Multiple periodic solutions for superquadratic and subquadratic second-order Hamiltonian systems *

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Abstract: In this paper, a class of second-order Hamiltonian systems is studied. Under the assumption of superquadratic and subquadratic for the nonlinear term, the existence of six periodic solutions and nine periodic solutions are obtained by using the variational method and space decomposition. Finally, two examples are given to verify the feasibility of the new criteria.

Keywords: Hamiltonian system; Variational approach; Critical point theorem; Space decomposition.

1 Introduction

Since Rabinowitz published his pioneer paper [12] in 1978, more and more mathematicians have paid more attention to the periodic solutions for the first-order or second-order Hamiltonian systems. There has been a lot of literature on the study of periodic solutions for Hamiltonian systems via critical point theory, such as [5, 13, 15–20] and the references therein. In [12], Rabinowitz consider the following second-order Hamiltonian systems

$$\ddot{u} + V'(u) = 0, \quad u \in R^N. \tag{1.1}$$

He studied the existence of periodic solutions of system (1.1) under the superquadratic condition, i.e., (AR): there exist $\mu > 2$ and $L > 0$ such that $0 < \mu F(t,u) \le (\nabla F(t,u), u)$ for $t \in [0,T]$

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and $|u| \geq L$. Observe that (AR) plays an important role to show that Palais-Smale sequence is bounded. Such condition was first introduced by Ambrosetti and Rabinowitz [1], and are useful in solving superlinear problems such as elliptic equations, dirac equations and wave equations. Tang and Wu [14] studied the existence of periodic solutions of system (1.1) with subquadratic and convex potentials, which unifies and generalizes the results in [11, 13, 15, 23]. In [8], Long proved the existence of period solution for system (1.1) without any convexity assumptions. This is one of the only papers on the assumption of nonconvexity, see [4, 6, 8–10, 21]. Inspired by some of ideas of [8], Li [7] obtained the existence of two minimal periodic solutions of system (1.1) by using a generalized version of the Weierstrass theorem and a new space decomposition in 2021. To our best knowledge, this is the first result of the existence of multiple minimal periodic solutions for Hamiltonian systems with subquadratic potentials. However, under the assumptions of superquadratic potentials and subquadratic potentials for system (1.1), the existence of periodic solutions with more properties has not been obtained.

Motivated by the above mentioned work, we will study the following second-order Hamiltonian systems with a parameter

$$\ddot{u} + \lambda V'(u) = 0, \tag{1.2}$$

where λ is a parameter, $V \in C^1(R^N, R)$, $V(0) = 0$ and $V(u) = \int_0^u V'(s)ds$.

Next, we assume the following conditions, in which condition (V_2) is the superquadratic assumption for the nonlinear term and condition (V_3) is the subquadratic assumption for the nonlinear term.

(V_1) $V(-u) = V(u)$ for any $u \in R^N$.

(V_2) $0 < \mu V(u) \leq V'(u)u$ for $u \geq M$, where μ and M are two positive constants and $\mu > 2$.

(V_3) there is a constants $1 < \beta < 2$ such that $V(u) \leq A|u|^\beta + p(t)$, where $p(t) \in L^1[0, T]$;

The new insights presented in the paper are as follows. Firstly, system (1.2) is a generalization of system (1.1). If $\lambda = 1$, system (1.2) reduces to system (1.1). Secondly, superquadratic and subquadratic assumptions are imposed on nonlinear term, respectively. In the two cases, the existence of six periodic solutions and nine periodic solutions are obtained. Finally, compared with [7], we also consider the existence of two odd $T/2$-antiperiodic nonconstant solutions with period T.

Our main results are as follows.

Theorem 1.1. *Assume that conditions (V_1), (V_2) hold and there exist a positive constants r such that (V_4) $\int_0^T V(\sin \frac{2\pi}{T} t)dt > \frac{\pi^2}{r} \max_{|u|<c_1} V(u)$, where $c_1 = \sqrt{\frac{(24+\pi^2)Tr}{24\pi^2}}$. Then for each*

$\lambda \in \left(\frac{\pi^2}{T\int_0^T V(\sin\frac{2\pi}{T}t)dt}, \frac{r}{T\max_{|u|<c_1} V(u)} \right)$, *system (1.2) has at least two odd $T/2$-antiperiodic nonconstant solutions with period T.*

Corollary 1.1. Assume that conditions (V_1), (V_2) hold and there exist a positive constants r such that (V_5) $\int_0^T V(\sin\frac{4\pi}{T}t)dt \geq \frac{4\pi^2}{r}\max_{|u|<c_2} V(u)$, where $c_2 = \sqrt{\frac{Tr}{24}}$. Then for each $\lambda \in \left(\frac{4\pi^2}{T\int_0^T V(\sin\frac{4\pi}{T}t)dt}, \frac{r}{T\max_{|u|<c_2} V(u)} \right)$, system (1.2) has at least two odd nonconstant periodic solutions with period $T/2$.

Corollary 1.2. Assume that conditions (V_1), (V_2) hold and there exist a positive constants r such that (V_6) $\int_0^T V(\cos\frac{2\pi}{T}t)dt \geq \frac{\pi^2}{r}\max_{|u|<c_3} V(u)$, where $c_3 = \sqrt{\frac{(24+\pi^2)Tr}{24\pi^2}}$. Then for each $\lambda \in \left(\frac{\pi^2}{T\int_0^T V(\cos\frac{2\pi}{T}t)dt}, \frac{r}{T\max_{|u|<c_3} V(u)} \right)$, system (1.2) has at least two even $T/2$-antiperiodic nonzero solutions with period T.

Theorem 1.2. *Assume that there is a positive constant r and a function $v \in X$ with $\Phi(v) > 2k_1$, where $k_1 = \sqrt{\frac{(24+\pi^2)Tr}{24\pi^2}}$. Suppose conditions (V_1) , (V_3) and (V_7) $\int_0^T V(\sin\frac{2\pi}{T}t)dt > \frac{3\pi^2}{2r}\max_{|u|<k_1} V(u)$ hold. Then, for each $\lambda \in \left(\frac{3\pi^2}{2T\int_0^T V(\sin\frac{2\pi}{T}t)dt}, \frac{r}{T\max_{|u|<k_1} V(u)} \right)$, the system (1.2) has at least three odd $T/2$-antiperiodic periodic solutions with period T.*

Corollary 1.3. Assume that there is a positive constant r and a function $v \in X$ with $\Phi(v) > 2k_2$, where $k_2 = \sqrt{\frac{Tr}{24}}$. Suppose conditions (V_1), (V_3) and (V_8) $\int_0^T V(\sin\frac{4\pi}{T}t)dt > \frac{6\pi^2}{r}\max_{|u|<k_2} V(u)$ hold. Then, for each $\lambda \in \left(\frac{6\pi^2}{T\int_0^T V(\sin\frac{4\pi}{T}t)dt}, \frac{r}{T\max_{|u|<k_2} V(u)} \right)$, the system (1.2) has at least three odd periodic solutions with period $T/2$.

Corollary 1.4. Assume that there is a positive constant r and a function $v \in X$ with $\Phi(v) > 2k_3$, where $k_3 = \sqrt{\frac{(24+\pi^2)Tr}{24\pi^2}}$. Suppose conditions (V_1), (V_3) and (V_9) $\int_0^T V(\cos\frac{2\pi}{T}t)dt > \frac{3\pi^2}{2r}\max_{|u|<k_3} V(u)$ hold. Then, for each $\lambda \in \left(\frac{3\pi^2}{2T\int_0^T V(\cos\frac{2\pi}{T}t)dt}, \frac{r}{T\max_{|u|<k_3} V(u)} \right)$, the system (1.2) has at least three even $T/2$-antiperiodic solutions with period T.

The following two lemmas are the latest Two-Critical-Point-Theorem and Three-Critical-Point-Theorem, which will be used to prove Theorem 1.1 and Theorem 1.2.

Lemma 1.3. *([2]) Let X be a reflexive real Banach space and let $\Phi, \Psi : X \to R$ be two functionals of class C^1 such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r \in R$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{1.3}$$

and for each $\lambda \in \wedge = \left(\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}\right)$*, the functional* $I_\lambda = \Phi - \lambda\Psi$ *satisfies the (PS)-condition and it is unbounded from below. Then, for each* $\lambda \in \wedge$*, the functional* $I_\lambda(x)$ *admits at least two nonzero critical points* $u_{\lambda,1}, u_{\lambda,2}$ *such that* $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$*.*

Lemma 1.4. *([3]) Let* X *be a reflexive real Banach space,* $\Phi : X \to R$ *be a coercive and continuous Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on* X^**, and* $\Psi : X \to R$ *be a continuous Gâteaux differentiable functional whose derivative is compact with* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$*. Assume that there a positive constant* r *and an element* $v \in X$*, with* $2r < \Phi(u)$*, such that*
(a_1) $\frac{\sup_{u\in\Phi^{-1}}\Psi(u)}{r} < \frac{2\Psi(v)}{3\Phi(v)}$*;*
(a_2) *for all* $\lambda \in \left(\frac{3\Phi(v)}{2\Psi(v)}, \frac{r}{\sup_{u\in\Phi^{-1}}\Psi(u)}\right)$*, the functional* $\Phi - \lambda\Psi$ *is coercive.*
Then, for each $\lambda \in \left(\frac{3\Phi(v)}{2\Psi(v)}, \frac{r}{\sup_{u\in\Phi^{-1}}\Psi(u)}\right)$*, the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical point.*

2 Preliminaries

In this section, we recall some essential definitions and several lemmas.

Let us consider the space $X = H_T^1 = W^{1,2}(S_T, R^N)$ with the norm $\|u\| = \left(\int_0^T |u|^2 + |\dot{u}|^2)dt\right)^{\frac{1}{2}}$, where $S_T = R/(TS), T > 0$. It is well known X is a reflexive Banach space. We can split X into $X = X_T \bigoplus Y_T$, where $X_T = \{u \in H_T^1 \big| u(-t) = -u(t)\}$ and $Y_T = \{u \in H_T^1 \big| u(-t) = u(t)\}$. X_T and Y_T are closed subspaces of X, then they are reflexive Banach space. Moreover, we define

$$X_T^1 = \{u \in X_T \big| u(t) = -u(t - T/2)\} \quad and \quad X_T^2 = \{u \in X_T \big| u(t) = u(t - T/2)\}$$

$$Y_T^1 = \{u \in Y_T \big| u(t) = -u(t - T/2)\} \quad and \quad Y_T^2 = \{u \in Y_T \big| u(t) = u(t - T/2)\},$$

where $X_T = X_T^1 \bigoplus X_T^2$ and $Y_T = Y_T^1 \bigoplus Y_T^2$. Obviously, for $x_1 \in X_T^1, x_2 \in X_T^2, y_1 \in Y_T^1$ and $y_2 \in Y_T^2$, we have the following Fourier expansions

$$x_1 = \sum_{k=0}^{+\infty} b_{2k+1}\sin((2k+1)\omega t) \quad and \quad x_2 = \sum_{k=1}^{+\infty} b_{2k}\sin(2k\omega t)$$

$$y_1 = \sum_{k=0}^{+\infty} a_{2k+1}\cos((2k+1)\omega t) \quad and \quad y_2 = \sum_{k=0}^{+\infty} a_{2k}\cos(2k\omega t),$$

where $\omega = \frac{2\pi}{T}$. In these spaces X_T^1, X_T^2 and Y_T^1, we defined the norm as follows $\|u\|_{X_T^1} = \|u\|_{X_T^2} = \|u\|_{Y_T^1} = \left(\int_0^T |\dot{u}|^2 dt\right)^{\frac{1}{2}}$, and these norms are equivalent to the normal norm $\|u\|$. In addition, let $\|u\|_{Y_T^2} = \left(\int_0^T |u|^2 + |\dot{u}|^2 dt\right)^{\frac{1}{2}} = \|u\|$. We define energy functional $I_\lambda : X \to R^N$ by

$$I_\lambda(u) = \frac{1}{2}\int_0^T |\dot{u}|^2 dt - \lambda\int_0^T V(u(t))dt. \tag{2.1}$$

And $I_\lambda(u)$ can also be represented as $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$, where the functionals $\Phi(u), \Psi(u) : X \to R^N$ are defined as follows

$$\Phi(u) = \frac{1}{2}\int_0^T |\dot{u}|^2 dt \tag{2.2}$$

and

$$\Psi(u) = \int_0^T V(u(t))dt. \tag{2.3}$$

Obviously, $I_\lambda(u)$ is a *Gâteaux* differentiable functional and its *Gâteaux* derivation is continuous in u. So its *Fréchet* derivative ia as follows at the point u is

$$\langle I'_\lambda(u), v\rangle = \int_0^T (\dot{u}(t), \dot{v}(t))dt - \lambda \int_0^T V'(u(t))v(t)dt. \tag{2.4}$$

Definition 2.1. A function $u \in X$ is said to be a weak solution of system (1.2) , if u satisfied $\langle I'_\lambda(u), v)\rangle = 0$ for all $v \in X$.

Definition 2.2. A funtion u is said to be a classical solution of system (1.2), if $u \in C^2(R, R)$ satisfies equation in system (1.2).

Lemma 2.1. *If $u \in X_T^1$ (or $u \in Y_T^1$), $\int_0^T |u(t)|^2 dt \le \frac{T^2}{4\pi^2}\int_0^T |\dot{u}(t)|^2 dt$ and $\|u\|_\infty^2 \le \frac{T}{2\pi^2}(1 + \frac{\pi^2}{24})\int_0^T |\dot{u}(t)|^2 dt$. In addition, if $u \in X_T^2$ and $\int_0^T u(t)dt = 0$, $\int_0^T |u(t)|^2 dt \le \frac{T^2}{16\pi^2}\int_0^T |\dot{u}(t)|^2 dt$ and $\|u\|_\infty^2 \le \frac{T}{48}\int_0^T |\dot{u}(t)|^2 dt$.*

Proof. If $u \in X_T^1$, we have

$$u(t) = \sum_{k=0}^{+\infty} b_{2k+1}\sin((2k+1)\omega t). \tag{2.5}$$

The Parseval equality implies that

$$\int_0^T |u(t)|^2 dt = \frac{T}{2}\sum_{k=0}^{+\infty} |b_{2k+1}|^2. \tag{2.6}$$

Since

$$\dot{u}(t) = \sum_{k=0}^{+\infty}(2k+1)\omega \cdot b_{2k+1}\cos((2k+1)\omega t) = \sum_{k=0}^{+\infty}\frac{2(2k+1)\pi}{T} b_{2k+1}\cos((2k+1)\omega t), \tag{2.7}$$

by (2.5)-(2.7), we have

$$\int_0^T |\dot{u}(t)|^2 dt = \sum_{k=0}^{+\infty}\frac{2(2k+1)^2\pi^2}{T}|b_{2k+1}|^2 \ge \frac{4\pi^2}{T^2}\sum_{k=0}^{+\infty}\frac{T}{2}|b_{2k+1}|^2 = \frac{4\pi^2}{T^2}\int_0^T |u(t)|^2 dt. \tag{2.8}$$

By Cauchy-Schwarz inequality, we obtain

$$|u(t)|^2 \leq \left(\sum_{k=0}^{+\infty}|b_{2k+1}|\right)^2 = \Big(\sum_{k=0}^{+\infty}\frac{T}{2(2k+1)^2\pi^2}\Big)\Big(\sum_{k=0}^{+\infty}\frac{2(2k+1)^2\pi^2}{T}|b_{2k+1}|^2\Big), \tag{2.9}$$

where

$$\begin{aligned}\sum_{k=0}^{+\infty}\frac{T}{2(2k+1)^2\pi^2} &= \frac{T}{2\pi^2}\sum_{k=0}^{+\infty}\frac{1}{(2k+1)^2} = \frac{T}{2\pi^2}\left(1+\sum_{k=1}^{+\infty}\frac{1}{(2k+1)^2}\right)\\ &\leq \frac{T}{2\pi^2}\left(1+\sum_{k=1}^{+\infty}\frac{1}{4k^2}\right) = \frac{T}{2\pi^2}\left(1+\frac{\pi^2}{4\times 6}\right) = \frac{T}{2\pi^2}\left(1+\frac{\pi^2}{24}\right).\end{aligned} \tag{2.10}$$

Bring (2.8) and (2.10) into (2.9), we get

$$|u(t)|^2 \leq \frac{T}{2\pi^2}\left(1+\frac{\pi^2}{24}\right)\int_0^T |\dot{u}(t)|^2 dt.$$

If $u \in X_T^2$,

$$u(t) = \sum_{k=1}^{+\infty} b_{2k}\sin(2k\omega t), \tag{2.11}$$

by Parseval equality,

$$\int_0^T |u(t)|^2 dt = \frac{T}{2}\sum_{k=1}^{+\infty}|b_{2k}|^2. \tag{2.12}$$

Since

$$\dot{u}(t) = \sum_{k=1}^{+\infty} 2k\omega b_{2k}\cos(2k\omega t) = \sum_{k=1}^{+\infty}\frac{4k\pi}{T}b_{2k}\cos(2k\omega t), \tag{2.13}$$

we have

$$\int_0^T |\dot{u}(t)|^2 dt = \sum_{k=1}^{+\infty}\frac{8k^2\pi^2}{T}|b_{2k}|^2 \geq \frac{16\pi^2}{T^2}\sum_{k=1}^{+\infty}\frac{T}{2}|b_{2k}|^2 = \frac{16\pi^2}{T^2}\int_0^T |u(t)|^2 dt. \tag{2.14}$$

According to Cauchy-Schwarz inequality and combined with (2.11) and (2.14), we obtain

$$\begin{aligned}|u(t)|^2 &\leq \left(\sum_{k=1}^{+\infty}|b_{2k}|\right)^2 \leq \Big(\sum_{k=1}^{+\infty}\frac{T}{8k^2\pi^2}\Big)\Big(\sum_{k=1}^{+\infty}\frac{8k^2\pi^2}{T}|b_{2k}|^2\Big)\\ &= \Big(\frac{T}{8\pi^2}\sum_{k=1}^{+\infty}\frac{1}{k^2}\Big)\Big(\sum_{k=1}^{+\infty}\frac{8k^2\pi^2}{T}|b_{2k}|^2\Big) \leq \frac{T}{48}\int_0^T |\dot{u}(t)|^2 dt.\end{aligned}$$

If $u \in Y_T^1$, we get

$$u(t) = \sum_{k=0}^{+\infty} a_{2k+1}\cos((2k+1)\omega t). \tag{2.15}$$

According to Parseval equality, it is obvious that

$$\int_0^T |u(t)|^2 dt = \frac{T}{2}\sum_{k=0}^{+\infty} |a_{2k+1}|^2. \tag{2.16}$$

According to

$$\dot{u}(t) = -\sum_{k=0}^{+\infty}(2k+1)\omega \cdot a_{2k+1}\sin((2k+1)\omega t) = -\sum_{k=0}^{+\infty}\frac{2(2k+1)\pi}{T}a_{2k+1}\sin((2k+1)\omega t), \tag{2.17}$$

we have

$$\int_0^T |\dot{u}(t)|^2 dt = \sum_{k=0}^{+\infty}\frac{2(2k+1)^2\pi^2}{T}|a_{2k+1}|^2 \geq \frac{4\pi^2}{T^2}\sum_{k=0}^{+\infty}\frac{T}{2}|a_{2k+1}|^2 = \frac{4\pi^2}{T^2}\int_0^T |u(t)|^2 dt. \tag{2.18}$$

By Cauchy-Schwarz inequality, we get

$$\begin{aligned} |u(t)|^2 &\leq \left(\sum_{k=0}^{+\infty}|a_{2k+1}|\right)^2 = \left(\sum_{k=0}^{+\infty}\frac{T}{2(2k+1)^2\pi^2}\right)\left(\sum_{k=0}^{+\infty}\frac{2(2k+1)^2\pi^2}{T}|a_{2k+1}|^2\right) \\ &\leq \frac{T}{2\pi^2}\left(1+\frac{\pi^2}{24}\right)\int_0^T |\dot{u}(t)|^2 dt. \end{aligned} \tag{2.19}$$

□

Lemma 2.2. *([4, 7]) Suppose that condition (V_1) holds. Then, $I_\lambda \in C^1(X_T, R)$, and $u \in X_T$ is a critical point of I_λ restricted to X_T if and only if it is a C^2-solution of system (1.2) (The result still holds if we replace X_T with Y_T.)*

Lemma 2.3. *([4, 7]) Suppose that (V_1) holds. Then, we have*
(i) $x^* \in X_T^1(X_T^2)$ is a critical point of I_λ restricted to $X_T^1(X_T^2)$ if and only if it is a critical point of I_λ in X_T, that is, x^* is an odd C^2-solution of system (1.2).
(i) $y^* \in Y_T^1(Y_T^2)$ is a critical point of I_λ restricted to $Y_T^1(Y_T^2)$ if and only if it is a critical point of I_λ in Y_T, that is, y^* is an odd C^2-solution of system (1.2).

Lemma 2.4. *The spaces $X_T^i (i = 1, 2)$ and Y_T^1 are compactly embedded to $C[0, T]$, i.e., $X_T^i \hookrightarrow\hookrightarrow C[0, T](i = 1, 2)$ and $Y_T^1 \hookrightarrow\hookrightarrow C[0, T]$.*

Proof. In order to prove space X_T^1 is compactly embedded to $C[0, T]$, it is sufficient to prove space X_T^1 is continuously embedded to space X. Since $X \hookrightarrow\hookrightarrow C[O, T]$, from Lemma 2.1, one has

$$\|u\|^2 = \|\dot{u}\|_{L^2}^2 + \|u\|_{L^2}^2 \leq \|\dot{u}\|_{L^2}^2 + \frac{T^2}{4\pi^2}\|\dot{u}\|_{L^2}^2 = \frac{T^2+4\pi^2}{4\pi^2}\|\dot{u}\|_{L^2}^2 = \frac{T^2+4\pi^2}{4\pi^2}\|u\|_{X_T^1}^2,$$

which means space X_T^1 is continuously embedded to space X. Therefore space X_T^1 is compactly embedded to $C[0, T]$. In the same way , we have $X_T^2 \hookrightarrow\hookrightarrow C[0, T]$ and $Y_T^1 \hookrightarrow\hookrightarrow C[0, T]$. □

Lemma 2.5. *The spaces $X_T^i (i = 1, 2)$ and Y_T^1 are reflexive real Banach spaces.*

Proof. It is enough to show X_T^1 is a closed subspace of X_T. Let $\{u_n\} \in X_T^1$ and $u_n \to u_0$ as $n \to \infty$. Following we will show $u_0 \in X_T^1$. Since $\{u_n\} \in X_T^1$ and $u_n \to u_0$ as $n \to \infty$, then $u_n(t) = -u_n(t - \frac{T}{2})$ and $\|u_n - u_0\|_{X_T^1} \to 0$ as $n \to \infty$. By Lemma 2.1 , $\|u_n - u_0\|_\infty \to 0$ as $n \to \infty$, which means $u_n(t) \to u_0(t)$ as $n \to \infty$, $t \in [0, T]$. So $u_0(t) = -u_0(t - \frac{T}{2})$. That is , $u_0 \in X_T^1$. Similarly, The spaces X_T^2 and Y_T^1 are reflexive real Banach space. □

Lemma 2.6. *If (V_2) holds and $\lambda > 0$, the functional I_λ is unbound from below on X_T^1(or X_T^2,Y_T^1), and it satisfies the (PS)-condition on X_T^1(or X_T^2,Y_T^1).*

Proof. Firstly, we discuss whether I_λ is unbound from below. By (V_2), one knows there exist two constants $\alpha, \beta > 0$ such that $V(u) \geq \alpha|u|^\mu - \beta$, where $\mu > 2$. For some $u_0 \in X_T^1/\{0\}, l \in R$, we obtain

$$
\begin{aligned}
I_\lambda(lu_0) &= \frac{1}{2}\int_0^T |l\dot{u}_0|^2 dt - \lambda \int_0^T V(lu_0)dt. \\
&\leq \frac{l^2}{2}\|\dot{u}_0\|_{L^2}^2 - \lambda l^\mu \int_0^T \alpha |u_0|^\mu dt + \lambda\beta T \to -\infty.
\end{aligned}
$$

Thus, the energy functional I_λ is unbound from below.

Secondly, we prove that I_λ satisfies the (PS)-condition. Let $\{u_n\} \in X_T^1$ be a sequence such that $|I_\lambda(u_n)| < M$ and $\langle I'_\lambda(u_n), u_n\rangle \to 0$ as $n \to \infty$. For n large enough, by (V_2), one evaluates

$$
\begin{aligned}
M + \frac{1}{\mu}\|u_n\|_{X_T^1} \geq I_\lambda(u_n) - \frac{1}{\mu}\langle I'_\lambda(u_n), u_n\rangle &= (\frac{1}{2} - \frac{1}{\mu})\|u_n\|_{X_T^1}^2 - \lambda(\int_0^T V(u_n)dt - \frac{1}{\mu}\int_0^T V'(u_n)u_n dt) \\
&\geq (\frac{1}{2} - \frac{1}{\mu})\|u_n\|_{X_T^1}^2,
\end{aligned}
$$

where $\mu > 2$. Thus $\{u_n\}$ is bounded in X_T^1. Since X_T^1 is a reflexive Banach space, the fact that $\{u_n\}$ is bounded in X_T^1 means that one has weakly convergent subsequence $\{u_{n_m}\}$ such that $u_{n_m} \rightharpoonup u$ in X_T^1. Moreover, one has

$$
0 \leftarrow \langle I'_\lambda(u_{n_m}) - I'_\lambda(u), u_{n_m} - u\rangle = \|u_{n_m} - u\|_{X_T^1}^2 - \lambda \int_0^T (V'(u_{n_m}) - V'(u))(u_{n_m}(t) - u(t))dt.
$$

By Lemma 2.4, then $(X_T^1, \|\cdot\|) \hookrightarrow\hookrightarrow C([0, T])$, which means

$$
\int_0^T (V'(u_{n_m}) - V'(u))(u_{n_m}(t) - u(t))dt \to 0
$$

as $m \to \infty$. We get $\|u_{n_m} - u\|_{X_T^1}^2 \to 0$ as $m \to +\infty$. Therefore I_λ satisfies (PS)-condition. Using the same proof method, if $u \in X_T^2$ or $u \in Y_T^1$, one knows the functional I_λ is unbound from below and satisfies the (PS)-condition. □

Lemma 2.7. *Φ is coercive on X_T^1(or X_T^2,Y_T^1) and Φ' has a continuous inverse on $(X_T^1)^*$(or$(X_T^2)^*$,$(Y_T^1)^*$).*

Proof. By (2.2), it is obvious that Φ is coercive. Moreover, from [[4], Theorem 26], we know Φ' has a continuous inverse on $(X_T^1)^*$ if Φ' is coercive and continuous monotone. Firstly, we know $\langle \Phi'(u), u\rangle = \int_0^T |\dot{u}(t)|^2 dt = \|u\|^2_{X_T^1}$, which yield that Φ' is coercive.

Secondly, in consideration of

$$\langle \Phi'(u) - \Phi(v), u - v\rangle = \|u(t) - v(t)\|^2_{X_T^1},$$

we get Φ' is continuous monotone. Hence, Lemma 2.7 hold. $\square$

Lemma 2.8. $\Psi' : X_T^1 \to (X_T^1)^*$ *is compact with* $\inf_{u\in X_T^1} \Phi(u) = \Phi(0) = \Psi(0) = 0$.

Proof. Firstly, by (V_1), it is clear that $\inf_{X_T^1} \Phi = \Phi(0) = \Psi(0) = 0$. Secondly, let $\{u_n\} \in X_T^1$ be a sequence such that $u_n \rightharpoonup u$ as $n \to \infty$. Since $X_T^1 \hookrightarrow\hookrightarrow C[0,T]$, we know $\{u_n\}$ is uniformly convergent to u in $C[0,T]$ as $n \to \infty$. According to the fact that $V(u) \in C^1(R^N, R)$, we get $\lim_{n\to\infty} V'(u_n) = V'(u)$, which obtains

$$\lim_{n\to\infty} \sup_{v\in X_T^1} \frac{< \Psi'(u_n) - \Psi'(u), v >}{\|v\|_{X_T^1}} = \lim_{n\to\infty} \sup_{v\in X_T^1} \frac{\int_0^T (V'(u_n) - V'(u), v)dt}{\|v\|_{X_T^1}} = 0.$$

Therefore, $\Psi'(u)_n$ is strongly continuous to $\Psi'(u)$ in X_T^1. By [[4], Proposition 26.2], Ψ' is compact. Similarly, Ψ' is compact in spaces X_T^2 and Y_T^1. In addition, we have $\inf_{u\in X_T^2} \Phi(u) = \Phi(0) = \Psi(0) = 0$ and $\inf_{u\in Y_T^1} \Phi(u) = \Phi(0) = \Psi(0) = 0$. $\square$

3 Proof of main results

Proof of Theorem 1.1

Proof. By (2.2) and Lemma 2.1, we deduce that

$$\begin{aligned}\Phi^{-1}(-\infty, r) &= \{u \in X_T^1 \big| \Phi(u) < r\} = \{u \in X_T^1 \big| \|\dot{u}\|^2_{L^2} < 2r\}\\ &\subseteq \{u \in X_T^1 \big| \|u\|^2_\infty < \frac{(24 + \pi^2)Tr}{24\pi^2}\} = \{u \in X_T^1 \big| \|u\|_\infty < c_1\}.\end{aligned}$$

Therefore

$$\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u) < \sup_{\|u\|_\infty<c_1} \int_0^T V(u(t))dt < T \max_{|u|<c_1} V(u).$$

Let $\tilde{u} = \sin(\frac{2\pi}{T}t)$, we have

$$\Phi(\tilde{u}) = \frac{1}{2}\int_0^T |\dot{\tilde{u}}|^2 dt = \frac{2\pi^2}{T^2}\int_0^T (\cos\frac{2\pi}{T}t)^2 dt = \frac{\pi^2}{T}.$$

From (V_4), we get

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}=\frac{T\int_0^T V(\sin\frac{2\pi}{T}t)dt}{\pi^2}>\frac{T\max_{|u|<c_1}V(u)}{r}>\frac{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}{r}.$$

Hence, inequality (1.2) of Lemma 1.3 is verified. Combining Lemma 2.6 and Lemma 2.8, we obtain that, for each $\lambda\in\left(\frac{\pi^2}{T\int_0^T V(\sin\frac{2\pi}{T}t)dt},\frac{r}{T\max_{|u|<c_1}V(u)}\right)$, system (1.2) has two nonzero critical points $u_{\lambda,1}$, $u_{\lambda,2}$.

In addition, it is obvious that $X_T^1\cap R^N=\{0\}$ from the definition of X_T^1. So $u_{\lambda,1}$ and $u_{\lambda,2}$ are not constants. By Lemma 2.3 , we conclude that there are at least two odd $T/2$-antiperiodic nonconstant solutions with period T of system (1.2). □

Proof of Theorem 1.2

Proof. By (2.2) and Lemma 2.1, one has

$$\begin{aligned}\Phi^{-1}(-\infty,r)&=\{u\in X_T^1\big|\Phi(u)<r\}=\{u\in X_T^1\big|\|\dot{u}\|_{L^2}^2<2r\}\\&\subseteq\{u\in X_T^1\big|\|u\|_\infty^2<\frac{(24+\pi^2)Tr}{24\pi^2}\}=\{u\in X_T^1\big|\|u\|_\infty<k_1\}.\end{aligned}$$

which deduces that

$$\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)<T\max_{|u|<k_1}V(u).$$

Let $v=\sin(\frac{2\pi}{T}t)$. One has

$$\Phi(v)=\frac{1}{2}\int_0^T|\dot{v}|^2dt=\frac{2\pi^2}{T^2}\int_0^T(\cos\frac{2\pi}{T}t)^2dt=\frac{\pi^2}{T}.$$

Following from (V_7), we have

$$\frac{2\Psi(v)}{3\Phi(v)}=\frac{2T\int_0^T V(\sin\frac{2\pi}{T}t)dt}{3\pi^2}>\frac{T\max_{|u|<k_1}V(u)}{r}>\frac{\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}{r}.$$

Then, combining (2.1) and (V_3), we have

$$I_\lambda(u)=\Phi(u)-\lambda\Psi(u)=\frac{1}{2}\int_0^T|\dot{u}|^2dt-\lambda\int_0^T V(u)dt\geq\frac{1}{2}\int_0^T|\dot{u}|^2dt-\lambda\int_0^T A|u|^\beta+p(t)dt,$$

which means $I_\lambda(u)\to+\infty$ as $\|u\|_X\to+\infty$. Thus, $\Phi-\lambda\Psi$ is coercive. In addition, by Lemma 2.7 and Lemma 2.8, the system (1.2) has at least three odd $T/2$-antiperiodic periodic solutions with period T. □

Remark 3.1. Similar to the proof of Theorem 1.1 and Theorem 1.2, we get Corollary 1.1-1.2 and Corollary 1.3-1.4

4 Main examples

Example 4.1. Consider the following second order Hamiltonian systems

$$\ddot{u} + \lambda V'(u) = 0, \tag{4.1}$$

where $V(u) = u^{100}$. Let $c_1 = \frac{1}{100}T = 2\pi$ and $r = \frac{12\pi c_1^2}{24+\pi^2}$. We have $\frac{\pi^2}{r}\max_{|u|<c_1} V(u) = \frac{24\pi+\pi^3}{12c_1^2}\max_{|u|<c_1} V(u) < \frac{24\pi+\pi^3}{12}(\frac{1}{100})^{98}$. If $v = \sin\frac{2\pi}{T}t \in X_T^1$, one obtains $\int_0^{2\pi}(\sin t)^{100}dt > \frac{24\pi+\pi^3}{12}(\frac{1}{100})^{98}$. Therefor for $v \in X_T^1$, the condition (V_4) of Theorem 1.1 is satisfied. According to Theorem 1.1, for each $\lambda \in \left(\frac{\pi}{2\int_0^{2\pi}(\sin t)^{100}dt}, \frac{6(100)^{98}}{24+\pi^2}\right)$, system (4.1) has at least two odd $T/2$-antiperiodic nonconstant solutions with period T. In addition, we take $v = \sin 2t$ in X_T^2, and take $v = \cos t$ in Y_T^1. Using the same proof method as in spaceX_T^1, we get the system (4.1) has at least two odd nonconstant periodic solutions with period $T/2$, and has at least two even $T/2$-antiperiodic nonzero solutions with period T.

Example 4.2. Consider the following second order Hamiltonian systems

$$\ddot{u} + \lambda V'(u) = 0, \tag{4.2}$$

where $V(u) = u^4$. Let $k_1 = \frac{1}{100}T = 2\pi$, $r = \frac{12\pi k_1^2}{24+\pi^2}$,$A = 2\beta = \frac{4}{3}$, $p(t) = u^4$. Take $v(t) = \sin t(t \in (\frac{\pi}{4}+k\pi, \frac{3\pi}{4}+k\pi))$, which means that $\Phi(v) > 2k_1$ and $V(u) \leq A|u|^\beta + p(t)$. If $t \in (\frac{\pi}{4}+k\pi, \frac{3\pi}{4}+k\pi)$, we have $v(t) > \frac{\sqrt{2}}{2}$. Further we have$\int_0^{2\pi} V(\sin t) > \frac{\pi}{2} > \frac{3\pi^2}{2r}\max_{|u|<k_1} V(u)$, which means condition (V_7) of Theorem 1.2 is satisfied. Therefore for $\lambda \in \left(\frac{3\pi}{4\int_0^T(\sin t)^4 dt}, \frac{6\pi(100)^2}{24+\pi^2}\right)$, the system (4.2) has at least three odd $T/2$-antiperiodic periodic solutions with period T. Further, we take $v = \sin 2t(t \in (\frac{\pi}{8} + k\pi, \frac{3\pi}{8} + k\pi))$ in X_T^2 and take $v = \cos t(t \in (-\frac{\pi}{4} + k\pi, \frac{\pi}{4} + k\pi))$ in Y_T^1. Using the same proof method as in space X_T^1, one know the system (4.2) has at least three odd periodic solutions with period $T/2$, and has at least three even $T/2$-antiperiodic solutions with period T.

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