Stability analysis for neural networks with discrete and leakage time-varying delay systems with delay-range-dependence and delay-derivative-dependence

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Abstract

The paper deals with the stability problem of neural networks with discrete and leakage interval time-varying delays. Firstly, a novel Lyapunov-Krasovskii functional was constructed based on the neural networks leakage time-varying delay systems model. The delayed decomposition approach (DDA) and integral inequality techniques (IIA) were altogether employed, which can help to estimate the derivative of Lyapunov-Krasovskii functional and effectively extend the application area of the results. Secondly, by taking the lower and upper bounds of time-delays and their derivatives, a criterion on asymptotical was presented in terms of linear matrix inequality (LMI), which can be easily checked by resorting to LMI in Matlab Toolbox. Thirdly, the resulting criteria can be applied for the case when the delay derivative is lower and upper bounded, when the lower bound is unknown, and when no restrictions are cast upon the derivative characteristics. Finally, through numerical examples, the criteria will be compared with relative ones. The smaller delay upper bound was obtained by the criteria, which demonstrates that our stability criterion can reduce the conservatism more efficiently than those earlier ones.

Keywords: Neural networks; leakage time delay; delay decomposition approach; delay-interval- dependence; delay-derivative- dependence

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1. Introduction

In real-world problems, neural networks (NNs) play an increasingly important role in terms of both theory and applications [13]. In recent years, neural networks have been extensively studied and successfully applied in many areas such as combinatorial optimization, signal processing, associative memory, affine invariant matching, and pattern recognition [36]. What is more, the existence of time delays may destroy the stability or weaken the performance, which is harmful to the applications of neural networks. Thus, stability analysis with time delays has developing more and more rapidly to create neural networks with high quality. It is generally known that the stability of NNs plays a significant role in their designs for solving practical problems. In such applications, it is major importance to ensure that the designed neural network is stable [27, 28]. Therefore, the issue of stability analysis of neural networks with time delay attracts many researchers[4, 8, 11, 15, 16, 17, 18, 20, 22, 23, 26, 27, 28, 34, 37, 38, 41, 24, 47, 48, 50, 51], and a number of remarkable results have been built up in the open literature.

On the other hand, the time-delay in the leakage term has a great impact on the dynamical behavior of neural networks, and time-delay in the leakage term is always not easy to handle, only little attention has been paid towards the stability analysis of neural networks and dynamic systems involving time-delay in the leakage term [2, 6, 8, 12, 24, 25, 26, 31, 38, 39, 46]. There are two typical types of time delays for incorporating time delays into neural networks: (i) introduce transmission delays into the neural networks, and consider discrete delays; (ii) consider the delays in the leakage term. All of the above two types of time delays may alter the dynamics of the neural network under consideration. In [39], Shan et al. derived a new stability analysis of Delayed neural networks (DNNs) with constant leakage delay. In practice, the leakage delay is not a constant. Gopalsamy [12] initially discussed the problem of bidirectional associative memory neural networks with constant delays in the leakage term by using model transformation technique. In Jiang and Zou [14], the problem of asymptotic stability criteria for neural networks with leakage time-varying delays, introducing free-weighting matrices, is considered. The derived stability condition is dependent on the upper bounds of the transmission delay, the leakage delay as well as their derivation. In [2], Banu and Balasubramaniam established the robust stability of DNNs with time-varying leakage delay and time-varying delay. The authors also given a simple two-neuron networks model involving leakage delay and shown that the system is stable when leakage delay is 0 and it is unstable when leakage delay is 0.2 through geometrical interpretation [26]. In [8], Chen et al. investigated the problem of passivity analysis for neural networks with time delay in the leakage term. In [26], Li et al. further studied the stability analysis of recurrent neural networks neural networks with time delay in the leakage term and impulsive perturbations. Manivannan et al. in [34] considered delay-dependent stability criteria for neutral- type neural networks with interval time-varying delay signals under the effects of leakage delay. In [38], Qiu et al. based on the delay-partitioning approach, robust stability analysis for uncertain recurrent neural networks with leakage delay was derived. The lower bounds of the transmission delay and leakage delay derivation which is ignored in the existing results is taken into account in [2, 6, 8, 12, 24, 25, 26, 31, 38, 39, 46]. Until now, the neural networks model with the leakage time-varying delay almost has not been fully investigated.

Recently, Manivannanc et al. [35] addresses an improved stability criterion for an interval time-delayed neural networks (NNs) including neutral delay and leakage delay. By proposing a suitable Lyapunov–Krasovskii functionals (LKFs) together with the Auxiliary function-based integral inequality (AFBII) and reciprocally convex approach (RCC) approach. Cao et al. [5] deals with the robust passivity analysis problem for uncertain neural networks with both leakage delay and additive time-varying delays by using a more general activation function technique. The information of activation function which is ignored in the existing results is taken into account in this paper. Moreover, it has been shown in Chen et al. [8] that the leakage delays are difficult to handle because it has quick tendency to destabilize the system performance. For simple circuits with a small number of cells, the use of fixed constant delays may provide a good approximation when modeling them. However, in practical implementation, neural networks usually have a spatial nature due to the presence of an amount of parallel pathways with a variety of axon sizes and lengths. As a consequence, the time-delay in neural networks is usually time-varying and belongs to an interval the lower bound of which is restricted to be zero. Therefore, stability analyses of neural networks with time-varying delays have been widely studied in recent years, and a variety of results have been established using the Lyapunov-Krasovskii functional (LKF) method in the framework of linear matrix inequality (LMI) [33]. Therefore, the major contribution of this study lies in a consideration of new integral inequalities and improved LKFs, fully taking the relationship between the terms in the Leibniz-Newton formula within the framework of linear matrix inequalities (LMIs). Moreover, we assume that the lower bound of interval time-varying delay is not restricted to zero. To the authors' best knowledge, the problem of removing some restrictive conditions on lower bound of interval time-varying delay when studying neural networks (NNs) with leakage term and discrete interval time-varying delays has not been well probed yet, which is still an open problem.

Very recently, to obtain the less conservative results, the delay decomposition approach was successfully introduced in [48] for the neural networks with constant delay. Followed this, in [7], the authors discussed the problem of stability analysis for neural networks with time-varying delay via delay decomposition approach. In [1], the authors studied a delay decomposition approach to delay-dependent passivity analysis for interval neural networks with time-varying delay. To apply delay decomposition technique in our article, discrete and leakage delay intervals are divided into finitely many equidistant subintervals. To obtain some less conservative LMI-based stability conditions, various kinds of important approaches have been explored, such as , model transformation method [9], free-weighting matrix technique [14, 32], delay-partitioning method [10, 23, 38, 40, 42, 43], the delay-decomposition approach [1, 4, 7, 11, 28, 29, 48], and the Wirtinger integral inequality [5]. The problems of improved delay-dependent robust stability criteria for recurrent neural networks (RNNs) with time-varying delays [27] and neutral-type recurrent neural networks (NRNNs) [28] are also investigated. In [29], author has further discussed the problem of delay-range-dependent stability analysis of recurrent neural networks with time- varying delay belonging to a given interval. Although, Liu [29] has studied the stability of neural networks with interval time-varying delay and derived some stability criteria, but the obtained criteria are all delay-range-dependent which also do not include the information on lower bound of delay-derivative. Therefore, Liu in [30] present a novel stability analysis for systems with time-varying delay and its derivative varying within intervals. Nevertheless, the leakage term effects on neural networks are neglected in these works [27-30], which forms one of the motivation of this paper.

Motivated by the above discussion, in this paper, we extend the recent results [27 -30] for the stability problem for neural networks leakage delay systems with interval time-varying delays. The main problem is to derive maximum admissible upper bounds

(MAUBs) of the time-varying delays such that the concerned systems are asymptotical stability for any delay size less than the MAUBs. Accordingly, the obtained MAUBs become a key performance index to estimate the conservatism of a delay-derivative-dependent stability conditions. Compared with recently published article, this paper features:

1) We investigate delay-range-dependent and delay-derivative-dependent stability condition for neural networks interval time-varying delay systems with leakage terms by considering an augmented system model and utilizing the delayed-decomposition and integral inequality approach jointly.
2) We introduce a novel LKF which depends on both lower and upper bounds of time derivative are fully taken into account in the derivation of the delay-derivative-dependent stability condition, as a result of which less conservative stability criteria are obtained.
3) Compared with the criteria obtained by different methods, the proposed criterion provides bigger MAUBs but requires a smaller number of decision variables.

2. Problem formulation

Consider the following neural networks (NNs) with leakage term and discrete interval time-varying delays:

$$\dot{z}(t) = -Cz(t-\tau(t)) + Ag(z(t)) + Bg(z(t-h(t))) + J, \tag{1a}$$

$$z(t) = \phi(t),\ t \in [-\bar{h}, 0], \bar{h} = \max\{h_2, \tau_2\}, \tag{1b}$$

where $z(\cdot) = [z_1(\cdot), z_2(\cdot), \cdots, z_n(\cdot)]^{\mathrm{T}} \in R^n$ is the state vector with the n neurons; $g(z(t)) = [g(z_1(t)), g(z_2(t)), ..., g(z_n(t))] \in R^n$ is called an activation function indicating how the j-th neuron responses to its input; $J = [J_1, J_2, \cdots, J_n]^{\mathrm{T}} \in R^n$ is the external bias vector; $C = diag(c_1, ..., c_n)$ is a diagonal matrix with each $c_i > 0$ controlling the rate with which the i-th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs; $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are interconnection matrices representing weight coefficient of the neurons. $h(t)$ and $\tau(t)$ denote the time-varying discrete delay and leakage delay, respectively.

$\phi(t) \in R^{n\times n}$ is the initial state function, $\phi_i(t)(i \in N)$ is continuous on $[-\bar{h},0]$ where $\bar{h} = \max\{h_2, \tau_2\}$, $h_1 \le h(t) \le h_2$ and $\tau_1 \le \tau(t) \le \tau_2$.

The time-varying discrete delay $h(t)$ and leakage delay $\tau(t)$ are differentiable function, which satisfies for all $t \ge 0$:

$$h(t) \in [h_1, h_2],\ \dot{h}(t) \in [h_{1d}, h_{2d}], \tau(t) \in [\tau_1, \tau_2],\ \dot{\tau}(t) \in [\tau_{1d}, \tau_{2d}] \tag{2}$$

is satisfied, where $h_1, h_2, \tau_1, \tau_2, h_{1d}, h_{2d}, \tau_{1d},$ and τ_{2d} are some positive scalars and $h_{1d} \le h_{2d}, \tau_{1d} \le \tau_{2d}$. Moreover, we assume that $\tau_1 \le \tau(t) \le \tau_1 + \rho\delta, h_1 \le h(t) \le h_1 + \rho\sigma,$; that is $[\tau_1, \tau_1 + \rho\delta], [\tau_1 + \rho\delta, \tau_2]$ and $[h_1, h_1 + \rho\sigma], [h_1 + \rho\delta, h_2]$ $(0 < \rho < 1, \delta = \tau_2 - \tau_1,$ $\sigma = h_2 - h_1)$.

Assumption 1: It is assumed that each of the activation functions $g_i(\cdot), i = 1, 2, ..., n$ possess the following condition

$$k_i^- \le \frac{g_i(a_1) - g_i(a_2)}{a_1 - a_2} \le k_i^+, i = 1, 2, ..., n \tag{3}$$

where $g_i(0) = 0, a_1, a_2 \in R, a_1 \ne a_2,$ and k_i^+, k_i^- are positive scalars.

We note that the existence of an equilibrium point of system (1a) is guaranteed by the fixed point theorem. Now letting $z^* = [z_1^*, z_2^*, \cdots, z_n^*]^{\mathrm{T}}$ be an equilibrium of (1a), that is $\dot{z}^*(t) = 0,\quad \dot{z}^*(t - h(t)) = 0,$ implies from (1) that

$$0 = -Cz^*(t - \tau(t)) + Ag(z^*(t)) + Bg(z^*(t - h(t))) + J, \tag{4}$$

Introducing the state deviation from equilibrium

$$x(t) = z(t) - z^* \tag{5}$$

where $x(\cdot)=[x_1(\cdot),\cdots,x_n(\cdot)]^{\mathrm{T}}$, with $f(x(\cdot))=[f_1(x_1(\cdot)),\cdots,f_n(x_n(\cdot))]^{\mathrm{T}}$, and

$$f_i(x_i(\cdot))=g_i(x_i(\cdot)+z_i^*)-g_i(z_i^*),\ f_i(0)=0,\quad i=1,2,\ldots,n. \tag{6}$$

Now subtracting (4) from (1) with some algebraic manipulations using (5) and (6), it is easy to see that the dynamics of the state deviation is governed by

$$\dot{x}(t)=-Cx(t-\tau(t))+Af(x(t))+Bf(x(t-h(t))), \tag{7}$$

According to the inequality (3), one can obtain that

$$k_i^-\le\frac{f_i(x_i)}{x_i}\le k_i^+,\ f_i(0)=g_i(0)=0,\forall x_i\ne 0, i=1,2,\ldots,n. \tag{8}$$

In the following, we will develop some practically computable stability criteria for the system described (7). The following lemmas are useful in deriving the criteria. First, we introduce the following integral inequality approach (IIA), which be used in the proof of ours.

Lemma 1 [27-30]. For any positive semi-definite matrices
$X=\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33}\end{bmatrix}\ge 0$ the following integral inequality holds

$$-\int_{t-h(t)}^{t}\dot{x}^T(s)X_{33}\dot{x}(s)ds\le\int_{t-h(t)}^{t}\xi_1^T(t,s)\overline{X}\,\xi_1(t,s)ds \tag{9}$$

where $\overline{X}=\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & 0\end{bmatrix}$ and $\xi_1^{\mathrm{T}}(t,s)=\begin{bmatrix} x^{\mathrm{T}}(t) & x^{\mathrm{T}}(t-h(t)) & \dot{x}^{\mathrm{T}}(s)\end{bmatrix}$.

Secondary, the following Schur complement result, which is essential in the proofs of Theorems 1, 2, and Corollaries 1- 3 are introduced.

Lemma 2 [3]. The following matrix inequality

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} < 0, \tag{10a}$$

where $Q(x) = Q^T(x), R(x) = R^T(x)$ and $S(x)$ depend on affine on x, is equivalent to

$$R(x) < 0, \tag{10b}$$

$$Q(x) < 0, \tag{10c}$$

and

$$Q(x) - S(x)R^{-1}(x)S^T(x) < 0. \tag{10d}$$

3. Main results

The main aim is to derive maximum admissible upper bounds (MAUBs) of the time-delay such that the concerned system is asymptotically stable for any delay size less than the MAUBs. Accordingly, the obtained MAUBs become a key performance index to measure the conservatism of delay-range-dependent and delay-derivative dependent stability conditions. The discrete delay interval $[h_1, h_2]$ is divided into two subintervals as $[h_1, h_1 + \rho\sigma]$ and $[h_1 + \rho\sigma, h_2]$ $(0 < \rho < 1, \sigma = h_2 - h_1)$. The leakage delay interval $[\tau_1, \tau_2]$ is divided into two subintervals as $[\tau_1, \tau_1 + \rho\delta]$ and $[\tau_1 + \rho\delta, \tau_2]$ $(0 < \rho < 1, \delta = \tau_2 - \tau_1)$. Then, the sufficient condition, which is depend on both the upper and lower bounds of the delay derivative, will be given in terms of linear matrix inequalities is derived in the following Theorems 1 and 2.

Theorem 1: If $\tau_1 \le \tau(t) \le \tau_1 + \rho\delta, h_1 \le h(t) \le h_1 + \rho\sigma$, $(0 < \rho < 1)$, for given τ_{1d}, τ_{2d}, h_{1d}, h_{2d}, $K_1 = diag(k_1^-, k_2^-, ..., k_m^-)$, and $K_2 = diag(k_1^+, k_2^+, ..., k_m^+)$ $(i=1,2,...,m)$, the system (7) is asymptotically stable if there exist symmetry positive-definite matrices $P = P^T > 0$, $Q_i = Q_i^T > 0$, $R_j = R_j^T > 0$ $(i = 1, 2, ..., 10;\quad j = 1, 2, ..., 6)$, positive diagonal matrices $D > 0$, $E > 0$, $\Lambda_1 > 0$, $\Lambda_2 > 0$, positive semi-definite matrices

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \ge 0,\ Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{bmatrix} \ge 0,\ Z = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12}^T & Z_{22} & Z_{23} \\ Z_{13}^T & Z_{23}^T & Z_{33} \end{bmatrix} \ge 0,$$

$$U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{12}^T & U_{22} & U_{23} \\ U_{13}^T & U_{23}^T & U_{33} \end{bmatrix} \ge 0, V = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{12}^T & V_{22} & V_{23} \\ V_{13}^T & V_{23}^T & V_{33} \end{bmatrix} \ge 0,\quad W = \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{12}^T & W_{22} & W_{23} \\ W_{13}^T & W_{23}^T & W_{33} \end{bmatrix} \ge 0,$$ such

that the following LMIs hold:

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & 0 & 0 & 0 & \Omega_{19} & 0 & 0 & 0 \\ \Omega_{12}^T & \Omega_{22} & \Omega_{23} & 0 & \Omega_{25} & \Omega_{26} & 0 & 0 & 0 & 0 & 0 & \Omega_{212} \\ \Omega_{13}^T & \Omega_{23}^T & \Omega_{33} & \Omega_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Omega_{312} \\ \Omega_{14}^T & 0 & \Omega_{34}^T & \Omega_{44} & 0 & 0 & 0 & \Omega_{48} & 0 & 0 & 0 & \Omega_{412} \\ \Omega_{15}^T & \Omega_{25}^T & 0 & 0 & \Omega_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Omega_{26}^T & 0 & 0 & 0 & \Omega_{66} & \Omega_{67} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Omega_{67}^T & \Omega_{77} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_{48}^T & 0 & 0 & 0 & \Omega_{88} & \Omega_{89} & \Omega_{810} & 0 & 0 \\ \Omega_{19}^T & 0 & 0 & 0 & 0 & 0 & 0 & \Omega_{89}^T & \Omega_{99} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Omega_{810}^T & 0 & \Omega_{1010} & \Omega_{1011} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Omega_{1011}^T & \Omega_{1111} & 0 \\ 0 & \Omega_{212}^T & \Omega_{312}^T & \Omega_{412}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Omega_{1212} \end{bmatrix} < 0 \tag{11a}$$

$$R_1 - X_{33} \geq 0, R_2 - Y_{33} \geq 0, R_3 - Z_{33} \geq 0, R_4 - U_{33} \geq 0, R_5 - V_{33} \geq 0, R_6 - W_{33} \geq 0 \tag{11b}$$

where

$\Omega_{11} = -2K_2\Lambda_1K_1 + Q_1 + Q_4 + Q_6 + Q_9 + \tau_1X_{11} + X_{13} + X_{13}^T + h_1U_{11} + U_{13} + U_{13}^T,$

$\Omega_{12} = -PC + K_1^TDC - K_2^TEC, \Omega_{13} = PA - K_1^TDA + K_2^TEA + \Lambda_1(K_1 + K_2),$

$\Omega_{14} = PB - K_1^TDB + K_2^TEB, \Omega_{15} = \tau_1X_{12} - X_{13} + X_{23}^T, \Omega_{19} = h_1U_{12} - U_{13} + U_{23}^T,$

$\Omega_{22} = (1-\tau_{1d})Q_5 - (1-\tau_{2d})Q_4 + \rho\delta Y_{11} + Y_{13} + Y_{13}^T + \rho\delta Y_{22} - Y_{23} - Y_{23}^T,$

$\Omega_{23} = -C^TD + C^TE, \Omega_{25} = \rho\delta Y_{12}^T - Y_{13}^T + Y_{23}, \Omega_{26} = \rho\delta Y_{12} - Y_{13} + Y_{23}^T, \Omega_{212} = -C^T\Psi,$

$\Omega_{33} = DA + A^TD^T - EA - A^TE^T - 2\Lambda_1, \Omega_{34} = DB - EB, \Omega_{312} = A^T\Psi, \Omega_{44} = -2\Lambda_2,$

$\Omega_{48} = \Lambda_2(K_1 + K_2), \Omega_{412} = B^T\Psi, \Omega_{55} = Q_2 - Q_1 + \tau_1X_{22} - X_{23} + X_{23}^T + \rho\delta Y_{11} + Y_{13} + Y_{13}^T,$

$\Omega_{66} = Q_3 - Q_2 + \rho\delta Y_{22} - Y_{23} - Y_{23}^T + (1-\rho)\delta Z_{11} + Z_{13} + Z_{13}^T,$

$\Omega_{67} = (1-\rho)\delta Z_{12} - Z_{13} + Z_{23}^T, \Omega_{77} = -Q_3 - Q_5 + (1-\rho)\delta Z_{22} - Z_{23} - Z_{23}^T,$

$\Omega_{88} = -2K_2\Lambda_2K_1 + (1-h_{1d})Q_{10} - (1-h_{2d})Q_9 + \rho\sigma V_{11} + V_{13} + V_{13}^T + \rho\sigma V_{22} - V_{23} - V_{23}^T,$

$\Omega_{89} = \rho\sigma V_{12}^T - V_{13}^T + V_{23}, \Omega_{810} = \rho\sigma V_{12} - V_{13} + V_{23}^T,$

$\Omega_{99} = Q_7 - Q_6 + h_1U_{22} - U_{23} - U_{23}^T + \rho\sigma V_{11} + V_{13} + V_{13}^T,$

$\Omega_{1010} = Q_8 - Q_7 + \rho\sigma V_{22} - V_{23} - V_{23}^T + (1-\rho)\sigma W_{11} + W_{13} + W_{13}^T,$

$\Omega_{1011} = (1-\rho)\sigma W_{12} - W_{13} + W_{23}^T, \Omega_{1111} = -Q_8 - Q_{10} + (1-\rho)\sigma W_{22} - W_{23} - W_{23}^T, \Omega_{1212} = -\Psi,$

$\Psi = \tau_1R_1 + \rho\delta R_2 + (1-\rho)\delta R_3 + h_1R_4 + \rho\sigma R_5 + (1-\rho)\sigma R_6.$

Proof: The main problem of stability analysis is how to construct appropriate Lyapunov functions which are widely used in various fields [26, 49].We extend the recent results [27-30] for the stability problem for neural networks leakage delay systems with interval time-varying delays. Since we construct the Lyapunov- Krasovskii functional candidate $V(x_t)$ for the neural networks (NNs) with leakage term and discrete interval time-varying delays as follows:

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) \tag{12}$$

where

$$\begin{aligned}
V_1(x_t) &= x^T(t)Px(t) + 2\sum_{j=1}^{n}[d_j\int_0^{x_j(t)}(f_j(s) - k_j^- s)ds + e_j\int_0^{x_j(t)}(k_j^+ s - f_j(s))ds],\\
V_2(x_t) &= \int_{t-\tau_1}^{t} x^T(s)Q_1x(s)ds + \int_{t-\tau_1-\rho\delta}^{t-\tau_1} x^T(s)Q_2x(s)ds + \int_{t-\tau_2}^{t-\tau_1-\rho\delta} x^T(s)Q_3x(s)ds\\
&\quad + \int_{t-\tau(t)}^{t} x^T(s)Q_4x(s)ds + \int_{t-\tau_2}^{t-\tau(t)} x^T(s)Q_5x(s)ds\\
&\quad + \int_{t-h_1}^{t} x^T(s)Q_6x(s)ds + \int_{t-h_1-\rho\sigma}^{t-h_1} x^T(s)Q_7x(s)ds + \int_{t-h_2}^{t-h_1-\rho\sigma} x^T(s)Q_8x(s)ds\\
&\quad + \int_{t-h(t)}^{t} x^T(s)Q_9x(s)ds + \int_{t-h_2}^{t-h(t)} x^T(s)Q_{10}x(s)ds,\\
V_3(x_t) &= \int_{-\tau_1}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R_1\dot{x}(s)dsd\theta + \int_{-\tau_1-\rho\delta}^{-\tau_1}\int_{t+\theta}^{t} \dot{x}^T(s)R_2\dot{x}(s)dsd\theta\\
&\quad + \int_{-\tau_2}^{-\tau_1-\rho\delta}\int_{t+\theta}^{t} \dot{x}^T(s)R_3\dot{x}(s)dsd\theta + \int_{-h_1}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R_4\dot{x}(s)dsd\theta\\
&\quad + \int_{-h_1-\rho\sigma}^{-h_1}\int_{t+\theta}^{t} \dot{x}^T(s)R_5\dot{x}(s)dsd\theta + \int_{-h_2}^{-h_1-\rho\sigma}\int_{t+\theta}^{t} \dot{x}^T(s)R_6\dot{x}(s)dsd\theta.
\end{aligned}$$

The time derivative of $V(x_t)$ with respect to time along the trajectory of system (7) is as follows. First, the derivative of $V_1(x_t)$ is

$$\dot{V}_1(x_t) = 2\dot{x}^T(t)Px(t) + 2[f(x(t)) - K_1x(t)]^T D\dot{x}(t) + 2[K_2x(t) - f(x(t))]^T E\dot{x}(t) \tag{13}$$

Second, the time-derivative of $V_2(x_t)$ can be obtained as

$$\begin{aligned}
\dot{V}_2(x_t) &= x^T(t)Q_1x(t) - x^T(t-\tau_1)Q_1x(t-\tau_1) + x^T(t-\tau_1)Q_2x(t-\tau_1)\\
&-x^T(t-\tau_1-\rho\delta)Q_2x(t-\tau_1-\rho\delta)\\
&+x^T(t-\tau_1-\rho\delta)Q_3x(t-\tau_1-\rho\delta) - x^T(t-\tau_2)Q_3x(t-\tau_2) + x^T(t)Q_4x(t)\\
&-x^T(t-\tau(t))(1-\dot{\tau}(t))Q_4x(t-\tau(t)) + x^T(t-\tau(t))(1-\dot{\tau}(t))Q_5x(t-\tau(t))\\
&-x^T(t-\tau_2)Q_5x(t-\tau_2) + x^T(t)Q_6x(t) - x^T(t-h_1)Q_6x(t-h_1) + x^T(t-h_1)Q_7x(t-h_1)\\
&-x^T(t-h_1-\rho\sigma)Q_7x(t-h_1-\rho\delta) + x^T(t-h_1-\rho\sigma)Q_8x(t-h_1-\rho\sigma)\\
&-x^T(t-h_2)Q_8x(t-h_2) + x^T(t)Q_9x(t) - x^T(t-h(t))(1-\dot{h}(t))Q_9x(t-h(t))\\
&+x^T(t-h(t))(1-\dot{h}(t))Q_{10}x(t-h(t)) - x^T(t-h_2)Q_{10}x(t-h_2)\\
&\le x^T(t)(Q_1+Q_4+Q_6+Q_9)x(t) + x^T(t-\tau_1)(Q_2-Q_1)x(t-\tau_1)\\
&+x^T(t-\tau_1-\rho\delta)(Q_3-Q_2)x(t-\tau_1-\rho\delta)\\
&+x^T(t-\tau_2)(-Q_3-Q_5)x(t-\tau_2) + x^T(t-h(t))((1-\tau_{1d})Q_5-(1-\tau_{2d})Q_4)x(t-\tau(t))\\
&+x^T(t-h_1)(Q_7-Q_6)x(t-h_1) + x^T(t-h_1-\rho\sigma)(Q_8-Q_7)x(t-h_1-\rho\delta)\\
&+x^T(t-h_2)(-Q_8-Q_{10})x(t-h_2) + x^T(t-h(t))((1-h_{1d})Q_{10}-(1-h_{2d})Q_9)x(t-h(t))
\end{aligned} \tag{14}$$

Finally, calculating the time-derivative of $V_3(x_t)$ lead to

$$
\begin{aligned}
\dot{V}_3(x_t) &= \dot{x}^T(t)\tau_1 R_1\dot{x}(t) - \int_{t-\tau_1}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds + \dot{x}^T(t)\rho\delta R_2\dot{x}(t) - \int_{t-\tau_1-\rho\delta}^{t-\tau_1}\dot{x}^T(s)R_2\dot{x}(s)ds \\
&+ \dot{x}^T(t)(1-\rho)\delta R_3\dot{x}(t) - \int_{t-\tau_2}^{t-\tau_1-\rho\delta}\dot{x}^T(s)R_3\dot{x}(s)ds + \dot{x}^T(t)h_1R_4\dot{x}(t) \\
&- \int_{t-h_1}^{t}\dot{x}^T(s)R_4\dot{x}(s)ds + \dot{x}^T(t)\rho\sigma R_5\dot{x}(t) - \int_{t-h_1-\rho\sigma}^{t-h_1}\dot{x}^T(s)R_5\dot{x}(s)ds \\
&+ \dot{x}^T(t)(1-\rho)\sigma R_6\dot{x}(t) - \int_{t-h_2}^{t-h_1-\rho\sigma}\dot{x}^T(s)R_6\dot{x}(s)ds \\
&= \dot{x}^T(t)[\tau_1R_1 + \rho\delta R_2 + (1-\rho)\delta R_3 + h_1R_4 + \rho\sigma R_5 + (1-\rho)\sigma R_6]\dot{x}(t) - \int_{t-\tau_1}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds \\
&- \int_{t-\tau_1-\rho\delta}^{t-\tau_1}\dot{x}^T(s)R_2\dot{x}(s)ds - \int_{t-\tau_2}^{t-\tau_1-\rho\delta}\dot{x}^T(s)R_3\dot{x}(s)ds - \int_{t-h_1}^{t}\dot{x}^T(s)R_4\dot{x}(s)ds - \int_{t-h_1-\rho\sigma}^{t-h_1}\dot{x}^T(s)R_5\dot{x}(s)ds \\
&- \int_{t-h_2}^{t-h_1-\rho\sigma}\dot{x}^T(s)R_6\dot{x}(s)ds \\
&= \dot{x}^T(t)[\tau_1R_1 + \rho\delta R_2 + (1-\rho)\delta R_3 + h_1R_4 + \rho\sigma R_5 + (1-\rho)\sigma R_6]\dot{x}(t) \\
&- \int_{t-\tau_1}^{t}\dot{x}^T(s)(R_1 - X_{33})\dot{x}(s)ds - \int_{t-\tau_1-\rho\delta}^{t-\tau_1}\dot{x}^T(s)(R_2 - Y_{33})\dot{x}(s)ds - \int_{t-\tau_2}^{t-\tau_1-\rho\delta}\dot{x}^T(s)(R_3 - Z_{33})\dot{x}(s)ds \\
&- \int_{t-h_1}^{t}\dot{x}^T(s)(R_4 - U_{33})\dot{x}(s)ds - \int_{t-h_1-\rho\sigma}^{t-h_1}\dot{x}^T(s)(R_5 - V_{33})\dot{x}(s)ds - \int_{t-h_2}^{t-h_1-\rho\sigma}\dot{x}^T(s)(R_6 - W_{33})\dot{x}(s)ds \\
&- \int_{t-\tau_1}^{t}\dot{x}^T(s)X_{33}\dot{x}(s)ds - \int_{t-\tau_1-\rho\delta}^{t-\tau_1}\dot{x}^T(s)Y_{33}\dot{x}(s)ds - \int_{t-\tau_2}^{t-\tau_1-\rho\delta}\dot{x}^T(s)Z_{33}\dot{x}(s)ds \\
&- \int_{t-h_1}^{t}\dot{x}^T(s)U_{33}\dot{x}(s)ds - \int_{t-h_1-\rho\sigma}^{t-h_1}\dot{x}^T(s)V_{33}\dot{x}(s)ds - \int_{t-h_2}^{t-h_1-\rho\sigma}\dot{x}^T(s)W_{33}\dot{x}(s)ds \qquad (15)
\end{aligned}
$$

For $\tau_1 \le \tau(t) \le \tau_1 + \rho\delta$ and $h_1 \le h(t) \le h_1 + \rho\sigma$, that is $[\tau_1, \tau_1 + \rho\delta]$ and $[h_1, h_1 + \rho\sigma]$ $(0 < \rho < 1,\ \ \delta = \tau_2 - \tau_1, \sigma = h_2 - h_1)$, by utilizing Lemma 1 and the Leibniz–Newton formula, we have

$$
\begin{aligned}
&- \int_{t-\tau_1}^{t}\dot{x}^T(s)X_{33}\dot{x}(s)ds - \int_{t-\tau_1-\rho\delta}^{t-\tau_1}\dot{x}^T(s)Y_{33}\dot{x}(s)ds - \int_{t-\tau_2}^{t-\tau_1-\rho\delta}\dot{x}^T(s)Z_{33}\dot{x}(s)ds \\
&- \int_{t-h_1}^{t}\dot{x}^T(s)U_{33}\dot{x}(s)ds - \int_{t-h_1-\rho\sigma}^{t-h_1}\dot{x}^T(s)V_{33}\dot{x}(s)ds - \int_{t-h_2}^{t-h_1-\rho\sigma}\dot{x}^T(s)W_{33}\dot{x}(s)ds \\
&= -\int_{t-\tau_1}^{t}\dot{x}^T(s)X_{33}\dot{x}(s)ds - \int_{t-\tau-\rho\delta}^{t-\tau(t)}\dot{x}^T(s)Y_{33}\dot{x}(s)ds - \int_{t-\tau(t)}^{t-\tau_1}\dot{x}^T(s)Y_{33}\dot{x}(s)ds \\
&- \int_{t-\tau_2}^{t-\tau_1-\rho\delta}\dot{x}^T(s)Z_{33}\dot{x}(s)ds - \int_{t-h_1}^{t}\dot{x}^T(s)U_{33}\dot{x}(s)ds - \int_{t-h_1-\rho\sigma}^{t-h(t)}\dot{x}^T(s)V_{33}\dot{x}(s)ds \\
&- \int_{t-h(t)}^{t-h_1}\dot{x}^T(s)V_{33}\dot{x}(s)ds - \int_{t-h_2}^{t-h_1-\rho\sigma}\dot{x}^T(s)W_{33}\dot{x}(s)ds \\
&\le \xi_2^T(t)\Pi\xi_2(t) \qquad (16)
\end{aligned}
$$

where

$$
\begin{aligned}
\xi_2^T(t) = \big[\, & x^T(t) \quad x^T(t-\tau(t)) \quad x^T(t-\tau_1) \quad x^T(t-\tau_1-\rho\delta) \quad x^T(t-\tau_2) \\
& x^T(t-h(t)) \quad x^T(t-h_1) \quad x^T(t-h_1-\rho\sigma) \quad x^T(t-h_2) \,\big],
\end{aligned}
$$

$$\Pi = \begin{bmatrix} \Pi_{11} & 0 & \Pi_{13} & 0 & 0 & 0 & \Pi_{17} & 0 & 0 \\ 0 & \Pi_{22} & \Pi_{23} & \Pi_{24} & 0 & 0 & 0 & 0 & 0 \\ \Pi_{13}^T & \Pi_{23}^T & \Pi_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Pi_{24}^T & 0 & \Pi_{44} & \Pi_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Pi_{45}^T & \Pi_{55} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Pi_{66} & \Pi_{67} & \Pi_{68} & 0 \\ \Pi_{17}^T & 0 & 0 & 0 & 0 & \Pi_{67}^T & \Pi_{77} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Pi_{68}^T & 0 & \Pi_{88} & \Pi_{89} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Pi_{89}^T & \Pi_{99} \end{bmatrix}, \text{ and}$$

$\Pi_{11} = \tau_1 X_{11} + X_{13} + X_{13}^T + h_1 U_{11} + U_{13} + U_{13}^T, \Pi_{13} = \tau_1 X_{12} - X_{13} + X_{23}^T,$
$\Pi_{17} = h_1 U_{12} - U_{13} + U_{23}^T, \Pi_{22} = \rho\delta Y_{11} + Y_{13} + Y_{13}^T + \rho\delta Y_{22} - Y_{23} - Y_{23}^T,$
$\Pi_{23} = \rho\delta Y_{12}^T - Y_{13}^T + Y_{23}, \Pi_{24} = \rho\delta Y_{12} - Y_{13} + Y_{23}^T,$
$\Pi_{33} = \tau_1 X_{22} - X_{23} - X_{23}^T + \rho\delta Y_{11} + Y_{13} + Y_{13}^T,$
$\Pi_{44} = \rho\delta Y_{22} - Y_{23} - Y_{23}^T + (1-\rho)\delta Z_{11} + Z_{13} + Z_{13}^T,$
$\Pi_{45} = (1-\rho)\delta Z_{12} - Z_{13} + Z_{23}^T, \Pi_{55} = (1-\rho)\delta Z_{22} - Z_{23} - Z_{23}^T,$
$\Pi_{66} = \rho\sigma V_{11} + V_{13} + V_{13}^T + \rho\sigma V_{22} - V_{23} - V_{23}^T, \Pi_{67} = \rho\sigma V_{12}^T - V_{13}^T + V_{23},$
$\Pi_{68} = \rho\sigma V_{12} - V_{13} + V_{23}^T, \Pi_{77} = h_1 U_{22} - U_{23} - U_{23}^T + \rho\sigma V_{11} + V_{13} + V_{13}^T,$
$\Pi_{88} = \rho\sigma V_{22} - V_{23} - V_{23}^T + (1-\rho)\sigma W_{11} + W_{13} + W_{13}^T,$
$\Pi_{89} = (1-\rho)\sigma W_{12} - W_{13} + W_{23}^T, \Pi_{99} = (1-\rho)\sigma W_{22} - W_{23} - W_{23}^T.$

With the operator for the term

$\dot{x}^T(t)[\tau_1 R_1 + \rho\delta R_2 + (1-\rho)\delta R_3 + h_1 R_4 + \rho\sigma R_5 + (1-\rho)\sigma R_6]\dot{x}(t)$ as follows:

$$\begin{aligned} &\dot{x}^T(t)[\tau_1 R_1 + \rho\delta R_2 + (1-\rho)\delta R_3 + h_1 R_4 + \rho\sigma R_5 + (1-\rho)\sigma R_6]\dot{x}(t) \\ &= \xi_3^T(t)\Sigma\xi_3(t) \end{aligned} \tag{17}$$

where $\xi_3^T(t) = \begin{bmatrix} x^T(t-\tau(t)) & f^T(x(t)) & f^T(x(t-h(t))) \end{bmatrix}, \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{12}^T & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{13}^T & \Sigma_{23}^T & \Sigma_{33} \end{bmatrix}$, and

$\Sigma_{11} = C^T\Psi C, \Sigma_{12} = -C^T\Psi A, \Sigma_{13} = -C^T\Psi B, \Sigma_{22} = A^T\Psi A, \Sigma_{23} = A^T\Psi B, \Sigma_{33} = B^T\Psi B$
$\Psi = \tau_1 R_1 + \rho\delta R_2 + (1-\rho)\delta R_3 + h_1 R_4 + \rho\sigma R_5 + (1-\rho)\sigma R_6.$

In addition, from the Assumption 2, there exist diagonal matrices $\Lambda_1 \geq 0$ and $\Lambda_2 \geq 0$, the following inequalities can be deduced:

$$\begin{aligned} &2[K_1 x(t) - f(x(t))]^T \Lambda_1 [f(x(t)) - K_2 x(t)] \\ &= -2 f^T(x(t))\Lambda_1 f(x(t)) + 2x^T(t)\Lambda_1(K_1 + K_2) f(x(t)) \\ &-2x^T(t) K_2 \Lambda_1 K_1 x(t) \geq 0 \end{aligned} \tag{18}$$

and

$$2[K_1x(t-h(t))-f(x(t-h(t)))]^T\Lambda_2[f(x(t-h(t)))-K_2x(t-h(t))]$$
$$=-2f^T(x(t-h(t)))\Lambda_2 f(x(t-h(t)))+2x^T(t-h(t))\Lambda_2(K_1+K_2)f(x(t-h(t)))$$
$$-2x^T(t-h(t))K_2\Lambda_2K_1x(t-h(t))\geq 0 \tag{19}$$

Combining (13)-(19), we obtain

$$\dot{V}(x_t)\leq \xi^T(t)\Xi\xi(t)-\int_{t-\tau_1}^{t}\dot{x}^T(s)(R_1-X_{33})\dot{x}(s)ds-\int_{t-\tau_1-\rho\delta}^{t-\tau_1}\dot{x}^T(s)(R_2-Y_{33})\dot{x}(s)ds$$
$$-\int_{t-\tau_2}^{t-\tau_1-\rho\delta}\dot{x}^T(s)(R_3-Z_{33})\dot{x}(s)ds-\int_{t-h_1}^{t}\dot{x}^T(s)(R_4-U_{33})\dot{x}(s)ds-\int_{t-h_1-\rho\sigma}^{t-h_1}\dot{x}^T(s)(R_5-V_{33})\dot{x}(s)ds$$
$$-\int_{t-h_2}^{t-h_1-\rho\sigma}\dot{x}^T(s)(R_6-W_{33})\dot{x}(s)ds \tag{20}$$

where

$$\xi^T(t)=[x^T(t)\quad x^T(t-\tau(t))\quad f^T(x(t))\quad f^T(x(t-h(t)))\quad x^T(t-\tau_1)$$
$$x^T(t-\tau_1-\rho\delta)\quad x^T(t-\tau_2)\quad x^T(t-h(t))\quad x^T(t-h_1)\quad x^T(t-h_1-\rho\sigma)\quad x^T(t-h_2)],$$

$$\Xi=\begin{bmatrix}
\Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & 0 & 0 & 0 & \Xi_{19} & 0 & 0\\
\Xi_{12}^T & \Xi_{22} & \Xi_{23} & \Xi_{24} & \Xi_{25} & \Xi_{26} & 0 & 0 & 0 & 0 & 0\\
\Xi_{13}^T & \Xi_{23}^T & \Xi_{33} & \Xi_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\Xi_{14}^T & \Xi_{24}^T & \Xi_{34}^T & \Xi_{44} & 0 & 0 & 0 & \Xi_{48} & 0 & 0 & 0\\
\Xi_{15}^T & \Xi_{25}^T & 0 & 0 & \Xi_{55} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \Xi_{26}^T & 0 & 0 & 0 & \Xi_{66} & \Xi_{67} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \Xi_{67}^T & \Xi_{77} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \Xi_{48}^T & 0 & 0 & 0 & \Xi_{88} & \Xi_{89} & \Xi_{810} & 0\\
\Xi_{19}^T & 0 & 0 & 0 & 0 & 0 & 0 & \Xi_{89}^T & \Xi_{99} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Xi_{810}^T & 0 & \Xi_{1010} & \Xi_{1011}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Xi_{1011}^T & \Xi_{1111}
\end{bmatrix}<0, \text{and}$$

$$\Xi_{11}=-2K_2\Lambda_1K_1+Q_1+Q_4+Q_6+Q_9+\tau_1X_{11}+X_{13}+X_{13}^T+h_1U_{11}+U_{13}+U_{13}^T,$$
$$\Xi_{12}=-PC+K_1^TDC-K_2^TEC, \Xi_{13}=PA-K_1^TDA+K_2^TEA+\Lambda_1(K_1+K_2),$$
$$\Xi_{14}=PB-K_1^TDB+K_2^TEB, \Xi_{15}=\tau_1X_{12}-X_{13}+X_{23}^T, \Xi_{19}=h_1U_{12}-U_{13}+U_{23}^T,$$
$$\Xi_{22}=(1-\tau_{1d})Q_5-(1-\tau_{2d})Q_4+\rho\delta Y_{11}+Y_{13}+Y_{13}^T+\rho\delta Y_{22}-Y_{23}-Y_{23}^T+C^T\Psi C,$$
$$\Xi_{23}=-C^TD+C^TE-C^T\Psi A, \Xi_{24}=-C^T\Psi B, \Xi_{25}=\rho\delta Y_{12}^T-Y_{13}^T+Y_{23},$$
$$\Xi_{26}=\rho\delta Y_{12}-Y_{13}+Y_{23}^T, \Xi_{33}=DA+A^TD^T-EA-A^TE^T-2\Lambda_1+A^T\Psi A,$$
$$\Xi_{34}=DB-EB+A^T\Psi B, \Xi_{44}=-2\Lambda_2+B^T\Psi B, \Xi_{48}=\Lambda_2(K_1+K_2),$$
$$\Xi_{55}=Q_2-Q_1+\tau_1X_{22}-X_{23}+X_{23}^T+\rho\delta Y_{11}+Y_{13}+Y_{13}^T,$$
$$\Xi_{66}=Q_3-Q_2+\rho\delta Y_{22}-Y_{23}-Y_{23}^T+(1-\rho)\delta Z_{11}+Z_{13}+Z_{13}^T,$$
$$\Xi_{67}=(1-\rho)\delta Z_{12}-Z_{13}+Z_{23}^T, \Xi_{77}=-Q_3-Q_5+(1-\rho)\delta Z_{22}-Z_{23}-Z_{23}^T,$$
$$\Xi_{88}=-2K_2\Lambda_2K_1+(1-h_{1d})Q_{10}-(1-h_{2d})Q_9+\rho\sigma V_{11}+V_{13}+V_{13}^T+\rho\sigma V_{22}-V_{23}-V_{23}^T,$$

$\Xi_{89} = \rho\sigma V_{12}^T - V_{13}^T + V_{23}, \Xi_{810} = \rho\sigma V_{12} - V_{13} + V_{23}^T,$
$\Xi_{99} = Q_7 - Q_6 + h_1 U_{22} - U_{23} - U_{23}^T + \rho\sigma V_{11} + V_{13} + V_{13}^T,$

$\Xi_{1010} = Q_8 - Q_7 + \rho\sigma V_{22} - V_{23} - V_{23}^T + (1-\rho)\sigma W_{11} + W_{13} + W_{13}^T,$
$\Xi_{1011} = (1-\rho)\sigma W_{12} - W_{13} + W_{23}^T, \Xi_{1111} = -Q_8 - Q_{10} + (1-\rho)\sigma W_{22} - W_{23} - W_{23}^T,$
$\Psi = \tau_1 R_1 + \rho\delta R_2 + (1-\rho)\delta R_3 + h_1 R_4 + \rho\sigma R_5 + (1-\rho)\sigma R_6.$

Theorem 2: If $\tau_1 + \rho\delta \le \tau(t) \le \tau_2, h_1 + \rho\sigma \le h(t) \le h_2$, $(0 < \rho < 1)$, for given τ_{1d}, τ_{2d}, h_{1d}, h_{2d}, $K_1 = diag(k_1^-, k_2^-, ..., k_m^-)$, and $K_2 = diag(k_1^+, k_2^+, ..., k_m^+)$ $(i=1,2,...,m)$, the system (7) is asymptotically stable if there exist symmetry positive-definite matrices $P = P^T > 0$, $Q_i = Q_i^T > 0$, $R_j = R_j^T > 0$ $(i = 1,2,...,10;\ j = 1,2,...,6)$, positive diagonal matrices $D > 0$, $E > 0$, $\Lambda_1 > 0$, $\Lambda_2 > 0$, positive semi-definite matrices

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \ge 0,\ Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{bmatrix} \ge 0,\ Z = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12}^T & Z_{22} & Z_{23} \\ Z_{13}^T & Z_{23}^T & Z_{33} \end{bmatrix} \ge 0,$$

$$U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{12}^T & U_{22} & U_{23} \\ U_{13}^T & U_{23}^T & U_{33} \end{bmatrix} \ge 0, V = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{12}^T & V_{22} & V_{23} \\ V_{13}^T & V_{23}^T & V_{33} \end{bmatrix} \ge 0,\quad W = \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{12}^T & W_{22} & W_{23} \\ W_{13}^T & W_{23}^T & W_{33} \end{bmatrix} \ge 0,$$ such

that the following LMIs hold:

$$\overline{\Omega} = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & 0 & 0 & 0 & \Omega_{19} & 0 & 0 & 0 \\
\Omega_{12}^T & \overline{\Omega}_{22} & \Omega_{23} & 0 & 0 & \overline{\Omega}_{26} & \overline{\Omega}_{27} & 0 & 0 & 0 & 0 & \Omega_{212} \\
\Omega_{13}^T & \Omega_{23}^T & \Omega_{33} & \Omega_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Omega_{312} \\
\Omega_{14}^T & 0 & \Omega_{34}^T & \Omega_{44} & 0 & 0 & 0 & \Omega_{48} & 0 & 0 & 0 & \Omega_{412} \\
\Omega_{15}^T & 0 & 0 & 0 & \Omega_{55} & \Omega_{56} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \overline{\Omega}_{26}^T & 0 & 0 & \overline{\Omega}_{56}^T & \Omega_{66} & \Omega_{67} & 0 & 0 & 0 & 0 & 0 \\
0 & \overline{\Omega}_{27}^T & 0 & 0 & 0 & \Omega_{67}^T & \Omega_{77} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Omega_{48}^T & 0 & 0 & 0 & \Omega_{88} & 0 & \overline{\Omega}_{810} & 0 & 0 \\
\Omega_{19}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Omega_{99} & \overline{\Omega}_{910} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{\Omega}_{810}^T & \overline{\Omega}_{910}^T & \Omega_{1010} & \Omega_{1011} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Omega_{1011}^T & \Omega_{1111} & 0 \\
0 & \Omega_{212}^T & \Omega_{312}^T & \Omega_{412}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Omega_{1212}
\end{bmatrix} < 0$$

(21a)

$$R_1 - X_{33} \ge 0, R_2 - Y_{33} \ge 0, R_3 - Z_{33} \ge 0, R_4 - U_{33} \ge 0, R_5 - V_{33} \ge 0, R_6 - W_{33} \ge 0 \qquad (21b)$$

where

$$\overline{\Omega}_{22} = (1-\tau_{1d})Q_5 - (1-\tau_{2d})Q_4 + (1-\rho)\delta Z_{11} + Z_{13} + Z_{13}^T + (1-\rho)\delta Z_{22} - Z_{23} - Z_{23}^T,$$
$$\overline{\Omega}_{26} = (1-\rho)\delta Z_{12}^T - Z_{13}^T + Z_{23}, \overline{\Omega}_{27} = (1-\rho)\delta Z_{12} - Z_{13} + Z_{23}^T,$$
$$\overline{\Omega}_{56} = \rho\delta Y_{12} - Y_{13} + Y_{23}^T, \overline{\Omega}_{810} = (1-\rho)\sigma W_{12}^T - W_{13}^T + W_{23}, \overline{\Omega}_{910} = \rho\sigma V_{12} - V_{13} + V_{23}^T,$$

Proof: By $\tau_1 + \rho\delta \le \tau(t) \le \tau_2, h_1 + \rho\sigma \le h(t) \le h_2$. Similar to (16), straightening out the integral term $-\int_{t-\tau_2}^{t-\tau_1-\rho\delta} \dot{x}^T(s) Z_{33}\dot{x}(s)ds$ and $-\int_{t-h_2}^{t-h_1-\rho\sigma} \dot{x}^T(s) W_{33}\dot{x}(s)ds$ into two parts as $-\int_{t-\tau_2}^{t-\tau(t)} \dot{x}^T(s) Z_{33}\dot{x}(s)ds$, $-\int_{t--\tau(t)}^{t-\tau_1-\rho\delta} \dot{x}^T(s) Z_{33}\dot{x}(s)ds$ and $-\int_{t-h_2}^{t-h(t)} \dot{x}^T(s) W_{33}\dot{x}(s)ds$, $-\int_{t-h(t)}^{t-h_1-\rho\sigma} \dot{x}^T(s) W_{33}\dot{x}(s)ds$, we have

$$\begin{aligned}
&-\int_{t-\tau_1}^{t} \dot{x}^T(s) X_{33}\dot{x}(s)ds - \int_{t-\tau_1-\rho\delta}^{t-\tau_1} \dot{x}^T(s) Y_{33}\dot{x}(s)ds - \int_{t-\tau_2}^{t-\tau_1-\rho\delta} \dot{x}^T(s) Z_{33}\dot{x}(s)ds \\
&-\int_{t-h_1}^{t} \dot{x}^T(s) U_{33}\dot{x}(s)ds - \int_{t-h_1-\rho\sigma}^{t-h_1} \dot{x}^T(s) V_{33}\dot{x}(s)ds - \int_{t-h_2}^{t-h_1-\rho\sigma} \dot{x}^T(s) W_{33}\dot{x}(s)ds \\
&= -\int_{t-\tau_1}^{t} \dot{x}^T(s) X_{33}\dot{x}(s)ds - \int_{t-\tau_1-\rho\delta}^{t-\tau_1} \dot{x}^T(s) Y_{33}\dot{x}(s)ds - \int_{t-\tau_2}^{t-\tau(t)} \dot{x}^T(s) Z_{33}\dot{x}(s)ds \\
&-\int_{t--\tau(t)}^{t-\tau_1-\rho\delta} \dot{x}^T(s) Z_{33}\dot{x}(s)ds - \int_{t-h_1}^{t} \dot{x}^T(s) U_{33}\dot{x}(s)ds - \int_{t-h_1-\rho\sigma}^{t-h_1} \dot{x}^T(s) V_{33}\dot{x}(s)ds \\
&-\int_{t-h_2}^{t-h(t)} \dot{x}^T(s) W_{33}\dot{x}(s)ds - \int_{t-h(t)}^{t-h_1-\rho\sigma} \dot{x}^T(s) W_{33}\dot{x}(s)ds
\end{aligned} \tag{22}$$

Proof: Based on Theorem 1, it is easy to obtain Theorem 2 by applying the same procedures of Theorem 1.

Furthermore, this criterion is the leakage delay is constant, and at the same time, is dependent on the derivative of the discrete time delay. This criterion can easily be extended to neural networks (NNs) with leakage term and discrete interval time-varying delay. The following Corollaries 1 and 2 have presented criteria that depend only on the size of the discrete delays derivation, and not on the size of the leakage delay derivation.

Corollary 1: If $h_1 \le h(t) \le h_1 + \rho\sigma$, $(0 < \rho < 1)$, for given h_{1d}, h_{2d}, $K_1 = diag(k_1^-, k_2^-, ..., k_m^-)$, and $K_2 = diag(k_1^+, k_2^+, ..., k_m^+)$ (i=1,2,...,m), the system (7) is asymptotically stable if there exist symmetry positive-definite matrices $P = P^T > 0$, $Q_i = Q_i^T > 0$, $R_j = R_j^T > 0$ $(i = 5, 6, ..., 10;\ j = 4, 5, 6)$, positive diagonal matrices $D > 0$, $E > 0$, $\Lambda_1 > 0$, $\Lambda_2 > 0$, positive semi-definite matrices $U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{12}^T & U_{22} & U_{23} \\ U_{13}^T & U_{23}^T & U_{33} \end{bmatrix} \ge 0,$

$V=\begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{12}^T & V_{22} & V_{23} \\ V_{13}^T & V_{23}^T & V_{33} \end{bmatrix} \geq 0, W=\begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{12}^T & W_{22} & W_{23} \\ W_{13}^T & W_{23}^T & W_{33} \end{bmatrix} \geq 0$, such that the following LMIs hold:

$$\Theta=\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & 0 & \Theta_{16} & 0 & 0 & 0 \\ \Theta_{12}^T & \Theta_{22} & \Theta_{23} & 0 & 0 & 0 & 0 & 0 & \Theta_{29} \\ \Theta_{13}^T & \Theta_{23}^T & \Theta_{33} & \Theta_{34} & 0 & 0 & 0 & 0 & \Theta_{39} \\ \Theta_{14}^T & 0 & \Theta_{34}^T & \Theta_{44} & \Theta_{45} & 0 & 0 & 0 & \Theta_{49} \\ 0 & 0 & 0 & \Theta_{45}^T & \Theta_{55} & 0 & 0 & 0 & 0 \\ \Theta_{16}^T & 0 & 0 & 0 & 0 & \Theta_{66} & \Theta_{67} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Theta_{67}^T & \Theta_{77} & \Theta_{78} & 0 \\ 0 & 0 & 0 & \Theta_{48}^T & 0 & 0 & \Theta_{78}^T & \Theta_{88} & 0 \\ 0 & \Theta_{29}^T & \Theta_{39}^T & \Theta_{49}^T & 0 & 0 & 0 & 0 & \Theta_{99} \end{bmatrix}<0 \tag{23a}$$

$$R_4-U_{33}\geq 0, R_5-V_{33}\geq 0, R_6-W_{33}\geq 0, \tag{23b}$$

where

$\Theta_{11}=-2K_2\Lambda_1K_1+Q_5+Q_6+Q_9+\tau Z_{11}+Z_{13}+Z_{13}^T+h_1U_{11}+U_{13}+U_{13}^T$,

$\Theta_{12}=-PC+K_1^TDC-K_2^TEC+\tau Z_{12}-Z_{13}+Z_{23}^T, \Theta_{13}=PA-K_1^TDA+K_2^TEA+\Lambda_1(K_1+K_2)$,

$\Theta_{14}=PB-K_1^TDB+K_2^TEB, \Theta_{16}=h_1U_{12}-U_{13}+U_{23}^T$,

$\Theta_{22}=-Q_5+\tau Z_{22}-Z_{23}-Z_{23}^T, \Theta_{23}=-C^TD+C^TE$,

$\Theta_{29}=-C^T[\tau R_3+h_1R_4+\rho\sigma R_5+(1-\rho)\sigma R_6]$,

$\Theta_{33}=DA+A^TD^T-EA-A^TE^T-2\Lambda_1, \Theta_{34}=DB-EB$,

$\Theta_{39}=A^T[\tau R_3+h_1R_4+\rho\sigma R_5+(1-\rho)\sigma R_6], \Theta_{44}=-2\Lambda_2, \Theta_{45}=\Lambda_2(K_1+K_2)$,

$\Theta_{49}=B^T[\tau R_3+h_1R_4+\rho\sigma R_5+(1-\rho)\sigma R_6]$,

$\Theta_{55}=(1-h_{1d})Q_{10}-(1-h_{2d})Q_9-2K_2\Lambda_2K_1+\rho\sigma V_{11}+V_{13}+V_{13}^T+\rho\sigma V_{22}-V_{23}-V_{23}^T$,

$\Theta_{56}=\rho\sigma V_{12}^T-V_{13}^T+V_{23}, \Theta_{57}=\rho\sigma V_{12}-V_{13}+V_{23}^T$,

$\Theta_{66}=Q_7-Q_6+h_1U_{22}-U_{23}-U_{23}^T+\rho\sigma V_{11}+V_{13}+V_{13}^T$,

$\Theta_{77}=Q_8-Q_7+\rho\sigma V_{22}-V_{23}-V_{23}^T+(1-\rho)\sigma W_{11}+W_{13}+W_{13}^T$,

$\Theta_{78}=(1-\rho)\sigma W_{12}-W_{13}+W_{23}^T, \Theta_{88}=-Q_8-Q_{10}+(1-\rho)\sigma W_{22}-W_{23}-W_{23}^T$,

$\Theta_{99}=-[\tau R_3+h_1R_4+\rho\sigma R_5+(1-\rho)\sigma R_6]$.

Proof: Choose the following Lyapunov-Krasovskii functional candidate to be

$$V_{1a}(x_t)=x^T(t)Px(t)+2\sum_{j=1}^{n}[d_j\int_0^{x_j(t)}(f_j(s)-k_j^-s)ds+e_j\int_0^{x_j(t)}(k_j^+s-f_j(s))ds] \tag{24}$$

$$V_{2b}(x_t) = \int_t^{t-\tau} x^T(s)Q_5x(s)ds + \int_{t-h_1}^{t} x^T(s)Q_6x(s)ds + \int_{t-h_1-\rho\sigma}^{t-h_1} x^T(s)Q_7x(s)ds$$
$$+\int_{t-h_2}^{t-h_1-\rho\sigma} x^T(s)Q_8x(s)ds + \int_{t-h(t)}^{t} x^T(s)Q_9x(s)ds + \int_{t-h_2}^{t-h(t)} x^T(s)Q_{10}x(s)ds \quad (25)$$

$$V_{3c}(x_t) = \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R_3\dot{x}(s)dsd\theta + \int_{-h_1}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R_4\dot{x}(s)dsd\theta$$
$$+\int_{-h_1-\rho\sigma}^{-h_1}\int_{t+\theta}^{t} \dot{x}^T(s)R_5\dot{x}(s)dsd\theta + \int_{-h_2}^{-h_1-\rho\sigma}\int_{t+\theta}^{t} \dot{x}^T(s)R_6\dot{x}(s)dsd\theta \quad (26)$$

The proof can be completed in a similar formulation to Theorem 1.

Corollary2: If $h_1+\rho\sigma \le h(t) \le h_2$ $(0<\rho<1)$, for given h_{1d}, h_{2d}, $K_1 = diag(k_1^-, k_2^-, ..., k_m^-)$, and $K_2 = diag(k_1^+, k_2^+, ..., k_m^+)$ $(i=1,2,...,m)$, the system (7) is asymptotically stable if there exist symmetry positive-definite matrices $P = P^T > 0$, $Q_i = Q_i^T > 0$, $R_j = R_j^T > 0$ $(i = 5,6,...,10;\ \ j = 4,5,6)$, positive diagonal matrices $D > 0$, $E > 0$, $\Lambda_1 > 0$, $\Lambda_2 > 0$, positive semi-definite matrices $U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{12}^T & U_{22} & U_{23} \\ U_{13}^T & U_{23}^T & U_{33} \end{bmatrix} \ge 0,$

$V = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{12}^T & V_{22} & V_{23} \\ V_{13}^T & V_{23}^T & V_{33} \end{bmatrix} \ge 0, W = \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{12}^T & W_{22} & W_{23} \\ W_{13}^T & W_{23}^T & W_{33} \end{bmatrix} \ge 0,$ such that the following LMIs hold:

$$\overline{\Theta} = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & 0 & \Theta_{16} & 0 & 0 & 0 \\ \Theta_{12}^T & \Theta_{22} & \Theta_{23} & 0 & 0 & 0 & 0 & 0 & \Theta_{29} \\ \Theta_{13}^T & \Theta_{23}^T & \Theta_{33} & \Theta_{34} & 0 & 0 & 0 & 0 & \Theta_{39} \\ \Theta_{14}^T & 0 & \Theta_{34}^T & \Theta_{44} & \Theta_{45} & 0 & 0 & 0 & \Theta_{49} \\ 0 & 0 & 0 & \Theta_{45}^T & \overline{\Theta}_{55} & 0 & \overline{\Theta}_{57} & \overline{\Theta}_{58} & 0 \\ \Theta_{16}^T & 0 & 0 & 0 & 0 & \Theta_{66} & \overline{\Theta}_{67} & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{\Theta}_{57}^T & \overline{\Theta}_{67}^T & \Theta_{77} & 0 & 0 \\ 0 & 0 & 0 & \Theta_{48}^T & \overline{\Theta}_{58}^T & 0 & 0 & \Theta_{88} & 0 \\ 0 & \Theta_{29}^T & \Theta_{39}^T & \Theta_{49}^T & 0 & 0 & 0 & 0 & \Theta_{99} \end{bmatrix} < 0 \quad (27a)$$

$$R_4 - U_{33} \ge 0, R_5 - V_{33} \ge 0, R_6 - W_{33} \ge 0, \quad (27b)$$

where

$\overline{\Theta}_{55} = (1-h_{1d})Q_{10} - (1-h_{2d})Q_9 - 2K_2\Lambda_2K_1 + (1-\rho)\sigma W_{11} + W_{13} + W_{13}^T$
$+(1-\rho)\sigma W_{22} - W_{23} - W_{23}^T, \overline{\Theta}_{57} = (1-\rho)\sigma W_{12}^T - W_{13}^T + W_{23},$

$\overline{\Theta}_{58} = (1-\rho)\sigma W_{12} - W_{13} + W_{23}^T, \overline{\Theta}_{67} = \rho\sigma V_{12} - V_{13} + V_{23}^T.$

When both the leakage delay and discrete delays are constant, $\tau(t)=\tau, h(t)=h,$ neural networks (NNs) with leakage term and discrete delays (1) will become the following model:

$$\dot{x}(t)=-Cx(t-\tau)+Af(x(t))+Bf(x(t-h)), \tag{28}$$

Then, for system (36), we have the following Corollary 3.

Corollary 3: For given τ, h, $K_1=diag(k_1^-,k_2^-,...,k_m^-)$, and $K_2=diag(k_1^+,k_2^+,...,k_m^+)$ $(i=1,2,...,m)$, the system (36) is asymptotically stable if there exist symmetry positive-definite matrices $P=P^T>0,\ Q_i=Q_i^T>0,\ R_j=R_j^T>0\ (i=j=a,b)$ positive diagonal matrices $D>0,\quad E>0,\quad \Lambda_1>0,\quad \Lambda_2>0,$ and positive semi-definite matrices $X=\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix}\geq 0,\ Y=\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{bmatrix}\geq 0,$ such that the following LMIs hold:

$$\Phi=\begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} & 0 \\ \Phi_{12}^T & \Phi_{22} & \Phi_{23} & 0 & 0 & \Phi_{26} \\ \Phi_{13}^T & \Phi_{23}^T & \Phi_{33} & \Phi_{34} & 0 & \Phi_{36} \\ \Phi_{14}^T & 0 & \Phi_{34}^T & \Phi_{44} & \Phi_{45} & \Phi_{46} \\ \Phi_{15}^T & 0 & 0 & \Phi_{45}^T & \Phi_{55} & 0 \\ 0 & \Phi_{26}^T & \Phi_{36}^T & \Phi_{46}^T & 0 & \Phi_{66} \end{bmatrix}<0, \tag{29a}$$

$$R_a-X_{33}\geq 0, R_b-Y_{33}\geq 0, \tag{29b}$$

where

$\Phi_{11}=Q_a+Q_b+\tau X_{11}+X_{13}+X_{13}^T+hY_{11}+Y_{13}+Y_{13}^T-2K_2\Lambda_1K_1,$
$\Phi_{12}=-PC+K_1^TDC-K_2EC+\tau X_{12}-X_{13}+X_{23}^T, \Phi_{13}=PA-K_1^TDA+K_2EA+\Lambda_1(K_1+K_2),$
$\Phi_{14}=PB-K_1^TDB+K_2EB, \Phi_{15}=hY_{12}-Y_{13}+Y_{23}^T, \Phi_{22}=-Q_a+\tau X_{22}-X_{23}-X_{23}^T,$
$\Phi_{23}=-C^TD+C^TE, \Phi_{26}=-C^T(\tau R_a+hR_b), \Phi_{33}=DA+A^TD-EA-A^TE-2\Lambda_1,$
$\Phi_{34}=B^TD-B^TE, \Phi_{36}=A^T(\tau R_a+hR_b), \Phi_{44}=-2\Lambda_2, \Phi_{45}=\Lambda_2(K_1+K_2),$
$\Phi_{46}=B^T(\tau R_a+hR_b), \Phi_{55}=-Q_b-2K_2\Lambda_1K_1+hY_{22}-Y_{23}-Y_{23}^T, \Phi_{66}=-(\tau R_a+hR_b).$

4. Examples

In this section, four examples will be used to check the feasibility and improvement of the proposed stability criteria.

Example 1. Consider the following neural networks (NNs) with leakage term and discrete interval time-varying delays:

$$\dot{x}(t) = -Cx(t-\tau(t) + Af(x(t)) + Bf(x(t-h(t))), \tag{30}$$

where

$C = \begin{bmatrix} 8 & 0 \\ 0 & 7 \end{bmatrix}, A = \begin{bmatrix} 0.5 & -0.2 \\ 0.7 & 0.5 \end{bmatrix}, B = \begin{bmatrix} 0.6 & -0.1 \\ -1.2 & -0.8 \end{bmatrix}, K_1 = diag\{0,0\} K_2 = diag\{1,1\}.$

Solution: We assume $h_{2d} = 0.5, h_{1d} = -0.5,$ for different τ and h_1 using Corollary 1 in this example, some MAUBs h_2 can be obtained for guaranteeing the asymptotic stability, which are listed in Table 1. From Table 1, the lower bound discrete delay h_1 increases the MAUBs h_2 increases. If leakage delay τ increases the MAUBs h_2 decreases. When $\tau > 0.157$ for different h_1 the maximum allowable values of h_2 as same $\tau = 0.157$.

Moreover, for the fast time-varying delay, let $h_{2d} = 1.5, h_{1d} = -1.5,$ the MAUBs h_2 for different τ and h_1 using Corollary 1is listed in Table 2. When $\tau > 0.157$ for different h_1 the maximum allowable values of h_2 as same $\tau = 0.157$.

Furthermore, the leakage delay and discrete delays are constant ($\tau(t) = \tau$ and $h(t) = h$), the neural networks (NNs) with leakage term and discrete delays (1) will become the following model:

$$\dot{x}(t) = -Cx(t-\tau) + Af(x(t)) + Bf(x(t-h)), \tag{31}$$

As in [24], we assume that the time-varying delay $h(t)$ is not differentiable. Using Corollary 3 in this paper, some maximum allowable values of τ can be obtained for guaranteeing the asymptotic stability, which is listed in Table 3.

To confirm the obtained result, when $h = 0.1$ and $\tau = 0.1771,$ under the initial condition $x(0) = [-1, \;\; 1]^{\mathrm{T}}$ is given in Fig. 1.

Let $\tau = 0.156$ and $h_{1d} = 0,$ the MAUBs h_2 for different h_{2d} by different methods is listed in Table 4. From Table 4, if h_{2d} increases the MAUBs h_2 decreases.

For comparison with some existing stability criteria, let $\tau = 0.17$ the MAUBs h_2 for different h_{2d} by different methods is listed in Table 5. It is seen from this Table 5 that our criterion in this paper is less conservative than those in Chen et al. [8]

Example 2. Consider the following neural networks (NNs) with leakage term and discrete interval time-varying delays:

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t-h(t))), \tag{32}$$

where

$$C = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.3 \end{bmatrix}, A = \begin{bmatrix} 0.5 & 0.2 \\ 0.4 & 0.3 \end{bmatrix}, B = \begin{bmatrix} 0.4 & -0.1 \\ 0.1 & 0.2 \end{bmatrix}, K_1 = diag\{0,0\} K_2 = diag\{1,1\}.$$

Solution: To compare of our result with existing ones, the maximum allowable leakage delay τ for different discrete delay h obtained is listed in Table 6. It is clear that the obtained results are significantly better than those in Chen et al. [8]. To confirm the obtained result, when $h = 0.1$ and $\tau = 0.7329,$ under the initial condition $x(0) = [-1,\ \ 1]^{\mathrm{T}}$ is given in Fig. 2.

Taking different τ_{2d} and h_{2d} ($h_{1d} = \tau_{1d} = h_1 = \tau_1 = 0, \rho = 0.5$), and from Theorem 1, we obtain maximum allowable τ_2 is shown in Table 7. From Table 7, if τ_{2d} increases the maximum allowable τ_2 decreases, if h_{2d} increases the maximum allowable τ_2 also decreases. Similarly, for various $\tau,$ the maximum allowable discrete delay h for different leakage delay τ obtained is summarized in Table 8. From Table 8, if leakage delay τ increases the maximum allowable discrete delay h decreases.

Example 3 Consider the following neural networks (NNs) with leakage term and discrete interval time-varying delays:

$$\dot{x}(t) = -Cx(t-\tau(t)) + Af(x(t)) + Bf(x(t-h(t))), \tag{33}$$

where

$$C=\begin{bmatrix}1.3 & 0\\ 0 & 0.9\end{bmatrix}, A=\begin{bmatrix}0.5 & -0.2\\ 0.7 & 0.5\end{bmatrix}, B=\begin{bmatrix}0.6 & -0.1\\ -1.2 & -0.8\end{bmatrix}, K_1=diag\{0,0\}, K_2=diag\{1,1\}.$$

Solution: With the condition $h(t)\in[h_2,h_1]$ and $\dot{h}(t)\in[h_{2d},h_{1d}]$ for the fast fast-varying delay case, maximum allowable h_2 for different h_1 and k_1 with $h_{1d}=-1.5$, $h_{2d}=1.5$, $\tau=0.1$, $K_2=diag\{1,1\}$, $\rho=0.5$, $K_1=diag\{k_1,k_1\}$, our results obtained by Theorem 1 to system (41) is shown in Table 9. From Table 9, if activation function lower bound k_1 increases the maximum allowable discrete delay MAUBs h_2 increases. If the lower bound discrete delay h_1 increases the maximum allowable discrete delay MAUBs h_2 also increases.

To confirm the obtained result, when discrete delay $h_2=0.5051$ sec and leakage delay $\tau=0.1$ sec, under the initial condition $x(0)=[-1,\ \ 1]^{\mathrm{T}}$ is given in Fig. 3.

With the condition $h(t)\in[h_2,h_1]$ and $\dot{h}(t)\in[h_{2d},h_{1d}]$ for the fast slow-varying delay case, maximum allowable τ for different h_1 and k_1 with $h_{1d}=-0.5$, $h_{2d}=0.5$, $\tau=0.1$, $\rho=0.5$, $K_2=diag\{1,1\}$, and $K_1=diag\{k_1,k_1\}$ is described in Table 10. From Table 10, if activation function lower bound k_1 increases the maximum allowable discrete delay MAUBs h_2 increases. If the lower bound discrete delay h_1 increases the maximum allowable discrete delay MAUBs h_2 decreases. However, from Table 10, one may observe that the MAUBs h_2 are same, when the change of k_1 is small.

For $h_{1d}=-h_{2d}$, maximum allowable h_2 for different τ with $h_1=0.1$, $K_1=diag\{0.5,0.5\}$, $K_2=diag\{1,1\}$, $\rho=0.5$, by using Theorem 1 is listed in Table 11. From Table 11, if leakage delay τ increases the maximum allowable discrete delay MAUBs h_2 decreases.

Let $h_1=0.1$, for given $h_{1d}=-1.5$, $h_{2d}=1.5$, maximum allowable h_2 for different k_1 with $h_1=0.1$, $h_{1d}=-1.5$, $h_{2d}=1.5$, $\tau=0.1$,, $K_2=diag\{1,1\}$, $K_1=diag\{k_1,k_1\}$, $\rho=0.1$, by using Corollary 1 is listed in Table 12. From Table 12, if activation function lower bound k_1 increases the maximum allowable discrete delay MAUBs h_2 decreases.

Let $h_1=1$, for given $h_{1d}=-1.5$, $h_{2d}=1.5$, maximum allowable h_2 for different k_1 with $h_1=1$, $\tau=0.1$,, $K_2=diag\{1,1\}$, $K_1=diag\{k_1,k_1\}$, $\rho=0.1$, by using Corollary 1 is listed in Table 13. From Table 13, if activation function lower bound k_1 increases the maximum allowable discrete delay MAUBs h_2 increases. However, from Table 13, one may observe that the MAUBs h_2 are same with different values of k_1 when $k_1 \le 0.4$, which shows the limited adaptive ranges of the development results in this paper. So there is still some room for us to develop and explore. In the future, we will do some further studies on this problem.

Example 4. Consider a delayed recurrent neural network with parameters as follows:

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t-h(t))), \tag{34}$$

where

$$C = \begin{bmatrix} 1.2769 & 0 & 0 & 0 \\ 0 & 0.6321 & 0 & 0 \\ 0 & 0 & 0.9230 & 0 \\ 0 & 0 & 0 & 0.4480 \end{bmatrix}, A = \begin{bmatrix} -0.0370 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix}, K = \text{diag}(0.1137, 0.1279, 0.7994, 0.2368).$$

Solution: For comparison with some existing stability criteria, let $\tau = 0, h_1 = 0, h_{1d} = 0$, the results obtained in [11, 15-22, 27, 28, 37, 38, 41-48, 50, 51]. In this paper, we have used the maximum possible number of decision variables in our LMIs. In Table 14, we also give a comparative result on the number of decision variables to be determined to obtain the MAUBs h_2 of the time-varying delays. From Table 14, the criterion of Corollary 1 reduces decision variables comparing to those of [17, 19, 20, 21, 38, 41, 44, 47]. Although the number of decision variables of Corollary 1 is bigger than those of [11, 15, 16, 22, 27, 28, 37, 42, 43, 45, 48, 51], the MAUBs h_2 provided by Corollary 1 are obviously larger than those reported in [11, 15, 16, 22, 27, 28, 37, 42, 43, 45, 48, 51]. Therefore, the proposed methods are superior to the existing ones [11, 15-22, 27, 28, 37, 38, 41-48, 50, 51]. Furthermore, taking different h_{2d} ($\tau = 0.8, h_1 = 0, h_{1d} = 0$), and from Corollary1, we obtain MAUBs h_2 is shown in Table 15. From Table 15, if τ_{2d} increases the maximum allowable h_2 decreases.

To confirm the obtained result, when discrete delay $h_2 = 2.9382 + 0.1\sin t$ sec and leakage delay $\tau = 0.8$ sec., under the initial condition $x(0) = [-1, \ 1, -1, 1]^{\mathrm{T}}$ is given in

Fig. 4. Moreover, for Comparison, let $\rho = 0.5, h_1 = 3,$ and $h_{1d} = \tau = 0$, for different h_{2d} the MAUBs h_2 is shown in Table 16. It can be seen from the Table 16 that our results are significantly better than those in [4, 18, 23, 34, 48].

5、Conclusion

This paper mainly discusses the stability of neural networks with discrete and leakage time-varying delay systems with delay-range-dependence and delay-derivative -dependence. The main results include the following four parts:

Firstly, through various inequality transformations based on interval matrix, using time-decomposition method, LMI is used to discuss and analyze the stability analysis of time-varying time-delay derivative of neural network with time-varying time-delay system. The Lyapunov function or functional of a neural network with leaky time-varying time delay system is constructed, and the selection of matrix parameters such as positive definite matrix and integral term coefficients in the Lyapunov function or functional is attributed to the solution of a set of LMI values to obtain The existence of Lyapunov function or functional of neural network with leaky time-varying time-delay system guarantees the relative stability of the time-delay interval of the control system. This makes the selection of the matrix parameters of the Lyapunov function or the functional no longer blind, thus greatly reducing the conservativeness of stable judgment. At the same time, the delay decomposition approach is proposed by the researchers. The selection of Lyapunov function matrix and the decomposition of intervals are all measures to generalize Lyapunov function, and the introduction of the concept of extended Lyapunov function method is to increase the degrees of freedom of the Lyapunov function. The calculation complexity (calculation load and variable quantity) is reduced, which is more convenient in practical applications such as system analysis and controller design.

Secondly, based on the differential system Lyapunov-Krasovskii stability theory combined with linear matrix inequalities, integral inequalities and matrix decomposition processing methods, we obtain sufficient conditions to ensure that the neural network with leakage time-varying time delay system is asymptotically stable and related to the time delay interval . At present, the treatment of discrete delays in literature methods requires that its derivative must be less than 1, namely, which to a certain extent makes the application of the obtained results have certain limitations.

This project breaks through the method proposed in the existing literature for the first time for a neural network with leaky time-varying time-delay system and proposes a stability analysis with both time-varying discrete time-delay derivatives and both upper and lower bounds can be measured simultaneously, which undoubtedly expands its application. In order to better solve the stability analysis problem of the time-delay control system, research and establish a less conservative stability criterion, for the mathematical model of the Markov jump saturation actuator system with time-varying time delay, by constructing an innovation The Lyapunov-Krasovskii functional, which combines the generalized convex set with integral inequality and other methods to estimate the upper bound of the derivative function of the Lyapunov-Krasovskii functional, effectively widens the scope of the conclusion.

Thirdly, in practical applications, due to the existence of parameter uncertainties, the stability study of the time-delay correlation of time-varying time-delay systems with time-varying time delay derivatives and neural network-like neural networks is more important. In the previous steps, respectively. Two methods for stability analysis of time-delay correlation of time-varying time-delay systems with time-varying time-delay derivatives and neural network-like neural networks are introduced: inequality method and endpoint matrix method. They have their own advantages and disadvantages. The advantages and disadvantages of the inequality method have been introduced above. For the endpoint matrix method, because of its calculation process, the endpoint matrix of the symmetric interval matrix is involved. According to the characteristics of the endpoint matrix, for the LMI, the calculation difficulty will increase correspondingly in terms of complexity. In order to reduce the above-mentioned disadvantages, in recent years, experts and scholars have obtained an equivalent description based on interval matrix. Through this equivalent description, the uncertainty system described by the interval matrix is transformed into a general deterministic system can greatly reduce the dimensions and computational complexity of linear matrix inequalities (LMI).

Finally, in control theory research and practice, system analysis software packages such as Matlab are more and more widely used, and the corresponding linear matrix inequality (LMI) is introduced in the research method of this project. Compared with the limitations of the general method, it makes it easy to solve some complex control problems, and the results obtained are less conservative than traditional norm estimation methods. In addition, the discussion of the system by each method proves the effectiveness of the proposed method through case analysis. Due to the introduction of a new Lyapunov-Krasovskii (LK) energy function with a smaller integral term, the number of matrix variables that appear in the proposed LMI condition is small, thereby reducing the computational complexity (calculation load) of the upper or lower bound of the estimated delay with variable quantity.

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