and $p > q \geq 3$;

ii) $f(t) = a|t|t\ln(1 + t^2)$ with $a > 0$.

As in (1.4), system (1.7) can be converted into an equivalent integro-differential equation

$$-\Delta u + 2\pi(\Gamma_2 * u^2)u = f(u), \qquad x \in \mathbb{R}^2. \tag{1.8}$$

Denote by $\phi_{2,u}(x) = 2\pi(\Gamma_2 * u^2)(x)$. Then at least formally, the energy functional associated with (1.8) is

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^2} \phi_{2,u} u^2 \mathrm{d}x - \int_{\mathbb{R}^2} F(u)\mathrm{d}x. \tag{1.9}$$

Now we can state our result as follows.

Theorem 1.1. *Assume that f satisfy* (F1), (F2″) *and* (F3). *Then* (1.7) *has a radially symmetric solution $\bar{u} \in H^1(\mathbb{R}^2) \setminus \{0\}$.*

The paper is organized as follows. In Section 2, we give the variational setting and preliminaries. We complete the proof of Theorem 1.1 in Section 3.

Throughout the paper, we make use of the following notations:

- $H^1(\mathbb{R}^2)$ denotes the usual Sobolev space equipped with the inner product and norm

$$(u, v)_{H^1} = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + uv)\mathrm{d}x, \quad \|u\|_{H^1} = (u, u)^{1/2}, \quad \forall u, v \in H^1(\mathbb{R}^2);$$

- $L^s(\mathbb{R}^2)(1 \leq s < \infty)$ denotes the Lebesgue space with the norm $\|u\|_s = \left(\int_{\mathbb{R}^2} |u|^s \mathrm{d}x\right)^{1/s}$;
- For any $u \in H^1(\mathbb{R}^2)$ and $t > 0$, $u_t(x) := u(tx)$ for $t > 0$;
- For any $x \in \mathbb{R}^2$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^2 : |y - x| < r\}$;
- $C_1, C_2, \cdots$ denote positive constants possibly different in different places.

2 Variational framework and preliminaries

In view of the Gagliardo-Nirenberg inequality, one has

$$\|u\|_s^s \le \mathcal{C}_s \|u\|_2^2 \|\nabla u\|_2^{s-2} \quad \text{for} \quad s > 2, \tag{2.1}$$

where $\mathcal{C}_s > 0$ is a constant determined by s. In the sequel, we set $\Theta_0 := e^{32(l+1)^2\mathcal{C}_3^2}$.

We define the following symmetric bilinear forms

$$(u,v) \mapsto A_1(u,v) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(\Theta_0 + |x-y|\right) u(x)v(y)\mathrm{d}x\mathrm{d}y, \tag{2.2}$$

$$(u,v) \mapsto A_2(u,v) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1 + \frac{\Theta_0}{|x-y|}\right) u(x)v(y)\mathrm{d}x\mathrm{d}y, \tag{2.3}$$

$$(u,v) \mapsto A_0(u,v) := A_1(u,v) - A_2(u,v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| u(x)v(y)\mathrm{d}x\mathrm{d}y, \tag{2.4}$$

where the definition is restricted, in each case, to measurable functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ such that the corresponding double integral is well defined in Lebesgue sense. Noting that $0 \le \ln(1+r) \le r$ for $r \ge 0$, it follows from the Hardy-Littlewood-Sobolev inequality (see [22] or [23, page 98]) that

$$|A_2(u,v)| \le \Theta_0 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x-y|}|u(x)v(y)|\mathrm{d}x\mathrm{d}y \le \mathcal{S}_1\|u\|_{4/3}\|v\|_{4/3} \tag{2.5}$$

with a constant $\mathcal{S}_1 > 0$. Using (2.2), (2.3) and (2.4), we define the functionals as follows:

$$I_1 : H^1(\mathbb{R}^2) \to [0,\infty], \quad I_1(u) := A_1(u^2,u^2) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(\Theta_0 + |x-y|\right) u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y,$$

$$I_2 : L^{8/3}(\mathbb{R}^2) \to [0,\infty), \quad I_2(u) := A_2(u^2,u^2) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1 + \frac{\Theta_0}{|x-y|}\right) u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y,$$

$$I_0 : H^1(\mathbb{R}^2) \to \mathbb{R}\cup\{\infty\}, \quad I_0(u) := A_0(u^2,u^2) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| \ u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y.$$

Here I_2 only takes finite values on $L^{8/3}(\mathbb{R}^2)$. Indeed, (2.5) implies

$$|I_2(u)| \le \mathcal{S}_1\|u\|_{8/3}^4, \quad \forall\, u \in L^{8/3}(\mathbb{R}^2). \tag{2.6}$$

We define, for any measurable function $u : \mathbb{R}^2 \to \mathbb{R}$

$$\|u\|_*^2 = \int_{\mathbb{R}^2} \ln\left(\Theta_0 + |x|\right) u^2(x)\mathrm{d}x \in [0,\infty]. \tag{2.7}$$

Then $\|u\|_E := (\|\nabla u\|_2^2 + \|u\|_*^2)^{1/2}$ is a norm on $E := H^1_{\mathrm{loc}}(\mathbb{R}^2)$. In view of Rellich imbedding theorem [32], E is continuously and compactly embedded in $L^s(\mathbb{R}^2)$ for $s \in [2,+\infty)$. Since

$$\ln(\Theta_0 + |x-y|) \le \ln(\Theta_0 + |x| + |y|) \le \ln(\Theta_0 + |x|) + \ln(\Theta_0 + |y|), \quad \forall\, x, y \in \mathbb{R}^2, \tag{2.8}$$

we have

$$
\begin{aligned}
|A_1(uv, wz)| &\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} [\ln(\Theta_0 + |x|) + \ln(\Theta_0 + |y|)]\, |u(x)v(x)||w(y)z(y)|\mathrm{d}x\mathrm{d}y \\
&\leq \|u\|_*\|v\|_*\|w\|_2\|z\|_2 + \|u\|_2\|v\|_2\|w\|_*\|z\|_*, \qquad \forall\ u, v, w, z \in E. \qquad (2.9)
\end{aligned}
$$

According to [17, Lemma 2.2], we have I_0, I_1 and I_2 are of class $\mathcal{C}^1$ on E, and

$$
\langle I_i'(u), v\rangle = 4A_i(u^2, v), \qquad \forall\ u, v \in E,\ i = 0, 1, 2. \tag{2.10}
$$

Then, (F1), (F2$''$) and (2.10) imply that $\Phi \in \mathcal{C}^1(E, \mathbb{R})$, and that

$$
\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\mathrm{d}x + \frac{1}{4}I_0(u) - \int_{\mathbb{R}^2} F(u)\mathrm{d}x \tag{2.11}
$$

and

$$
\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^2} \nabla u \cdot \nabla v\mathrm{d}x + A_0(u^2, uv) - \int_{\mathbb{R}^2} f(u)v\mathrm{d}x. \tag{2.12}
$$

Hence, the solutions of (1.7) are the critical points of the reduced functional (2.11).

The following lemmas come from [14] which is very crucial for the proof of our theorem.

Lemma 2.1. ([14, Lemma 2.2]) *There holds*

$$
A_1(u^2, v^2) \geq \frac{1}{4}\|u\|_2^2\|v\|_*^2, \qquad \forall\ u, v \in E. \tag{2.13}
$$

Lemma 2.2. ([14, Corollary 2.3]) *There holds*

$$
I_1(u) \geq \frac{1}{4}\|u\|_2^2\|u\|_*^2, \qquad \forall\ u \in E. \tag{2.14}
$$

3 Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1.

Proposition 3.1. [21] *Let H be a Banach space and let $\Lambda \subset \mathbb{R}^+$ be an interval. We consider a family $\{\Phi_\lambda\}_{\lambda\in\Lambda}$ of $\mathcal{C}^1$-functional on H of the form*

$$
\Phi_\lambda(u) = A(u) - \lambda B(u), \qquad \forall\ \lambda \in \Lambda,
$$

where $B(u) \geq 0$, $\forall u \in H$, and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$, as $\|u\|_H \to \infty$. We assume that there are two points v_1, v_2 in H such that

$$
c_\lambda := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \Phi_\lambda(\gamma(t)) > \max\{\Phi_\lambda(v_1), \Phi_\lambda(v_2)\}, \tag{3.1}
$$

where

$$
\Gamma = \{\gamma \in \mathcal{C}([0, 1], H) : \gamma(0) = v_1, \gamma(1) = v_2\}.
$$

Then, for almost every $\lambda \in \Lambda$, there is a bounded $(\mathrm{PS})_{c_\lambda}$ sequence for Φ_λ, that is, there exists a sequence such that

i) $\{u_n(\lambda)\}$ *is bounded in* H*;*

ii) $\Phi_\lambda(u_n(\lambda)) \to c_\lambda$*;*

iii) $\Phi'_\lambda(u_n(\lambda)) \to 0$ *in* H^**, where* H^* *is the dual of* H*.*

Moreover, c_λ *is non-increasing on* $\lambda \in \Lambda$*.*

To apply Proposition 3.1, inspired by [14], we introduce a family of functional on E defined by

$$\Phi_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x + \frac{1}{2}\|u\|_*^2 + \frac{1}{4}I_0(u) - \lambda\left[\frac{1}{2}\|u\|_*^2 + \int_{\mathbb{R}^2} F(u)\mathrm{d}x\right], \tag{3.2}$$

for $\lambda \in [1/2, 1]$. Let

$$A(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x + \frac{1}{2}\|u\|_*^2 + \frac{1}{4}I_0(u), \qquad B(u) = \frac{1}{2}\|u\|_*^2 + \int_{\mathbb{R}^2} F(u)\mathrm{d}x. \tag{3.3}$$

Similar to the proof of [18, Lemma 2.4], we have the following lemma.

Lemma 3.2. *Assume that* (F1), (F2$''$) *and* (F3) *hold. Let* u *be a critical point of* Φ_λ *in* E*, then we have the following Pohozaev type identity*

$$\begin{aligned}\mathcal{P}_\lambda(u) \quad := \quad & \frac{1-\lambda}{2}\left(2\|u\|_*^2 + \int_{\mathbb{R}^2} \frac{|x|}{2+|x|}u^2 \mathrm{d}x\right) \\ & + I_0(u) + \frac{1}{4}\|u\|_2^4 - 2\lambda\int_{\mathbb{R}^2} F(u)\mathrm{d}x = 0.\end{aligned} \tag{3.4}$$

For $\lambda \in [0.5, 1]$, we set $J_\lambda(u) := 2\langle \Phi'_\lambda(u), u\rangle - \mathcal{P}_\lambda(u)$, then

$$\begin{aligned}J_\lambda(u) \quad = \quad & 2\|\nabla u\|_2^2 + \frac{1-\lambda}{2}\left(2\|u\|_*^2 - \int_{\mathbb{R}^2} \frac{|x|}{2+|x|}u^2 \mathrm{d}x\right) \\ & + I_0(u) - \frac{1}{4}\|u\|_2^4 - 2\lambda\int_{\mathbb{R}^2} [f(u)u - F(u)]\mathrm{d}x.\end{aligned} \tag{3.5}$$

In the following, we verify that (3.1) of Proposition 3.1 holds. To this end, we choose a fixed function $\hat{w} \in E \setminus \{0\}$. Note that

$$\begin{aligned}I_0\left(t^2 \hat{w}_t\right) \quad & = \quad t^4 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y| \hat{w}^2(tx)\hat{w}^2(ty)\mathrm{d}(tx)\mathrm{d}(ty) \\ & = \quad t^4 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(\ln|x-y| - \ln t\right)\hat{w}^2(x)\hat{w}^2(y)\mathrm{d}x\mathrm{d}y \\ & = \quad t^4 I_0(\hat{w}) - t^4 \ln t\|\hat{w}\|_2^4, \qquad \forall\ t > 0.\end{aligned} \tag{3.6}$$

From (3.2) and (3.6), one has

$$\begin{aligned}\Phi_\lambda\left(t^2 \hat{w}_t\right) \quad = \quad & \frac{t^4}{2}\|\nabla \hat{w}\|_2^2 + \frac{(1-\lambda)t^2}{2}\int_{\mathbb{R}^2} \ln(\Theta_0 + t^{-1}|x|)\hat{w}^2 \mathrm{d}x \\ & + \frac{t^4}{4}I_0(\hat{w}) - \frac{t^4 \ln t}{4}\|\hat{w}\|_2^4 - \frac{\lambda}{t^2}\int_{\mathbb{R}^2} F\left(t^2 \hat{w}\right)\mathrm{d}x\end{aligned}$$

$$\leq \frac{t^4}{2}\|\nabla \hat{w}\|_2^2 + \frac{t^2}{4}\|\hat{w}\|_*^2 + \frac{t^4}{4}I_0(\hat{w}) - \frac{t^4 \ln t}{4}\|\hat{w}\|_2^4, \quad \forall\ t \geq 1.$$

It follows that there exists $T > 0$ such that

$$\Phi_\lambda\left(t^2\hat{w}_t\right) < 0, \quad \forall\ \lambda \in [0.5, 1],\ t \geq T. \tag{3.7}$$

Lemma 3.3. *Assume that* (F1), (F2′′) *and* (F3) *hold. Then there exists a positive constant* κ_0 *independent of* λ *such that for all* $\lambda \in [0.5, 1]$*,*

$$c_\lambda := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Phi_\lambda(\gamma(t)) \geq \kappa_0 > \max\{\Phi_\lambda(0), \Phi_\lambda(T^2\hat{w}_T)\}. \tag{3.8}$$

Proof. Let $H = E$, $\Lambda = [0.5, 1]$, $v_1 = 0$ and $v_2 = T^2\hat{w}_T$ in Proposition 3.1 and let

$$\Gamma = \left\{\gamma \in \mathcal{C}([0, 1], E) : \gamma(0) = 0, \gamma(1) = T^2\hat{w}_T\right\}. \tag{3.9}$$

By (F1) and (F2′′), we have

$$|F(t)| \leq (l + 1)|t|^3 + C_1|t|^{p+2}, \quad \forall\ t \in \mathbb{R}, \tag{3.10}$$

which yields

$$\int_{\mathbb{R}^2} F(u)\mathrm{d}x \leq (l + 1)\|u\|_3^3 + C_1\|u\|_{p+2}^{p+2}, \quad \forall\ u \in E. \tag{3.11}$$

Then it follows from (2.1) and Young inequality that

$$\|u\|_{8/3}^4 \leq \mathcal{C}_{8/3}^{3/2}\|u\|_2^3\|\nabla u\|_2 \leq \mathcal{C}_{8/3}^{3/2}\left(\frac{2}{3}\|u\|_2^{9/2} + \frac{1}{3}\|\nabla u\|_2^3\right), \tag{3.12}$$

$$\|u\|_3^3 \leq \mathcal{C}_3\|u\|_2^2\|\nabla u\|_2 \leq (l + 1)\mathcal{C}_3^2\|u\|_2^4 + \frac{1}{4(l + 1)}\|\nabla u\|_2^2 \tag{3.13}$$

and

$$\|u\|_{p+2}^{p+2} \leq \mathcal{C}_{p+2}\|u\|_2^2\|\nabla u\|_2^p \leq \mathcal{C}_{p+2}\left(\frac{1}{3}\|u\|_2^6 + \frac{2}{3}\|\nabla u\|_2^{3p/2}\right). \tag{3.14}$$

Combining (2.6), (2.7), (2.11), (2.14), (3.11), (3.12), (3.13) and (3.14), we have

$$\begin{aligned}
\Phi_\lambda(u) &= \frac{1}{2}\|\nabla u\|_2^2 + \frac{1-\lambda}{2}\|u\|_*^2 + \frac{1}{4}I_0(u) - \lambda\int_{\mathbb{R}^2} F(u)\mathrm{d}x \\
&\geq \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}[I_1(u) - I_2(u)] - \int_{\mathbb{R}^2} F(u)\mathrm{d}x \\
&\geq \frac{1}{2}\|\nabla u\|_2^2 + 2(l + 1)^2\mathcal{C}_3^2\|u\|_2^4 - \frac{\mathcal{S}_1}{4}\|u\|_{8/3}^4 - (l + 1)\|u\|_3^3 - C_1\|u\|_{p+2}^{p+2} \\
&\geq \frac{1}{2}\|\nabla u\|_2^2 + 2(l + 1)^2\mathcal{C}_3^2\|u\|_2^4 - \frac{\mathcal{S}_1\mathcal{C}_{8/3}^{3/2}}{4}\left(\frac{2}{3}\|u\|_2^{9/2} + \frac{1}{3}\|\nabla u\|_2^3\right) \\
&\quad -(l + 1)\left[(l + 1)\mathcal{C}_3^2\|u\|_2^4 + \frac{1}{4(l + 1)}\|\nabla u\|_2^2\right] \\
&\quad -C_1\mathcal{C}_{p+2}\left(\frac{1}{3}\|u\|_2^6 + \frac{2}{3}\|\nabla u\|_2^{3p/2}\right)
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{4}\min\{1, 4(l+1)^2\mathcal{C}_3^2\}\left[\|\nabla u\|_2^2 + \|u\|_2^4\right] - C_3\left(\|u\|_2^{9/2} + \|u\|_2^6\right) \\
&\quad -C_4\left(\|\nabla u\|_2^3 + \|\nabla u\|_2^{3p/2}\right), \qquad \forall\, u \in E.
\end{aligned} \tag{3.15}$$

Let $\rho(u) := \|\nabla u\|_2^2 + \|u\|_2^4$. Then (3.15) gives

$$\begin{aligned}
\Phi_\lambda(u) \;\; &\geq \;\; \frac{1}{4}\min\{1, 4(l+1)^2\mathcal{C}_3^2\}\rho(u) - C_3\left[\rho^{9/8}(u) + \rho^{3/2}(u)\right] \\
&\quad -C_4\left[\rho^{3/2}(u) + \rho^{3p/4}(u)\right], \qquad \forall\, u \in E.
\end{aligned}$$

Therefore, there exist $\kappa_0 > 0$ and $\rho_0 > 0$ such that

$$\Phi_\lambda(u) \geq \kappa_0, \qquad \forall\, u \in S := \{u \in E : \|\nabla u\|_2^2 + \|u\|_2^4 = \rho_0\}. \tag{3.16}$$

(3.7), (3.9) and (3.16) show that Φ_λ satisfies all conditions of Proposition 3.1 with $H = E$ and $\Lambda = [0.5, 1]$.

□

Lemma 3.4. *Assume that* (F1), (F2$''$) *and* (F3) *hold. Then* c_λ *is non-increasing on* $\lambda \in [0.5, 1]$. *Moreover, for almost every* $\lambda \in [0.5, 1]$, *there exists* $u_\lambda \in E \setminus \{0\}$ *such that*

$$u_n \to u_\lambda \;\; \text{in } E, \qquad \Phi'_\lambda(u_\lambda) = 0, \qquad \Phi_\lambda(u_\lambda) = c_\lambda. \tag{3.17}$$

Proof. Lemma 3.3 implies that $\Phi_\lambda(u)$ satisfies the assumptions of Proposition 3.1 with $H = E$ and $\Lambda = [0.5, 1]$. So c_λ is non-increasing on $\lambda \in [0.5, 1]$, and for almost every $\lambda \in [0.5, 1]$, there exists a sequence $\{u_n(\lambda)\} \subset E$ (for simplicity, we denote $\{u_n\}$ instead of $\{u_n(\lambda)\}$) such that

$$\|u_n\|_E \leq C_1, \qquad \Phi_\lambda(u_n) \to c_\lambda > 0, \qquad \|\Phi'_\lambda(u_n)\|_{E^*} \to 0. \tag{3.18}$$

If $\delta_0 := \limsup_{n\to\infty} \|u_n\|_2 = 0$, the from the Gagliardo-Nirenberg inequality (2.1), we derive that $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for $s \geq 2$. Hence it follows from (F1), (F2$''$) and (2.6) that $I_2(u_n) \to 0$ and

$$\limsup_{n\to\infty} \int_{\mathbb{R}^2} \left|\frac{1}{2}f(u_n)u_n - F(u_n)\right| \mathrm{d}x = 0. \tag{3.19}$$

Now from (3.2), (3.8) and (3.19), one has

$$\begin{aligned}
\kappa_0 + o(1) \;\; &\leq \;\; c_\lambda + o(1) = \Phi_\lambda(u_n) - \frac{1}{2}\langle \Phi'_\lambda(u_n), u_n\rangle \\
&= \;\; -\frac{1}{4}I_1(u_n) + \frac{1}{4}I_2(u_n) + \lambda\int_{\mathbb{R}^2}\left[\frac{1}{2}f(u_n)u_n - F(u_n)\right]\mathrm{d}x \\
&\leq \;\; o(1).
\end{aligned} \tag{3.20}$$

This contradiction shows that $\delta_0 > 0$. Since $\{\|u_n\|_E\}$ is bounded, we may thus assume, passing to a subsequence again if necessary, that $u_n \rightharpoonup u_\lambda$ in E, $u_n \to u_\lambda$ in $L^s(\mathbb{R}^2)$, $s \in [2, \infty)$ and $u_n(x) \to u_\lambda(x)$ a.e. on $\mathbb{R}^2$. Hence it follows from (F1), (F2$''$) and (2.5) that

$$A_2(u_n^2, u_n(u_n - u_\lambda)) = o(1), \qquad A_2(u_\lambda^2, u_n(u_n - u_\lambda)) = o(1) \tag{3.21}$$

and

$$\int_{\mathbb{R}^2}[f(u_n)-f(u_\lambda)](u_n-u_\lambda)\mathrm{d}x=o(1). \tag{3.22}$$

Since $\{\|u_n\|_*\}$ and $\{\|u_n\|_2\}$ are bounded, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\ln(\Theta_0+|y|)|u_\lambda(y)||u_n(y)-u_\lambda(y)|\mathrm{d}y\\
\leq\ & \ln(\Theta_0+|R|)\|u_\lambda\|_2\|u_n-u_\lambda\|_2+\|u_n-u_\lambda\|_*\left[\int_{\mathbb{R}^2\setminus B_R(0)}\ln(\Theta_0+|y|)u_\lambda^2(y)\mathrm{d}y\right]^{1/2}\\
=\ & o_n(1)+o_R(1),\quad \text{as } n\to\infty, R\to\infty,
\end{aligned}\tag{3.23}$$

which implies

$$\int_{\mathbb{R}^2}\ln(\Theta_0+|y|)|u_\lambda(y)||u_n(y)-u_\lambda(y)|\mathrm{d}y=o(1). \tag{3.24}$$

By (2.8), (3.24) and the fact that $\|u_n-u_\lambda\|_2\to 0$, we have

$$\begin{aligned}
&A_1(u_n^2,u_\lambda(u_n-u_\lambda))\\
\leq\ & \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\ln(\Theta_0+|x|)+\ln(\Theta_0+|y|)\right]u_n^2(x)|u_\lambda(y)||u_n(y)-u_\lambda(y)|\mathrm{d}x\mathrm{d}y\\
\leq\ & \|u_n\|_*^2\|u_\lambda\|_2\|u_n-u_\lambda\|_2+\|u_n\|_2^2\int_{\mathbb{R}^2}\ln(\Theta_0+|y|)|u_\lambda(y)||u_n(y)-u_\lambda(y)|\mathrm{d}y\\
=\ & o(1).
\end{aligned}\tag{3.25}$$

Similarly, we have

$$A_1(u_\lambda^2,u_\lambda(u_n-u_\lambda))=o(1). \tag{3.26}$$

From (2.13), (3.2), (3.21), (3.22), (3.25) and (3.26), one has

$$\begin{aligned}
o(1) =\ & \langle\Phi_\lambda'(u_n)-\Phi_\lambda'(u_\lambda),u_n-u_\lambda\rangle\\
=\ & \|\nabla(u_n-u_\lambda)\|_2^2+(1-\lambda)\|u_n-u_\lambda\|_*^2+A_1(u_n^2,(u_n-u_\lambda)^2)+A_1(u_n^2,u_\lambda(u_n-u_\lambda))\\
&-A_1(u_\lambda^2,u_\lambda(u_n-u_\lambda))-A_2(u_n^2,u_n(u_n-u_\lambda))+A_2(u_\lambda^2,u_\lambda(u_n-u_\lambda))\\
&-\lambda\int_{\mathbb{R}^2}[f(u_n)-f(u_\lambda)](u_n-u_\lambda)\mathrm{d}x\\
=\ & \|\nabla(u_n-u_\lambda)\|_2^2+(1-\lambda)\|u_n-u_\lambda\|_*^2+A_1(u_n^2,(u_n-u_\lambda)^2)+o(1)\\
\geq\ & \|\nabla(u_n-u_\lambda)\|_2^2+\frac{1}{4}\|u_n\|_2^2\|u_n-u_\lambda\|_*^2+o(1),
\end{aligned}$$

which, together with $\delta_0=\limsup_{n\to\infty}\|u_n\|_2>0$, implies that $u_n\to u_\lambda$ in E. Hence, $0<c_\lambda=\lim_{n\to\infty}\Phi_\lambda(u_n)=\Phi_\lambda(u_\lambda)$ and $\Phi_\lambda'(u_\lambda)=0$. □

In view of Lemmas 3.2 and 3.4, there exist two sequences $\{\lambda_n\}\subset[0.5,1]$ and $\{u_{\lambda_n}\}\subset E$, denoted by $\{u_n\}$ such that

$$\lambda_n\to 1,\qquad \Phi_{\lambda_n}'(u_n)=0,\qquad \Phi_{\lambda_n}(u_n)=c_{\lambda_n}\in[c_1,c_{0.5}],\qquad J_{\lambda_n}(u_n)=0. \tag{3.27}$$

Lemma 3.5. *Assume that* (F1), (F2$''$) *and* (F3) *hold. Let* $\{\lambda_n\} \subset [0.5, 1]$ *and* $\{u_n\} \subset E$ *be two sequences satisfying* (3.27)*. Then* $\{\|u_n\|_{H^1}\}$ *is bounded.*

Proof. By (3.2), (3.5) and (3.27), one has

$$\begin{aligned}
c_{\lambda_n} &= \Phi_{\lambda_n}(u_n) - \frac{1}{4}J_{\lambda_n}(u_n) \\
&= \frac{1-\lambda_n}{8}\left(2\|u_n\|_*^2 + \int_{\mathbb{R}^2} \frac{|x|}{2+|x|}u_n^2 \mathrm{d}x\right) \\
&\quad + \frac{1}{16}\|u_n\|_2^4 + \frac{\lambda_n}{2}\int_{\mathbb{R}^2}[f(u_n)u_n - 3F(u_n)]\mathrm{d}x \\
&\geq \frac{1}{16}\|u_n\|_2^4 + \frac{1-\lambda_n}{4}\|u_n\|_*^2 + \frac{\lambda_n}{2}\int_{\mathbb{R}^2}[f(u_n)u_n - 3F(u_n)]\mathrm{d}x, \qquad (3.28)
\end{aligned}$$

which, together with (F3), implies

$$c_{\lambda_n} \geq \frac{1}{16}\|u_n\|_2^4 - \frac{\alpha_0}{2}\|u_n\|_2^2. \qquad (3.29)$$

From (3.28) and (3.29), one has

$$\|u_n\|_2 \leq C_1, \qquad \int_{\mathbb{R}^2}\left[f(u_n)u_n - 3F(u_n) + \alpha_0 u_n^2\right]\mathrm{d}x \leq C_2. \qquad (3.30)$$

Now, we prove that $\{\|u_n\|_{H^1}\}$ is also bounded. Arguing by contradiction, suppose that $\|u_n\|_{H^1} \to \infty$. Let $v_n := \frac{u_n}{\|u_n\|_{H^1}}$. Then $\|v_n\|_{H^1} = 1$, and $\|v_n\|_2 \to 0$ due to (3.30). Set $\kappa' = \kappa/(\kappa-1)$. By the Gagliardo-Nirenberg inequality (2.1), one has

$$\|v_n\|_{2\kappa'}^{2\kappa'} \leq C_2\|v_n\|_2^2\|\nabla v_n\|_2^{2\kappa'-2} = o(1). \qquad (3.31)$$

Set

$$\Omega_n := \left\{x \in \mathbb{R}^2 : |u_n(x)| \leq R_0\right\}.$$

Then by (F1) and (F2$''$), we have

$$\int_{\Omega_n}\left|\frac{f(u_n)}{u_n}\right| v_n^2 \mathrm{d}x \leq C_3\|v_n\|_2^2 = o(1). \qquad (3.32)$$

Moreover, by (F3), (3.30), (3.31) and the Hölder inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^2\setminus\Omega_n}\left|\frac{f(u_n)}{u_n}\right| v_n^2\mathrm{d}x &\leq \left(\int_{\mathbb{R}^2\setminus\Omega_n}\left|\frac{f(u_n)}{u_n}\right|^\kappa \mathrm{d}x\right)^{1/\kappa}\left(\int_{\mathbb{R}^2\setminus\Omega_n}|v_n|^{2\kappa'}\mathrm{d}x\right)^{1/\kappa'} \\
&\leq c_0^{1/\kappa}\left(\int_{\mathbb{R}^2\setminus\Omega_n}\left[f(u_n)u_n - 3F(u_n) + \alpha_0 u_n^2\right]\mathrm{d}x\right)^{1/\kappa}\|v_n\|_{2\kappa'}^2 \\
&= o(1). \qquad (3.33)
\end{aligned}$$

From (2.6), (3.30) and the Gagliardo-Nirenberg inequality, we have

$$I_2(u_n) \leq \mathcal{C}_1\|u_n\|_{8/3}^4 \leq C_4\|u_n\|_2^3\|\nabla u_n\|_2 \leq C_5\|\nabla u_n\|_2. \qquad (3.34)$$

Thus, it follows from (2.11), (3.27), (3.32), (3.33) and (3.34) that

$$\begin{aligned} 1+o(1) &= \frac{\|\nabla u_n\|_2^2 - \langle \Phi'_{\lambda_n}(u_n), u_n\rangle}{\|u_n\|_{H^1}^2} \\ &= \frac{-I_1(u_n)+I_2(u_n)-(1-\lambda_n)\|u_n\|_*^2+\lambda_n\int_{\mathbb{R}^2} f(u_n)u_n\mathrm{d}x}{\|u_n\|_{H^1}^2} \\ &\le \frac{C_5}{\|u_n\|_{H^1}} + \lambda_n\int_{\Omega_n}\left|\frac{f(u_n)}{u_n}\right| v_n^2\mathrm{d}x + \lambda_n\int_{\mathbb{R}^2\setminus\Omega_n}\left|\frac{f(u_n)}{u_n}\right| v_n^2\mathrm{d}x = o(1), \end{aligned}$$

which is a contradiction. Hence, $\{\|u_n\|_{H^1}\}$ is bounded. $\square$

Proof of Theorem 1.1. In view of Lemmas 3.4 and 3.5, under assumptions of Theorem 1.1, there exists two sequences $\{\lambda_n\} \subset [0.5, 1]$ and $\{u_n\} \subset E$ satisfying (3.27) and $\{\|u_n\|_{H^1}\}$ is bounded. It follows from (2.6), (3.2), (F1) and (F2$''$) that $I_1(u_n)$ is bounded. Similar to the proof of Lemma 3.4, we can show that $\limsup_{n\to\infty} \|u_n\|_2 > 0$. Applying Lemma 2.2, we have $\{\|u_n\|_*\}$ is bounded. Hence $\{u_n\}$ is bounded in E. We may thus assume, passing to a subsequence if necessary, that $u_n \rightharpoonup \bar{u}$ in E, $u_n \to \bar{u}$ in $L^s(\mathbb{R}^2)$, $s \in [2,\infty)$ and $u_n(x) \to \bar{u}(x)$ a.e. on $\mathbb{R}^2$. Since $\lambda_n \to 1$, by (2.11), (2.12), (3.2) and (3.27), we have

$$\Phi'(u_n) \to 0, \qquad \Phi(u_n) \to c_* := \lim_{n\to\infty} c_{\lambda_n}, \qquad J(u_n) \to 0. \tag{3.35}$$

Similar to the proof of Lemma 3.4, we can deduce that $u_n \to \bar{u}$ in E. Hence, $0 < c_* = \lim_{n\to\infty} \Phi(u_n) = \Phi(\bar{u})$ and $\Phi'(\bar{u}) = 0$. This completes the proof. $\square$

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