On vanishing limits of the shear viscosity and Hall coefficients for the planar compressible Hall-MHD system

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Abstract. This paper deals with an initial-boundary value problem of the planar compressible Hall-magnetohydrodynamic (for short, Hall-MHD) equations. For the fixed shear viscosity and Hall coefficients, it is shown that the strong solutions of Hall-MHD equations and corresponding MHD equations are global. As both the shear viscosity and the Hall coefficients tend to zero, the convergence rate for the solutions from Hall-MHD equations to MHD equations is given. The thickness of boundary layer is discussed by spatial weighted estimation and the characteristic of boundary layer is described by constructing a boundary layer function.

Keywords Hall-MHD equations; global well-posedness; convergence rate; boundary layer

1 Introduction

In this paper, we consider the vanishing limits of the shear viscosity and Hall coefficients for the planar compressible Hall-magnetohydrodynamic system, which is governed by the following equations:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + [\rho u^2 + P + \frac{1}{2}(b_2^2 + b_3^2)]_x = u_{xx}, \\ (\rho v)_t + (\rho uv - b_2)_x = \mu v_{xx}, \\ (\rho w)_t + (\rho uw - b_3)_x = \mu w_{xx}, \\ b_{2t} + (ub_2 - v)_x - \kappa(\frac{b_{3x}}{\rho})_x = b_{2xx}, \\ b_{3t} + (ub_3 - w)_x + \kappa(\frac{b_{2x}}{\rho})_x = b_{3xx}, \end{cases} \tag{1.1}$$

where ρ is the density of the fluid, u and (v, w) are the longitudinal velocity and the transverse velocity, b_1 and (b_2, b_3) are longitudinal magnetic field and the transverse magnetic field, respectively. $P(\rho) = A\rho^\gamma$ is the pressure, where $\gamma \geq 1$ and A is a positive constant. The parameter μ denotes the shear viscosity coefficient and κ is the Hall coefficient.

The Hall-MHD system (1.1) can be derived from fluid mechanics with appropriate modifications to account for electrical forces and Hall effects. Since the Hall effect restores the influence of the electric current in the Lorentz force occurring in Ohms law, Hall-MHD system

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plays an important role in many physical fields such as magnetic reconnection in space plasmas, star formulation, neutron stars and geo-dynamo (see e.g., [2, 12, 16, 21, 24, 28]). When the Hall term is neglected, the system (1.1) is reduced to the classical MHD system. It is known that the MHD system has been studied widely, the global existence and asymptotic behavior of solutions can be found in [6, 7, 11, 15, 17, 18, 22, 27] and the references therein. Especially, The vanishing shear viscosity limit and the behavior of boundary layer for planar MHD equations were described in [11, 23, 30].

In the past few years, the Hall-MHD system has been studied by some authors (cf.[1, 3, 4, 5, 8, 9, 13, 19, 20]), For the compressible isentropic case, the local strong solutions with large data, the global strong solutions with small data, the global existence and asymptotic behavior under the initial data sufficiently close to the equilibrium, and the low Mach number limit of smooth solution were proved in [9, 13, 20]. Tao and his coauthors derived the global existence of solutions for the planar compressible Hall-MHD equations with Dirichlet boundary or free boundary in [25, 26]. Xiang [29] established the smooth solution of the compressible Hall-magnetohydrodynamics system converges to the solution of the compressible magneto-hydrodynamics system as the Hall coefficient to zero. For the compressible non-isentropic case, local well-posedness and blow-up criteria and the time decay of smooth solutions were established in [10, 14]. Then, Lai-Xu-Zhang proved the global existence and the optimal decay rates of solutions with the initial data sufficiently close to the non-vacuum equilibrium in H^1, and the vanishing limit of Hall coefficient is also justified in [19].

Motivated by the results as in [19, 23, 29, 30], the aim of this paper is to study the vanishing limits of the shear viscosity and Hall coefficients for the planar compressible Hall-MHD system (1.1) on $(0.1) \times (0, T)$ with the following initial and boundary conditions:

$$\begin{cases} (\rho, u, v, w, b_2, b_3)|_{t=0} = (\rho_0, u_0, v_0, w_0, b_{2_0}, b_{3_0})(x), \\ (v, w)|_{x=0} = (v_1, w_1)(t), \quad (v, w)|_{x=1} = (v_2, w_2)(t), \\ (u, b_2, b_3)|_{x=0,1} = (0, 0, 0), \end{cases} \tag{1.2}$$

For the convenience of statement below, we rewrite the system (1.1)-(1.2) with $\mu = 0$, $\kappa = 0$, i.e., the MHD system without the shear viscosity:

$$\begin{cases} \bar{\rho}_t + (\bar{\rho}\bar{u})_x = 0, \\ (\bar{\rho}\bar{u})_t + [\bar{\rho}(\bar{u})^2 + \bar{P} + \frac{1}{2}(\bar{b}_2^2 + \bar{b}_3^2)]_x = \bar{u}_{xx}, \\ (\bar{\rho}\bar{v})_t + (\bar{\rho}\bar{u}\bar{v} - \bar{b}_2)_x = 0, \\ (\bar{\rho}\bar{w})_t + (\bar{\rho}\bar{u}\bar{w} - \bar{b}_3)_x = 0, \\ \bar{b}_{2t} + (\bar{u}\bar{b}_2 - \bar{v})_x = \bar{b}_{2xx}, \\ \bar{b}_{3t} + (\bar{u}\bar{b}_3 - \bar{w})_x = \bar{b}_{3xx} \end{cases} \tag{1.3}$$

and

$$\begin{cases} (\bar{\rho}, \bar{u}, \bar{v}, \bar{w}, \bar{b}_2, \bar{b}_3)|_{t=0} = (\rho_0, u_0, v_0, w_0, b_{2_0}, b_{3_0})(x), \\ (\bar{u}, \bar{b}_2, \bar{b}_3)|_{x=0,1} = (0, 0, 0). \end{cases} \tag{1.4}$$

Our main results are as follows:

Theorem 1.1 *For any $T > 0$, assume that*

$$0 < \rho_0 \in H^2(\Omega), \quad (u_0, v_0, w_0, b_{2_0}, b_{3_0}) \in H^2(\Omega), \quad (v_1(t), v_2(t), w_1(t), w_2(t)) \in C^1([0, T]). \tag{1.5}$$

Then,

(i) for each fixed $\mu > 0$ and small $\kappa > 0$, the following estimates on the solution of the problem (1.1)-(1.2) *hold uniformly in μ and κ:*

$$0 < C_1 \leq \rho(x,t) \leq 1/C_1, \quad (x,t) \in [0,1] \times [0,T],$$

$$\sup_{0 \leq t \leq T} \Big(\|(\rho, u, b_2, b_3)\|_{H^1}^2 + \|(v,w)\|_{L^2}^2 + \mu^{1/2}\|(v_x, w_x)\|_{L^2}^2 + \|(v,w)\|_{L^\infty} \Big)$$
$$+ \int_0^T \Big(\|u_x\|_{H^1}^2 + \mu^{3/2}\|(v_{xx}, w_{xx})\|_{L^2}^2 + \|(b_{2t}, b_{3t})\|_{L^2}^2 + \|(b_{2x}, b_{3x})\|_{L^\infty}^2 \Big)\, dt \leq C;$$

Here and in what follows, these letters C and $C_i (i = 1,1)$ denote constants independent of μ and κ.

(ii) the following global estimates hold for the solution of the problem (1.3)-(1.4)*,*

$$0 < C_2 \leq \bar{\rho}(x,t) \leq 1/C_2, \quad (x,t) \in [0,1] \times [0,T], \tag{1.6}$$

$$\sup_{0 \leq t \leq T} \|(\bar{\rho}, \bar{u}, \bar{v}, \bar{w}, \bar{b}_2, \bar{b}_3)\|_{H^2}^2 + \int_0^T \big(\|(\bar{u}_x, \bar{b}_{2x}, \bar{b}_{3x})\|_{H^2}^2 + \|(\bar{u}_t, \bar{b}_{2t}, \bar{b}_{3t})\|_{L^2}^2 \big)\, dt \leq C. \tag{1.7}$$

Theorem 1.2 *Assume that $(\rho, u, v, w, b_2, b_3)$ and $(\bar{\rho}, \bar{u}, \bar{v}, \bar{w}, \bar{b}_2, \bar{b}_3)$ are the solutions of the problem* (1.1)-(1.2) *and* (1.3)-(1.4)*, respectively, then*
(i)

$$\sup_{0 \leq t \leq T} \big(\|v - \bar{v}\|_{L^2}^2 + \|w - \bar{w}\|_{L^2}^2 + \|\rho - \bar{\rho}\|_{H^1}^2 + \|u - \bar{u}\|_{H^1}^2 + \|b_2 - \bar{b}_2\|_{H^1}^2 + \|b_3 - \bar{b}_3\|_{H^1}^2 \big)$$
$$+ \int_0^T \big(\|(u - \bar{u})_x\|_{H^1}^2 + \|(b_2 - \bar{b}_2)_x\|_{L^2}^2 + \|(b_3 - \bar{b}_3)_x\|_{L^2}^2 \big)\, dt \leq C(\mu^{1/2} + \kappa^2).$$

(ii) To simplify the explanation, we assume $\kappa \leq \mu^{1/4}$, there exists a boundary-layer-thickness function $\delta(\mu)$ satisfying

$$\frac{\delta(\mu)}{\mu^{1/2}} \to \infty \quad \text{and} \quad \delta(\mu) \to 0 \quad \text{as} \quad \mu \to 0,$$

such that

$$\lim_{\mu \to 0} \|(v - \bar{v}, w - \bar{w})(t)\|_{C(\Omega_\delta)}^2 = 0,$$

and

$$\liminf_{\mu \to 0} \|(v - \bar{v}, w - \bar{w})(t)\|_{C(\Omega)}^2 > 0$$

with $\Omega_\delta \triangleq (\delta, 1-\delta) (0 < \delta < 1/2)$, provided the boundary values $(v_i, w_i) (i = 1,2)$ are not identically equivalent to the boundary values of $(\bar{v}, \bar{w})$ on the boundaries $x = 0, 1$.

Theorem 1.3 *Suppose $\kappa \leq \mu^{1/4}$, let v^* and w^* be the solutions to the following two initial-boundary value problems, respectively:*

$$\begin{cases} v^*_t = \mu \dfrac{v^*_{xx}}{\bar{\rho}} + \bar{u} v^*_x, \quad v^*(x,t)|_{t=0} = 0, \\ v^*(x,t)|_{x=0} = v_1(t) - \bar{v}_1(t), \quad v^*(x,t)|_{x=1} = v_2(t) - \bar{v}_2(t) \end{cases} \tag{1.8}$$

and

$$\begin{cases} w^*_t = \mu \dfrac{w^*_{xx}}{\bar{\rho}} + \bar{u} w^*_x, \quad w^*(x,t)|_{t=0} = 0, \\ w^*(x,t)|_{x=0} = w_1(t) - \bar{w}_1(t), \quad w^*(x,t)|_{x=1} = w_2(t) - \bar{w}_2(t). \end{cases} \tag{1.9}$$

Then under the conditions of Theorem 1.1, we have

$$\sup_{0 \leq t \leq T} (\|v - \bar{v} - v^*\|_{L^\infty} + \|w - \bar{w} - w^*\|_{L^\infty}) \leq C\mu^{1/8}. \tag{1.10}$$

Remark 1.1 *Compared the results in Theorem 1.1 with those in [26], the derivative estimates of the global solution to Hall-MHD system* (1.1)-(1.2) *obtained in [26] depend on μ and κ, while our estimates obtained in Theorem 1.1 are independent of μ and κ.*

Remark 1.2 *Theorem 1.2-(i) implies that the solution (ρ, u, b_2, b_3) of Hall-MHD system* (1.1)-(1.2) *converges uniformly to the solution of the corresponding MHD system* (1.3)-(1.4). *However, the appearance of the boundary layer for (v, w) leads to that (v, w) can't converge uniformly to $(\bar{v}, \bar{w})$ in the whole domain.*

Remark 1.3 *If we only consider the vanishing limits of the Hall coefficient for Hall-MHD system, since the diffusion terms of μv_{xx} and μw_{xx} are good terms in this case, it can be proved that the solution $(\rho, u, v, w, b_2, b_3)$ of Hall-MHD system* (1.1)-(1.2) *converges uniformly to the solution of the corresponding MHD system* (1.3)-(1.4) *as the Hall coefficient $\kappa \to 0$ in the whole domain.*

We now make some comments on the key ideas used in this paper. First, due to strong interaction between the velocity field and the magnetic field in the system (1.1), it is hard to estimate the first-order derivative of the solution. To overcome this difficulty, motivated by [30], we multiply $(1.1)_5$, $(1.1)_6$ by the material derivative $\dot{b}_2$ (i.e. $\dot{b}_2 = b_{2t} + ub_{2x}$) and $\dot{b}_3$, respectively, instead of the usual b_{2t}, b_{3t} (see(2.7)), then we only need to deal with $\|u_x\|_{L^\infty(0,T;L^2)}$. By utilizing the "effective viscous" flux and the construction of equations, we obtain the expected estimates(see (2.19)). Second, in the process of showing the convergence rate of b_{2x} and b_{3x} in L^2, we use a similar method as above to solve the difficulty caused by the presence of $\tilde{w}_x$ and $\tilde{v}_x$ in magnetic field(see (3.1)), that is, multiply $(3.1)_5$ and $(3.1)_6$ by $\tilde{b}_{2t} + u\tilde{b}_{2x}$ and $\tilde{b}_{3t} + u\tilde{b}_{3x}$, respectively (see 3.13). Third, the priori hypotheses method is used to overcome the difficulties brought by the Hall term, we establish the global estimates under the assumption that $\kappa^2\|b_{2xx}\|_{L^2(0,T;L^2)} \leq 1$ and $\kappa^2\|b_{3xx}\|_{L^2(0,T;L^2)} \leq 1$, then using the smallness condition of the Hall coefficient κ to close the priori hypotheses.

The rest of this paper is organized as follows. In section 2, we show the uniform estimates on the global smooth solutions of Hall-MHD and MHD system with respect to the shear viscosity and the Hall coefficients, namely, Theorem 1.1. In section 3, it is devoted to verify Theorem 1.2, in which, we first give the convergence rate for the solutions from Hall-MHD systems to MHD system as both the shear viscosity and the Hall coefficients tend to zero, then the boundary layer thickness is discussed. In section 4, Theorem 1.3 is proved by constructing boundary layer function.

2 Proof of Theorem 1.1

The aim of this section is to prove Theorem 1.1. To do this, we will establish the global estimates on the solutions to the problems (1.1)-(1.2) and (1.3)-(1.4), respectively. Throughout this section, for any $T > 0$, assume that

$$0 < \rho_0 \in H^2(\Omega), \quad (u_0, v_0, w_0, b_{2_0}, b_{3_0}) \in H^2(\Omega), \quad (v_1(t), v_2(t), w_1(t), w_2(t)) \in C^1(t).$$

2.1 Global estimates on Hall-MHD system

Now we show the global estimates independent of μ and κ for the solutions to Hall-MHD system (1.1)-(1.2).

First, for small κ, we give the following a priori estimate.

Proposition 2.1 *Assume that* $(\rho, u, v, w, b_2, b_3)$ *be a smooth solution of* (1.1)-(1.2) *on* $(0,1) \times [0,T)$*, and satisfy*

$$\kappa^2 \int_0^T \left(\|b_{2xx}\|_{L^2}^2 + \|b_{3xx}\|_{L^2}^2\right) dt \leq 1, \tag{2.1}$$

then one has

$$\kappa^2 \int_0^T \left(\|b_{2xx}\|_{L^2}^2 + \|b_{3xx}\|_{L^2}^2\right) dt \leq 1/2, \tag{2.2}$$

provided

$$\kappa \leq \mu^{1/4} \min\left\{1/\sqrt{2M_1}, 1/\sqrt{2M_2}, 1\right\},$$

here $M_i (i = 1, 2)$ *are positive constants independent of* κ *and* μ.

The proof of Proposition 2.1 comprises the following four lemmas. First of all, we give the following L^2-estimate of $(\sqrt{\rho}, u, v, w, b_2, b_3, \rho_x)$, the positive lower and upper bounds of the density, which can be proved similar to the method in [26], and the details are omitted here for simplicity.

Lemma 2.1 ([26]) *Let* $(\rho, u, v, w, b_2, b_3)$ *be a smooth solution of* (1.1)-(1.2) *on* $(0, 1) \times [0, T)$. *Then it holds that,*

$$\begin{aligned} &\sup_{0 \leq t \leq T} \left(\|\rho\|_{L^1} + \|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} v\|_{L^2}^2 + \|\sqrt{\rho} w\|_{L^2}^2 + \|b_2\|_{L^2}^2 + \|b_3\|_{L^2}^2\right) \\ &\quad + \int_0^T \left(\|u_x\|_{L^2}^2 + \mu\|v_x\|_{L^2}^2 + \mu\|w_x\|_{L^2}^2 + \|b_{2x}\|_{L^2}^2 + \|b_{3x}\|_{L^2}^2\right) dt \leq C \end{aligned} \tag{2.3}$$

and

$$0 < C^{-1} \leq \rho(x,t) \leq C, \qquad \sup_{0 \leq t \leq T} \|\rho_x\|_{L^2}^2 \leq C. \tag{2.4}$$

Lemma 2.2 *Under the conditions of Proposition 2.1, let $(\rho, u, v, w, b_2, b_3)$ be a smooth solution of* (1.1)-(1.2) *on* $(0,1) \times [0,T)$*, then,*

$$\sup_{0\le t\le T}\left(\|u_x\|_{L^2}^2+\|b_{2x}\|_{L^2}^2+\|b_{3x}\|_{L^2}^2+\mu\|v_x\|_{L^2}^2+\mu\|w_x\|_{L^2}^2\right)$$
$$+\int_0^T\left(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\dot{b}_2\|_{L^2}^2+\|\dot{b}_3\|_{L^2}^2+\mu^2\|v_{xx}\|_{L^2}^2+\mu^2\|w_{xx}\|_{L^2}^2\right)dt\le C, \tag{2.5}$$

here, $\dot{f} \triangleq f_t + u f_x$ denote the derivative of material.

Proof. Multiplying $(1.1)_5$ by $\dot{b}_2$ in L^2 and integrating it by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|b_{2x}\|_{L^2}^2+\|\dot{b}_2\|_{L^2}^2=-\int_0^1 u_x b_2\dot{b}_2dx+\int_0^1 v_x\dot{b}_2dx+\kappa\int_0^1(\frac{b_{3x}}{\rho})_x\dot{b}_2dx+\int_0^1 b_{2xx}ub_{2x}dx. \tag{2.6}$$

Definition of the derivative of material implies that

$$\int_0^1 v_x\dot{b}_2dx=\int_0^1 v_x(b_{2t}+ub_{2x})dx=-\frac{d}{dt}\int_0^1 vb_{2x}dx+\int_0^1\dot{v}b_{2x}dx. \tag{2.7}$$

Inserting (2.7) into (2.6) and integrating by parts again, using (2.4), Cauchy-Schwarz, Sobolev and Poincaré inequalities, we have

$$\begin{aligned}
&\frac{d}{dt}\|b_{2x}\|_{L^2}^2+\frac{d}{dt}\int_0^1 vb_{2x}dx+\|\dot{b}_2\|_{L^2}^2\\
&\le C\Big(\|b_2\|_{L^\infty}\|u_x\|_{L^2}\|\dot{b}_2\|_{L^2}+\|b_{2x}\|_{L^2}\|\dot{v}\|_{L^2}+\kappa(\|b_{3xx}\|_{L^2}\\
&\qquad+\|b_{3x}\|_{L^\infty}\|\rho_x\|_{L^2})\|\dot{b}_2\|_{L^2}+\|u_x\|_{L^\infty}\|b_{2x}\|_{L^2}^2\Big)\\
&\le C\left(\|b_{2x}\|_{L^2}^2\|u_x\|_{L^2}^2+\|b_{2x}\|_{L^2}^2+\kappa^2\|b_{3x}\|_{H^1}^2+\|u_x\|_{L^\infty}\|b_{2x}\|_{L^2}^2\right)\\
&\quad+\frac{1}{4}\left(\|\dot{v}\|_{L^2}^2+\|\dot{b}_2\|_{L^2}^2\right)\\
&\le C\left(\|b_{2x}\|_{L^2}^4+\|u_x\|_{L^2}^4+\kappa^2\|b_{3x}\|_{H^1}^2+\|u_x\|_{L^\infty}^2+1\right)+\frac{1}{4}\left(\|\dot{v}\|_{L^2}^2+\|\dot{b}_2\|_{L^2}^2\right).
\end{aligned} \tag{2.8}$$

Similarly, it follows that

$$\begin{aligned}
&\frac{d}{dt}\|b_{3x}\|_{L^2}^2+\|\dot{b}_3\|_{L^2}^2+2\frac{d}{dt}\int_0^1 wb_{3x}dx\\
&\le C\left(\|b_{3x}\|_{L^2}^4+\|u_x\|_{L^2}^4+\kappa^2\|b_{2x}\|_{H^1}^2+\|u_x\|_{L^\infty}^2+1\right)+\frac{1}{4}\left(\|\dot{w}\|_{L^2}^2+\|\dot{b}_3\|_{L^2}^2\right).
\end{aligned} \tag{2.9}$$

Rewrite the equation $(1.1)_3$ as

$$\rho^{1/2}\dot{v}-\mu\rho^{-1/2}v_{xx}=\rho^{-1/2}b_{2x},$$

it is easy to deduce

$$\mu\frac{d}{dt}\|v_x\|_{L^2}^2+\int_0^1\rho|\dot{v}|^2dx+\mu^2\int_0^1\rho^{-1}|v_{xx}|^2dx$$

$$= 2\mu(v_t v_x)(x,t)\big|_{x=0}^{x=1} + 2\mu\int_0^1 u v_x v_{xx} dx + \int_0^1 \rho^{-1}|b_{2x}|^2 dx. \tag{2.10}$$

To deal with the boundary term in (2.10), integrating $(1.1)_3$ about x over $(0,x)$ yields

$$\mu v_x(0,t) = \mu v_x(x,t) - \int_0^x \rho\dot{v} dx + b_2(x,t).$$

Integrating the above equation about x over (0,1), we get

$$\mu v_x(0,t) = \mu\left(v_2(t) - v_1(t)\right) - \int_0^1\int_0^x \rho\dot{v}(\eta,t) d\eta dx + \int_0^1 b_2(x,t) dx. \tag{2.11}$$

Similarly,

$$\mu v_x(1,t) = \mu(v_1(t) - v_2(t)) + \int_0^1\int_x^1 \rho\dot{v}(\eta,t) d\eta dx + \int_0^1 b_2(x,t) dx. \tag{2.12}$$

Hence, from (1.5), (2.11) and (2.12), it holds that

$$2\mu(v_t v_x)(x,t)\big|_{x=0}^{x=1} \le \frac{1}{4}\|\rho^{1/2}\dot{v}\|_{L^2}^2 + C,$$

which, combining with (2.4) and (2.10), we arrive at

$$\begin{aligned}
&\mu\frac{d}{dt}\|v_x\|_{L^2}^2 + \int_0^1 \rho|\dot{v}|^2 dx + \mu^2\int_0^1 \rho^{-1}|v_{xx}|^2 dx \\
&\quad\le C\left(\mu\|u_x\|_{L^\infty}\|v_x\|_{L^2}^2 + \|b_{2x}\|_{L^2}^2 + 1\right) \\
&\quad\le C\left(\|u_x\|_{L^\infty}^2 + \mu^2\|v_x\|_{L^2}^4 + \|b_{2x}\|_{L^2}^2 + 1\right).
\end{aligned} \tag{2.13}$$

In a similar manner,

$$\begin{aligned}
&\mu\frac{d}{dt}\|w_x\|_{L^2}^2 + \int_0^1 \rho|\dot{w}|^2 dx + \mu^2\int_0^1 \rho^{-1}|w_{xx}|^2 dx \\
&\quad\le C\left(\|u_x\|_{L^\infty}^2 + \mu^2\|w_x\|_{L^2}^4 + \|b_{3x}\|_{L^2}^2 + 1\right).
\end{aligned} \tag{2.14}$$

Summing up the estimates of (2.8), (2.9), (2.13) and (2.14), we obtain from (2.4) that

$$\begin{aligned}
&\frac{d}{dt}\left(\|b_{2x}\|_{L^2}^2 + \|b_{3x}\|_{L^2}^2 + \mu\|v_x\|_{L^2}^2 + \mu\|w_x\|_{L^2}^2\right) \\
&\quad + \left(\|\dot{b}_2\|_{L^2}^2 + \|\dot{b}_3\|_{L^2}^2 + \|\dot{v}\|_{L^2}^2 + \|\dot{w}\|_{L^2}^2\right) \\
&\quad + \mu^2\|v_{xx}\|_{L^2}^2 + \mu^2\|w_{xx}\|_{L^2}^2 + 2\frac{d}{dt}\int_0^1 (wb_{3x} + vb_{2x}) dx \\
&\le C\left(\|b_{2x}\|_{L^2}^4 + \|b_{3x}\|_{L^2}^4 + \|u_x\|_{L^2}^4 + \kappa^2(\|b_{2x}\|_{H^1}^2 + \|b_{3x}\|_{H^1}^2)\right. \\
&\quad \left. + \|u_x\|_{L^\infty}^2 + \mu^2(\|w_x\|_{L^2}^4 + \|v_x\|_{L^2}^4) + 1\right).
\end{aligned} \tag{2.15}$$

Now we deal with $\|\nabla u\|_{L^\infty}$. Define

$$F(x,t) \triangleq u_x(x,t) - P(x,t) - \frac{(b_2^2 + b_3^2)}{2}(x,t).$$

By $(1.1)_2$, we have

$$\rho\dot{u} = F_x. \tag{2.16}$$

According to the definition of $F(x,t)$, (2.3), (2.4) and (2.16), we deduce

$$\|F\|_{L^2}^2 \le C\left(\|u_x\|_{L^2}^2 + \|b_2\|_{L^4}^2 + \|b_3\|_{L^4}^2 + 1\right) \le C\left(\|u_x\|_{L^2}^2 + \|b_{2x}\|_{L^2} + \|b_{3x}\|_{L^2} + 1\right) \tag{2.17}$$

and

$$\|F_x\|_{L^2}^2 \le C\|\sqrt{\rho}\dot{u}\|_{L^2}^2. \tag{2.18}$$

We infer from (2.17) and (2.18) that

$$\begin{aligned}
\|u_x\|_{L^\infty} &\le C\left(\|F\|_{L^\infty} + \|P\|_{L^\infty} + \|b_2\|_{L^\infty}^2 + \|b_3\|_{L^\infty}^2\right) \\
&\le C\left(\|F\|_{L^2} + \|F\|_{L^2}^{1/2}\|F_x\|_{L^2}^{1/2} + \|b_{2x}\|_{L^2}^2 + \|b_{3x}\|_{L^2}^2 + 1\right) \\
&\le \frac{1}{4}\|\sqrt{\rho}\dot{u}\|_{L^2} + C\left(\|u_x\|_{L^2} + \|b_{2x}\|_{L^2}^2 + \|b_{3x}\|_{L^2}^2 + 1\right).
\end{aligned} \tag{2.19}$$

Substituting (2.19) into (2.15), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|b_{2x}\|_{L^2}^2 + \|b_{3x}\|_{L^2}^2 + \mu\|v_x\|_{L^2}^2 + \mu\|w_x\|_{L^2}^2 + 2\int_0^1 (wb_{3x} + vb_{2x})dx\right) \\
&\quad + \|\dot{b}_2\|_{L^2}^2 + \|\dot{b}_3\|_{L^2}^2 + \|\dot{v}\|_{L^2}^2 + \|\dot{w}\|_{L^2}^2 + \mu^2\|v_{xx}\|_{L^2}^2 + \mu^2\|w_{xx}\|_{L^2}^2 \\
&\le C\left(\|b_{2x}\|_{L^2}^4 + \|b_{3x}\|_{L^2}^4 + \|u_x\|_{L^2}^4 + \kappa^2(\|b_{2x}\|_{H^1}^2 + \|b_{3x}\|_{H^1}^2)\right. \\
&\quad \left. + \mu^2(\|w_x\|_{L^2}^4 + \|v_x\|_{L^2}^4) + 1\right) + \frac{1}{4}\|\sqrt{\rho}\dot{u}\|_{L^2}^2.
\end{aligned} \tag{2.20}$$

To control $\|\sqrt{\rho}\dot{u}\|_{L^2}^2$, multiplying $(1.1)_2$ by $\dot{u}$, integrating the resulting equality by parts, using (2.3), (2.4) and Sobolev inequality, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2}^2 + \|\sqrt{\rho}\dot{u}\|_{L^2}^2 &= -\int_0^1 P_x\dot{u}dx - \frac{1}{2}\int_0^1 (b_2^2 + b_3^2)_x\dot{u}dx + \int_0^1 u_{xx}uu_xdx \\
&\le C\left(\|\rho_x\|_{L^2}^2 + \|b_2\|_{L^\infty}^2\|b_{2x}\|_{L^2}^2 + \|b_3\|_{L^\infty}^2\|b_{3x}\|_{L^2}^2\right) \\
&\quad + \frac{1}{2}\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + C\|u_x\|_{L^\infty}\|u_x\|_{L^2}^2 \\
&\le C\left(1 + \|b_{2x}\|_{L^2}^4 + \|b_{3x}\|_{L^2}^4 + \|u_x\|_{L^\infty}^2 + \|u_x\|_{L^2}^4\right) + \frac{1}{2}\|\sqrt{\rho}\dot{u}\|_{L^2}^2,
\end{aligned}$$

we observe from the above equality and (2.19) that

$$\frac{d}{dt}\|u_x\|_{L^2}^2 + \|\sqrt{\rho}\dot{u}\|_{L^2}^2 \le C\left(1 + \|b_{2x}\|_{L^2}^4 + \|b_{3x}\|_{L^2}^4 + \|u_x\|_{L^2}^4\right), \tag{2.21}$$

this, together with (2.20), we get

$$\begin{aligned}
&\frac{d}{dt}\left(\|b_{2x}\|_{L^2}^2 + \|b_{3x}\|_{L^2}^2 + \mu\|v_x\|_{L^2}^2 + \mu\|w_x\|_{L^2}^2 + \|u_x\|_{L^2}^2\right) \\
&\quad + \mu^2\|v_{xx}\|_{L^2}^2 + \mu^2\|w_{xx}\|_{L^2}^2 + (\|\dot{b}_2\|_{L^2}^2 + \|\dot{b}_3\|_{L^2}^2 + \|\dot{v}\|_{L^2}^2
\end{aligned}$$

$$
\begin{aligned}
&+\|\dot w\|_{L^2}^2+\|\sqrt{\rho}\dot u\|_{L^2}^2)+2\frac{d}{dt}\int_0^1(wb_{3x}+vb_{2x})dx\\
&\le C\left(\|b_{2x}\|_{L^2}^4+\|b_{3x}\|_{L^2}^4+\|u_x\|_{L^2}^4+\kappa^2(\|b_{2x}\|_{H^1}^2+\|b_{3x}\|_{H^1}^2)\right.\\
&\quad\left.+\mu^2(\|w_x\|_{L^2}^4+\|v_x\|_{L^2}^4)+1\right). \qquad (2.22)
\end{aligned}
$$

Thus, using the above inequality, the estimates of (2.5) readily follows from Gronwall inequality, (2.1) and (2.3).

Lemma 2.3 *Under the conditions of Proposition 2.1, let (ρ,u,v,w,b_2,b_3) be a smooth solution of* (1.1)-(1.2) *on* $(0,1)\times[0,T)$. *Then,*

$$
\sup_{0\le t\le T}(\|v\|_{L^\infty}+\|w\|_{L^\infty})\le C \quad \text{and} \quad \int_0^T\left(\|b_{2x}\|_{L^\infty}^2+\|b_{3x}\|_{L^\infty}^2\right)dt\le C. \qquad (2.23)
$$

Proof. Multiplying $(1.1)_3$ by $|v|^{n-2}v$ and integrating by parts, we have

$$
\begin{aligned}
&\frac{1}{n}\frac{d}{dt}\int_0^1\rho v^n dx+\mu\int_0^1|v_x|^2|v|^{n-2}dx+\mu(n-2)\int_0^1|v|^{n-2}(|v|_x)^2dx\\
&=\mu\left[(v_x|v|^{n-2}v)(x,t)\right]\Big|_{x=0}^{x=1}+\int_0^1 b_{2x}|v|^{n-2}v dx\\
&\le C\left(\|\sqrt{\rho}\dot u\|_{L^2}^2+1+\|b_{2x}\|_{L^n}\|v\|_{L^n}^{n-1}\right). \qquad (2.24)
\end{aligned}
$$

In order to estimate $\|b_{2x}\|_{L^n}$, we give the definition

$$
G\triangleq b_{2x}+v+\frac{\kappa b_{3x}}{\rho}, \qquad (2.25)
$$

then from $(1.1)_5$, G satisfies the following equation:

$$
\dot b_2+u_xb_2=G_x. \qquad (2.26)
$$

In view of (2.5), (2.25), (2.26) and Sobolev inequality, direct calculation yields that

$$
\|G\|_{L^2}\le C\left(\|b_{2x}\|_{L^2}+\|v\|_{L^2}+\kappa\|b_{3x}\|_{L^2}\right)\le C \qquad (2.27)
$$

and

$$
\|G_x\|_{L^2}\le C\left(\|\dot b_2\|_{L^2}+\|u_x\|_{L^2}\|b_2\|_{L^\infty}\right)\le C\left(\|\dot b_2\|_{L^2}+1\right). \qquad (2.28)
$$

Due to (2.5), (2.6), (2.27) and (2.28), it follows that

$$
\begin{aligned}
\|b_{2x}\|_{L^n}&\le C\left(\|G\|_{L^n}+\|v\|_{L^n}+\|\kappa\rho^{-1}b_{3x}\|_{L^n}\right)\\
&\le C\left(\|G\|_{L^2}+\|G_x\|_{L^2}+\|v\|_{L^n}+\kappa\left(\|b_{3x}\|_{L^2}+\|b_{3xx}\|_{L^2}\right)\right)\\
&\le C\left(\|\dot b_2\|_{L^2}+\|v\|_{L^n}+\kappa\|b_{3xx}\|_{L^2}+1\right). \qquad (2.29)
\end{aligned}
$$

Substitute it into (2.24), we obtain

$$
\frac{1}{n}\frac{d}{dt}\int_0^1\rho v^n dx+\mu\int_0^1|v_x|^2|v|^{n-2}dx
$$

$$\leq C\left(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\left(\|\dot{b}_2\|_{L^2}+\kappa\|b_{3xx}\|_{L^2}+1\right)(\|v\|_{L^n}^n+1)\right),$$

which, by Gronwall inequality, we infer from (2.1), (2.5) that

$$\sup_{0\leq t\leq T}\|\rho^{1/n}v\|_{L^n}\leq(nC)^{1/n}.$$

Taking $n\to\infty$, from (2.4), we deduce that

$$\sup_{0\leq t\leq T}\|v\|_{L^\infty}\leq C. \tag{2.30}$$

From (2.29) and (2.30), it is easy to get

$$\int_0^T\|b_{2x}\|_{L^\infty}^2dt\leq C.$$

In the same manner, we obtain

$$\sup_{0\leq t\leq T}\|w\|_{L^\infty}\leq C \quad \textit{and} \quad \int_0^T\|b_{3x}\|_{L^\infty}^2dt\leq C.$$

Lemma 2.4 *Under the conditions of Proposition 2.1, let* (ρ,u,v,w,b_2,b_3) *be a smooth solution of* (1.1)-(1.2) *on* $(0,1)\times[0,T)$. *Then,*

$$\mu^{1/2}\sup_{0\leq t\leq T}\left(\|v_x\|_{L^2}^2+\|w_x\|_{L^2}^2\right)+\mu^{3/2}\int_0^T\left(\|v_{xx}\|_{L^2}^2+\|w_{xx}\|_{L^2}^2\right)dt\leq C. \tag{2.31}$$

Proof. Multiplying $(1.1)_3$ by μv_{xx} and integrating it by parts, we have

$$\begin{aligned}\frac{\mu}{2}\frac{d}{dt}\|v_x\|_{L^2}^2+\mu^2\int_0^1\rho^{-1}|v_{xx}|^2dx&=\mu\int_0^1uv_xv_{xx}dx-\mu\int_0^1\rho^{-1}b_{2x}v_{xx}dx+\mu\left[(v_tv_x)(x,t)\right]|_{x=0}^{x=1}\\&\triangleq\sum_{i=1}^3E_i.\end{aligned} \tag{2.32}$$

Now we estimate E_2 and E_3, from integrating by parts, (2.4), (2.5), (2.23) and Cauchy-Schwarz inequality imply that

$$\begin{aligned}E_2&=-\mu\int_0^1\rho^{-2}\rho_xb_{2x}v_xdx+\mu\int_0^1\rho^{-1}b_{2xx}v_xdx-\mu(\rho^{-1}b_{2x}v_x)|_{x=0}^{x=1}\\&\leq C\left(\mu\|\rho_x\|_{L^2}\|b_{2x}\|_{L^\infty}\|v_x\|_{L^2}+\mu\|b_{2xx}\|_{L^2}\|v_x\|_{L^2}+\mu\|b_{2x}\|_{L^\infty}\|v_x\|_{L^\infty}\right).\end{aligned} \tag{2.33}$$

By $(1.1)_4$, (2.3), (2.4) and (2.23), we have

$$\begin{aligned}\|b_{2xx}\|_{L^2}&\leq C\left(\|\dot{b}_2\|_{L^2}+\|u_xb_2\|_{L^2}+\|v_x\|_{L^2}+\kappa^2\|(\rho^{-1}b_{3x})_x\|_{L^2}\right)\\&\leq C\left(\|\dot{b}_2\|_{L^2}+\|v_x\|_{L^2}+\kappa^2\|\rho_x\|_{L^2}\|b_{3x}\|_{L^\infty}+\kappa^2\|b_{3xx}\|_{L^2}+1\right)\\&\leq C\left(\|\dot{b}_2\|_{L^2}+\|v_x\|_{L^2}+\kappa^2\|b_{3xx}\|_{L^2}+1\right),\end{aligned} \tag{2.34}$$

which, putted into (2.33), we find

$$E_2 \le C\left(\mu(1+\|b_{2x}\|_{L^\infty}^2)\|v_x\|_{L^2}^2 + \mu\|\dot{b}_2\|_{L^2}^2 + \mu\kappa^2\|b_{3x}\|_{H^1}^2 + \mu\|b_{2x}\|_{L^\infty}\|v_x\|_{L^2}^{1/2}\|v_{xx}\|_{L^2}^{1/2} + \mu\right)$$
$$\le C\left(\mu\|\dot{b}_2\|_{L^2}^2 + \mu(1+\|b_{2x}\|_{L^\infty}^2)\|v_x\|_{L^2}^2 + \mu\kappa^2\|b_{3x}\|_{H^1}^2 + \mu^{1/2}\right) + \varepsilon\mu^2\|v_{xx}\|_{L^2}^2. \tag{2.35}$$

It follows from (1.5) and Cauchy-Schwarz inequality that

$$E_3 \le C\mu\|v_{1t}\|_{L^\infty}\|v_x\|_{L^\infty} \le C\left(\mu\|v_x\|_{L^2}^2 + \mu^{1/2}\right) + \varepsilon\mu^2\|v_{xx}\|_{L^2}^2. \tag{2.36}$$

Substituting (2.35) and (2.36) into (2.32), choosing $\varepsilon > 0$ sufficiently small, we have

$$\frac{\mu}{2}\frac{d}{dt}\|v_x\|_{L^2}^2 + \mu^2\|v_{xx}\|_{L^2}^2 \le C\left(\mu\|u_x\|_{L^\infty}\|v_x\|_{L^2}^2 + \mu\|\dot{b}_2\|_{L^2}^2 + \mu\|v_x\|_{L^2}^2 + \mu\kappa^2\|b_{3x}\|_{H^1}^2 + \mu^{1/2}\right),$$

which, by Gronwall inequality, we infer from (2.5), (2.18) and (2.19) that

$$\mu \sup_{0\le t\le T}\|v_x\|_{L^2}^2 + \mu^2\int_0^T \|v_{xx}\|_{L^2}^2 dt \le C\mu^{1/2}. \tag{2.37}$$

Similarly, we have

$$\mu \sup_{0\le t\le T}\|w_x\|_{L^2}^2 + \mu^2\int_0^T \|w_{xx}\|_{L^2}^2 dt \le C\mu^{1/2}. \tag{2.38}$$

The inequalities (2.34), (2.37) and (2.38) indicate that

$$\mu^{1/2}\int_0^T \|b_{2xx}\|_{L^2}^2 dt \le M_1 \quad \text{and} \quad \mu^{1/2}\int_0^T \|b_{3xx}\|_{L^2}^2 dt \le M_2. \tag{2.39}$$

here, $M_i(i=1,1)$ are positive constants independent of μ and κ.

Proof of Proposition 2.1. Thanks to (2.39), just choosing $\kappa \le \mu^{1/4}\min\{1/\sqrt{2M_1},\, 1/\sqrt{2M_2}\}$, it is easy to deduce that (2.2) hold. Hence we finish the proof of Proposition 2.1. $\square$

Proof of Theorem 1.1-*(i)*. Frist, the local existence of regular solution can be obtained by the Banach theorem and the contractivity of the operator defined by the linearized the problem (1.1). Second, by virtue of the global a priori established in Lemma 2.1-2.4, we can extend the local solutions to global strong solution of (1.1)-(1.2).

2.2 Global estimates on MHD system

This subsection is devoted to the global estimates for the solution of the initial-boundary value problem (1.3)-(1.4). Due to these estimates in Lemmas 2.1-2.4 independent of μ and κ, hence the lemmas 2.1-2.4 still hold for the solution of the problem (1.3)-(1.4), we summarize them in the following lemma.

Lemma 2.5 *Assume* $(\bar\rho, \bar u, \bar v, \bar w, \bar b_2, \bar b_3)$ *be a smooth solution of* (1.3)-(1.4) *on* $(0,1)\times[0,T)$, *then*

$$0 < C_3 \le \bar\rho(x,t) \le C_4, \qquad (x,t)\in(0,1)\times[0,T), \tag{2.40}$$

$$\sup_{0\le t\le T}\left(\|\bar v\|_{L^2}^2 + \|\bar w\|_{L^2}^2 + \|\bar\rho\|_{H^1}^2 + \|\bar u\|_{H^1}^2 + \|\bar b_2\|_{H^1}^2 + \|\bar b_3\|_{H^1}^2\right) + \int_0^T \left(\|\bar u_x\|_{H^1}^2\right.$$
$$\left. + \|\bar b_{2x}\|_{L^2}^2 + \|\bar b_{2x}\|_{L^\infty}^2 + \|\dot{\bar b}_2\|_{L^2}^2 + \|\bar b_{3x}\|_{L^2}^2 + \|\bar b_{3x}\|_{L^\infty}^2 + \|\dot{\bar b}_3\|_{L^2}^2\right) dt \le C. \tag{2.41}$$

In order to discuss the convergence rate of the solution and the boundary layer, we need stronger regularity of the solution of (1.3)-(1.4) on $(0,1)\times[0,T]$, so we give the following higher order estimates.

Lemma 2.6 *Assume that* $(\bar\rho,\bar u,\bar v,\bar w,\bar b_2,\bar b_3)$ *be smooth solution of* (1.3)-(1.4) *on* $(0,1)\times[0,T)$, *then*

$$\sup_{0\le t\le T}\left(\|\bar v_x\|_{L^2}^2+\|\bar w_x\|_{L^2}^2+\|\bar v_t\|_{L^2}^2+\|\bar w_t\|_{L^2}^2\right)+\int_0^T\left(\|\bar b_{2xx}\|_{L^2}^2+\|\bar b_{3xx}\|_{L^2}^2\right)dt\le C. \tag{2.42}$$

Proof. First, we rewrite $(1.3)_3$ into the form

$$\bar v_t+\bar u\bar v_x-\frac{\bar b_{2x}}{\bar\rho}=0. \tag{2.43}$$

Differentiating (2.43) with respect to x, multiplying it by $\bar v_x$ in L^2, after integrating by parts, by (2.40), (2.41), Sobolev and Cauchy-Schwarz inequalities, we have

$$\begin{aligned}\frac{d}{dt}\|\bar v_x\|_{L^2}^2&\le C\|\bar u_x\|_{L^\infty}\|\bar v_x\|_{L^2}^2+\varepsilon\|\bar b_{2xx}\|_{L^2}^2+C\|\bar b_{2x}\|_{L^\infty}^2\|\bar\rho_x\|_{L^2}^2+C\|\bar v_x\|_{L^2}^2\\&\le C\left(\|\bar u_x\|_{H^1}\|\bar v_x\|_{L^2}^2+\|\bar v_x\|_{L^2}^2+1\right)+\varepsilon\|\bar b_{2xx}\|_{L^2}^2,\end{aligned} \tag{2.44}$$

we have from $(1.3)_5$ that

$$\|\bar b_{2xx}\|_{L^2}^2\le C\left(\|\dot{\bar b}_2\|_{L^2}^2+\|\bar u_x\|_{L^2}^2\|\bar b_2\|_{L^\infty}+\|\bar v_x\|_{L^2}^2\right),$$

which, adding (2.44), by Gronwall inequality and (2.41), we get

$$\sup_{0\le t\le T}\|\bar v_x\|_{L^2}^2+\int_0^T\|\bar b_{2xx}\|_{L^2}^2dt\le C. \tag{2.45}$$

In a similar manner, we have

$$\sup_{0\le t\le T}\|\bar w_x\|_{L^2}^2+\int_0^T\|\bar b_{3xx}\|_{L^2}^2dt\le C,$$

which, together with $(1.3)_3$, $(1.3)_4$, (2.40), (2.41) and (2.45), we also have

$$\sup_{0\le t\le T}\left(\|\bar v_t\|_{L^2}^2+\|\bar w_t\|_{L^2}^2\right)\le C.$$

Lemma 2.7 *Assume that* $(\bar\rho,\bar u,\bar v,\bar w,\bar b_2,\bar b_3)$ *be a smooth solution of* (1.3)-(1.4) *on* $(0,1)\times[0,T)$, *then*

$$\sup_{0\le t\le T}\left(\|\sqrt{\bar\rho}\bar u_t\|_{L^2}^2+\|\bar u_{xx}\|_{L^2}^2\right)+\int_0^T\|\bar u_{xt}\|_{L^2}^2dt\le C. \tag{2.46}$$

Proof. Differentiating $(1.3)_2$ with respect to t, and multiplying it by $\bar u_t$ in L^2, in view of (2.40), (2.41), Sobolev and Cauchy-Schwarz inequalities, we deduce

$$\frac12\frac{d}{dt}\|\sqrt{\bar\rho}\bar u_t\|_{L^2}^2+\|\bar u_{xt}\|_{L^2}^2=-\int_0^1\bar\rho_t\bar u\bar u_x\bar u_tdx-\int_0^1\rho\bar u_x|\bar u_t|^2dx$$

$$
\begin{aligned}
&-2\int_0^1 \bar\rho\bar u\bar u_{xt}\bar u_t dx+\int_0^1 \bar P_t\bar u_{xt}dx+\frac{1}{2}\int_0^1(\bar b_2^2+\bar b_3^2)_t\bar u_{xt}dx\\
&\le C\left(\|\bar\rho_t\|_{L^2}\|\bar u\|_{L^\infty}\|\bar u_x\|_{L^\infty}\|\bar u_t\|_{L^2}+\|\bar u_x\|_{L^\infty}\|\bar u_t\|_{L^2}^2+\|\bar u\|_{L^\infty}^2\|\bar u_t\|_{L^2}^2\right.\\
&\quad\left.+\|\bar\rho_t\|_{L^2}^2+\|\bar b_2\|_{L^\infty}^2\|\bar b_{2t}\|_{L^2}^2+\|\bar b_3\|_{L^\infty}^2\|\bar b_{3t}\|_{L^2}^2\right)+\varepsilon\|\bar u_{xt}\|_{L^2}^2\\
&\le C\left(\|\bar u_x\|_{L^\infty}^2+\|\bar u_t\|_{L^2}^2+\|\bar u_x\|_{L^\infty}\|\bar u_t\|_{L^2}^2+\|\bar b_{2t}\|_{L^2}^2+\|\bar b_{3t}\|_{L^2}^2+1\right)+\varepsilon\|\bar u_{xt}\|_{L^2}^2.
\end{aligned}
$$

From the above equality, by Gronwall inequality and (2.41), one has

$$
\sup_{0\le t\le T}\|\sqrt{\bar\rho}\bar u_t\|_{L^2}^2+\int_0^T\|\bar u_{xt}\|_{L^2}^2dt\le C. \tag{2.47}
$$

By (2.40), (2.41) and (2.47), we have

$$
\sup_{0\le t\le T}\|\bar u_{xx}\|_{L^2}^2\le C.
$$

The proof of Lemma 2.7 is therefore completed. □

Lemma 2.8 *Assume that* $(\bar\rho,\bar u,\bar v,\bar w,\bar b_2,\bar b_3)$ *be a smooth solution of* (1.3)-(1.4) *on* $(0,1)\times[0,T)$, *then*

$$
\sup_{0\le t\le T}\left(\|\bar b_{2t}\|_{L^2}^2+\|\bar b_{3t}\|_{L^2}^2+\|\bar b_{2xx}\|_{L^2}^2+\|\bar b_{3xx}\|_{L^2}^2\right)+\int_0^T\left(\|\bar b_{2xt}\|_{L^2}^2+\|\bar b_{3xt}\|_{L^2}^2\right)dt\le C. \tag{2.48}
$$

Proof. Differentiating $(1.3)_5$ with respect to t, and multiplying it by $\bar b_{2t}$ in L^2, by (2.41), we deduce

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\bar b_{2t}\|_{L^2}^2+\|\bar b_{2xt}\|_{L^2}^2&=-\int_0^1\bar u_{xt}\bar b_2\bar b_{2t}dx-\frac{1}{2}\int_0^1\bar u_x|\bar b_{2t}|^2dx\\
&\quad-\int_0^1\bar u_t\bar b_{2x}b_{2t}dx+\int_0^1\bar v_{xt}\bar b_{2t}dx\\
&\le C\left(\|\bar b_2\|_{L^\infty}^2\|\bar b_{2t}\|_{L^2}^2+\|\bar u_{xt}\|_{L^2}^2+\|\bar u_x\|_{L^\infty}\|\bar b_{2t}\|_{L^2}^2\right.\\
&\quad\left.+\|\bar b_{2x}\|_{L^\infty}^2\|\bar b_{2t}\|_{L^2}^2+\|\bar u_t\|_{L^2}^2+\|\bar v_t\|_{L^2}^2\right)+\varepsilon\|\bar b_{2xt}\|_{L^2}^2\\
&\le C\left(\|\bar b_{2t}\|_{L^2}^2+\|\bar u_{xt}\|_{L^2}^2+\|\bar u_x\|_{L^\infty}\|\bar b_{2t}\|_{L^2}^2+\|\bar b_{2x}\|_{L^\infty}^2\|\bar b_{2t}\|_{L^2}^2\right.\\
&\quad\left.+\|\bar u_t\|_{L^2}^2+\|\bar v_t\|_{L^2}^2\right)+\varepsilon\|\bar b_{2xt}\|_{L^2}^2.
\end{aligned} \tag{2.49}
$$

Similarly, we have

$$
\begin{aligned}
\frac{d}{dt}\|\bar b_{3t}\|_{L^2}^2+\|\bar b_{3xt}\|_{L^2}^2&\le C\left(\|\bar b_{3t}\|_{L^2}^2+\|\bar u_{xt}\|_{L^2}^2+\|\bar u_x\|_{L^\infty}\|\bar b_{3t}\|_{L^2}^2\right.\\
&\quad\left.+\|\bar b_{3x}\|_{L^\infty}^2\|\bar b_{3t}\|_{L^2}^2+\|\bar u_t\|_{L^2}^2+\|\bar w_t\|_{L^2}^2\right)+\varepsilon\|\bar b_{3xt}\|_{L^2}^2.
\end{aligned} \tag{2.50}
$$

Collecting (2.49) and (2.50) together, and choosing ε small enough, we get

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\bar b_{2t}\|_{L^2}^2+\|\bar b_{3t}\|_{L^2}^2\right)+\|\bar b_{2xt}\|_{L^2}^2+\|\bar b_{3xt}\|_{L^2}^2\\
&\quad\le C\left(\|\bar b_{2t}\|_{L^2}^2+\|\bar b_{3t}\|_{L^2}^2+\|\bar u_{xt}\|_{L^2}^2+\|\bar u_x\|_{L^\infty}(\|\bar b_{2t}\|_{L^2}^2\right.
\end{aligned}
$$

$$+\|\bar{b}_{3t}\|_{L^2}^2) + \|\bar{u}_t\|_{L^2}^2 + \|\bar{v}_t\|_{L^2}^2 + \|\bar{w}_t\|_{L^2}^2\big)\,, \tag{2.51}$$

which, by Gronwall inequality, we deduce from (2.41) and (2.42) that

$$\sup_{0\le t\le T}\left(\|\bar{b}_{2t}\|_{L^2}^2 + \|\bar{b}_{3t}\|_{L^2}^2\right) + \int_0^T \left(\|\bar{b}_{2xt}\|_{L^2}^2 + \|\bar{b}_{3xt}\|_{L^2}^2\right) dt \le C. \tag{2.52}$$

In view of (2.41), (2.42), (2.52), it follows $(1.3)_5$ and $(1.3)_6$ that

$$\sup_{0\le t\le T}\left(\|\bar{b}_{2xx}\|_{L^2}^2 + \|\bar{b}_{3xx}\|_{L^2}^2\right) \le C.$$

Hence, we finish the proof of the Lemma 2.8. $\square$

Lemma 2.9 *Assume* $(\bar{\rho}, \bar{u}, \bar{v}, \bar{w}, \bar{b}_2, \bar{b}_3)$ *be a solution of* (1.3)-(1.4) *on* $(0,1)\times[0,T)$, *then*

$$\sup_{0\le t\le T}\left(\|\bar{v}_{xx}\|_{L^2}^2 + \|\bar{w}_{xx}\|_{L^2}^2 + \|\bar{\rho}_{xx}\|_{L^2}^2\right) + \int_0^T \left(\|\bar{u}_{xxx}\|_{L^2}^2 + \|\bar{b}_{2xxx}\|_{L^2}^2 + \|\bar{b}_{3xxx}\|_{L^2}^2\right) \le C, \tag{2.53}$$

$$\sup_{0\le t\le T}\left(\|\dot{\bar{v}}_x\|_{L^2}^2 + \|\dot{\bar{w}}_x\|_{L^2}^2\right) \le C. \tag{2.54}$$

Proof. Applying ∂_{xx} to (2.43), and multiplying it by $\bar{v}_{xx}$ in L^2, and integrating by parts, by (2.40), (2.41) and (2.46), Sobolev and Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned}
\frac{d}{dt}\|\bar{v}_{xx}\|_{L^2}^2 \le\ & C\left(\|\bar{u}_x\|_{L^\infty}\|\bar{v}_{xx}\|_{L^2} + \|\bar{u}_{xx}\|_{L^2}\|\bar{v}_x\|_{L^\infty} + \|\bar{b}_{2xxx}\|_{L^2}\right.\\
&\left. + \|\bar{b}_{2xx}\|_{L^\infty}\|\bar{\rho}_x\|_{L^2} + \|\bar{b}_{2x}\|_{L^\infty}\|\bar{\rho}_{xx}\|_{L^2}\right)\|\bar{v}_{xx}\|_{L^2}\\
\le\ & C\left(\|\bar{v}_{xx}\|_{L^2}^2 + \|\bar{b}_{2xxx}\|_{L^2}^2 + \|\bar{b}_{2x}\|_{H^1}^2\|\bar{\rho}_x\|_{H^1}^2 + 1\right).
\end{aligned} \tag{2.55}$$

To deal with $\|\bar{b}_{2xxx}\|_{L^2}$, from $(1.3)_4$, $(1.3)_5$, (2.40) and (2.46), we obtain

$$\begin{aligned}
\|\bar{b}_{2xxx}\|_{L^2}^2 &\le C\left(\|\bar{b}_{2xt}\|_{L^2}^2 + \|\bar{u}\bar{b}_2\|_{H^2} + \|\bar{v}_{xx}\|_{L^2}^2\right)\\
&\le C\left(\|\bar{b}_{2xt}\|_{L^2}^2 + \|\bar{b}_{2x}\|_{H^1}^2 + \|\bar{v}_{xx}\|_{L^2}^2 + 1\right).
\end{aligned} \tag{2.56}$$

Putting (2.56) into (2.55), we get

$$\frac{d}{dt}\|\bar{v}_{xx}\|_{L^2}^2 \le C\left(\|\bar{v}_{xx}\|_{L^2}^2 + (\|\bar{b}_{2x}\|_{H^1}^2 + 1)\|\bar{\rho}_x\|_{H^1}^2 + \|\bar{b}_{2xt}\|_{L^2}^2 + 1\right), \tag{2.57}$$

similarly,

$$\frac{d}{dt}\|\bar{w}_{xx}\|_{L^2}^2 \le C\left(\|\bar{w}_{xx}\|_{L^2}^2 + (\|\bar{b}_{3x}\|_{H^1}^2 + 1)\|\bar{\rho}_x\|_{H^1}^2 + \|\bar{b}_{3xt}\|_{L^2}^2 + 1\right). \tag{2.58}$$

Applying ∂_{xx} to $(1.3)_1$, multiplying it by $\bar{\rho}_{xx}$ in L^2, and integrating by parts, due to Sobolev and Cauchy-Schwarz inequalities, (2.41), (2.46), we deduce

$$\frac{d}{dt}\|\bar{\rho}_{xx}\|_{L^2}^2 \le C\left(\|\bar{u}_x\|_{L^\infty}\|\bar{\rho}_{xx}\|_{L^2}^2 + \|\bar{\rho}_x\|_{L^\infty}^2\|\bar{u}_{xx}\|_{L^2}^2 + \|\bar{u}_{xxx}\|_{L^2}^2 + \|\bar{\rho}_{xx}\|_{L^2}^2\right)$$

$$\leq C\left(\|\bar{u}_{xxx}\|_{L^2}^2+\|\bar{\rho}_{xx}\|_{L^2}^2+1\right). \tag{2.59}$$

To estimate $\|\bar{u}_{xxx}\|_{L^2}$, from $(1.3)_2$, Sobolev inequality, (2.40), (2.41) and (2.46) imply that

$$\begin{aligned}\|\bar{u}_{xxx}\|_{L^2}^2 &\leq C\left(\|\bar{\rho}\bar{u}_t\|_{H^1}^2+\|\bar{\rho}\bar{u}\bar{u}_t\|_{H^1}^2+\|\bar{P}_x\|_{H^1}^2+\|\bar{b}_2\|_{H^2}^2+\|\bar{b}_3\|_{H^2}^2\right)\\ &\leq C\left(\|\bar{\rho}_{xx}\|_{L^2}^2+\|\bar{u}_{xt}\|_{L^2}^2+\|\bar{b}_2\|_{H^2}^2+\|\bar{b}_3\|_{H^2}^2\right).\end{aligned} \tag{2.60}$$

Substituting (2.60) into (2.59), we arrive at

$$\frac{d}{dt}\|\bar{\rho}_{xx}\|_{L^2}^2\leq C\left(\|\bar{\rho}_{xx}\|_{L^2}^2+\|\bar{u}_{xt}\|_{L^2}^2+\|\bar{b}_{2x}\|_{H^1}^2+\|\bar{b}_{3x}\|_{H^1}^2\right). \tag{2.61}$$

Combining with (2.57), (2.58) and (2.61), with the help of (2.41), (2.46), (2.48), by Gronwall inequality, one can derive

$$\sup_{0\leq t\leq T}\left(\|\bar{v}_{xx}\|_{L^2}^2+\|\bar{w}_{xx}\|_{L^2}^2+\|\bar{\rho}_{xx}\|_{L^2}^2\right)\leq C. \tag{2.62}$$

Taking operator ∂_x to $(1.3)_3$, and multiplying it by $\dot{\bar{v}}_x$ in L^2, integrating by parts, we have

$$\|\dot{\bar{v}}_x\|_{L^2}^2\leq C\left(\|\bar{b}_{2xx}\|_{L^2}^2+\|\bar{b}_{2x}\|_{L^\infty}^2\|\bar{\rho}_x\|_{L^2}^2\right)+\frac{1}{2}\|\dot{\bar{v}}_x\|_{L^2}^2.$$

From the above inequality, using (2.41), (2.48), we get

$$\sup_{0\leq t\leq T}\|\dot{\bar{v}}_x\|_{L^2}^2\leq C. \tag{2.63}$$

Similar to the analysis process of (2.63), we receive

$$\sup_{0\leq t\leq T}\|\dot{\bar{w}}_x\|_{L^2}^2\leq C. \tag{2.64}$$

Therefore, collecting (2.56), (2.60), (2.62), (2.63) and (2.64) together finishes the proof of Lemma 2.9.

Proof of Theorem 1.1-(ii). Combining the local existence result, which can be established similar to the local regular solution of problem (1.1) and the global a priori established in Lemma 2.5-2.9, the existence of global strong solution for (1.3)-(1.4) is obtained and the global solution satisfies the regularity result in (ii)-Theorem 1.1.

3 Proof of Theorem 1.2

In this section, our goal is to prove Theorem 1.2. We first discuss the vanishing limits of shear viscosity and Hall term coefficients, then give the thickness of boundary layer.

3.1 The vanishing limits of shear viscosity and Hall term coefficients

In order to investigate the limits as both the shear viscosity and Hall term coefficients tend to zero, let $\tilde{\rho}\triangleq\rho-\bar{\rho}$, $\tilde{u}\triangleq u-\bar{u}$, $\tilde{v}\triangleq v-\bar{v}$, $\tilde{w}\triangleq w-\bar{w}$, $\tilde{b}_2\triangleq b_2-\bar{b}_2$, $\tilde{b}_3\triangleq b_3-\bar{b}_3$. Then equations

(1.1) and (1.3) imply that $(\tilde\rho,\tilde u,\tilde v,\tilde w,\tilde b_2,\tilde b_3)$ satisfies the following equations on $(0,1)\times[0,T)$:

$$\begin{cases}\tilde\rho_t+(\bar\rho\tilde u)_x+(\tilde\rho u)_x=0,\\ \rho\tilde u_t+\rho u\tilde u_x+\tilde P_x-\tilde u_{xx}=-\tilde\rho\bar u_t-\tilde\rho\bar u\bar u_x-\rho\tilde u\bar u_x-\dfrac12[(b_2+\bar b_2)\tilde b_2+(b_3+\bar b_3)\tilde b_3]_x,\\ \rho\tilde v_t+\rho u\tilde v_x=\mu v_{xx}-\tilde\rho\bar v_t-\tilde\rho\bar u\bar v_x-\rho\tilde u\bar v_x+\tilde b_{2x},\\ \rho\tilde w_t+\rho u\tilde w_x=\mu w_{xx}-\tilde\rho\bar w_t-\tilde\rho\bar u\bar w_x-\rho\tilde u\bar w_x+\tilde b_{3x},\\ \tilde b_{2t}+(u\tilde b_2)_x+(\tilde u\bar b_2)_x-\tilde v_x-\kappa\left(\dfrac{b_{3x}}{\rho}\right)_x=\tilde b_{2xx},\\ \tilde b_{3t}+(u\tilde b_3)_x+(\tilde u\bar b_3)_x-\tilde w_x-\kappa\left(\dfrac{b_{2x}}{\rho}\right)_x=\tilde b_{3xx}.\end{cases}\tag{3.1}$$

Multiplying $(3.1)_1$ by $\tilde\rho$, from (2.40) and (2.41), we have

$$\begin{aligned}\frac{d}{dt}\|\tilde\rho\|_{L^2}^2&\le C\left(\|\bar\rho_x\|_{L^2}\|\tilde u\|_{L^\infty}\|\tilde\rho\|_{L^2}+\|\bar\rho\|_{L^\infty}\|\tilde u_x\|_{L^2}\|\tilde\rho\|_{L^2}+\|u_x\|_{L^\infty}\|\tilde\rho\|_{L^2}^2\right)\\&\le C\left(1+\|u_x\|_{L^\infty}\right)\|\tilde\rho\|_{L^2}^2+\varepsilon\|\tilde u_x\|_{L^2}^2.\end{aligned}\tag{3.2}$$

Multiplying $(3.1)_2$ by $\tilde u$, from (2.4), (2.5), (2.41) and (2.46), we get

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\rho}\tilde u\|_{L^2}^2+\|\tilde u_x\|_{L^2}^2\le C\Big(\|\tilde P\|_{L^2}\|\tilde u_x\|_{L^2}+\|\tilde\rho\|_{L^2}\|\dot{\bar u}\|_{L^2}\|\tilde u\|_{L^\infty}+\|\bar u_x\|_{L^\infty}\|\sqrt{\rho}\tilde u\|_{L^2}^2\\&\quad+(\|b_2\|_{L^\infty}+\|\bar b_2\|_{L^\infty})\|\tilde b_2\|_{L^2}\|\tilde u_x\|_{L^2}+(\|b_3\|_{L^\infty}+\|\bar b_3\|_{L^\infty})\|\tilde b_3\|_{L^2}\|\tilde u_x\|_{L^2}\Big)\\&\le C\left(1+\|\dot{\bar u}\|_{L^2}^2\right)\Big(\|\tilde\rho\|_{L^2}^2+\|\tilde b_2\|_{L^2}^2+\|\tilde b_3\|_{L^2}^2+\|\sqrt{\rho}\tilde u\|_{L^2}^2\Big)+\varepsilon\|\tilde u_x\|_{L^2}^2.\end{aligned}\tag{3.3}$$

Multiplying $(3.1)_3$ by $\tilde v$, by (2.4), (2.42) and Poincaré inequality, we arrive at

$$\begin{aligned}\frac{d}{dt}\|\sqrt{\rho}\tilde v\|_{L^2}^2&\le C\Big(\|\dot{\bar v}\|_{L^\infty}\|\tilde\rho\|_{L^2}\|\tilde v\|_{L^2}+\|\rho\|_{L^\infty}\|\bar v_x\|_{L^2}\|\tilde u\|_{L^\infty}\|\tilde v\|_{L^2}\\&\quad+\|\tilde b_{2x}\|_{L^2}\|\tilde v\|_{L^2}+\mu\|\bar v_{xx}\|_{L^2}\|\tilde v\|_{L^2}\Big)\\&\le C\left(\|\dot{\bar v}\|_{H^1}^2\|\tilde\rho\|_{L^2}^2+\|\tilde v\|_{L^2}^2+\mu^2\|\bar v_{xx}\|_{L^2}^2\right)+\varepsilon\|\tilde u_x\|_{L^2}^2+\varepsilon\|\tilde b_{2x}\|_{L^2}^2.\end{aligned}\tag{3.4}$$

Similarly,

$$\frac{d}{dt}\|\sqrt{\rho}\tilde w\|_{L^2}^2\le C\left(\|\dot{\bar w}\|_{H^1}^2\|\tilde\rho\|_{L^2}^2+\|\tilde w\|_{L^2}^2+\mu^2\|\bar w_{xx}\|_{L^2}^2\right)+\varepsilon\|\tilde u_x\|_{L^2}^2+\varepsilon\|\tilde b_{3x}\|_{L^2}^2.\tag{3.5}$$

Multiplying $(3.1)_4$ by $\tilde b_2$, from (2.4), (2.5) and (2.41), then it holds that

$$\begin{aligned}\frac12\frac{d}{dt}\|\tilde b_2\|_{L^2}^2+\|\tilde b_{2x}\|_{L^2}^2&\le C\Big(\|u_x\|_{L^\infty}\|\tilde b_2\|_{L^2}^2+\|\bar b_2\|_{L^\infty}\|\tilde b_2\|_{L^2}\|\tilde u_x\|_{L^2}+\|\tilde u\|_{L^\infty}\|\bar b_{2x}\|_{L^2}\|\tilde b_2\|_{L^2}\\&\quad+\|\tilde v\|_{L^2}\|\tilde b_{2x}\|_{L^2}+\kappa\|b_{3x}\rho^{-1}\|_{L^2}\|\tilde b_{2x}\|_{L^2}\Big)\\&\le C\Big((\|u_x\|_{L^\infty}+1)\|\tilde b_2\|_{L^2}^2+\|\tilde v\|_{L^2}^2+\kappa^2\Big)+\varepsilon\|\tilde b_{2x}\|_{L^2}^2+\varepsilon\|\tilde u_x\|_{L^2}^2\end{aligned}\tag{3.6}$$

and

$$\frac12\frac{d}{dt}\|\tilde b_3\|_{L^2}^2+\|\tilde b_{3x}\|_{L^2}^2\le C\Big((\|u_x\|_{L^\infty}+1)\|\tilde b_3\|_{L^2}^2+\|\tilde w\|_{L^2}^2+\kappa^2\Big)+\varepsilon\|\tilde b_{3x}\|_{L^2}^2+\varepsilon\|\tilde u_x\|_{L^2}^2.\tag{3.7}$$

Hence, collecting with (3.2)-(3.7), by Gronwall inequality, (2.19), (2.31), (2.46), (2.54), it is easy to get the L^2-convergence rate from $(\rho, u, v, w, b_2, b_3)$ to $(\bar\rho, \bar u, \bar v, \bar w, \bar b_2, \bar b_3)$,

$$\begin{aligned}\sup_{0\le t\le T}&\left(\|\tilde\rho\|_{L^2}^2+\|\tilde u\|_{L^2}^2+\|\tilde v\|_{L^2}^2+\|\tilde w\|_{L^2}^2+\|\tilde b_2\|_{L^2}^2+\|\tilde b_3\|_{L^2}^2\right)\\&+\int_0^T\left(\|\tilde u_x\|_{L^2}^2+\|\tilde b_{2x}\|_{L^2}^2+\|\tilde b_{3x}\|_{L^2}^2\right)dt\le C(\kappa^2+\mu^{1/2}).\end{aligned}\tag{3.8}$$

Next, we estimate the L^2-convergence rate of $(\rho_x, u_x, b_{2x}, b_{3x})$ to $(\bar\rho_x, \bar u_x, \bar b_{2x}, \bar b_{3x})$. Differentiating $(3.1)_1$ respect to x, multiplying it by $\tilde\rho_x$ in L^2 and integrating by parts, using Cauchy-Schwarz inequality, (2.41) and (2.54), we have

$$\begin{aligned}\frac{d}{dt}\|\tilde\rho_x\|_{L^2}^2\le{}& C\left(\|\bar\rho_{xx}\|_{L^2}\|\tilde u\|_{L^\infty}\|\tilde\rho_x\|_{L^2}+\|\bar\rho_x\|_{L^\infty}\|\tilde u_x\|_{L^2}\|\tilde\rho_x\|_{L^2}\right.\\&\left.+\|\bar\rho\|_{L^\infty}\|\tilde u_{xx}\|_{L^2}\|\tilde\rho_x\|_{L^2}+\|u_x\|_{L^\infty}\|\tilde\rho_x\|_{L^2}^2+\|\tilde\rho\|_{L^\infty}\|u_{xx}\|_{L^2}\|\tilde\rho_x\|_{L^2}\right)\\\le{}& C\left(\|\tilde u_x\|_{L^2}^2+(\|u_x\|_{H^1}^2+1)\|\tilde\rho\|_{H^1}^2\right)+\varepsilon\|\tilde u_{xx}\|_{L^2}^2.\end{aligned}\tag{3.9}$$

Multiplying $(3.1)_1$ by $\tilde u_t$ and integrating by parts, in view of (2.4), (2.5), (2.41) and (2.46), we get

$$\frac{d}{dt}\|\tilde u_x\|_{L^2}^2+\|\tilde u_t\|_{L^2}^2\le C(\|\tilde u\|_{H^1}^2+\|\tilde\rho\|_{H^1}^2+\|\tilde b_2\|_{H^1}^2+\|\tilde b_3\|_{H^1}^2),\tag{3.10}$$

then, thanks to $(3.1)_2$, (2.4), (2.5), (2.41) and (2.46), we obtain

$$\|\tilde u_{xx}\|_{L^2}\le C\left(\|\tilde u_t\|_{L^2}+\|\tilde u\|_{H^1}+\|\tilde\rho\|_{H^1}+\|\tilde b_2\|_{H^1}+\|\tilde b_3\|_{H^1}\right).\tag{3.11}$$

Collecting with (3.9)-(3.11), in view of (2.5), choosing ε sufficiently small, we deduce

$$\sup_{0\le t\le T}\left(\|\tilde\rho_x\|_{L^2}^2+\|\tilde u_x\|_{L^2}^2\right)+\int_0^T\left(\|\tilde u_t\|_{L^2}^2+\|\tilde u_{xx}\|_{L^2}^2\right)dt\le C(\kappa^2+\mu^{1/2}).\tag{3.12}$$

Now, in order to obtain the convergence rate of $(\tilde b_{2x}, \tilde b_{3x})$ in L^2, multiplying (3.1) by $\tilde b_{2t}+u\tilde b_{2x}$, using Sobolev inequality, (2.3), (2.5) and (2.41), we obtain

$$\frac{d}{dt}\|\tilde b_{2x}\|_{L^2}^2+\|\tilde b_{2t}+u\tilde b_{2x}\|_{L^2}^2\le C\left((\|u_x\|_{L^\infty}^2+1)\|\tilde b_2\|_{H^1}^2+\|\tilde u\|_{H^1}^2\right)+\int_0^1\tilde v_x(\tilde b_{2t}+u\tilde b_{2x})dx.\tag{3.13}$$

Using $(3.1)_3$, we can get the following equation

$$\begin{aligned}\int_0^1\tilde v_x(\tilde b_{2t}+u\tilde b_{2x})dx&=-\frac{d}{dt}\int_0^1\tilde v\tilde b_{2x}dx+\int_0^1(\tilde v_t+u\tilde v_x)\tilde b_{2x}dx\\&=-\frac{d}{dt}\tilde v\tilde b_{2x}dx+\int_0^1\rho^{-1}(\kappa v_{xx}-\tilde\rho\dot{\bar v}-\rho\tilde u\bar v_x+\tilde b_{2x})\tilde b_{2x}dx,\end{aligned}$$

which inserted into (3.13), with the help of (2.4), (2.31) and (2.42), we have

$$\frac{d}{dt}\|\tilde b_{2x}\|_{L^2}^2+\frac{d}{dt}\int_0^1\tilde v\tilde b_{2x}dx+\|\tilde b_{2t}+u\tilde b_{2x}\|_{L^2}^2$$

$$\leq C\left((\|u_x\|_{L^\infty}^2+1)\|\tilde{b}_2\|_{H^1}^2+\|\tilde{u}\|_{H^1}^2+\|\tilde{\rho}\|_{H^1}^2+\kappa^{1/2}\right),$$

from this and (3.8), using Gronwall inequality, we have

$$\sup_{0\leq t\leq T}\|\tilde{b}_{2x}\|_{L^2}^2+\int_0^T\|\tilde{b}_{2t}+u\tilde{b}_{2x}\|_{L^2}^2dt\leq C(\kappa^2+\mu^{1/2}). \tag{3.14}$$

Similarly, we also get

$$\sup_{0\leq t\leq T}\|\tilde{b}_{3x}\|_{L^2}^2+\int_0^T\|\tilde{b}_{3t}+u\tilde{b}_{3x}\|_{L^2}^2dt\leq C(\kappa^2+\mu^{1/2}). \tag{3.15}$$

Proof of Theorem 1.2-(i). Combining the convergence rate for the solution in L^2 with (3.8) and the convergence rate for the derivative of solution in L^2 with (3.12), (3.14) and (3.15), we finish the proof of Theorem 1.2-(i).

3.2 Thickness of boundary layer

In this section, we will discuss the thickness of boundary layer. To this end, we give the following spatially weighted estimates on $(v-\bar{v})_x$ and $(w-\bar{w})_x$ in L^2.

Lemma 3.1 *Assume that* (ρ,u,v,w,b_2,b_3) *and* $(\bar{\rho},\bar{u},\bar{v},\bar{w},\bar{b}_2,\bar{b}_3)$ *be the solution of the problems* (1.1)-(1.2) *and* (1.3)-(1.4) *on* $(0,1)\times[0,T)$*, then*

$$\sup_{0\leq t\leq T}\left(\|\xi(x)(v-\bar{v})_x\|_{L^2}^2+\|\xi(x)(w-\bar{w})_x\|_{L^2}^2\right)\leq C\left(\mu^{1/2}+\kappa^2\right), \tag{3.16}$$

where $\xi(x)\triangleq x(1-x)$ *for* $x\in(0,1)$.

Proof. Differentiating $(3.1)_3$ with respect to x, then multiplying it by $\xi^2(x)\tilde{v}_x$, we obtain after integrating by parts that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1\xi^2(x)|\tilde{v}_x|^2dx=&-\int_0^1(u_x\tilde{v}_x+u\tilde{v}_{xx})\xi^2(x)\tilde{v}_xdx-\mu\int_0^1\frac{v_{xx}}{\rho}(\xi^2(x)\tilde{v}_x)_xdx\\&-\int_0^1(\frac{\tilde{\rho}\dot{\bar{v}}}{\rho})_x\xi^2(x)\tilde{v}_xdx-\int_0^1(\tilde{u}\bar{v}_x)_x\xi^2(x)\tilde{v}_xdx+\int_0^1(\frac{\tilde{b}_{2x}}{\rho})_x\xi^2(x)\tilde{v}_xdx\\\triangleq&\sum_{i=1}^5I_i.\end{aligned} \tag{3.17}$$

Thanks to

$$|u(x,t)|\leq|\int_0^xu_xdx|\leq Cx(1-x)\|u_x\|_{L^\infty}\qquad for\quad 0<x<1/2$$

and

$$|u(x,t)|\leq|\int_1^xu_xdx|\leq Cx(1-x)\|u_x\|_{L^\infty}\qquad for\quad 1/2<x<1,$$

hence

$$I_1\leq C\|u_x\|_{L^\infty}\|\xi(x)\tilde{v}_x\|_{L^2}^2. \tag{3.18}$$

From (2.4) and (2.54), we get

$$
\begin{aligned}
I_2 &= -\mu \int_0^1 \frac{v_{xx}}{\rho} \left(2\xi(x)\xi'(x)\tilde{v}_x + \xi^2(x)\tilde{v}_{xx}\right) dx \\
&\le -\mu \int_0^1 \frac{|\tilde{v}_{xx}|^2}{\rho} \xi^2(x) dx + C\left(\|\xi(x)\tilde{v}_x\|_{L^2}^2 + \kappa^2 \|\bar{v}_{xx}\|_{L^2}^2\right) \\
&\le -\mu \int_0^1 \frac{|\tilde{v}_{xx}|^2}{\rho} \xi^2(x) dx + C\left(\|\xi(x)\tilde{v}_x\|_{L^2}^2 + \kappa^2\right). \qquad (3.19)
\end{aligned}
$$

Using (2.4), (2.42) and (2.54), we have

$$
I_3 \le C\left(\|\dot{v}\|_{H^1}^2 \|\tilde{\rho}\|_{H^1}^2 + \|\xi(x)\tilde{v}_x\|_{L^2}^2\right); \qquad (3.20)
$$

$$
I_4 \le C\left(\|\tilde{u}\|_{H^1}^2 \|\bar{v}_x\|_{H^1}^2 + \|\xi(x)\tilde{v}_x\|_{L^2}^2\right); \qquad (3.21)
$$

$$
\begin{aligned}
I_5 &\le C\left(\|\tilde{b}_{2x}\|_{L^2}^2 + \|\tilde{b}_{2xx} + \tilde{v}_x\|_{L^2}^2 + \|\xi(x)\tilde{v}_x\|_{L^2}^2\right) \\
&\le C\left(\|\tilde{b}_2\|_{H^1}^2 + \|\tilde{b}_{2t} + u\tilde{b}_{2x}\|_{L^2}^2 + \|\tilde{u}\|_{H^1}^2 + \|\xi(x)\tilde{v}_x\|_{L^2}^2\right). \qquad (3.22)
\end{aligned}
$$

Here, we have used $(3.1)_5$ to get that

$$
\begin{aligned}
\|\tilde{b}_{2xx} + \tilde{v}_x\|_{L^2} &\le C\left(\|\tilde{b}_{2t} + u\tilde{b}_{2x}\|_{L^2} + \|u_x\|_{L^\infty}\|\tilde{b}_2\|_{L^2} + \|\bar{b}_2\|_{L^\infty}\|\tilde{u}_x\|_{L^2} + \|\bar{b}_{2x}\|_{L^\infty}\|\tilde{u}\|_{L^2}\right) \\
&\le C\left(\|\tilde{b}_{2t} + u\tilde{b}_{2x}\|_{L^2} + \|\tilde{b}_2\|_{L^2} + \|\tilde{u}\|_{H^1}\right).
\end{aligned}
$$

Inserting (3.18)-(3.22) into (3.17), by Gronwall inequality and (3.15), we obtain

$$
\sup_{0\le t\le T} \|\xi(x)\tilde{v}_x\|_{L^2}^2 \le C(\mu^{1/2} + \kappa^2). \qquad (3.23)
$$

In the same method as (3.23), we deduce

$$
\sup_{0\le t\le T} \|\xi(x)\tilde{w}_x\|_{L^2}^2 \le C(\mu^{1/2} + \kappa^2).
$$

Therefore, we complete the proof of Lemma 3.1.

Proof of Theorem 1.2-(ii). Firstly, in view of (3.16) and according to the assumption $\kappa \le \mu^{1/4}$ in Theorem 1.2, it is easy to deduce that

$$
\begin{aligned}
\delta^2 \int_\delta^{1-\delta} |\tilde{v}_x|^2 dx &= \delta^2 \int_\delta^{1/2} |\tilde{v}_x|^2 dx + \delta^2 \int_{1/2}^{1-\delta} |\tilde{v}_x|^2 dx \\
&\le \int_\delta^{1/2} x^2 |\tilde{v}_x|^2 dx + \int_{1/2}^{1-\delta} (1-x)^2 |\tilde{v}_x|^2 dx \\
&\le 4\int_\delta^{1/2} x^2(1-x)^2 |\tilde{v}_x|^2 dx + 4\int_{1/2}^{1-\delta} x^2(1-x)^2 |\tilde{v}_x|^2 dx \\
&\le 4\int_0^1 x^2(1-x)^2 |\tilde{v}_x|^2 dx \le C(\kappa^2 + \mu^{1/2}) \le C\mu^{1/2}. \qquad (3.24)
\end{aligned}
$$

Therefore, we obtain

$$\begin{aligned}\|\tilde{v}_x(t)\|^2_{C(\Omega_\delta)} &\leq C\left(\|\tilde{v}(t)\|^2_{L^2(\Omega_\delta)} + \|\tilde{v}(t)\|_{L^2(\Omega_\delta)}\|\tilde{v}_x(t)\|_{L^2(\Omega_\delta)}\right)\\ &\leq C\left(\mu^{1/2} + \kappa^2 + \delta^{-1}(\kappa^2 + \mu^{1/2})\right) \leq C\delta^{-1}\mu^{1/2},\end{aligned} \tag{3.25}$$

which provided that $\delta = \delta(\mu)$ satisfies

$$\delta(\mu) \to 0 \quad \textit{and} \quad \delta\mu^{-1/2} \to \infty \quad \textit{as} \quad \mu \to 0. \tag{3.26}$$

On the other hand, due to the continuity of the solution, we have

$$\liminf_{\mu\to 0} \|\tilde{v}(x,t)\|_{L^\infty(0,T;C(\bar{\Omega}))} > 0, \tag{3.27}$$

which provided the boundary data $v_1(t)$, $v_2(t)$ are not identically zero.

4 Proof of Theorem 1.3

In this section, we construct a boundary layer solution such that $(v - \bar{v})(x,t)$ convergence to $v^*(x,t)$ in L^∞ as $(\kappa,\mu) \to (0,0)$. Assume that $v^*(x,t)$ and $w^*(x,t)$ are solutions of (1.8) and (1.9), respectively, it is easily derived the following estimates:

$$\sup_{0\leq t\leq T}\left(\|v^*\|^2_{L^2} + \|w^*\|^2_{L^2}\right) \leq C\mu^{1/2}, \tag{4.1}$$

$$\mu^{1/2}\sup_{0\leq t\leq T}\left(\|v^*_x\|^2_{L^2} + \|w^*_x\|^2_{L^2}\right) + \mu^{3/2}\int_0^T \left(\|v^*_{xx}\|^2_{L^2} + \|w^*_{xx}\|^2_{L^2}\right)dt \leq C. \tag{4.2}$$

Let $\hat{v} \triangleq v - \bar{v} - v^*$, then $\hat{v}$ satisfies the equation:

$$\hat{v}_t - \frac{\mu\hat{v}_{xx}}{\rho} = -\tilde{u}\tilde{v}_x - \bar{u}\hat{v}_x - \tilde{u}\bar{v}_x - \mu\frac{\bar{v}_{xx}}{\rho} - \mu\frac{\tilde{\rho}v^*_{xx}}{\rho\bar{\rho}} - \frac{\tilde{\rho}\dot{\bar{v}}}{\rho} + \frac{\tilde{b}_{2x}}{\rho}. \tag{4.3}$$

Multiplying it by $\hat{v}_{xx}$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\hat{v}_x\|^2_{L^2} + \mu\int_0^1 \frac{|\hat{v}_{xx}|^2}{\rho}dx =& -\int_0^1 \tilde{u}\tilde{v}_x\hat{v}_{xx}dx - \int_0^1 \left(\bar{u}\hat{v}_x + \tilde{u}\bar{v}_x\right)\hat{v}_{xx}dx\\ &- \mu\int_0^1\left(\frac{\bar{\rho}\bar{v}_{xx} + \tilde{\rho}v^*_{xx}}{\rho\bar{\rho}}\right)\hat{v}_{xx}dx - \int_0^1 \frac{\tilde{\rho}\dot{\bar{v}}}{\rho}\hat{v}_{xx}dx\\ &+ \int_0^1 \frac{\tilde{b}_{2x}}{\rho}\hat{v}_{xx}dx \triangleq \sum_{i=1}^5 H_i.\end{aligned} \tag{4.4}$$

Noticing that

$$|\tilde{u}(x,t)| \leq |\int_0^x \tilde{u}_x dx| \leq Cx(1-x)\|\tilde{u}_x\|_{L^\infty} \qquad \textit{for} \quad 0 < x < 1/2$$

and

$$|\tilde{u}(x,t)| \leq |\int_1^x \tilde{u}_x dx| \leq Cx(1-x)\|\tilde{u}_x\|_{L^\infty} \qquad \textit{for} \quad 1/2 < x < 1,$$

we get

$$\begin{aligned}
H_1 &\le C\|\tilde{u}_x\|_{L^\infty}\|x(1-x)\tilde{v}_x\|_{L^2}\|\hat{v}_{xx}\|_{L^2} \\
&\le C\mu^{-1}\|\tilde{u}_x\|_{L^\infty}^2\|x(1-x)\tilde{v}_x\|_{L^2}^2+\varepsilon\mu\|\hat{v}_{xx}\|_{L^2}^2 \\
&\le C\mu^{-1/2}\|\tilde{u}_x\|_{L^\infty}^2+\varepsilon\mu\|\hat{v}_{xx}\|_{L^2}^2;
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
H_2 &\le C\left(\|\bar{u}_x\|_{L^\infty}\|\hat{v}_x\|_{L^2}+\|\bar{v}_x\|_{L^\infty}\|\tilde{u}_x\|_{L^2}+\|\bar{v}_{xx}\|_{L^2}\|\tilde{u}\|_{L^\infty}\right)\|\hat{v}_x\|_{L^2} \\
&\le C\left((\|\bar{u}_x\|_{L^\infty}+1)\|\hat{v}_x\|_{L^2}^2+\|\bar{v}_x\|_{H^1}^2\|\tilde{u}\|_{H^1}^2\right) \\
&\le C\left((\|\bar{u}_x\|_{L^\infty}+1)\|\hat{v}_x\|_{L^2}^2+\mu^{1/2}\right);
\end{aligned} \tag{4.6}$$

$$H_3 \le C\left(\mu\|\bar{v}_{xx}\|_{L^2}^2+\mu\|\tilde{\rho}\|_{L^\infty}^2\|v_{xx}^*\|_{L^2}^2\right)+\varepsilon\mu\|\hat{v}_{xx}\|_{L^2}^2; \tag{4.7}$$

$$\begin{aligned}
H_4+H_5 &\le C\left(\|\tilde{\rho}\|_\infty\|\dot{\bar{v}}_x\|_{L^2}+\|\tilde{\rho}_x\|_{L^2}\|\dot{\bar{v}}\|_{L^\infty}\right)\|\hat{v}_x\|_{L^2} \\
&\le C\left(\|\tilde{\rho}\|_{H^1}^2\|\dot{\bar{v}}\|_{H^1}^2+\|\hat{v}_x\|_{L^2}^2\right).
\end{aligned} \tag{4.8}$$

Substituting (4.5)-(4.8) into (4.4), choosing ε sufficiently small, we have

$$\sup_{0\le t\le T}\|\hat{v}_x\|_{L^2}^2+\mu\int_0^T\|\hat{v}_{xx}\|_{L^2}^2dt \le C. \tag{4.9}$$

In similar manner, we also have

$$\sup_{0\le t\le T}\|\hat{w}_x\|_{L^2}^2+\mu\int_0^T\|\hat{w}_{xx}\|_{L^2}^2dt \le C. \tag{4.10}$$

Proof of Theorem 1.3. It is easily deduced from Sobolev inequality, (3.8), (4.1), (4.9) and (4.10) that

$$\|\hat{v}\|_{L^\infty} \le C\left(\|\hat{v}\|_{L^2}+\|\hat{v}\|_{L^2}^{1/2}\|\hat{v}_x\|_{L^2}^{1/2}\right)\le C\mu^{1/8} \tag{4.11}$$

and

$$\|\hat{w}\|_{L^\infty} \le C\left(\|\hat{w}\|_{L^2}+\|\hat{w}\|_{L^2}^{1/2}\|\hat{w}_x\|_{L^2}^{1/2}\right)\le C\mu^{1/8}. \tag{4.12}$$

Therefore, collecting (4.11) and (4.12) together, we complete the proof of Theorem 1.3.

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