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Advanced controller design for uncertain linear systems with time-varying delays via augmented zero equality approach

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Summary

This paper deals with the stability analysis and controller design for linear systems with time-varying delays and parameter uncertainties. By choosing appropriate augmented Lyapunov-Krasovskii functionals, a set of Linear Matrix inequalities is derived to get advanced feasible region of stability, and controller gain matrices which guarantee the asymptotic stability of the concerned systems within maximum bound of time-delays and its time-derivative. To further reduce the conservatism of stabilization criterion a recently developed mathematical technique which constructed a new augmented zero equality is applied. Finally, two numerical examples are utilized to show the validity and superiority of the proposed methods.

KEYWORDS:
Time-varying delay, Linear systems, LMIs, Uncertainty, Controller design

1 | INTRODUCTION

In practical engineering fields, like networking, robotics, chemical, and mechanic, time-delays are unavoidable phenomena which occur in various systems.[1,2] Sometimes these delays lead to the system instability, poor performances, and aging.[3,4] Time-delays can be found in system states or inputs. The former one is found in dynamic operation systems and chemical mechanic chamber due to external disturbances like friction and temperature. And the latter is found in network communication systems, heavy equipment, and sensing feedback environments due to internal causes by hardware limitation and synchronization problem.[5,6] Therefore, stability analysis for dynamic systems with time-delays has been one of the hot issues in a control engineering field. Furthermore, unlike systems without time-delays, a controller design for systems with time-delays has an important meaning.[7,8,9,10]

Time-delays have infinite dimensional states. So, the analytic methods can't be found in frequency domains. Also, characteristic function of time-delay systems, which has infinite poles, has difficulty in finding poles by analytic methods.[11] Unlike limitation in frequency analysis, Lyapunov-Krasovskii functionals (LKFs) based stability analysis has the advantages of that can analyze the stability of time-delay systems.[12] Therefore, selecting appropriate LKFs is an important work for getting improved results. After integral terms are introduced for expressing LKFs, deriving the Linear Matrix inequalities (LMIs) condition for asymptotical stability of systems becomes complicated. By that reasons, some mathematical techniques are developed for relieving conservative conditions of stability.[13] Jensen's inequality[14], Wirtinger-based integral inequality (WBII)[15] techniques are used for handling integral terms in LKFs with less conservatism. And some other techniques like free-weighting matrix methods[16], delay-partitioning approach[11], and convex combination approach[17] helped to find tight upper bound for time-derivative of LKFs.

The above mentioned techniques proved its effectiveness not only in time-delay linear systems, but also in uncertain systems and nonlinear systems.[18,19,20] It should be noted that disturbance or parameter uncertainties are essential elements in many systems.[21] So, finding advanced stability and stabilization criteria of the linear system with time-varying delays and parameter uncertainties are worthy of investigation by many researchers for a long time. In this paper, to get superior results comparing with the previous ones, the following three techniques are utilized.

- *Auxiliary function-based integral inequalities*
- *Extended Reciprocally convex approach*
- *Augmented zero equality approach*

First one is the Auxiliary function-based integral inequalities (AFII)[22], which is developed from the Jensen's inequality and Wirtinger-based integral inequality[14,15]. This method is utilized for getting more tight lower bounds and handling single integral terms from time-derivative of LKFs. And then, the extended reciprocally convex inequality approach[23] (ERCA) is used for treating inverses convex parameters, and getting less conservatism. ERCA was developed from the reciprocally convex approach[24] by calculating more decision variables. The last one is the Augmented zero equality approach[25] (AZEA). AZEA, the applicated form of zero equality[26] and finsler's lemma[27], gives effective calculating costs by removing decision variables. It is confirmed that AZEA is a notable tool to get a high guaranteed delay bound in the systems with time-varying delays. So, it becomes an inevitable choice to use above whole three techniques to get the better results.

By combining above three main techniques, in this paper, the problems of stability criteria and designing controller for uncertain systems with time-varying delays are investigated. Theorem 1 is introduced for getting advanced feasible region of stability by using some Lemmas and constructing an appropriate LKFs. As results of solving the stability problem, improved maximum delay bounds h which guarantee the asymptotic stability of concerned systems are obtained. Corollary 1 focuses on controller design with constructing the same LKFs in Theorem 1. Based on Theorem 1, the AZEA method for reducing computational burden and getting more improved maximum delay bounds is utilized in Theorem 2. In Corollary 2, utilizing AZEA method for stabilization problem of uncertain systems is introduced. By applying AZEA for controller design of uncertain linear systems, which have not been discussed in any other literatures, Finsler's lemma for deriving enhanced stabilization conditions can be applied. As a result, an advanced asymptotic stabilization region and controller gain matrices can be obtained. Through two numerical examples, the superiority and effectiveness of above idea will be shown. Notation : $\mathbf{R}^n$ and $\mathbf{R}^{m\times n}$ are the n-dimensional Euclidean space and the set of all $m\times n$ real matrix, respectively. $\mathbf{S}^n$ and $\mathbf{S}^n_+$ denote the sets of symmetric and positive definite $n\times n$ matrices. $X > 0$ means that X is the positive definite matrix. $diag\{\cdots\}$ is the block diagonal matrix. I_n and 0_n are $n\times n$ sizes identity and zero matrices, respectively. $0_{m\times n}$ is $m\times n$ zeros matrices. $col\{\cdots\}$ is the column matrices. $Sym\{X\}$ denotes $X + X^T$. $M^\perp \in \mathbf{R}^{n\times(n-r)}$ is the null matrix of the $M \in \mathbf{R}^{m\times n}$ with rank r; e.g., $MM^\perp = 0$. $X_{[\alpha(t)]}$ means the value of function X is dependent on the scalar function $\alpha(t)$. The symmetric terms will be denoted by $*$ when necessary.

2 | PROBLEM STATEMENT AND PRELIMINARIES

Consider the following linear system with parameter uncertainties and time-varying delays:

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t - h(t)) + (B + \Delta B(t))u(t), \\ x(s) &= \phi(s),\ s \in [-h,\ 0], \end{aligned} \tag{1}$$

where $\phi(t)$ is an initial function, $x(t) \in R^n$ is the state vector, $A \in R^{n\times n},\ A_d \in R^{n\times n},\ B \in R^{n\times m}$ are known system matrices, and $u(t) \in R^m$ is control input. Uncertainties $\Delta A(t),\ \Delta A_d(t)$ and $\Delta B(t)$ are of the form

$$\left[\Delta A(t)\ \Delta A_d(t)\ \Delta B(t)\right] = DF(t)\left[E_s\ E_d\ E_u\right],$$

where $D \in \mathbf{R}^{n\times l},\ E_s,\ E_d \in \mathbf{R}^{k\times n},\ E_u \in \mathbf{R}^{k\times m}$ are known constant matrices and $F(t) \in R^{l\times k}$, which satisfies $F^T(t)F(t) \le I_k$, is nonlinear time-varying function. The delay $h(t)$ is a time-varying continuous function satisfying $0 \le h(t) \le h$ and $\dot{h}(t) \le \mu$, where h and μ are known constant values.

The purpose of this paper is to find delay-dependent criteria which guarantee the asymptotic stability of system (1) under $u(t) = 0_{m\times 1}$ and stabilization criteria of system (1) under $u(t) = Kx(t)$. The controller gain K will be obtained and system under

$u(t) = Kx(t)$ can be described as

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A(t) + BK + \Delta B(t)K)\, x(t) + (A_d + \Delta A_d(t))x(t - h(t)), \\ x(s) &= \phi(s),\ s \in [-h,\ 0]. \end{aligned} \tag{2}$$

The following lemmas are introduced for deriving our main results.

Lemma 1: For a given positive-definite matrix $R > 0$ and an integral function $w(u) \in [a, b]$, the following inequality holds:[22]

$$\int_a^b w^T(s)Rw(s)ds \geq \frac{1}{b-a}\left(\int_a^b w(s)ds\right)^T R\left(\int_a^b w(s)ds\right) + \frac{3}{b-a}\gamma_1 R\gamma_1 + \frac{5}{b-a}\gamma_2 R\gamma_2,$$

where $\gamma_1 = \int_a^b w(s)ds - \frac{2}{b-a}\int_a^b \int_s^b w(u)duds$, and $\gamma_2 = \int_a^b w(s)ds - \frac{6}{b-a}\int_a^b \int_s^b w(u)duds + \frac{12}{(b-a)^2}\int_a^b \int_s^b \int_u^b w(v)dvduds$.

Lemma 2: For a scalar α $(0 < \alpha < 1)$, symmetric matrices $M_1,\ M_2 \in \mathbf{S}_+^n$, if there exists symmetric matrices $S_1,\ S_2 \in \mathbf{S}^n$ and any matrices $F_1,\ F_2 \in \mathbf{R}^{n\times n}$ such that $\begin{bmatrix} M_1 - \alpha S_1 & -\alpha F_1 - (1-\alpha)F_2 \\ * & M_2 - (1-\alpha)S_2 \end{bmatrix} \geq 0$ for $\alpha = 0, 1$, then for all $\alpha \in (0,\ 1)$ the following inequality holds:[23]

$$\begin{bmatrix} \frac{1}{\alpha}M_1 & 0 \\ 0 & \frac{1}{1-\alpha}M_2 \end{bmatrix} \geq \begin{bmatrix} M_1 + (1-\alpha)S_1 & \alpha F_1 + (1-\alpha)F_2 \\ * & M_2 + \alpha S_2 \end{bmatrix}.$$

Lemma 3: Let $D \in \mathbf{R}^{n\times k},\ E \in \mathbf{R}^{k\times n}$ and $F(t) \in \mathbf{R}^{k\times k}$ be real matrices, and assume that $F(t)$ satisfies $F(t)^T F(t) \leq I_k$. Then, for any diagonal matrix $\Theta \in \mathbf{R}^{k\times k} > 0$, the following matrix inequality holds:[28]

$$DF(t)E + E^T F^T(t)D^T \leq E^T \Theta E + D\Theta^{-1}D^T.$$

Lemma 4: Let $\zeta \in \mathbf{R}^n,\ \Phi = \Phi^T \in \mathbf{R}^{n\times n}$, and $B \in \mathbf{R}^{m\times n}$ such that $rank(B) < n$. The following statements are equivalent:[27]

$$\begin{aligned} &\text{(i)}\ \ \zeta^T\,\Phi\,\zeta\ < 0,\ \ \forall B\zeta = 0,\ \zeta \neq 0, \\ &\text{(ii)}\ \ \exists L\ \in\ R^{n\times m}\ :\ \Phi\ + LB\ + B^T L^T\ <\ 0, \\ &\text{(iii)}\ \ B^{\perp T}\,\Phi\,B^{\perp}\ < 0. \end{aligned}$$

3 | MAIN RESULTS

In this section, improved stability and stabilization criteria for the system (1) are derived based on the Lyapunov-Krasovskii method. The following Lyapunov-Krasovskii functional is used in Theorems 1, 2, and Corollaries 1, 2:

$$V = \sum_{i=1}^{4} V_i, \tag{3}$$

where

$$V_1 = \begin{bmatrix} x(t) \\ x(t-h) \\ \eta_1(t) \end{bmatrix}^T R \begin{bmatrix} x(t) \\ x(t-h) \\ \eta_1(t) \end{bmatrix},$$

$$V_2 = \int_{t-h}^{t} \begin{bmatrix} \dot{x}(s) \\ x(s) \\ \eta_2(t,s) \end{bmatrix}^T N \begin{bmatrix} \dot{x}(s) \\ x(s) \\ \eta_2(t,s) \end{bmatrix} ds,$$

$$V_3 = \int_{t-h(t)}^{t} \begin{bmatrix} x(s) \\ \eta_2(t,s) \end{bmatrix}^T G \begin{bmatrix} x(s) \\ \eta_2(t,s) \end{bmatrix} ds,$$

$$V_4 = h \int_{t-h}^{t} \int_{s}^{t} \begin{bmatrix} \dot{x}(u) \\ x(u) \end{bmatrix}^T Q \begin{bmatrix} \dot{x}(u) \\ x(u) \end{bmatrix} du ds,$$

where R, N, G, and Q are positive definite matrices, $\eta_1(t)$, $\eta_2(t,s)$ are defined as

$$\eta_1(t) = col\left\{ \int_{t-h}^{t} x(s)ds, \int_{t-h}^{t}\int_{s}^{t} x(u)duds, \int_{t-h}^{t}\int_{t-h}^{s} x(u)duds \right\},$$

$$\eta_2(t,s) = col\left\{ \int_{s}^{t} \dot{x}(u)du, \int_{s}^{t} x(u)du, \int_{t-h}^{s} \dot{x}(u)du, \int_{t-h}^{s} x(u)du \right\}.$$

For simplicity of expression, the following notations of several matrices are defined as:

$$\zeta(t) = col\left\{ \begin{bmatrix} x(t) \\ \hline x(t-h(t)) \\ \hline x(t-h) \\ \hline \dot{x}(t) \\ \hline \dot{x}(t-h) \\ \hline \int_{t-h(t)}^{t} x(s)ds \\ \hline \int_{t-h}^{t-h(t)} x(s)ds \end{bmatrix}, \begin{bmatrix} \frac{1}{h(t)}\int_{t-h(t)}^{t}\int_{s}^{t} x(u)duds \\ \hline \frac{1}{h-h(t)}\int_{t-h}^{t-h(t)}\int_{s}^{t-h(t)} x(u)duds \\ \hline \frac{1}{h(t)^2}\int_{t-h(t)}^{t}\int_{s}^{t}\int_{u}^{t} x(v)dvduds \\ \hline \frac{1}{(h-h(t))^2}\int_{t-h}^{t-h(t)}\int_{s}^{t-h(t)}\int_{u}^{t-h(t)} x(v)dvduds \\ \hline \frac{1}{h(t)}\int_{t-h(t)}^{t} x(s)ds \\ \hline \frac{1}{h-h(t)}\int_{t-h}^{t-h(t)} x(s)ds \\ \hline \frac{1}{h(t)^2}\int_{t-h(t)}^{t}\int_{s}^{t} x(u)duds \\ \hline \frac{1}{(h-h(t))^2}\int_{t-h}^{t-h(t)}\int_{s}^{t-h(t)} x(u)duds \end{bmatrix}, \begin{bmatrix} \int_{t-h(t)}^{t}\int_{s}^{t} x(u)duds \\ \hline \int_{t-h}^{t-h(t)}\int_{s}^{t-h(t)} x(u)duds \end{bmatrix} \right\},$$

$$\begin{aligned}
Q_{aug1} &= Q + \begin{bmatrix} 0_n & P_1 \\ * & 0_n \end{bmatrix}, \quad Q_{aug2} = Q + \begin{bmatrix} 0_n & P_2 \\ * & 0_n \end{bmatrix}, \quad P = diag\left\{P_1,\ P_2 - P_1,\ -P_2\right\}, \\
\Omega_1 &= diag\left\{Q_{aug1},\ 3Q_{aug1},\ 5Q_{aug1}\right\},\ \Omega_2 = diag\left\{Q_{aug2},\ 3Q_{aug2},\ 5Q_{aug2}\right\}, \\
\Omega_{[h(t)]} &= \begin{bmatrix} \Omega_1 + (1-\frac{h(t)}{h})S_1 & \frac{h(t)}{h}F_1 + (1-\frac{h(t)}{h})F_2 \\ * & \Omega_2 + \frac{h(t)}{h}S_2 \end{bmatrix}, \\
e_i &= [0_{(i-1)n\times n},\ I_n,\ 0_{(17-i)n\times n}]\ (i = 1,2,\dots,17),\ e_0 = 0_{17n\times n},\ E = [e_1,\ e_2,\ \dots,\ e_{17}], \\
\Lambda_1 &= \left[e_1 - e_2,\ e_6,\ -e_1 - e_2 + 2e_{12},\ e_6 - 2e_8,\ e_1 - e_2 + 6e_{12} - 12e_{14},\ e_6 - 6e_8 + 12e_{10}\right], \\
\Lambda_2 &= \left[e_2 - e_3,\ e_7,\ -e_2 - e_3 + 2e_{13},\ e_7 - 2e_9,\ e_2 - e_3 + 6e_{13} - 12e_{15},\ e_7 - 6e_9 + 12e_{11}\right], \\
\Pi_{11[h(t)]} &= \left[e_1,\ e_3,\ e_6 + e_7,\ e_{16} + e_{17} + (h - h(t))e_6,\ h(t)e_6 + he_7 - e_{16} - e_{17}\right], \\
\Pi_{12} &= \left[e_4,\ e_5,\ e_1 - e_3,\ he_1 - (e_6 + e_7),\ (e_6 + e_7) - he_3\right], \\
\Pi_{21} &= \left[e_4,\ e_1,\ e_0,\ e_0,\ e_1 - e_3,\ e_6 + e_7\right],\ \Pi_{22} = \left[e_5,\ e_3,\ e_1 - e_3,\ e_6 + e_7,\ e_0,\ e_0\right], \\
\Pi_{23[h(t)]} &= \left[e_1 - e_3,\ e_6 + e_7,\ he_1 - (e_6 + e_7),\ e_{16} + e_{17} + (h - h(t))e_6,\ (e_6 + e_7) - he_3,\ h(t)e_6 + he_7 - e_{16} - e_{17}\right], \\
\Pi_{24} &= \left[e_0,\ e_0,\ e_4,\ e_1,\ -e_5,\ -e_3\right], \\
\Pi_{31} &= \left[e_1,\ e_0,\ e_0,\ e_1 - e_3,\ e_6 + e_7\right],\ \Pi_{32} = \left[e_2,\ e_1 - e_2,\ e_6,\ e_2 - e_3,\ e_7\right],
\end{aligned}$$

$$
\begin{aligned}
\Pi_{33[h(t)]} &= \left[e_6,\ h(t)e_1 - e_6,\ e_{16},\ e_6 - h(t)e_3,\ -e_{16} + h(t)(e_6 + e_7)\right], \Pi_{34} = \left[e_0,\ e_4,\ e_1,\ -e_5,\ -e_3\right],\\
\Pi_{41} &= \left[e_4,\ e_1\right],\ \Pi_{42} = \left[e_1,\ e_2,\ e_3\right],\ \Pi_{43} = \left[\Lambda_1,\ \Lambda_2\right],\\
\Xi_{1[h(t)]} &= Sym\left\{\Pi_{11[h(t)]} R \Pi_{12}^T\right\},\\
\Xi_{2[h(t)]} &= \Pi_{21} N \Pi_{21}^T - \Pi_{22} N \Pi_{22}^T + Sym\left\{\Pi_{23[h(t)]} N \Pi_{24}^T\right\},\\
\Xi_{3[h(t)]} &= \Pi_{31} G \Pi_{31}^T - (1-\mu)\Pi_{32} G \Pi_{32}^T + Sym\left\{\Pi_{33[h(t)]} G \Pi_{34}^T\right\},\\
\Xi_{41} &= h\Pi_{42} P \Pi_{42}^T, \quad \Xi_{42[h(t)]} = -\Pi_{43}\Omega_{[h(t)]}\Pi_{43}^T,\\
\Xi_{4[h(t)]} &= h^2 \Pi_{41} Q \Pi_{41}^T + \Xi_{41} + \Xi_{42[h(t)]},\\
\Xi_{5[h(t)]} &= Sym\left\{[h(t)e_{12} - e_6]\Psi_1 E^T + [(h - h(t))e_{13} - e_7]\Psi_2 E^T\right.\\
&\quad + [h(t)e_{14} - e_8]\Psi_3 E^T + [(h - h(t))e_{15} - e_9]\Psi_4 E^T\\
&\quad \left. + [h(t)e_8 - e_{16}]\Psi_5 E^T + [(h - h(t))e_9 - e_{17}]\Psi_6 E^T\right\},\\
\Xi_6 &= Sym\left\{\left[e_1 X_1 + e_2 X_2 + e_4 X_3\right]\left[-e_4^T + A e_1^T + A_d e_2^T\right]\right\},\\
\Xi_7 &= \left[e_1,\ e_2,\ e_4\right]\left[E_s,\ E_d,\ 0_n\right]^T \Theta \left[E_s,\ E_d,\ 0_n\right]\left[e_1,\ e_2,\ e_4\right]^T,\\
\Xi_{[h(t)]} &= \sum_{i=1}^{5} \Xi_{i[h(t)]} + \Xi_6 + \Xi_7,\\
\Xi_8 &= \left[e_1,\ e_2,\ e_4\right]\begin{bmatrix} X_1 D\\ X_2 D\\ X_3 D\end{bmatrix}\Theta^{-1}\begin{bmatrix} X_1 D\\ X_2 D\\ X_3 D\end{bmatrix}^T \left[e_1,\ e_2,\ e_4\right]^T.
\end{aligned}
\tag{4}
$$

Then, the following theorem for finding asymptotically stable region when the system (1) under $u(t) = 0_{m\times 1}$ is given as a main result.

Theorem 1. For given positive scalars h and μ, system (1) under $u(t) = 0_{m\times 1}$ is asymptotically stable for $0 \le h(t) \le h$ and $\dot{h}(t) \le \mu$, if there exist positive-definite matrices $R \in \mathbf{S}_+^{5n}$, $N \in \mathbf{S}_+^{6n}$, $G \in \mathbf{S}_+^{5n}$, $Q \in \mathbf{S}_+^{2n}$, positive-definite diagonal matrix $\Theta \in \mathbf{S}_+^{n}$, symmetric matrices $P_i \in \mathbf{S}^n (i = 1,\ 2)$, $S_i \in \mathbf{S}^{6n}(i{=}1, 2)$, any matrices $X_i \in \mathbf{R}^{n\times n}(i{=}1, 2, 3)$, $\Psi_i \in \mathbf{R}^{n\ \times 17n}(i{=}1, 2, 3, 4, 5, 6)$, $F_i \in \mathbf{R}^{6n\ \times 6n}(i{=}1, 2)$ satisfying the following LMIs:

$$
\begin{bmatrix}
\Xi_{[h(t)=0]} & \left[e_1,\ e_2,\ e_4\right]\begin{bmatrix} X_1 D\\ X_2 D\\ X_3 D\end{bmatrix}\\
\left[D^T X_1^T,\ D^T X_2^T,\ D^T X_3^T\right]\begin{bmatrix} e_1^T\\ e_2^T\\ e_4^T\end{bmatrix} & -\Theta
\end{bmatrix} < 0,
\tag{5}
$$

$$
\begin{bmatrix}
\Xi_{[h(t)=h]} & \left[e_1,\ e_2,\ e_4\right]\begin{bmatrix} X_1 D\\ X_2 D\\ X_3 D\end{bmatrix}\\
\left[D^T X_1^T,\ D^T X_2^T,\ D^T X_3^T\right]\begin{bmatrix} e_1^T\\ e_2^T\\ e_4^T\end{bmatrix} & -\Theta
\end{bmatrix} < 0,
\tag{6}
$$

$$
\begin{bmatrix} \Omega_1 & -F_2\\ * & \Omega_2 - S_2 \end{bmatrix} \ge 0,
\tag{7}
$$

$$
\begin{bmatrix} \Omega_1 - S_1 & -F_1\\ * & \Omega_2 \end{bmatrix} \ge 0.
\tag{8}
$$

Proof. Let us consider Lyapunov-Krasovskii functional (3). The $\dot{V}_i$ (i=1,2,3,4) can be expressed as

$$\dot{V}_1 = 2\begin{bmatrix} x(t) \\ x(t-h) \\ \eta_1(t) \end{bmatrix}^T R \begin{bmatrix} \dot{x}(t) \\ \hline \dot{x}(t-h) \\ \hline x(t)-x(t-h) \\ \hline hx(t)-\int_{t-h}^{t} x(s)ds \\ \hline \int_{t-h}^{t} x(s)ds - hx(t-h) \end{bmatrix} = \zeta^T(t)\Xi_{1[h(t)]}\zeta(t), \tag{9}$$

$$\begin{aligned}
\dot{V}_2 &= \frac{d}{dt}\left(\int_{t-h}^{t} \begin{bmatrix} \dot{x}(s) \\ x(s) \\ \eta_2(t,s) \end{bmatrix}^T N \begin{bmatrix} \dot{x}(s) \\ x(s) \\ \eta_2(t,s) \end{bmatrix} ds\right) \\
&= \begin{bmatrix} \dot{x}(t) \\ x(t) \\ \eta_2(t,t) \end{bmatrix}^T N \begin{bmatrix} \dot{x}(t) \\ x(t) \\ \eta_2(t,t) \end{bmatrix} - \begin{bmatrix} \dot{x}(t-h) \\ x(t-h) \\ \eta_2(t,t-h) \end{bmatrix}^T N \begin{bmatrix} \dot{x}(t-h) \\ x(t-h) \\ \eta_2(t,t-h) \end{bmatrix} + \int_{t-h}^{t} \frac{d}{dt}\left(\begin{bmatrix} \dot{x}(s) \\ x(s) \\ \eta_2(t,s) \end{bmatrix}^T N \begin{bmatrix} \dot{x}(s) \\ x(s) \\ \eta_2(t,s) \end{bmatrix}\right) ds \\
&= \begin{bmatrix} \dot{x}(t) \\ x(t) \\ 0_{n\times 1} \\ 0_{n\times 1} \\ \int_{t-h}^{t} \dot{x}(s)ds \\ \int_{t-h}^{t} x(s)ds \end{bmatrix}^T N \begin{bmatrix} \dot{x}(t) \\ x(t) \\ 0_{n\times 1} \\ 0_{n\times 1} \\ \int_{t-h}^{t} \dot{x}(s)ds \\ \int_{t-h}^{t} x(s)ds \end{bmatrix} - \begin{bmatrix} \dot{x}(t-h) \\ x(t-h) \\ \int_{t-h}^{t} \dot{x}(s)ds \\ \int_{t-h}^{t} x(s)ds \\ 0_{n\times 1} \\ 0_{n\times 1} \end{bmatrix}^T N \begin{bmatrix} \dot{x}(t-h) \\ x(t-h) \\ \int_{t-h}^{t} \dot{x}(s)ds \\ \int_{t-h}^{t} x(s)ds \\ 0_{n\times 1} \\ 0_{n\times 1} \end{bmatrix} + 2\begin{bmatrix} x(t)-x(t-h) \\ \int_{t-h}^{t} x(s)ds \\ hx(t)-\int_{t-h}^{t} x(s)ds \\ \int_{t-h}^{t}\int_{s}^{t} x(u)duds \\ \int_{t-h}^{t} x(s)ds - hx(t-h) \\ \int_{t-h}^{t}\int_{t-h}^{s} x(u)duds \end{bmatrix}^T N \\
&\quad \times \begin{bmatrix} 0_{n\times 1} \\ 0_{n\times 1} \\ \dot{x}(t) \\ x(t) \\ -\dot{x}(t-h) \\ -x(t-h) \end{bmatrix} = \zeta^T(t)\Xi_{2[h(t)]}\zeta(t),
\end{aligned} \tag{10}$$

$$\begin{aligned}
\dot{V}_3 &= \frac{d}{dt}\left(\int_{t-h(t)}^{t} \begin{bmatrix} x(s) \\ \eta_2(t,s) \end{bmatrix}^T G \begin{bmatrix} x(s) \\ \eta_2(t,s) \end{bmatrix} ds\right) \\
&= \begin{bmatrix} x(t) \\ 0_{n\times 1} \\ 0_{n\times 1} \\ \int_{t-h}^{t} \dot{x}(s)ds \\ \int_{t-h}^{t} x(s)ds \end{bmatrix}^T G \begin{bmatrix} x(t) \\ 0_{n\times 1} \\ 0_{n\times 1} \\ \int_{t-h}^{t} \dot{x}(s)ds \\ \int_{t-h}^{t} x(s)ds \end{bmatrix} - \left(1-\dot{h}(t)\right) \begin{bmatrix} x(t-h(t)) \\ \int_{t-h(t)}^{t} \dot{x}(s)ds \\ \int_{t-h(t)}^{t} x(s)ds \\ \int_{t-h}^{t-h(t)} \dot{x}(s)ds \\ \int_{t-h}^{t-h(t)} x(s)ds \end{bmatrix}^T G \begin{bmatrix} x(t-h(t)) \\ \int_{t-h(t)}^{t} \dot{x}(s)ds \\ \int_{t-h(t)}^{t} x(s)ds \\ \int_{t-h}^{t-h(t)} \dot{x}(s)ds \\ \int_{t-h}^{t-h(t)} x(s)ds \end{bmatrix} \\
&\quad + 2\int_{t-h(t)}^{t} \begin{bmatrix} x(s) \\ \eta_2(t,s) \end{bmatrix}^T G \begin{bmatrix} 0_{n\times 1} \\ \dot{x}(t) \\ x(t) \\ -\dot{x}(t-h) \\ -x(t-h) \end{bmatrix} ds \le \zeta^T(t)\Xi_{3[h(t)]}\zeta(t),
\end{aligned} \tag{11}$$

$$\begin{aligned}
\dot{V}_4 &= \frac{d}{dt}\left(h\int_{t-h}^{t}\int_{s}^{t} \begin{bmatrix} \dot{x}(u) \\ x(u) \end{bmatrix}^T Q \begin{bmatrix} \dot{x}(u) \\ x(u) \end{bmatrix} duds\right) \\
&= h\int_{t-h}^{t} \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix}^T Q \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix} ds - h\int_{t-h}^{t} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}^T Q \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix} ds + h\int_{t-h}^{t}\int_{s}^{t} \frac{d}{dt}\left(\begin{bmatrix} \dot{x}(u) \\ x(u) \end{bmatrix}^T Q \begin{bmatrix} \dot{x}(u) \\ x(u) \end{bmatrix}\right) duds
\end{aligned}$$

$$= h^2 \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix}^T Q \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix} - h \int_{t-h}^{t} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}^T Q \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix} ds. \tag{12}$$

By the way, the integral term which is composed in (12) can be divided into integral interval from t to $t - h(t)$ and from $t - h(t)$ to $t - h$ as

$$- h \int_{t-h}^{t} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}^T Q \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix} ds = -h \int_{t-h(t)}^{t} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}^T Q \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix} ds - h \int_{t-h}^{t-h(t)} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}^T Q \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix} ds. \tag{13}$$

In this progress, zero equations are introduced with symmetric matrices $P_1,\ P_2$ as

$$0 = h\left[x^T(t)P_1x(t) - x^T(t-h(t))P_1x(t-h(t)) - 2\int_{t-h(t)}^{t} \dot{x}^T(s)P_1x(s)ds \right], \tag{14}$$

$$0 = h\left[x^T(t-h(t))P_2x(t-h(t)) - x^T(t-h)P_2x(t-h) - 2\int_{t-h}^{t-h(t)} \dot{x}^T(s)P_2x(s)ds \right]. \tag{15}$$

By adding zero equations (14) and (15) into the integral terms (13), respectively, the following equations can be obtained as

$$-h \int_{t-h(t)}^{t} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}^T Q \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix} ds = -h \int_{t-h(t)}^{t} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}^T \underbrace{\left(Q + \begin{bmatrix} 0_n & P_1 \\ * & 0_n \end{bmatrix} \right)}_{Q_{aug1}} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix} ds + h \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} P_1 & 0_n \\ * & -P_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}, \tag{16}$$

$$-h \int_{t-h}^{t-h(t)} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}^T Q \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix} ds = -h \int_{t-h}^{t-h(t)} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}^T \underbrace{\left(Q + \begin{bmatrix} 0_n & P_2 \\ * & 0_n \end{bmatrix} \right)}_{Q_{aug2}} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix} ds + h \begin{bmatrix} x(t-h(t)) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} P_2 & 0_n \\ * & -P_2 \end{bmatrix} \begin{bmatrix} x(t-h(t)) \\ x(t-h) \end{bmatrix}. \tag{17}$$

The integral terms of Equation (16), (17) are bounded by Lemma 1 as follow:

$$- h \int_{t-h(t)}^{t} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}^T Q_{aug1} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix} ds \le -\frac{h}{h(t)} \Lambda_1(t)^T \Omega_1 \Lambda_1(t), \tag{18}$$

$$- h \int_{t-h}^{t-h(t)} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}^T Q_{aug2} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix} ds \le -\frac{h}{h - h(t)} \Lambda_2(t)^T \Omega_2 \Lambda_2(t), \tag{19}$$

where

$$\Lambda_1(t) = \left[\begin{array}{c} x(t) - x(t-h(t)) \\ \hline \int_{t-h(t)}^{t} x(s)ds \\ \hline -x(t) - x(t-h(t)) + \frac{2}{h(t)} \int_{t-h(t)}^{t} x(s)ds \\ \hline \int_{t-h(t)}^{t} x(s)ds - \frac{2}{h(t)} \int_{t-h(t)}^{t} \int_{s}^{t} x(u)duds \\ \hline x(t) - x(t-h(t)) + \frac{6}{h(t)} \int_{t-h(t)}^{t} x(s)ds - \frac{12}{(h(t))^2} \int_{t-h(t)}^{t} \int_{s}^{t} x(u)duds \\ \hline \int_{t-h(t)}^{t} x(s)ds + \frac{6}{h(t)} \int_{t-h(t)}^{t} \int_{s}^{t} x(u)duds - \frac{12}{(h(t))^2} \int_{t-h(t)}^{t} \int_{s}^{t} \int_{u}^{t} x(v)dvduds \end{array} \right],$$

$$
\Lambda_2(t) = \left[\begin{array}{c}
x(t-h(t)) - x(t-h) \\ \hline
\int_{t-h}^{t-h(t)} x(s)ds \\ \hline
-x(t-h(t)) - x(t-h) + \frac{2}{h-h(t)} \int_{t-h}^{t-h(t)} x(s)ds \\ \hline
\int_{t-h}^{t-h(t)} x(s)ds - \frac{2}{h-h(t)} \int_{t-h}^{t-h(t)} \int_{s}^{t-h(t)} x(u)duds \\ \hline
x(t-h(t)) - x(t-h) + \frac{6}{h-h(t)} \int_{t-h}^{t-h(t)} x(s)ds - \frac{12}{(h-h(t))^2} \int_{t-h}^{t-h(t)} \int_{s}^{t-h(t)} x(u)duds \\ \hline
\int_{t-h}^{t-h(t)} x(s)ds + \frac{6}{h-h(t)} \int_{t-h}^{t-h(t)} \int_{s}^{t-h(t)} x(u)duds - \frac{12}{(h-h(t))^2} \int_{t-h}^{t-h(t)} \int_{s}^{t-h(t)} \int_{u}^{t-h(t)} x(v)dvduds
\end{array}\right].
$$

As a result, by Lemma 2 if (7) and (8) holds, then the following inequality holds:

$$
\begin{aligned}
-h \int_{t-h}^{t} \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix}^T Q \begin{bmatrix} \dot{x}(s) \\ x(s) \end{bmatrix} ds \leq & \begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h) \end{bmatrix}^T \underbrace{\begin{bmatrix} P_1 & 0_n & 0_n \\ * & P_2 - P_1 & 0_n \\ * & * & -P_2 \end{bmatrix}}_{P} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h) \end{bmatrix} \\
& - \begin{bmatrix} \Lambda_1(t) \\ \Lambda_2(t) \end{bmatrix}^T \underbrace{\begin{bmatrix} \Omega_1 + (1 - \frac{h(t)}{h}) S_1 & \frac{h(t)}{h} F_1 + (1 - \frac{h(t)}{h} F_2) \\ * & \Omega_2 + \frac{h(t)}{h} S_2 \end{bmatrix}}_{\Omega_{[h(t)]}} \begin{bmatrix} \Lambda_1(t) \\ \Lambda_2(t) \end{bmatrix} \\
& = \zeta^T(t) \left(\Xi_{41} + \Xi_{42[h(t)]} \right) \zeta(t).
\end{aligned} \tag{20}
$$

Therefore, $\dot{V}_4$ can be bounded as

$$
\dot{V}_4 \leq \zeta^T(t) \underbrace{\left(h^2 \Pi_{41} Q \Pi_{41}^T + \Xi_{41} + \Xi_{42[h(t)]} \right)}_{\Xi_{4[h(t)]}} \zeta(t). \tag{21}
$$

By combining the augmented vector and free-weighting matrices $\Psi_i (i = 1, \dots, 6)$, zero equalities can be considered as

$$
\begin{aligned}
0 &= \zeta^T(t) \left(Sym \left\{ [h(t) e_{12} - e_6] \Psi_1 E^T \right\} \right) \zeta(t), && (22) \\
0 &= \zeta^T(t) \left(Sym \left\{ [(h - h(t)) e_{13} - e_7] \Psi_2 E^T \right\} \right) \zeta(t), && (23) \\
0 &= \zeta^T(t) \left(Sym \left\{ [h(t) e_{14} - e_8] \Psi_3 E^T \right\} \right) \zeta(t), && (24) \\
0 &= \zeta^T(t) \left(Sym \left\{ [(h - h(t)) e_{15} - e_9] \Psi_4 E^T \right\} \right) \zeta(t), && (25) \\
0 &= \zeta^T(t) \left(Sym \left\{ [h(t) e_8 - e_{16}] \Psi_5 E^T \right\} \right) \zeta(t), && (26) \\
0 &= \zeta^T(t) \left(Sym \left\{ [(h - h(t)) e_9 - e_{17}] \Psi_6 E^T \right\} \right) \zeta(t). && (27)
\end{aligned}
$$

Summing the above equalities leads to

$$
0 = \zeta^T(t) \Xi_{5[h(t)]} \zeta(t). \tag{28}
$$

From system (1) under $u(t) = 0_{m \times 1}$ for any free-weighting matrices $X_i (i = 1, 2, 3)$, the following zero equality holds:

$$
0 = 2 \left[x^T(t) X_1 + x^T(t - h(t)) X_2 + \dot{x}^T(t) X_3 \right] \left[-\dot{x}(t) + (A + \Delta A) x(t) + \left(A_d + \Delta A_d(t) \right) x(t - h(t)) \right].
$$

The zero equality with parameter uncertainties $\Delta A(t),\ \Delta A_d(t)$ is bounded as follows by Lemma 3

$$
\begin{aligned}
0 &= 2 \left[x^T(t) X_1 + x^T(t - h(t)) X_2 + \dot{x}^T(t) X_3 \right] \left[-\dot{x}(t) + (A + \Delta A) x(t) + \left(A_d + \Delta A_d(t) \right) x(t - h(t)) \right] \\
&\leq \underbrace{2 \left[x^T(t) X_1 + x^T(t - h(t)) X_2 + \dot{x}^T(t) X_3 \right] \left[-\dot{x}(t) + A x(t) + A_d x(t - h(t)) \right]}_{\zeta^T(t) \Xi_6 \zeta(t)}
\end{aligned}
$$

$$
+\underbrace{\begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} E_s^T \\ E_d^T \\ 0_n \end{bmatrix} \Theta \begin{bmatrix} E_s^T \\ E_d^T \\ 0_n \end{bmatrix}^T \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}}_{\zeta^T(t)\Xi_7\zeta(t)} + \underbrace{\begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} X_1 D \\ X_2 D \\ X_3 D \end{bmatrix} \Theta^{-1} \begin{bmatrix} X_1 D \\ X_2 D \\ X_3 D \end{bmatrix}^T \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}}_{\zeta^T(t)\Xi_8\zeta(t)}
$$

$$
= \zeta^T(t) \left(\Xi_6 + \Xi_7 + \Xi_8\right) \zeta(t). \tag{29}
$$

From (9) - (29), an upper bound of $\dot{V}$ is obtained as

$$
\dot{V} \leq \zeta^T(t) \left(\Xi_{[h(t)]} + \Xi_8\right) \zeta(t). \tag{30}
$$

Therefore, condition for asymptotic stability of system (1) is

$$
\Xi_{[h(t)]} + \left[e_1,\ e_2,\ e_4\right] \begin{bmatrix} X_1 D \\ X_2 D \\ X_3 D \end{bmatrix} \Theta^{-1} \begin{bmatrix} X_1 D \\ X_2 D \\ X_3 D \end{bmatrix}^T \left[e_1,\ e_2,\ e_4\right]^T < 0. \tag{31}
$$

By Schur's complement[29], inequality (31) is equivalent to

$$
\begin{bmatrix} \Xi_{[h(t)]} & [e_1,\ e_2,\ e_4] \begin{bmatrix} X_1 D \\ X_2 D \\ X_3 D \end{bmatrix} \\ [D^T X_1^T,\ D^T X_2^T,\ D^T X_3^T] \begin{bmatrix} e_1^T \\ e_2^T \\ e_4^T \end{bmatrix} & -\Theta \end{bmatrix} < 0. \tag{32}
$$

The left side of inequality (32) is affinely dependent on $h(t)$. Therefore, if (5), (6) are satisfied with (7), (8), the system (1) under $u(t) = 0_{m\times 1}$ is asymptotically stable for $0 \leq h(t) \leq h$ and $\dot{h}(t) \leq \mu$. This completes the proof of Theorem 1. □

Remark 1: Note that the equations $\int_{t-h(t)}^{t} x(s)ds = e_6^T\zeta(t)$, $\frac{1}{h(t)}\int_{t-h(t)}^{t} x(s)ds = e_{12}^T\zeta(t)$ and these can have equality $0 = h(t)\frac{1}{h(t)}\int_{t-h(t)}^{t} x(s)ds - \int_{t-h(t)}^{t} x(s)ds$. Above methods with (22) - (27) can effect in getting more tighter bound by utilizing free-weighting matrices Ψ_i. Similary processes of $0 = (h(t)e_{12}^T - e_6^T)\zeta(t)$ are used in (23) - (27). Second, when we applied AFII[22], non-convex forms of $h(t)^2$ and $(h - h(t))^2$ are expressed. Constructing augmented vector $\zeta(t)$ which includes vector like $\frac{1}{h(t)^2}\int_{t-h(t)}^{t}\int_{s}^{t} x(u)duds$ and considering (28) help to eleminate the non-convex expression. Therefore, by utilizing above methods, the stability conditions can be obtained.

Based on the proposed method in Theorem 1, this can be applied in finding advanced stabilization criterion for closed-loop system (2) under $u(t) = Kx(t)$. For proving that case, the following notations and Corollary are introduced.

$$
\begin{aligned}
\chi_i &= diag\{\underbrace{X, \dots, X}_{=\,i\ elements}\}, \\
\tilde{R} &= \chi_5^T R \chi_5,\ \tilde{N} = \chi_6^T N \chi_6,\ \tilde{G} = \chi_5^T G \chi_5,\ \tilde{P} = \chi_3^T P \chi_3,\ \tilde{F}_i = \chi_6^T F_i \chi_6, \\
\tilde{S}_i &= \chi_6^T S_i \chi_6,\ \tilde{\Omega}_i = \chi_6^T \Omega_i \chi_6,\ \tilde{\Omega}_{[h(t)]} = \chi_{12}^T \Omega_{[h(t)]} \chi_{12},\ \tilde{\Psi}_i = \chi_1^T \Psi \chi_{17}, \\
\tilde{\Xi}_{1[h(t)]} &= Sym\left\{\Pi_{11[h(t)]}\tilde{R}\Pi_{12}^T\right\}, \\
\tilde{\Xi}_{2[h(t)]} &= \Pi_{21}\tilde{N}\Pi_{21}^T - \Pi_{22}\tilde{N}\Pi_{22}^T + Sym\left\{\Pi_{23[h(t)]}\tilde{N}\Pi_{24}^T\right\}, \\
\tilde{\Xi}_{3[h(t)]} &= \Pi_{31}\tilde{G}\Pi_{31}^T - (1-\mu)\Pi_{32}\tilde{G}\Pi_{32}^T + Sym\left\{\Pi_{33[h(t)]}\tilde{G}\Pi_{34}^T\right\},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Xi}_{41} &= h\Pi_{42}\tilde{P}\Pi_{42}^{T}, \\
\tilde{\Xi}_{42[h(t)]} &= -\Pi_{43}\tilde{\Omega}_{[h(t)]}\Pi_{43}^{T}, \\
\tilde{\Xi}_{4[h(t)]} &= h^{2}\Pi_{41}\tilde{Q}\Pi_{41}^{T} + \tilde{\Xi}_{41} + \tilde{\Xi}_{42[h(t)]}, \\
\tilde{\Xi}_{5[h(t)]} &= Sym\left\{[h(t)e_{12} - e_{6}]\tilde{\Psi}_{1}E^{T} + [(h-h(t))e_{13} - e_{7}]\tilde{\Psi}_{2}E^{T}\right. \\
&\quad + [h(t)e_{14} - e_{8}]\tilde{\Psi}_{3}E^{T} + [(h-h(t))e_{15} - e_{9}]\tilde{\Psi}_{4}E^{T} \\
&\quad \left. + [h(t)e_{8} - e_{16}]\tilde{\Psi}_{5}E^{T} + [(h-h(t))e_{9} - e_{17}]\tilde{\Psi}_{6}E^{T}\right\}, \\
\tilde{\Xi}_{6} &= Sym\left\{\left[e_{1} + \delta_{1}e_{2} + \delta_{2}e_{4}\right]\left[-Xe_{4}^{T} + (AX + BY)e_{1}^{T} + A_{d}Xe_{2}^{T}\right]\right\}, \\
\tilde{\Xi}_{7} &= \left[e_{1},\ e_{2},\ e_{4}\right]\left[D,\ \delta_{1}D,\ \delta_{2}D\right]^{T}\Theta\left[D,\ \delta_{1}D,\ \delta_{2}D\right]\left[e_{1},\ e_{2},\ e_{4}\right]^{T}, \\
\tilde{\Xi}_{[h(t)]} &= \sum_{i=1}^{5}\tilde{\Xi}_{i[h(t)]} + \tilde{\Xi}_{6} + \tilde{\Xi}_{7}.
\end{aligned}
\tag{33}
$$

Corollary 1: For given any scalars $\delta_1,\ \delta_2$, positive scalars h and μ, system (2) is asymptotically stable for $0 \le h(t) \le h$ and $\dot{h}(t) \le \mu$, if there exist positive-definite matrices $\tilde{R} \in \mathbf{S}_{+}^{5n}$, $\tilde{N} \in \mathbf{S}_{+}^{6n}$, $\tilde{G} \in \mathbf{S}_{+}^{5n}$, $\tilde{Q} \in \mathbf{S}_{+}^{2n}$ and positive-definite diagonal matrix $\Theta \in \mathbf{S}_{+}^{n}$, symmetric matrices $\tilde{P}_i \in \mathbf{S}^{n}(i = 1, 2)$, $\tilde{S}_i \in \mathbf{S}_{+}^{6n}(i = 1, 2)$, any matrices $X \in \mathbf{R}^{n\times n}$, $Y \in \mathbf{R}^{m\times n}$, $\tilde{\Psi}_i \in \mathbf{R}^{n\times 17n}(i = 1, \dots, 6)$, $\tilde{F}_i \in \mathbf{R}^{6n\times 6n}(i = 1, 2)$ satisfying the following LMIs:

$$
\begin{bmatrix}
\tilde{\Xi}_{[h(t)=0]} & [e_1,\ e_2,\ e_4]\begin{bmatrix} X^{T}E_{s}^{T} + Y^{T}E_{u}^{T} \\ X^{T}E_{d}^{T} \\ 0_n \end{bmatrix} \\
[E_sX + E_uY,\ E_dX,\ 0_n]\begin{bmatrix} e_1^T \\ e_2^T \\ e_4^T \end{bmatrix} & -\Theta
\end{bmatrix} < 0,
\tag{34}
$$

$$
\begin{bmatrix}
\tilde{\Xi}_{[h(t)=h]} & [e_1,\ e_2,\ e_4]\begin{bmatrix} X^{T}E_{s}^{T} + Y^{T}E_{u}^{T} \\ X^{T}E_{d}^{T} \\ 0_n \end{bmatrix} \\
[E_sX + E_uY,\ E_dX,\ 0_n]\begin{bmatrix} e_1^T \\ e_2^T \\ e_4^T \end{bmatrix} & -\Theta
\end{bmatrix} < 0,
\tag{35}
$$

$$
\begin{bmatrix} \tilde{\Omega}_1 & -\tilde{F}_2 \\ * & \tilde{\Omega}_2 - \tilde{S}_2 \end{bmatrix} \ge 0,
\tag{36}
$$

$$
\begin{bmatrix} \tilde{\Omega}_1 - \tilde{S}_1 & -\tilde{F}_1 \\ * & \tilde{\Omega}_2 \end{bmatrix} \ge 0.
\tag{37}
$$

Then, the controller gain K can be obtained as $K = YX^{-1}$.

Proof. From Equation (28) in Theorem 1, the following zero equality can be added to (28)

$$
\begin{aligned}
0 &= 2\left[x^{T}(t)X_1 + x^{T}(t-h(t))X_2 + \dot{x}^{T}(t)X_3\right]\left[-\dot{x}(t) + (A + \Delta A(t) + BK + \Delta B(t)K)\,x(t) + \left(A_d + \Delta A_d(t)\right)x(t-h(t))\right] \\
&\le \underbrace{2\left[x^{T}(t)X_1 + x^{T}(t-h(t))X_2 + \dot{x}^{T}(t)X_3\right]\left[\dot{x}(t) + (A + BK)x(t) + A_d x(t-h(t))\right]}_{\zeta^{T}(t)\Xi_{9}\zeta(t)} \\
&\quad + \underbrace{\begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}^{T}\begin{bmatrix} X_1D \\ X_2D \\ X_3D \end{bmatrix}\Theta\begin{bmatrix} X_1D \\ X_2D \\ X_3D \end{bmatrix}^{T}\begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}}_{\zeta^{T}(t)\Xi_{10}\zeta(t)} + \underbrace{\begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}^{T}\begin{bmatrix} E_s^T + K^TE_u^T \\ E_d^T \\ 0_n \end{bmatrix}\Theta^{-1}\begin{bmatrix} E_s^T + K^TE_u^T \\ E_d^T \\ 0_n \end{bmatrix}^{T}\begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(t) \end{bmatrix}}_{\zeta^{T}(t)\Xi_{11}\zeta(t)} \\
&= \zeta^{T}(t)\left(\Xi_9 + \Xi_{10} + \Xi_{11}\right)\zeta(t),
\end{aligned}
\tag{38}
$$

where

$$\begin{aligned}
\Xi_9 &= Sym\left\{\left[e_1X_1+e_2X_2+e_4X_3\right]\left[-e_4^T+(A+BK)e_1^T+A_de_2^T\right]\right\},\\
\Xi_{10} &= \left[e_1,\ e_2,\ e_4\right]\left[D^TX_1^T,\ D^TX_2^T,\ D^TX_3^T\right]^T\Theta\left[D^TX_1^T,\ D^TX_2^T,\ D^TX_3^T\right]\left[e_1,\ e_2,\ e_4\right]^T,\\
\Xi_{11} &= \left[e_1,\ e_2,\ e_4\right]\left[E_s+E_uK,\ E_d,\ 0_n\right]^T\Theta^{-1}\left[E_s+E_uK,\ E_d,\ 0_n\right]\left[e_1,\ e_2,\ e_4\right]^T.
\end{aligned} \tag{39}$$

Then, an upper bound of $\dot{V}$ can be obtained as

$$\dot{V} \le \zeta^T(t)\left(\Xi_{[h(t)]}-\Xi_6-\Xi_7+\Xi_9+\Xi_{10}+\Xi_{11}\right)\zeta(t). \tag{40}$$

If the above Equation (40) is negative definite, the system (2) is asymptotically stable. Thus the stability condition of system (2) is summarized as

$$\Xi_{[h(t)]}-\Xi_6-\Xi_7+\Xi_9+\Xi_{10}+\Xi_{11}<0. \tag{41}$$

Next, pre- and post- multiplying by χ_{17}^T and χ_{17} can be used in both sides of (41) with relations of $X=(X_1^{-1})^T$, $X_2=\delta_1X_1$, $X_3=\delta_2X_1$. Result of this process is expressed by the following inequality holds

$$\tilde{\Xi}_{[h(t)]}+\left[e_1,\ e_2,\ e_4\right]\left[E_sX+E_uY,\ E_dX,\ 0_n\right]^T\Theta^{-1}\left[E_sX+E_uY,\ E_dX,\ 0_n\right]\left[e_1,\ e_2,\ e_4\right]^T<0. \tag{42}$$

And the conditions (36) and (37) can be obtained by pre- and post- multiplying χ_{12}^T and χ_{12}^T to both sides (7) and (8), respectively.

By Schur's complement[29], inequality (42) is equivalent to

$$\begin{bmatrix}
\tilde{\Xi}_{[h(t)]} & [e_1,e_2,e_4]\begin{bmatrix}X^TE_s^T+Y^TE_u^T\\ X^TE_d^T\\ 0_n\end{bmatrix}\\
[E_sX+E_uY,E_dX,0_n]\begin{bmatrix}e_1^T\\ e_2^T\\ e_4^T\end{bmatrix} & -\Theta
\end{bmatrix}<0. \tag{43}$$

Since the left side of inequality (43) is affinely dependent on $h(t)$, inequalities (34) and (35) are satisfied. Therefore, if inequalities (34) and (35) are satisfied with (36), (37), the system (2) under $u(t)=YX^{-1}x(t)$ is asymptotically stable for $0\le h(t)\le h$ and $\dot{h}(t)\le\mu$. This completes our proof. □

In next theorem an important idea, which is based on Theorem 1 and Lemma 4, is utilized.

Theorem 2. For given positive scalars h and μ, system (1) under $u(t)=0_{m\times1}$ is asymptotically stable for $0\le h(t)\le h$ and $\dot{h}(t)\le\mu$, if there exist positive-definite matrices $R\in\mathbf{S}_+^{5n}$, $N\in\mathbf{S}_+^{6n}$, $G\in\mathbf{S}_+^{5n}$, $Q\in\mathbf{S}_+^{2n}$, positive-definite diagonal matrix $\Theta\in\mathbf{S}_+^n$, symmetric matrices $P_i\in\mathbf{S}^n(i=1,2)$, $S_i\in\mathbf{S}^{6n}(i=1,2)$, any matrices $X_i\in\mathbf{R}^{n\times n}(i=1,2,3)$, $F_i\in\mathbf{R}^{6n\times6n}(i=1,2)$ satisfying the following LMIs with (7) and (8):

$$\begin{bmatrix} (\Gamma^{\perp}_{[h(t)=0]})^T \Phi_{[h(t)=0]} \Gamma^{\perp}_{[h(t)=0]} & (\Gamma^{\perp}_{[h(t)=0]})^T \left\{ [e_1,\ e_2,\ e_4] \begin{bmatrix} X_1 D \\ X_2 D \\ X_3 D \end{bmatrix} \right\} \\ \left\{ [D^T X_1^T,\ D^T X_2^T,\ D^T X_3^T] \begin{bmatrix} e_1^T \\ e_2^T \\ e_4^T \end{bmatrix} \right\} \Gamma^{\perp}_{[h(t)=0]} & -\Theta \end{bmatrix} < 0, \tag{44}$$

$$\begin{bmatrix} (\Gamma^{\perp}_{[h(t)=h]})^T \Phi_{[h(t)=h]} \Gamma^{\perp}_{[h(t)=h]} & (\Gamma^{\perp}_{[h(t)=h]})^T \left\{ [e_1,\ e_2,\ e_4] \begin{bmatrix} X_1 D \\ X_2 D \\ X_3 D \end{bmatrix} \right\} \\ \left\{ [D^T X_1^T,\ D^T X_2^T,\ D^T X_3^T] \begin{bmatrix} e_1^T \\ e_2^T \\ e_4^T \end{bmatrix} \right\} \Gamma^{\perp}_{[h(t)=h]} & -\Theta \end{bmatrix} < 0, \tag{45}$$

where

$$\Phi_{[h(t)]} = \Xi_{[h(t)]} - \Xi_{5[h(t)]},$$

and

$$\Gamma_{[h(t)]} = \begin{bmatrix} h(t)e_{12}^T - e_6^T \\ (h - h(t))e_{13}^T - e_7^T \\ h(t)e_{14}^T - e_8^T \\ (h - h(t))e_{15}^T - e_9^T \\ h(t)e_8^T - e_{16}^T \\ (h - h(t))e_9^T - e_{17}^T \end{bmatrix}.$$

Proof. Let us choose the same LKFs in Theorem 1. $\Xi_{5[h(t)]}$ can be alternated by utilizing $\Gamma_{[h(t)]}$ and Ψ. Then, the inequality (31) is equivalent to

$$\Phi_{[h(t)]} + \Gamma^T_{[h(t)]} \Psi^T + \Psi \Gamma_{[h(t)]} + \Xi_8 < 0, \tag{46}$$

where

$$\Psi = E \left[\Psi_1^T,\ \Psi_2^T,\ \Psi_3^T,\ \Psi_4^T,\ \Psi_5^T,\ \Psi_6^T \right].$$

Because the left side of inequality (46) is affinely dependent on $h(t)$, the following inequalities hold

$$\Phi_{[h(t)=0]} + \Gamma^T_{[h(t)=0]} \Psi^T + \Psi \Gamma_{[h(t)=0]} + \Xi_8 < 0, \tag{47}$$

$$\Phi_{[h(t)=h]} + \Gamma^T_{[h(t)=h]} \Psi^T + \Psi \Gamma_{[h(t)=h]} + \Xi_8 < 0. \tag{48}$$

By relations of (ii) and (iii) in Lemma 4 with $0 = \Gamma_{[h(t)]}\zeta(t)$, below inequalities can be derived as

$$(\Gamma^{\perp}_{[h(t)=0]})^T \left(\Phi_{[h(t)=0]} + [e_1,\ e_2,\ e_4] \begin{bmatrix} X_1 D \\ X_2 D \\ X_3 D \end{bmatrix} \Theta^{-1} \begin{bmatrix} X_1 D \\ X_2 D \\ X_3 D \end{bmatrix}^T [e_1,\ e_2,\ e_4]^T \right) (\Gamma^{\perp}_{[h(t)=0]}) < 0,$$

$$(\Gamma^{\perp}_{[h(t)=h]})^T \left(\Phi_{[h(t)=h]} + \begin{bmatrix} e_1, & e_2, & e_4 \end{bmatrix} \begin{bmatrix} X_1 D \\ X_2 D \\ X_3 D \end{bmatrix} \Theta^{-1} \begin{bmatrix} X_1 D \\ X_2 D \\ X_3 D \end{bmatrix}^T \begin{bmatrix} e_1, & e_2, & e_4 \end{bmatrix}^T \right) (\Gamma^{\perp}_{[h(t)=h]}) < 0. \tag{49}$$

The last, LMI conditons (44), (45) for system (1) under $u(t) = 0_{m\times 1}$ can be described by Schur's complement similar in Theorem 1, so rest proof is omitted. □

Remark 2: The zero equalities $0=\zeta^T(t)\Psi\Gamma_{[h(t)]}\zeta(t)$ can be utilized by expanding the sizes of augmented vectors and free-weighting matrices. However there exists limitations, which give more burdens in calculating costs. To overcome that disadvantages, *Kwon et al* proposed Augmented zero equalities approach.[25] The AZEA not only reduces mentioned calculating costs by eliminating decision variables from Ψ, but also provides less conservatism in stability criteria with improved maximum upper bounds of time-delays.

In Theorem 2, advanced method for finding stability criteria by AZEA was introduced. So, in Corollary 2, method for finding stabilization criteria of closed-loop system (2) with AZEA will be introduced.

Corollary 2: For given any scalars δ_1, δ_2, positive scalars h and μ, system (2) is asymptotically stable for $0 \leq h(t) \leq h$ and $\dot{h}(t) \leq \mu$, if there exist positive-definite matrices $\tilde{R} \in \mathbf{S}^{5n}_{+}$, $\tilde{N} \in \mathbf{S}^{6n}_{+}$, $\tilde{G} \in \mathbf{S}^{5n}_{+}$, $\tilde{Q} \in \mathbf{S}^{2n}_{+}$ and positive-definite diagonal matrix $\Theta \in \mathbf{S}^{n}_{+}$, symmetric matrices $\tilde{P}_i \in \mathbf{S}^{n}(i = 1, 2)$, $\tilde{S}_i \in \mathbf{S}^{6n}(i = 1, 2)$ any matrices $X \in \mathbf{R}^{n\times n}$, $Y \in \mathbf{R}^{m\times n}$, $\tilde{F}_i \in \mathbf{R}^{6n\times 6n}(i = 1, 2)$ satisfying the following LMIs with (36) and (37):

$$\begin{bmatrix} (\Gamma^{\perp}_{[h(t)=0]})^T \tilde{\Phi}_{[h(t)=0]} \Gamma^{\perp}_{[h(t)=0]} & (\Gamma^{\perp}_{[h(t)=0]})^T \left\{ [e_1, \ e_2, \ e_4] \begin{bmatrix} X^T E_s^T + Y^T E_u^T \\ X^T E_d^T \\ 0_n \end{bmatrix} \right\} \\ \left\{ [E_s X + E_u Y, \ E_d X, \ 0_n] \begin{bmatrix} e_1^T \\ e_2^T \\ e_4^T \end{bmatrix} \right\} \Gamma^{\perp}_{[h(t)=0]} & -\Theta \end{bmatrix} < 0, \tag{50}$$

$$\begin{bmatrix} (\Gamma^{\perp}_{[h(t)=h]})^T \tilde{\Phi}_{[h(t)=h]} \Gamma^{\perp}_{[h(t)=h]} & (\Gamma^{\perp}_{[h(t)=h]})^T \left\{ [e_1, \ e_2, \ e_4] \begin{bmatrix} X^T E_s^T + Y^T E_u^T \\ X^T E_d^T \\ 0_n \end{bmatrix} \right\} \\ \left\{ [E_s X + E_u Y, \ E_d X, \ 0_n] \begin{bmatrix} e_1^T \\ e_2^T \\ e_4^T \end{bmatrix} \right\} \Gamma^{\perp}_{[h(t)=h]} & -\Theta \end{bmatrix} < 0, \tag{51}$$

where

$$\tilde{\Phi}_{[h(t)]} = \tilde{\Xi}_{[h(t)]} - \tilde{\Xi}_5.$$

Proof. Similar with (46) in Theorem 2, inequality (41) from Corollary 1 is equivalent to

$$\bar{\Phi}_{[h(t)]} + \Gamma^T_{[h(t)]}\Psi^T + \Psi\Gamma_{[h(t)]} + \Xi_{11} < 0, \tag{52}$$

where

$$\bar{\Phi}_{[h(t)]} = \Phi_{[h(t)]} - \Xi_6 - \Xi_7 + \Xi_9 + \Xi_{10}.$$

Like the processes of (42), (47) and (48), following inequalities are derived as

$$\tilde{\Phi}_{[h(t)=0]} + \Gamma^T_{[h(t)=0]}\tilde{\Psi}^T + \tilde{\Psi}\Gamma_{[h(t)=0]} + \left[e_1,\ e_2,\ e_4\right]\left[E_sX + E_uY,\ E_dX,\ 0_n\right]^T \Theta^{-1}\left[E_sX + E_uY,\ E_dX,\ 0_n\right]\left[e_1,\ e_2,\ e_4\right]^T < 0, \tag{53}$$

$$\tilde{\Phi}_{[h(t)=h]} + \Gamma^T_{[h(t)=h]}\tilde{\Psi}^T + \tilde{\Psi}\Gamma_{[h(t)=h]} + \left[e_1,\ e_2,\ e_4\right]\left[E_sX + E_uY,\ E_dX,\ 0_n\right]^T \Theta^{-1}\left[E_sX + E_uY,\ E_dX,\ 0_n\right]\left[e_1,\ e_2,\ e_4\right]^T < 0. \tag{54}$$

And then, by using the Lemma 4 with $0 = \Gamma_{[h(t)]}\zeta(t)$ following inequalities can be obtained as

$$(\Gamma^{\perp}_{[h(t)=0]})^T\left(\tilde{\Phi}_{[h(t)=0]} + \left[e_1,\ e_2,\ e_4\right]\left[E_sX + E_uY,\ E_dX,\ 0_n\right]\Theta^{-1}\left[E_sX + E_uY,\ E_dX,\ 0_n\right]^T\left[e_1,\ e_2,\ e_4\right]^T\right)(\Gamma^{\perp}_{[h(t)=0]}) < 0, \tag{55}$$

$$(\Gamma^{\perp}_{[h(t)=h]})^T\left(\tilde{\Phi}_{[h(t)=h]} + \left[e_1,\ e_2,\ e_4\right]\left[E_sX + E_uY,\ E_dX,\ 0_n\right]\Theta^{-1}\left[E_sX + E_uY,\ E_dX,\ 0_n\right]^T\left[e_1,\ e_2,\ e_4\right]^T\right)(\Gamma^{\perp}_{[h(t)=h]}) < 0. \tag{56}$$

The other processes are same to the proof of Corollary 1 and Theorem 2, so it is omitted. □

Remark 3: Different from Theorems 1 and 2, Corollaries 1 and 2 are related to design a controller gain for system (2). Unlike the previous results, it is a first trial to design a controller for uncertain linear systems with time-varying delays via augmented approach introduced in (54) and (54) where $\Gamma_{[h(t)]}$ contains zero equalities generated from (22) and (27). By utilizing form like $\Gamma^{\perp}_{[h(t)]}$, this Corollary showed that Finsler's lemma can be applied in the closed-loop system which has feedback control gain K and uncertainties. So applying AZEA for system stabilization gives advantages not only reducing calculating costs, also getting improved stabilization criteria region in controller design.

To confirm the superiority and validity of the proposed methods, maximum delay bounds obtained in other literatures are compared in next section.

4 | NUMERICAL EXAMPLES

Example 1. Consider the system (1) under $u(t) = 0$ with following information

$$A = \begin{bmatrix} -0.4 & 0 \\ 0 & -1 \end{bmatrix},\ A_d = \begin{bmatrix} -0.9 & 0 \\ -1 & -0.7 \end{bmatrix},$$
$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ E_s = E_d = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

In Table 1, our results from Theorem 1 and Theorem 2 are compared with other literatures.[30,31,32,33,34,35] Theroem 1, which utilzed mentioned Lemmas and new zero equlity approaches with expanding augmented vectors, gives advanced results about system (1). By applying Theorem 2, which is utilized AZEA, notable improvement in maximum delay bounds are given with lower decision variables.

Example 2. Consider the system (2) with the following information,

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_d = \begin{bmatrix} -2 & -0.5 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$D = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},\ E_s = E_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E_u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

TABLE 1 Comparison of h with the condition about $\mu = unknown$(Example 1).

Method	h
Jiang[30]	0.9442
Ramakrishnan[31]	1.0571
Ramakrishnan[32] (N=2)	1.1030
Zhang[33] (N=2)	1.3213
He[34]	1.4127
Kwon[35]	1.4209
Theorem 1	1.4270
Theorem 2	1.6720

*IN=Iteration num.

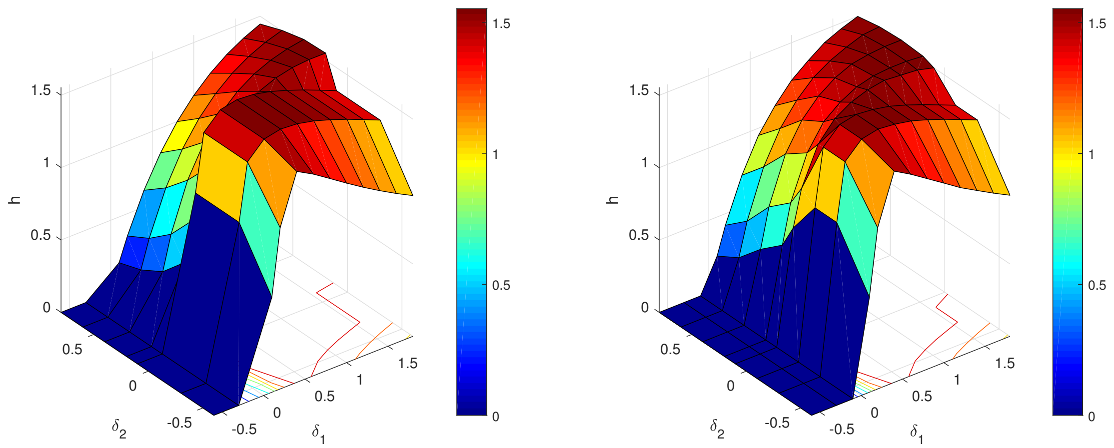

FIGURE 1 Contour figures of Table 1(left), Table 2(right).

TABLE 2 Comparison of h with the condition about Corollary 1 (Example 2).

$\delta_1 \backslash \delta_2$	**0.1**	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**	**1.2**	**1.4**	**1.6**	**1.8**
-0.6	0.657	1.114	1.457	1.393	1.317	1.240	1.166	1.097	1.032	0.972
-0.3	1.021	1.416	1.528	1.552	1.543	1.517	1.480	1.437	1.390	1.341
0.1	0.621	0.888	1.187	1.353	1.452	1.509	1.540	1.551	1.549	1.538
0.2	0.490	0.759	1.082	1.272	1.392	1.468	1.515	1.541	1.551	1.550

The value of h is highest when $\delta_1 = -0.3,\ \delta_2 = 0.6$

In the past studies[18 35], finding proper δ scale was limited in positive region. But we don't have to be limited positive region by giving more free-weighting with matrices $X_1,\ X_2,\ X_3$. Table 2 and Table 3 show the results of comparing maximum delays bound h by scaling $\delta_1,\ \delta_2$ with choosing $\mu = 0$ in Example 2. And the best result h is 1.553 with $\delta_1 = -0.3,\ \delta_2 = 0.6$ with choosing $\mu = 0$ when utilize Corollary 2 in Example 2. Table 4 shows the comparison of the controller gains K with their maximum delays bound h by choosing $\mu = 0,\ 0.5$ and *unknown*. From the results, Corollary 2 which utilized AZEA provides more improved results than those of the works lists in Table 4. Furthermore, effectiveness of the controller gains by Corollary 2 is proved by simulation result in Figure 2 with the maximum delays bound $h = 1.552$ under $\mu = 0.5$ and time-varying delays are assumed to be $h(t) = 0.5cos(t) + h - 0.5$ in Figure 2.

TABLE 3 Comparison of h with the condition about Corollary 2(Example 2)

$\delta_1 \backslash \delta_2$	**0.1**	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**	**1.2**	**1.4**	**1.6**	**1.8**
-0.6	0.657	1.114	1.460	1.393	1.317	1.241	1.164	1.099	1.035	0.975
-0.3	1.021	1.418	1.529	1.553	1.541	1.518	1.481	1.438	1.391	1.343
-0.1	0.982	1.184	1.390	1.490	1.533	1.552	1.548	1.531	1.505	1.475
0.1	1.113	1.324	1.472	1.533	1.552	1.547	1.526	1.495	1.459	1.418

The value of h is highest when $\delta_1 = -0.3,\ \delta_2 = 0.6$

TABLE 4 Upper bounds of time-varying delays and controller gains when $\mu = 0,\ 0.5$ and *unknown* (Example 2).

Method($\mu = 0$)	h	**Controller gains**
Wu[36]	0.6548	[-24.5739 -17.6699]
Li[37]	0.84	[-34.72 -18.41]
Dey[38] (IN=150)	0.9	[-27.367 -26.249]
Lee[18]	0.9949	$10^4\times$[-2.32 -0.85]
Kwon[35] ($\delta = 1.3$)	1.5500	$10^5\times$ [-2.1061 -0.6915]
Corollary 1 ($\delta_1 = -0.3,\ \delta_2 = 0.6$)	1.5500	$10^5\times$ [-1.1892 -0.3902]
	1.5524	$10^5\times$ [-4.7968 -1.5730]
Corollary 2 ($\delta_1 = -0.3,\ \delta_2 = 0.6$)	0.9	[-155.1987 -9.0114]
	1.5500	$10^5\times$ [-1.1935 -0.3916]
	1.5531	$10^5\times$ [-9.8494 -3.2294]
Method($\mu = 0.5$)	h	**Controller gains**
Fridman[7]	0.4960	[-0.34 -5.168]
Alpaslan[39] (IN=54)	0.6000	[-9.5735 -2.9742]
Dey[38] (IN=54)	0.7	[-4.8123 -7.2495]
Lee[18]	0.9847	$10^4\times$[-6.83 -2.48]
Kwon[35]	1.5290	$10^5\times$[-2.1146 -0.6881]
Corollary 1($\delta_1 = -0.3,\ \delta_2 = 0.6$)	1.5290	$10^4\times$[-4.0522 -1.3214]
	1.5401	$10^6\times$[-1.1388 -0.3704]
Corollary 2($\delta_1 = -0.3,\ \delta_2 = 0.6$)	1.5290	$10^3\times$[-4.4800 -1.4774]
	1.5524	$10^5\times$[-4.9029 -1.6078]
Method($\mu = unknown$)	h	**Controller gains**
Kwon[35]	1.4623	$10^6\times$[-1.5494 -0.5123]
Corollary 2($\delta_1 = 0,\ \delta_2 = 1.2$)	1.4623	$10^4\times$[-1.7081 -0.5798]
	1.5421	$10^5\times$[-9.1869 -3.0322]

*IN=Iteration num.

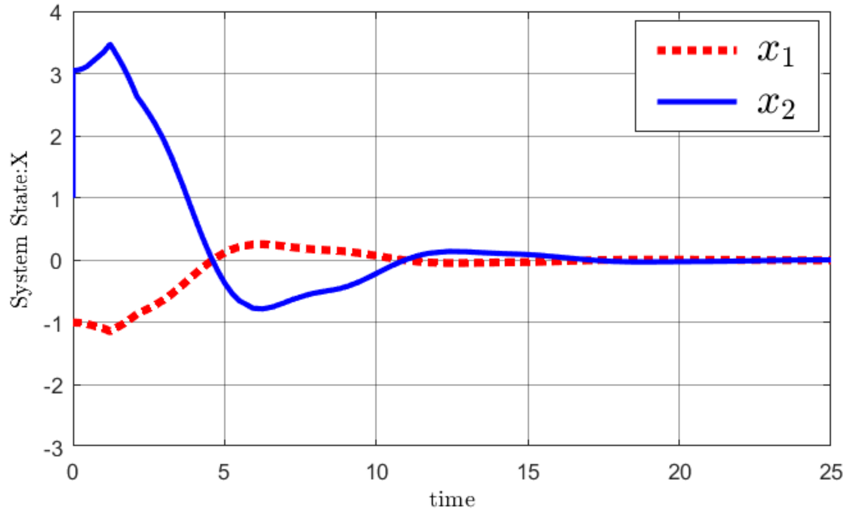

FIGURE 2 The trajectories under value h, μ, $h(t)$, K : h=1.552, μ=0.5, $h(t)$=0.5$cos(t)$+h−0.5, Control gain K=10^5×[-4.9029 -1.6078]. (Example 2, Corollary 2)

5 | CONCLUSIONS

In this paper, the LKFs methods and LMI frameworks for stability and stabilization problem about uncertain linear systems with time-varying delays were proposed. In Theorem 1 and Corollary 1, AFII and ERCA are utilized. And the sufficient stability and stabilization conditions were derived by applying the proposed methods and constructing the appropriate augmented LKFs. By AZEA with Lemma 4, Theorem 2 and Corollary 2 derived advanced conditions for guaranteeing the asymptotic stability of system (1) under $u(t) = 0_{m\times 1}$ and stabilization of system (2). The effectiveness and superiority of proposed results were proved though numerical examples by comparing with previous works. Based on the proposed methods, expanding and applying the proposed methods will be focused on other systems like nonlinear[40 41], sampled data systems[42], switched system[43], and Neural network[44] and so on.

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Conflict of interest

This work does not have any conflicts of interest.

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