Abundant novel solitary wave Solutions of two-mode Sawada-Kotera model and its Applications

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Abstract: The Sawada-Kotera equations illustrate the non-linear wave phenomena in shallow water, ion-acoustic waves in plasmas, fluid dynamics, etc. In this article, the two-mode Sawada-Kotera equation (tmSKE) occurring in fluid dynamics is addressed. The improved F-expansion and generalized $\exp(-\phi(\zeta))$-expansion methods are utilized in this model and abundant of solitary wave solutions of different kinds such as bright and dark solitons, multi-peak soliton, breather type waves, periodic solutions, and other wave results are obtained. These achieved abundant novel solitary and other wave results have significant applications in fluid dynamics, applied sciences and engineering. By granting appropriate values to parameters, the structures of few results are presented in which many structures are novel. The graphical moments of few solutions help the engineers and scientists for understanding the physical phenomena of this model. To explain the novelty between the present results and the previously attained results, a comparative study has been carried out. Furthermore, the executed techniques can be employed for further studies to explain the realistic phenomena arising in fluid dynamics correlated with any physical and engineering problems.
Keywords: Improve F-expansion method; Generalized $\exp(-\phi(\zeta))$-expansion method; Two-mode Sawada-Kotera equation; Traveling and dual wave solutions, breather waves, Periodic solitons.

1 Introduction

The dynamic complexity of physical phenomena in the real world can be expressed by the changes in temporal and spatial events. The temporal and spatial changes of physical phenomena are greatest articulated by partial differential equations (PDEs). The nonlinear PDEs are utilized for expressing various physical phenomena in the real world. Nonlinear wave phenomena emerge in plasma physics, fluid mechanics, solid-state physics, dynamics of chemical, non-linear optics, population model and other fields of science and engineering [1–11]. The analytical solutions of non-linear PDEs play a decisive part in non-linear science as they inform us deep imminent into the physical characteristics of the model and can provide further physical informations to help in other applications. In recent years, the approximate and exact results of non-linear PDEs have attracted more and more attention, as they are utilized to illustrate the complex non-linear phenomena in dissimilar scientific areas. Numerous real world problems are altered into equations mathematically by differential equations. Thus, the finding wave results of all kinds of PDEs are a major problem, such as the present direction of non-linear science, which originated from the research of chemistry, physics, material science, biology and many more, and has a burly practical backdrop. They have significant realistic applications and theoretical study in mathematics.

Lately, in terms of time and space derivatives, novel families of nonlinear PDEs have been recognized in the name of "dual-mode" or "two-mode". With regard to this curiosity, researchers have established some dual mode nonlinear PDEs, namely two-mode (tm) mKdV [12, 13], tm KdV [7, 14], tm Sharma-Tasso-Olver [9], tm fifth order KdV [4, 15], two-mode Burger equation (tmBE) [16], tm Ostrovsky [17], tm perturbed Burger (tmPB) [17], tm KdV Burgers (tmKdVB) [18], tm Kadomtsev Petviashvili (tmKP) [19, 20], two-mode dispersive Fisher (tmdF) [21], tm Kuramoto-Sivashinsky (tmKS) [22], tm Boussinesq Burgers (tmBB) [23], two-mode coupled KdV (tmKdV) and mKdV (tmCmKdV) [24, 25], wo-mode non-linear Schrödinger (tmNLS) [26], and tm Hirota Satsuma coupled KdV (tmHSKdV) [27] equations and the related dual-wave solutions are analyzed by different methods, such as Tanh expansion technique, (G'/G)-expansion technique, rational Sine-Cosine technique, Kudryshov technique, simplified Hirota technique, Tanh-Coth tachnque, Sech-Csch technique, Fourier spectral technique, Bcklund transformation scheme and Trigonometric function technique [12–27]. As results, few solitons results in the form Kink, Kinks type of multiple soliton, periodic wave of singular kind, dark and bright solitons have been conceded out for the aforementioned models.

The researcher Wazwaz [4] developed the tmSKE from the tmfKdV equation, and few multiple solitons results were determined by the simplified Hirota technique. Later on, the researchers in [15] investigated the tmfKdV model and established some Kink, bright and periodic solutions in singular form by sine-cosine function and Kudryashov techniques. The authors in [11] were modified Kudryashov and auxiliary equation methods, and dual wave solutions were constructed. It should be pointed out that the tmSKE is a special case of the tmfKdV equation. As far as the author is aware, although some two-mode PDEs have been extensively studied, the contribution to the above tmSKE is limited. It can be seen from the literature that there is room for further study of the tmSKE through the improved F-expansion and generalized $\exp(-\phi(\zeta))$-expansion methods, as well as the illustrating their physical explanations. The results executing by the projected methods will be new in method applications.

Numerous powerful methods (analytic, semi-analytic, and numerical methods) for studying non-linear PDEs [21–48], such as modified direct algebraic technique, Hirota bilinear technique, modified simple equation technique, Bcklund transformation scheme, F-expansion method, modified Kudryashov , Darboux transform technique, (G'/G)-expansion technique, rational Sine-Cosine technique, inverse scattering scheme, auxiliary equation method, painlev analysis method, trigonometric function technique, tanh/coth method, sine and sinh Gordon equation expansion methods, general symmetry technique, variational iteration technique, reduced differential transform method, Fourier spectral technique, finite difference technique, Adomian decomposition technique, finite element technique, the wavelet technique and other techniques.

This work intends to attain solitons and other wave results of tmSKE. The described generalized $\exp(-\phi(\zeta))$-expansion and improved F-expansion methods are employed for obtaining wave solutions. The constructed results are novel and more general. To our best knowledge, these approaches are not utilized to address the early work on this equation.

This paper is structured as follows. Section 1, specifies the introduction. In Section 2, a summary of the general type of tm standard and SK equations are summarized. In Section 3, the appraisal of the improved F-expansion and generalized $\exp(-\phi(\zeta))$-expansion techniques are depicted. The constructed results of the exploration are given in Section 4. In Section 5, a general discussion and graphical illustrations of some acquired solutions are described. Finally, the conclusion of the article is illustrated in Section 6.

2 Formulation of Mathematical Models

2.1 General type of dual-mode standard model

The general type of the dual-mode model proposed by Korsonski [7] is

$$\frac{\partial^2 u}{\partial t^2} - \nu\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial}{\partial t} - \beta\nu\frac{\partial}{\partial x}\right) G\left(u, u\frac{\partial u}{\partial x}, ...\right) + \left(\frac{\partial}{\partial t} - \gamma\nu\frac{\partial}{\partial x}\right) N\left(\frac{\partial^2 u}{\partial r \partial x}, r \geq 2\right) = 0, \tag{1}$$

the above equation (1) is recognized from the equation of standard mode:

$$\frac{\partial u}{\partial t} + N\left(u, u\frac{\partial u}{\partial x}, ...\right) + L\left(\frac{\partial^2 u}{\partial r \partial x}, r \geq 2\right) = 0.$$

In equation (1), the function $u(x,t)$ is an unknown with $(t,x) \in (-\infty, \infty)$, and $\nu > 0$ is velocity of the phase, $\beta \leq 1, \gamma \leq 1$, β and γ symbolize nonlinearity and dispersion parameters respectively. The terms $L\left(\frac{\partial^2 u}{\partial r \partial x}, r \geq 2\right)$ and $N\left(u, u\frac{\partial u}{\partial x}, ...\right)$ signify the terms of linear and nonlinear respectively.

2.2 Dual-mode Sawada-Kotera model

The SKE in standard form having two non-linear terms [4] has as

$$\frac{\partial u}{\partial t} + 5\frac{\partial}{\partial x}\left(\frac{u^3}{3} + u\frac{\partial^2 u}{\partial x^2}\right) + \frac{\partial^5 u}{\partial x^5} = 0, \tag{2}$$

in above equation, the terms $\frac{\partial^5 u}{\partial x^5}$ and $\frac{\partial}{\partial x}\left(\frac{u^3}{3} + u\frac{\partial^2 u}{\partial x^2}\right)$ are linear and nonlinear respectively.

Merging the sense of Korsunsky [7], and follow Wazwaz [4], the tmSKE of the standard SKE precises by equation (2) is presented as

$$\frac{\partial^2 u}{\partial t^2} - \nu\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial}{\partial t} - \beta\nu\frac{\partial}{\partial x}\right)\frac{\partial}{\partial x}\left(\frac{5u^3}{3} + 5u\frac{\partial^2 u}{\partial x^2}\right) + \left(\frac{\partial}{\partial t} - \gamma\nu\frac{\partial}{\partial x}\right)\frac{\partial^5 u}{\partial x^5} = 0. \tag{3}$$

Obviously, for $\nu = 0$, the tmSKE specified through equation (3) after integrating the relevant time t has been simplified to the standard mode SKE given through equation (2).
The equation (3) illustrates the proliferation of two moving waves under the persuade of phase velocity ν, dispersion (γ), and non-linearity (β) factors.

3 Portrayal of Proposed Methods

Here, we reveal the algorithms of suggested techniques namely as improved F-expansion and generalized $\exp(-\phi(\zeta))$-expansion methods for constructing the wave results of two-mode Sawada-Kotera model. The general non-linear PDE has as

$$G\left(v, v_x, vv_x, v_t, v_{xx}, vv_{xx}, w_{tt}, ...\right) = 0, \tag{4}$$

where the polynomial function G having unknown function $v(x,t)$ with respect to a few specific independent variables x and t, that also having derivative terms of linear and non-linear. Assuming the transformation for changing independent variables into sole variable has as

$$v(x,t) = U(\zeta), \quad \zeta = kx - \omega t + \theta, \tag{5}$$

where the constant k and ω are wave length and frequency. Utilizing (5), the equation (4) is converting into ODE as

$$F\left(U, U', U'', UU'', ...\right) = 0, \tag{6}$$

where $U' = \frac{dU}{d\zeta}$ and F is a polynomial of U and its derivatives.

3.1 Improved F-expantion Method

The main steps are as

1st step: Consider the solution of Eq.(6) has as

$$U(\zeta) = \sum_{i=0}^{N} A_i\left(\mu + F(\zeta)\right)^i + \sum_{j=-1}^{-N} B_{-j}\left(\mu + F(\zeta)\right)^j, \tag{7}$$

where the constants A_i, B_{-j}, μ are real and the function $F(\zeta)$ in equation (7) pledges the below ODE

$$F'(\zeta) = \delta_0 + \delta_1 F(\zeta) + \delta_2 F^2(\zeta) + \delta_3 F^3(\zeta), \tag{8}$$

where $\delta_0, \delta_1, \delta_2$ and δ_3 are real constants.

2nd step: By utilizing homogeneous principle on Eq.(6), the positive integer N is obtained.

3rd step: Deputizing Eq.(7) into Eq.(6) and taking the various coefficients of $\frac{F^i(\zeta)}{(\mu+F(\zeta))^j}$ to zero, capitulate a system of equation. By using Mathematica, this system is solved and constant values can be achieved. After substituting constant values and solutions of Eq.(6), the wave solutions of Eq.(7) are constructed.

3.2 Generalized $\exp(-\phi(\zeta))$-expansion Method

The main steps are as

1st step: Assume the solution of Eq.(6) has the form as

$$U(\zeta) = \sum_{i=0}^{N} A_i \left(\exp(-\phi(\zeta))\right)^i, \tag{9}$$

where A_i $(0 \leq i \leq N)$ are real constants such that $A_N \neq 0$ and $\phi = \phi(\zeta)$ pledges the ODE as

$$\phi'(\zeta) = a\exp(-\phi(\zeta)) + b\exp(\phi(\zeta)) + c, \tag{10}$$

where a, b, c are real constants.

2nd step: Utilizing homogeneous principle on Eq.(6), the positive integer N is obtained.

3rd step: By Deputizing equation (9) into (6) and polynomial obtained in $e^{(-\phi(\zeta))}$, and taking diverse powers of $(e^{(-\phi(\zeta))})^i$ to zero, capitulate a system of equation. By resolving this system and reverse substitution, we construct many exact solutions for Eq.(4).

4 Applications

In this segment, we construct the solitons and other wave solutions of two-mode Sawada-Kotera equation by employing described methods. By employing the transformation described in Eq.(5), the Eq.(3) is converted into ODE as

$$\left(\omega^2 - k^2\nu^2\right)U'' - 5k\left(\omega + \beta k\nu\right)\left(k^2 U U^{(iv)} + 2k^2 U' U''' + k^2\left(U''\right)^2 + U^2 U'' + 2U\left(U'\right)^2\right) \\ -k^5\left(\omega + \gamma k\nu\right)U^{(vi)} = 0. \tag{11}$$

4.1 Application of improved F-expansion Method

Employing balancing principle on Eq.(11) and solution of equation (11) assumed as

$$U(\zeta) = A_0 + A_1\left(\mu + F(\zeta)\right) + A_2\left(\mu + F(\zeta)\right)^2 + \frac{B_1}{\mu + F(\zeta)} + \frac{B_2}{\left(\mu + F(\zeta)\right)^2}. \tag{12}$$

By substituting Eq.(12) into Eq.(11) and deputing the coefficients of $\frac{F^i(\zeta)}{(\mu+F(\zeta))^j}$ to zero, we attained a equations system $A_0, A_1, A_2, B_1, B_2, \delta_0, \delta_1, \delta_2, \delta_3, \beta, \gamma, k, \nu, \omega$ and θ. Mathematica 9 was utilized for solving this equation system. We attain the families of wave results as:

1st Family: here assume $\delta_0 = \delta_3 = 0$,
Set 1:

$$A_0 = -\frac{\sqrt{3\left(\gamma^2-1\right)\nu}\left(12\delta_2^2\mu^2 - 12\delta_2\delta_1\mu + \delta_1^2\right)}{\delta_1^2\sqrt{5(\beta-\gamma)}},\ A_1 = -\frac{12\delta_2\sqrt{3\left(\gamma^2-1\right)\nu}\left(\delta_1 - 2\delta_2\mu\right)}{\delta_1^2\sqrt{5(\beta-\gamma)}},$$
$$A_2 = -\frac{12\delta_2^2\sqrt{3\left(\gamma^2-1\right)\nu}}{\delta_1^2\sqrt{5(\beta-\gamma)}},\ B_1 = B_2 = 0,\ k = \mp\frac{\sqrt[4]{4\left(\gamma^2-1\right)\nu}}{\delta_1\sqrt[4]{15(\beta-\gamma)}},\ \omega = \pm\frac{\gamma\nu\sqrt[4]{4\left(\gamma^2-1\right)\nu}}{\delta_1\sqrt[4]{15(\beta-\gamma)}}. \tag{13}$$

Set 2:

$$A_0 = -\frac{3k^2\left(12\delta_2^2\mu^2 - 12\delta_2\delta_1\mu + \delta_1^2\right)}{2},\ A_1 = 18\delta_2 k^2\left(2\delta_2\mu - \delta_1\right),\ A_2 = -18\delta_2^2 k^2, B_1 = 0,$$
$$B_2 = 0,\ \nu = \frac{15\delta_1^4 k^4(\beta-\gamma)}{4\left(\gamma^2-1\right)},\ \omega = \frac{15\gamma\delta_1^4 k^5(\gamma-\beta)}{4\left(\gamma^2-1\right)}. \tag{14}$$

Set 3:

$$A_0 = \frac{\sqrt{3\left(\gamma^2-1\right)\nu}\left(12\delta_2^2\mu^2 - 12\delta_2\delta_1\mu + \delta_1^2\right)}{\delta_1^2\sqrt{5(\beta-\gamma)}},\ A_1 = \frac{12\delta_2\sqrt{3\left(\gamma^2-1\right)\nu}\left(\delta_1 - 2\delta_2\mu\right)}{\delta_1^2\sqrt{5(\beta-\gamma)}},\ B_1 = 0,$$
$$B_2 = 0,\ A_2 = \frac{12\delta_2^2\sqrt{3\left(\gamma^2-1\right)\nu}}{\delta_1^2\sqrt{5(\beta-\gamma)}},\ k = \pm\frac{\sqrt[4]{4\left(1-\gamma^2\right)\nu}}{\delta_1\sqrt[4]{15(\beta-\gamma)}},\ \omega = \mp\frac{\gamma\nu\sqrt[4]{4\left(1-\gamma^2\right)\nu}}{\delta_1\sqrt[4]{15(\beta-\gamma)}}. \tag{15}$$

The soliton results of Eq.(3) from sets 1 and 2 are constructed in the form as

$$u_{1,2}(x,t) = -\frac{\sqrt{3\left(\gamma^2-1\right)\nu}\left(\delta_2 e^{\delta_1(\zeta+\zeta_0)}\left(\delta_2 e^{\delta_1(\zeta+\zeta_0)} + 10\right) + 1\right)}{\sqrt{5(\beta-\gamma)}\left(\delta_2 e^{\delta_1(\zeta+\zeta_0)} - 1\right)^2}, \quad \delta_1 > 0. \tag{16}$$

$$u_{3,4}(x,t) = -\frac{\sqrt{3\left(\gamma^2-1\right)\nu}\left(\delta_2 e^{\delta_1(\zeta+\zeta_0)}\left(\delta_2 e^{\delta_1(\zeta+\zeta_0)} - 10\right) + 1\right)}{\sqrt{5(\beta-\gamma)}\left(\delta_2 e^{\delta_1(\zeta+\zeta_0)} + 1\right)^2}, \quad \delta_1 < 0. \tag{17}$$

$$u_5(x,t) = -\frac{3\delta_1^2 k^2\left(\delta_2 e^{\delta_1(\zeta+\zeta_0)}\left(\delta_2 e^{\delta_1(\zeta+\zeta_0)} + 10\right) + 1\right)}{2\left(\delta_2 e^{\delta_1(\zeta+\zeta_0)} - 1\right)^2}, \qquad \delta_1 > 0. \tag{18}$$

$$u_6(x,t) = -\frac{3\delta_1^2 k^2\left(\delta_2 e^{\delta_1(\zeta+\zeta_0)}\left(\delta_2 e^{\delta_1(\zeta+\zeta_0)} - 10\right) + 1\right)}{2\left(\delta_2 e^{\delta_1(\zeta+\zeta_0)} + 1\right)^2}, \qquad \delta_1 < 0. \tag{19}$$

Similar-way, one can construct more wave results of Eq.(3) from set 3.

2nd Family: In this family, we assume as $\delta_1 = \delta_3 = 0$,
Set 1:

$$A_0 = -\frac{\sqrt{3\left(1-\gamma^2\right)\nu}\left(3\delta_2\mu^2 + 2\delta_0\right)}{\delta_0\sqrt{5(\gamma-\beta)}},\ A_1 = \frac{6\mu\delta_2\sqrt{3\left(1-\gamma^2\right)\nu}}{\delta_0\sqrt{5(\gamma-\beta)}},\ A_2 = -\frac{3\delta_2\sqrt{3\left(1-\gamma^2\right)\nu}}{\delta_0\sqrt{5(\gamma-\beta)}}$$
$$B_1 = 0,\ B_2 = 0,\ k = \mp\frac{\sqrt[4]{\left(1-\gamma^2\right)\nu}}{\sqrt[4]{60\delta_0^2\delta_2^2(\gamma-\beta)}},\ \omega = \pm\frac{\gamma\nu\sqrt[4]{\left(1-\gamma^2\right)\nu}}{\sqrt[4]{60\delta_0^2\delta_2^2(\gamma-\beta)}}. \tag{20}$$

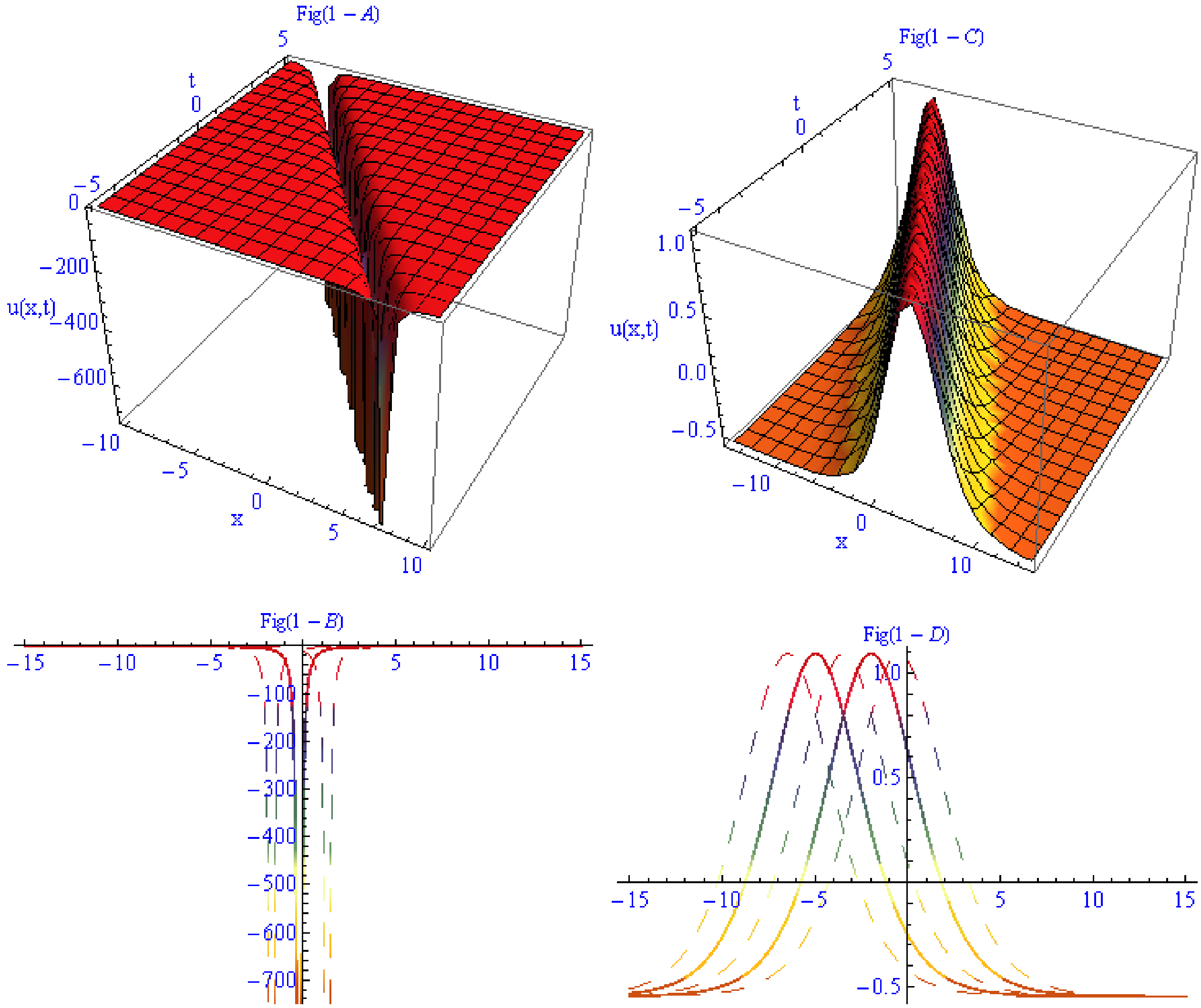

Figure 1: By granting appropriate values to parameters, the formation of solutions (16) and (17) are revealed as: Fig(1-A) Dark solitary wave and its 2-dimensional (2D) in Fig(1-B), Fig(1-C) bright soliton and its 2D in Fig(1-D).

Set 2:

$$A_0 = \frac{\sqrt{3(1-\gamma^2)\nu}\,(3\delta_2\mu^2 + 2\delta_0)}{\delta_0\sqrt{5(\gamma-\beta)}},\ A_1 = -\frac{6\mu\delta_2\sqrt{3(1-\gamma^2)\nu}}{\delta_0\sqrt{5(\gamma-\beta)}},\ A_2 = \frac{3\delta_2\sqrt{3(1-\gamma^2)\nu}}{\delta_0\sqrt{5(\gamma-\beta)}}$$

$$B_1 = 0,\ B_2 = 0,\ k = \pm\frac{(-1)^{3/4}\sqrt[4]{\gamma^2-1}\sqrt[4]{\nu}}{\sqrt{2}\sqrt[4]{15}\sqrt[4]{\delta_0^2\delta_2^2(\gamma-\beta)}}, \omega = \mp\frac{(-1)^{3/4}\gamma\sqrt[4]{\gamma^2-1}\nu^{5/4}}{\sqrt{2}\sqrt[4]{15}\sqrt[4]{\delta_0^2\delta_2^2(\gamma-\beta)}}. \tag{21}$$

The wave solutions of Eq.(3) are constructed from solution sets 1 and 2 as

$$u_{7,8}(x,t) = -\frac{\sqrt{3(1-\gamma^2)\nu}\left(3\tan^2\left(\sqrt{\delta_0\delta_2}(\zeta+\zeta_0)\right)+2\right)}{\sqrt{5(\gamma-\beta)}}, \qquad \delta_0\delta_2 > 0. \tag{22}$$

$$u_{9,10}(x,t) = \frac{\sqrt{3(1-\gamma^2)\nu}\left(3\tanh^2\left(\sqrt{-\delta_0\delta_2}(\theta+\xi)\right)-2\right)}{\sqrt{5(\gamma-\beta)}}, \qquad \delta_0\delta_2 < 0. \tag{23}$$

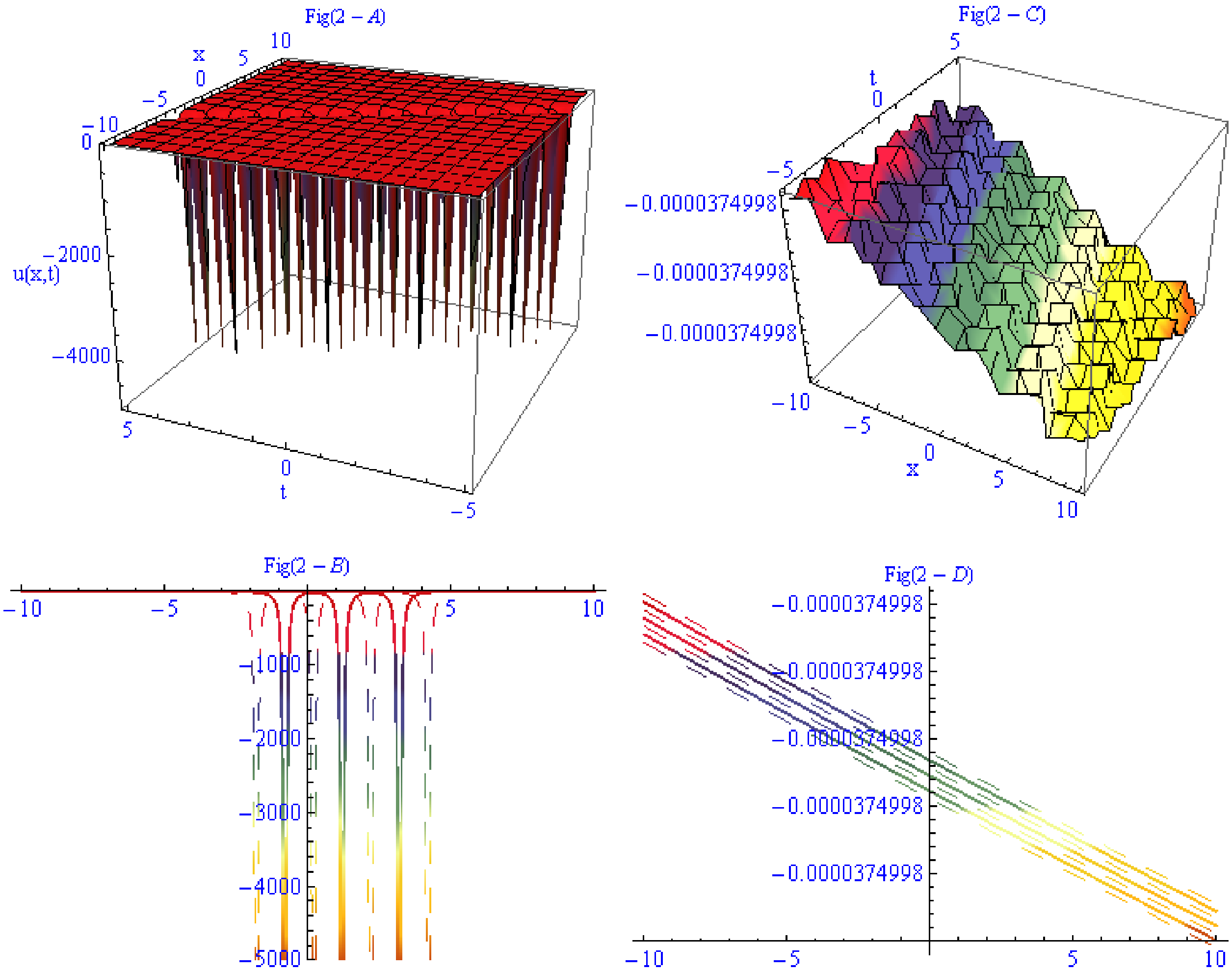

Figure 2: By granting appropriate values to parameters, the formation of solutions (18) and (19) are revealed as: Fig(2-A) is Multi-peak solitons and its 2D in Fig(2-B), Fig(2-C) is solitary wave of Kink type and its 2D in Fig(1-D).

$$u_{11,12}(x,t)=\frac{\sqrt{3\left(1-\gamma^2\right)\nu}\left(3\tan^2\left(\sqrt{\delta_0\delta_2}(\theta+\xi)\right)+2\right)}{\sqrt{5(\gamma-\beta)}},\qquad \delta_0\delta_2>0. \tag{24}$$

$$u_{13,14}(x,t)=\frac{\sqrt{3\left(1-\gamma^2\right)\nu}\left(2-3\tanh^2\left(\sqrt{-\delta_0\delta_2}(\theta+\xi)\right)\right)}{\sqrt{5(\gamma-\beta)}},\qquad \delta_0\delta_2<0. \tag{25}$$

3rd Family: In this family, we assume as $\delta_3=0$,
Set 1:

$$A_0=-\frac{3k^2}{2}\left(12\delta_2^2\mu^2+4\delta_2\left(2\delta_0-3\delta_1\mu\right)+\delta_1^2\right),\ A_1=-18\delta_2k^2\left(\delta_1-2\delta_2\mu\right),\ A_2=-18\delta_2^2k^2,\ B_1=0,$$
$$B_2=0,\omega=\frac{15k^5\left(\delta_1^2-4\delta_0\delta_2\right)^2\mp\sqrt{225\left(\delta_1^2-4\delta_0\delta_2\right){}^4k^{10}+16k^2\nu\left(15\beta\left(\delta_1^2-4\delta_0\delta_2\right){}^2k^4+4\nu\right)}}{8},$$
$$\gamma=-\frac{15k^5\left(\delta_1^2-4\delta_0\delta_2\right)^2\mp\sqrt{225\left(\delta_1^2-4\delta_0\delta_2\right){}^4k^{10}+16k^2\nu\left(15\beta\left(\delta_1^2-4\delta_0\delta_2\right){}^2k^4+4\nu\right)}}{8k\nu}. \tag{26}$$

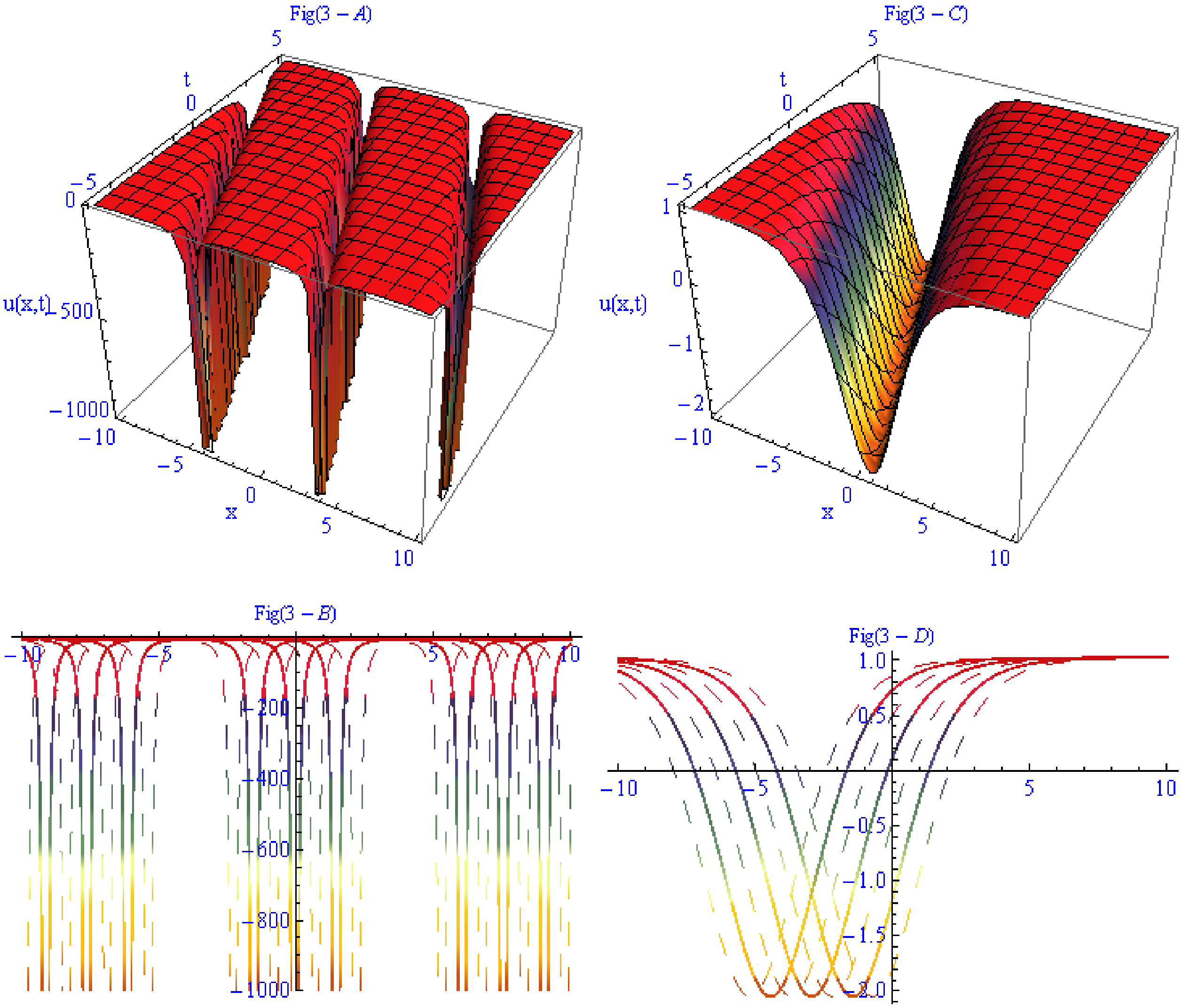

Figure 3: By granting appropriate values to parameters, the shape of solutions (22) and (23) are shown as: Fig(3-A) periodic solitary wave and its 2D in Fig(3-B), Fig(3-C) is dark soliton and its 2D in Fig(3-D).

Set 2:

$$A_0=-\frac{\sqrt{3\left(\gamma^2-1\right)\nu}\left(12\delta_2^2\mu^2+4\delta_2\left(2\delta_0-3\delta_1\mu\right)+\delta_1^2\right)}{\left(\delta_1^2-4\delta_0\delta_2\right)\sqrt{5(\beta-\gamma)}},\ A_1=-\frac{12\delta_2\sqrt{3\left(\gamma^2-1\right)\nu}\left(\delta_1-2\delta_2\mu\right)}{\left(\delta_1^2-4\delta_0\delta_2\right)\sqrt{5(\beta-\gamma)}},$$

$$A_2=-\frac{12\delta_2^2\sqrt{3\left(\gamma^2-1\right)\nu}}{\left(\delta_1^2-4\delta_0\delta_2\right)\sqrt{5(\beta-\gamma)}},\ B_1=0,\ B_2=0,\ k=\mp\frac{\sqrt{2}\sqrt[4]{\left(\gamma^2-1\right)\nu}}{\sqrt[4]{15\left(\delta_1^2-4\delta_0\delta_2\right)^2(\beta-\gamma)}},$$

$$\omega=\pm\frac{\gamma\nu\sqrt[4]{4\left(\gamma^2-1\right)\nu}}{\sqrt[4]{15\left(\delta_1^2-4\delta_0\delta_2\right)^2(\beta-\gamma)}}. \quad (27)$$

The wave results of Eq.(3) from sets 1 and 2 are constructed as follows

$$u_{15,16}(x,t)=\frac{3k^2}{2}\left(\delta_1^2\left(3\tan^2\left(\frac{\sqrt{4\delta_0\delta_2-\delta_1^2}}{2}(\zeta+\zeta_0)\right)-10\right)-4\delta_0\delta_2\left(3\tan^2\left(\frac{\sqrt{4\delta_0\delta_2-\delta_1^2}}{2}(\zeta+\zeta_0)\right)\right.\right.$$
$$\left.\left.+2)+12\sqrt{4\delta_0\delta_2-\delta_1^2}\delta_1\tan\left(\frac{\sqrt{4\delta_0\delta_2-\delta_1^2}}{2}(\zeta+\zeta_0)\right)\right),\qquad 4\delta_0\delta_2>\delta_1^2; \tag{28}$$

$$u_{17,18}(x,t)=\frac{\sqrt{3\left(\gamma^2-1\right)\nu}}{\left(\delta_1^2-4\delta_0\delta_2\right)\sqrt{5(\beta-\gamma)}}\left(\delta_1^2\left(3\tan^2\left(\frac{\sqrt{4\delta_0\delta_2-\delta_1^2}}{2}(\zeta+\zeta_0)\right)-10\right)\right.$$
$$+12\sqrt{4\delta_0\delta_2-\delta_1^2}\delta_1\tan\left(\frac{\sqrt{4\delta_0\delta_2-\delta_1^2}}{2}(\zeta+\zeta_0)\right)$$
$$\left.-4\delta_0\delta_2\left(3\tan^2\left(\frac{\sqrt{4\delta_0\delta_2-\delta_1^2}}{2}(\zeta+\zeta_0)\right)+2\right)\right),\qquad 4\delta_0\delta_2>\delta_1^2; \tag{29}$$

where ζ_0 is constant.

4.2 Application of Generalized $\exp(-\phi(\zeta))$-expansion Method

In this part, we employ generalized $\exp(-\phi(\zeta))$-expansion method on two-mode Sawada-Kotera for constructing the solitons and more waves solutions. Employing balancing principle of homogeneous on Eq.(11) and assume the wave solution as

$$\psi(\xi)=A_0+A_1\exp(-\phi(x))+A_2\left(\exp(-\phi(x))\right)^2. \tag{30}$$

By substituting Eq.(30) into Eq.(11) and deputing the coefficients of $\left(e^{(-\phi(\zeta))}\right)^i$ to zero, we achieved a equations system $A_0, A_1, A_2, a, b, c, k, \nu, \omega, \eta, \beta$. Mathematica 9 was utilized to resolve the equations set. We attained below families as:

1st Family:

$$A_0=-\frac{2\left(8abk^2+c^2k^2\right)}{3},\ A_1=-8ack^2,\ A_2=-8a^2k^2,\ \omega=\mp k\nu,\ \gamma=\frac{(10\beta\mp 1)}{9}. \tag{31}$$

2nd Family:

$$A_2=0,\ \omega=\mp k\nu,\ \gamma=\pm 1,\ \beta=\pm 1. \tag{32}$$

3rd Family:

$$A_0=-\left(\sqrt{5k(\beta k\nu+\omega)\left(16a^2b^2k^5(\beta k\nu+\omega)-8abc^2k^5(\beta k\nu+\omega)+c^4k^5(\beta k\nu+\omega)-4k^2\nu^2+4\omega^2\right)}\right.$$
$$\left.+40abk^3(\beta k\nu+\omega)+5\beta c^2k^4\nu+5c^2k^3\omega\right)/\left(10k(\beta k\nu+\omega)\right),\ A_1=-6ack^2,\ A_2=-6a^2k^2,\ \gamma=\beta. \tag{33}$$

4th Family:

$$A_1=-8ack^2,\ A_2=-8a^2k^2,\ \omega=\pm k\nu,\ \beta=\mp 1,\ \gamma=\mp 1. \tag{34}$$

From 1st family, the different forms of solitons and other solutions of Eq.(3) are obtained as
Type I: for $a=1, b\neq 0, c^2-4b>0$,

$$u_{1,2}(\zeta)=\frac{2k^2}{3}\left(8ab\left(\frac{3\left(c^2-2ab+c\sqrt{c^2-4b}\tanh\left(\frac{\sqrt{c^2-4b}}{2}(\zeta+\zeta_0)\right)\right)}{\left(\sqrt{c^2-4b}\tanh\left(\frac{\sqrt{c^2-4b}}{2}(\zeta+\zeta_0)\right)+c\right)^2}-1\right)-c^2\right). \tag{35}$$

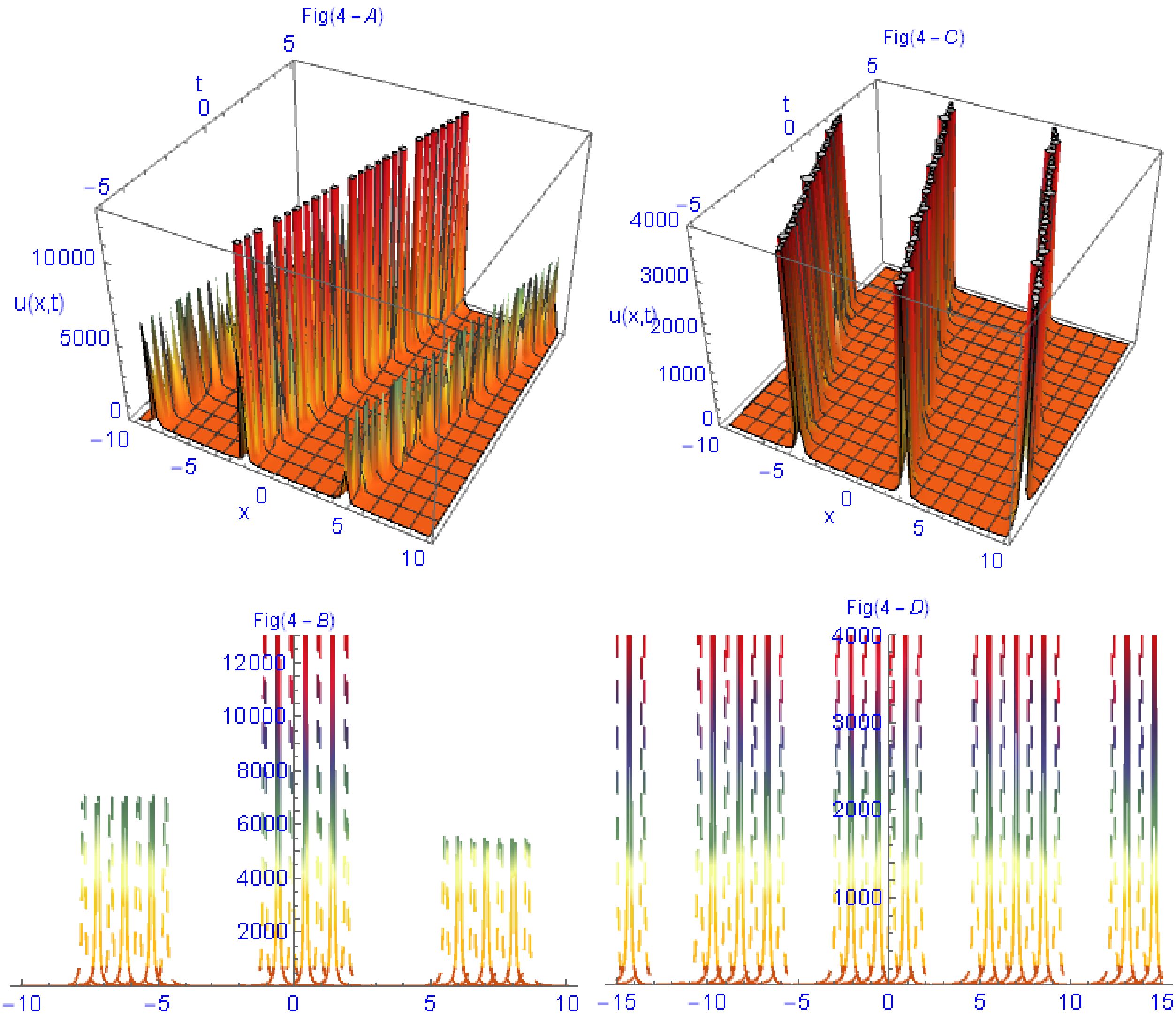

Figure 4: By granting appropriate values to parameters, the shape of solutions (24) and (29) are shown as: Fig(4-A) Multi peak soliton of different amplitude and its 2D in Fig(4-B), Fig(4-C) periodic solitary wave and its 2D in Fig(4-D).

Type II: for $a = 1,\ b \neq 0,\ c^2 - 4b < 0,$

$$u_{3,4}(\zeta) = \frac{2k^2}{3}\left(8ab\left(\frac{3\left(c^2 - 2ab - c\sqrt{4b - c^2}\tan\left(\frac{\sqrt{4b-c^2}}{2}(\zeta + \zeta_0)\right)\right)}{\left(c - \sqrt{4b - c^2}\tan\left(\frac{\sqrt{4b-c^2}}{2}(\zeta + \zeta_0)\right)\right)^2} - 1\right) - c^2\right). \tag{36}$$

Type III: for $a = 1,\ b = 0,\ c \neq 0,\ c^2 - 4b > 0,$

$$u_{5,6}(\zeta) = -\frac{2k^2}{3}\left(c^2 + 8ab + \frac{12ac^2}{e^{c(zeta+\zeta_0)} - 1} + \frac{12a^2c^2}{\left(e^{c(\zeta+\zeta_0)} - 1\right)^2}\right). \tag{37}$$

Type IV: for $a = 1,\ b \neq 0,\ c \neq 0,\ c^2 - 4b = 0,$

$$u_{7,8}(\zeta) = \frac{2k^2}{3}\left(\frac{6ac^3(\zeta + \zeta_0)}{c(\zeta + \zeta_0) + 1} - \frac{3a^2c^4(\zeta + \zeta_0)^2}{(c(\zeta + \zeta_0) + 1)^2} - 8ab - c^2\right). \tag{38}$$

Type V: for $c=0,\ a>0,\ b>0,$

$$u_{9,10}(\zeta)=-\frac{2k^2}{3}\left(12b\cot\left(\sqrt{ab}(\zeta+\zeta_0)\right)\left(c\sqrt{\frac{a}{b}}+a\cot\left(\sqrt{ab}(\zeta+\zeta_0)\right)\right)+8ab+c^2\right). \tag{39}$$

Type VI: for $c=0,\ a<0,\ b<0,$

$$u_{11,12}(\zeta)=-\frac{2k^2}{3}\left(8ab+c^2-12bc\sqrt{\frac{a}{b}}\cot\left(\sqrt{ab}(\zeta-\zeta_0)\right)+12ab\cot^2\left(\sqrt{ab}(\zeta-\zeta_0)\right)\right). \tag{40}$$

Type VII: for $c=0,\ a>0,\ b<0,$

$$u_{13,14}(\zeta)=\frac{2k^2}{3}\left(12bc\sqrt{-\frac{a}{b}}\coth\left(\sqrt{-ab}(\zeta-\zeta_0)\right)+12ab\coth^2\left(\sqrt{-ab}(\zeta-\zeta_0)\right)-8ab-c^2\right). \tag{41}$$

Type VIII: for $c=0,\ a<0,\ b>0,$

$$u_{15,16}(\zeta)=\frac{2k^2}{3}\left(12ab\coth^2\left(\sqrt{-ab}(\zeta+\zeta_0)\right)-12bc\sqrt{-\frac{a}{b}}\coth\left(\sqrt{-ab}(\zeta+\zeta_0)\right)-8ab-c^2\right). \tag{42}$$

Type IX: for $b=0,\ c=0,$

$$u_{17,18}(\zeta)=-\frac{2k^2}{3}\left(8ab+c^2+\frac{12c}{\zeta+\zeta_0}+\frac{12}{(\zeta+\zeta_0)^2}\right). \tag{43}$$

From 2nd family, the more solitons and other wave solutions of Eq.(3) are obtained as
Type I: for $a=1,b\neq 0,c^2-4b>0,$

$$u_{19,20}(\zeta)=A_0-\frac{2A_1b}{\sqrt{c^2-4b}\tanh\left(\frac{\sqrt{c^2-4b}}{2}(\zeta+\zeta_0)\right)+c}. \tag{44}$$

Type II: for $a=1,\ b\neq 0,\ c^2-4b<0,$

$$u_{21,22}(\zeta)=A_0-\frac{2A_1b}{c-\sqrt{4b-c^2}\tan\left(\frac{\sqrt{4b-c^2}}{2}(\zeta+\zeta_0)\right)}. \tag{45}$$

Type III: for $a=1,\ b=0,\ c\neq 0,\ c^2-4b>0,$

$$u_{23,24}(\zeta)=A_0-\frac{A_1c}{1-e^{c(\zeta+\zeta)}}. \tag{46}$$

Type IV: for $a=1,\ b\neq 0,\ c\neq 0,\ c^2-4b=0,$

$$u_{25,26}(\zeta)=A_0-\frac{A_1c^2(\zeta+\zeta_0)}{2c(\zeta+\zeta_0)+2}. \tag{47}$$

Type V: for $c=0,\ a>0,\ b>0,$

$$u_{27,28}(\zeta)=A_0+\frac{A_1\sqrt{b}\cot\left(\sqrt{ab}(\zeta+\zeta_0)\right)}{\sqrt{a}}. \tag{48}$$

Type VI: for $c=0,\ a<0,\ b<0,$

$$u_{29,30}(\zeta)=A_0-\frac{A_1\sqrt{b}\cot\left(\sqrt{ab}(\zeta-\zeta_0)\right)}{\sqrt{a}}. \tag{49}$$

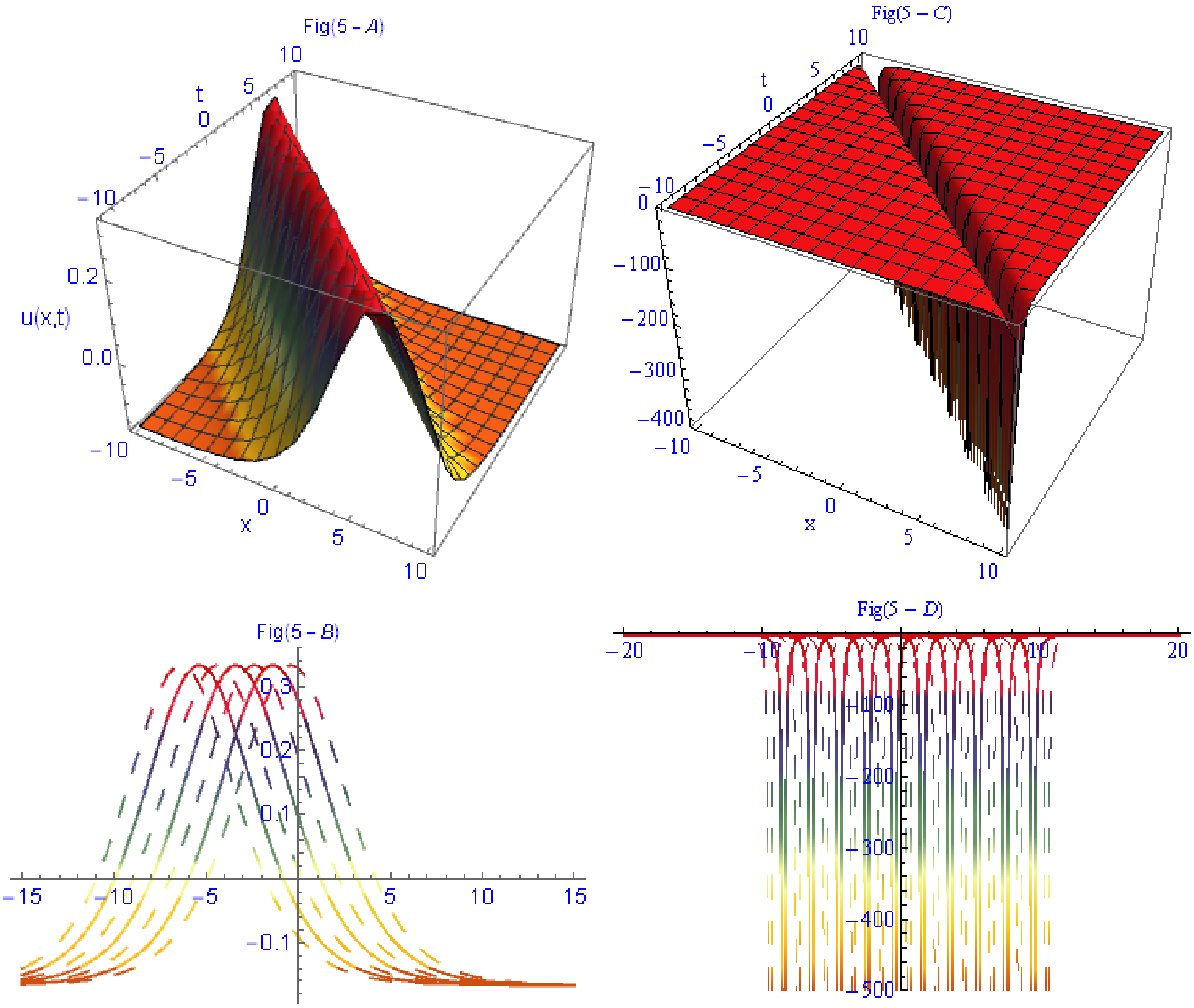

Figure 5: By granting appropriate values to parameters, the shape of solutions (35) and (37) are shown as: Fig(4-A) is bright soliton wave and its 2D in Fig(5-B), Fig(5-C) is dark solitary wave and its 2D in Fig(5-D).

Type VII: for $c = 0,\ a > 0,\ b < 0,$

$$u_{31,32}(\zeta) = A_0 + A_1\sqrt{-\frac{b}{a}}\coth\left(\sqrt{-ab}(\zeta - \zeta_0)\right). \tag{50}$$

Type VIII: for $c = 0,\ a < 0,\ b > 0,$

$$u_{33,34}(\zeta) = A_0 - A_1\sqrt{-\frac{b}{a}}\coth\left(\sqrt{-ab}(\zeta + \zeta_0)\right). \tag{51}$$

Type IX: for $b = 0,\ c = 0,$

$$u_{35,36}(\zeta) = A_0 + \frac{A_1}{a(\zeta + \zeta_0)}. \tag{52}$$

Similarly, more general soliton results can construct of equation (3) from families 3rd and 4th.

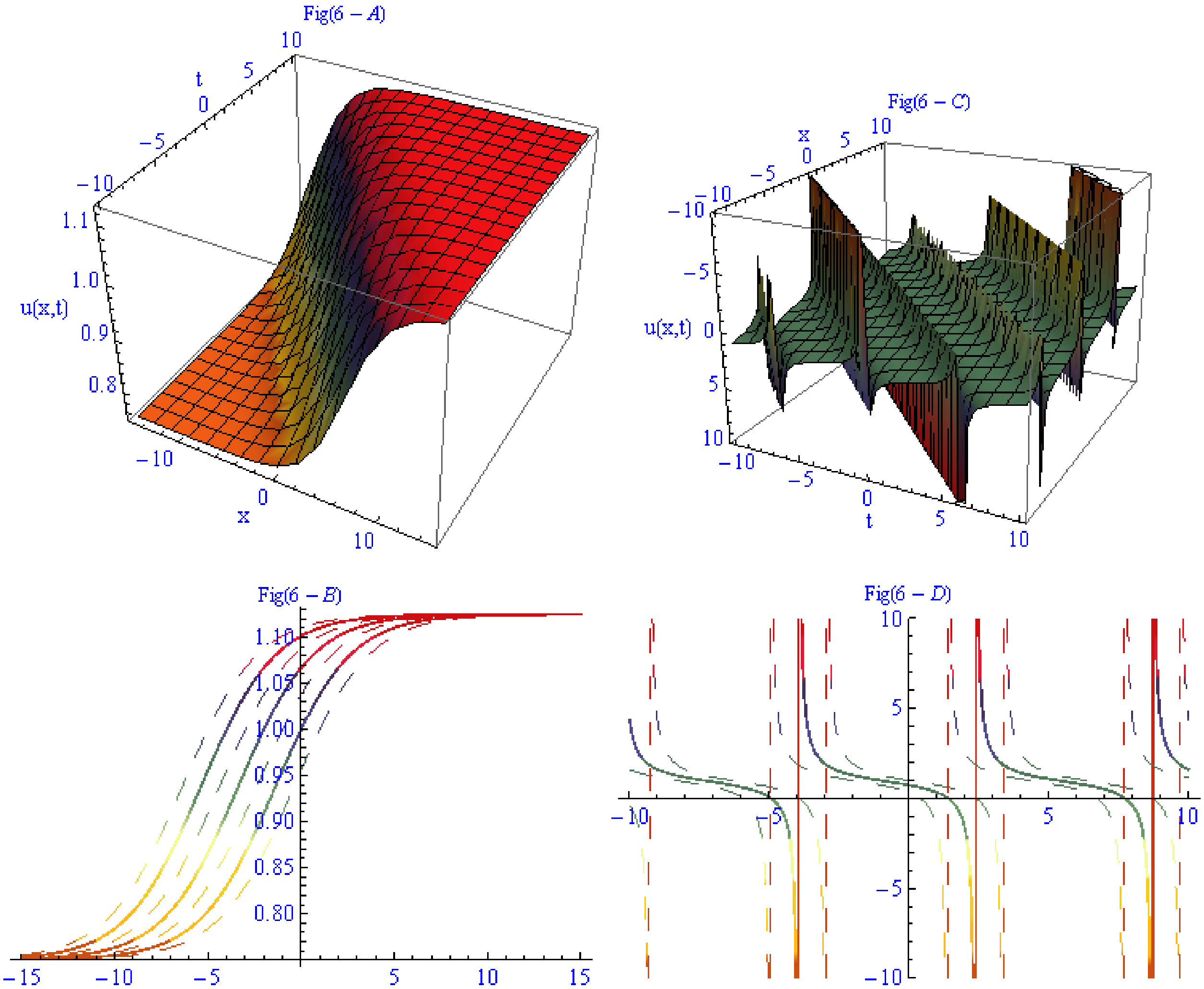

Figure 6: By granting appropriate values to parameters, the shape of solutions (44) and (45) are shown as: Fig(6-A) is Kink soliton wave and its 2D in Fig(6-B), Fig(6-C) is Breather wave of strange shape and its 2D in Fig(6-D).

5 Discussion of Results and Graphical Representation

The accomplished solutions are dissimilar from the results obtained by other researchers in the previous methods. The equations (8) and (10) present numerous dissimilar kinds of solutions by giving different values of parameters. It was announced earlier that the tmSKE was studied by the simplified Hirota technique [4], the Kudryashov and the modified Kudryashov methods [11, 15], auxiliary equation method [11] and sine-cosine technique [15]. Pedestal on the applications of these methods, the authors report some bright, dark, multi-solitons, singular periodic and kink structured results with the restricted conditions $\beta = \gamma = 1$. However in this article, eighteen wave solutions are constructed through the improved F-expansion method and thirty-six wave solutions are constructed through the generalized $\exp(-\phi(\zeta))$-expansion technique. The explored solutions demonstrate the dual-mode bright, dark, periodic, Kink, multi soliton and singular wave behaviors that are being classified as waves of right/left mode. Evaluated with published results [4, 11, 15], it is worth revealed that the constructed dual-wave solutions are new for the interests of applied methods. As a result, we have constructed several original results, which have not been explained before.

The Figures 1 to 4 indicate the solitons and other waves in dissimilar structures are described.

In the Figure 1, by granting appropriate values to parameters, the formation of solutions (16) and (17) are revealed as: Fig(1-A) Dark solitary wave and its 2-dimensional (2D) in Fig(1-B), Fig(1-C) bright soliton and its 2D in Fig(1-D). By granting appropriate values to parameters, the formation of solutions (18) and (19) in Figure 2 are revealed as: Fig(2-A) is Multi-peak solitons and its 2D in Fig(2-B), Fig(2-C) is solitary wave of Kink type and its 2D in Fig(1-D). In Figure 3, by granting appropriate values to parameters, the shape of solutions (22) and (23) are shown as: Fig(3-A) periodic solitary wave and its 2D in Fig(3-B), Fig(3-C) is dark soliton and its 2D in Fig(3-D). By granting appropriate values to parameters, the shape of solutions (24) and (29) in Figure 4 are shown as: Fig(4-A) Multi peak soliton of different amplitude and its 2D in Fig(4-B), Fig(4-C) periodic solitary wave and its 2D in Fig(4-D).

The Figures 5 and 6 illustrate the solitary waves in dissimilar structures are described. In the Figure 55, By granting appropriate values to parameters, the shape of solutions (35) and (37) are shown as: Fig(4-A) is bright soliton wave and its 2D in Fig(5-B), Fig(5-C) is dark solitary wave and its 2D in Fig(5-D). By granting appropriate values to parameters, the shape of solutions (44) and (45) in Figure 6 are shown as: Fig(6-A) is Kink soliton wave and its 2D in Fig(6-B), Fig(6-C) is Breather wave of strange shape and its 2D in Fig(6-D).

6 Conclusion

The described methods namely, improved F-expansion method and generalized $\exp(-\phi(\zeta))$-expansion method have been effectively employed on the tmSKE and as consequences, abundant of solitary wave solutions of different kinds such as bright and dark solitons, multi peak soliton, breather type waves, periodic solutions and other wave solutions are obtained. These obtained abundant novel solitons and other wave results have significant applications in fluid dynamics, applied sciences and engineering. The Sawada-Kotera equations illustrating the non-linear wave phenomena in shallow water, ion-acoustic waves in plasmas, fluid dynamics etc., and tmSKE also arising in fluid dynamics is addresses in this article. The graphical moments of few solutions are depicted that helps the engineers and scientist for understanding the physical phenomena of this model. To explain the novelty between the present results and the previously attained results, a comparative study has been presented. Furthermore, the executed techniques can be employed for further models arising in fluid dynamics correlated with any physical and engineering problems. The computational work approves the effectiveness, simplicity, and impact of described techniques.

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