Conservation Laws and Exact Series Solution of Fractional-Order Hirota-Satsoma Coupled KdV system by Symmetry Analysis

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Abstract: In this work, we investigated the invariance analysis of fractional-order Hirota-Satsoma coupled Korteveg-de-Vries (HSC-KdV) system of equations based on Riemann-Liouville (RL) derivatives. The Lie Symmetry analysis is considered to obtain infinitesimal generators; we reduced the system of coupled equations into nonlinear fractional ordinary differential equations (FODEs) with the help of Erdelyi's-Kober (EK) fractional differential and integral operators. The reduced system of FODEs solved by means of power series technique with its convergence. The conservation laws of the system constructed by the Noether's theorem.

Keywords: Lie symmetry, Time fractional differential equations, Conservation laws, Noether's theorem.

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1. Introduction

In this pandemic decade, the need for fractional modeling as the generalization of the classical differential equation of integer order has achieved great attention in the research field, which is feasible with the help of fractional calculus. The fractional derivatives play a vital role in the historic study of differential modeling. It is well known that fractional differential equations (FDEs) have applied to describe a large number of physical nonlinear phenomena in diffusion, solid mechanics, wave propagation, signal processing, optics, statistics, neural network, bioengineering, polymer science and other scientific research areas [1, 3, 4, 10, 18, 19, 39]. The main framework of evaluating FPDEs is to search for exact and approximate solutions of problem, which has been a great task for mathematicians. In order to find exact and approximate solutions of PDEs, researchers projected distinct methods such as the sine-cosine method, tanh method [6], reduced differential transform method [7, 8], homotopy analysis [18, 40], Lie symmetry analysis [9, 31,32, 33, 47] and variation Iteration method [20] etc.

Lie Symmetry Analysis [2] is powerful tool to generate explicit solution by reducing the given system of FPDEs into a nonlinear system of FODEs with EK fractional differential and integral operators. The Lie symmetry analysis method is to find continuous transformations of one or more parameters leaving the differential equation invariant in the new coordinate system wherein the resulting differential equation is easier to solve. Authors [5, 11-17, 21-26] explained the applications of Lie symmetries to the time fractional KdV equation and concluded that the fractional order differential equations can be transformed into FODEs by introducing new independent variable. Sneddon [27] introduced the applicability of EK fractional order operators which helps us to reduce the considered system into fractional ODEs and Huang et. al [28] emphasized the efficiency of Lie symmetry approach analysis of Harry-dym equation with Riemann Liouville derivatives. Authors [29, 45] also made a complete group classification of fourth and the fifth order KdV equations. Chauhan el al. [34] presented the Lie symmetry analysis and explicit series solution to the Date–Jimbo–Kashiwara–Miwa equation. Singla and Gupta [35] extended the symmetry approach from single time FPDEs to nonlinear system of time FODEs. Noether's theorem [30, 36, 37, 38] established a relation between conservation laws and symmetry of differential equations and applied on FPDEs without Lagrangian operators.

The coupled Hirota-Satsoma fractional order system is explained to study the flow of fluids in power system and extended the study of the propagation of shallow water waves. The Lie symmetry invariant analysis and conservation laws have made progress in FPDEs still the research field for coupled KdV fractional order system is not well exposed.

This system was proposed by Hirota and Satsoma to describe the relation between long waves with distinct dispersion interactions and its generalized behavior has led to relevance in various branches of applied mathematics and time fractional HSC-KdV system has been studied by using various methods [41-44]. The focus of this article is to investigate fractional order Lie symmetry analysis and new conservation laws via Noether's theorem for coupled time fractional HSC-KdV system [24] of fractional parameter $'\theta'$.

$$\begin{cases} \dfrac{\partial^\theta u}{\partial t^\theta} = \dfrac{1}{4}\dfrac{\partial^3 u}{\partial x^3} + 3u\dfrac{\partial u}{\partial x} - 6v\dfrac{\partial v}{\partial x} + 3\dfrac{\partial w}{\partial x}, \\ \dfrac{\partial^\theta v}{\partial t^\theta} = -\dfrac{1}{2}\dfrac{\partial^3 v}{\partial x^3} - 3u\dfrac{\partial v}{\partial x}, \\ \dfrac{\partial^\theta w}{\partial t^\theta} = -\dfrac{1}{2}\dfrac{\partial^3 w}{\partial x^3} - 3u\dfrac{\partial w}{\partial x}. \end{cases} \tag{1}$$

The article is summarised as follows: in section 2, we recall some concepts and fractional order derivatives and integral operators. In section 3, we discussed the Lie symmetry scheme and obtained infinitesimals and infinitesimal generators of set of equations (1). By the application of EK fractional operators, conversion of FPDEs into FODEs have been suggested in section 4 and in section 5, the power series expansion method is used to find the explicit solution of system (1). Section 6 and 7, dealt with convergence analysis of the solution and conservation laws of the system; respectively. Finally, section 8 concludes the article.

2. Preliminaries:

In this part, we would like to explain certain needful definitions for the sake of understanding the methodologies and concepts, concerned with fractional order derivatives and integrals and their applications in fractional calculus.

Definition 1 The Caputo explained the fractional order derivative of function *F(t)*as

$$D_t^\mu(F(t)) = \frac{1}{\Gamma(\lambda-\mu)}\int_0^t (t-\rho)^{\lambda-\mu-1}F^\lambda(\rho)d\rho \ \ for\ \lambda-1<\mu\le\lambda;\ \lambda\in N\ ;\ t>0 \tag{2}$$

Definition 2 The RL derived the definition of fractional order derivative of *F(t)* as

$$D_t^\mu(F(t)) = \frac{1}{\Gamma(\lambda-\mu)}\frac{d^\lambda}{dt^\lambda}\int_0^t (t-\rho)^{\lambda-\mu-1}F(\rho)d\rho \ \ for\ \lambda-1<\mu\le\lambda;\ \lambda\in N;\ t>0 \tag{3}$$

Definition 3 Let the function $u(x,t)$ with variables '*x'* and*t> 0*then RL fractional partial order derivative is proposed as

$$\partial_t^\mu(u(x,t)) = \begin{cases} \dfrac{1}{\Gamma(\lambda-\mu)}\dfrac{\partial^\lambda}{\partial t^\lambda}\displaystyle\int_0^t (t-\rho)^{\lambda-\mu-1}u(\rho,x)d\rho \ \ for\ \lambda-1<\mu<\lambda,\ \lambda\in N \\ \dfrac{\partial^\lambda u}{\partial t^\lambda} \quad for\ \mu=\lambda \end{cases} \qquad 4)$$

Definition 4 The Leibnitz described the product rule under application of RL fractional order derivatives in the form

$$D_t^\mu(U.V) = \sum_{\lambda=0}^{\infty}\binom{\mu}{\lambda}D_t^{\mu-\lambda}(U).D_t^\lambda(V) \ \ ;\mu>0\ with\ \binom{\mu}{\lambda} = \frac{(-1)^\lambda\,\mu\,\Gamma(n-\mu)}{\Gamma(1-\mu)\Gamma(\mu+1)} \tag{5}$$

Definition 5 The E-Kober generalized fractional differential operator $\left(E_\partial{}^{\tau,\mu}\omega\right)(\zeta)$ is given by

$$\left(E_{\partial}^{\tau,\mu}\omega\right)(\zeta)=\prod_{\lambda=0}^{m-1}\left(\tau+\lambda-\frac{1}{\partial}z\frac{d}{dz}\right)(K_{\partial}^{\tau+\mu,m-\mu}\omega)(z)\ with\ \zeta>0,\ \partial>0\ and\ \mu>0;$$
$$m=\begin{cases}[\mu]+1, & \mu\notin N\\ \mu\ , & \mu\in N\end{cases} \tag{6}$$

Definition 6 The E-Kober generalized fractional order integral operator $(K_{\partial}{}^{\tau,\mu}\omega)(\zeta)$ is

$$(K_{\partial}^{\tau,\mu}\omega)(\zeta)=\begin{cases}\dfrac{1}{\Gamma(\mu)}\displaystyle\int_{1}^{\infty}(\upsilon-1)^{\mu-1}\upsilon^{-(\tau+\mu)}\omega(\zeta\upsilon^{1/\partial})d\upsilon, & \mu>0\\ \omega(\zeta), & \mu=0\end{cases}. \tag{7}$$

3. Methodology:

In this section, we would like to pursue the steps and process of fractional Lie symmetry reduction to coupled time fractional system of FPDEs. Initially, Sophus Lie established the applications of Lie groups and symmetries in solution of ODEs. He remarked that Lie transformation maps every solution of system to other solution of same system and nowadays mathematicians worked on application of methodology on PDEs, FPDEs and system of linear and nonlinear FPDEs.

Let us assume the system of FPDEs with fractional order $'\theta'$

$$\begin{cases}\partial_t^{\theta}u=F_1(t,x,u,v,w,u_x,v_x,w_x,u_{xx},v_{xx},w_{xx}...),\\ \partial_t^{\theta}v=F_2(t,x,u,v,w,u_x,v_x,w_x,u_{xx},v_{xx},w_{xx}...),\\ \partial_t^{\theta}w=F_3(t,x,u,v,w,u_x,v_x,w_x,u_{xx},v_{xx},w_{xx}...).0<\theta<1\end{cases} \tag{9}$$

The infinitesimal transformations with single parametric notation in fractional Lie symmetry analysis is expressed as

$$\begin{cases}\bar{t}=\bar{t}(x,t,u,v,w;\varepsilon)=t+\varepsilon\tau(x,t,u,v,w)+o(\varepsilon^2),\\ \bar{x}=\bar{x}(x,t,u,v,w;\varepsilon)=x+\varepsilon\xi(x,t,u,v,w)+o(\varepsilon^2),\\ \bar{u}=\bar{u}(x,t,u,v,w;\varepsilon)=u+\varepsilon\eta(x,t,u,v,w)+o(\varepsilon^2),\\ \bar{v}=\bar{v}(x,t,u,v,w;\varepsilon)=v+\varepsilon\varphi(x,t,u,v,w)+o(\varepsilon^2),\\ \bar{w}=\bar{w}(x,t,u,v,w;\varepsilon)=w+\varepsilon\mu(x,t,u,v,w)+o(\varepsilon^2).\end{cases} \tag{10}$$

The vector field generated by infinitesimals is taken as

$$X=\tau\,\partial_t+\xi\,\partial_x+\eta\,\partial_u+\varphi\,\partial_v+\mu\,\partial_w, \tag{11}$$

with $\tau=\left.\dfrac{d\bar{t}}{d\varepsilon}\right|_{\varepsilon=0},\ \xi=\left.\dfrac{d\bar{x}}{d\varepsilon}\right|_{\varepsilon=0},\ \eta=\left.\dfrac{d\bar{u}}{d\varepsilon}\right|_{\varepsilon=0},\varphi=\left.\dfrac{d\bar{v}}{d\varepsilon}\right|_{\varepsilon=0},\mu=\left.\dfrac{dw}{d\varepsilon}\right|_{\varepsilon=0}.$

Here ξ, τ, η, φ and μ are obtained infinitesimals operators from (11), $\eta^{\sigma,t}$, $\varphi^{\sigma,t}$ and $\mu^{\sigma,t}$ are the fractional extended infinitesimals of order *'α'* and η^{x}, η^{xx}, η^{xxx}, φ^{x}, φ^{xx}, φ^{xxx}, μ^{x}, μ^{xx}, μ^{xxx} are extended infinitesimals of integer-order described

$$
\begin{aligned}
&\eta^{x}=D_x(\eta)-u_xD_x(\xi)-u_tD_x(\tau),\\
&\eta^{xx}=D_x(\eta^{x})-u_{xx}D_x(\xi)-u_{xt}D_x(\tau),\\
&\eta^{xxx}=D_x(\eta^{xx})-u_{xxx}D_x(\xi)-u_{xxt}D_x(\tau),\\
&\varphi^{x}=D_x(\varphi)-v_xD_x(\xi)-v_tD_x(\tau),\\
&\varphi^{xx}=D_x(\varphi^{x})-v_{xx}D_x(\xi)-v_{xt}D_x(\tau),\\
&\varphi^{xxx}=D_x(\varphi^{xx})-v_{xxx}D_x(\xi)-v_{xxt}D_x(\tau),\\
&\mu^{x}=D_x(\mu)-w_xD_x(\xi)-w_tD_x(\tau),\\
&\mu^{xx}=D_x(\mu^{x})-w_{xx}D_x(\xi)-w_{xt}D_x(\tau),\\
&\mu^{xxx}=D_x(\mu^{xx})-w_{xxx}D_x(\xi)-w_{xxt}D_x(\tau),
\end{aligned}
\tag{12}
$$

where 'D_x'is total derivative operator defined as

$$
D_x=\frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_{xx}}+...+v_x\frac{\partial}{\partial v_x}+v_{xx}\frac{\partial}{\partial v_{xx}}+...+w_x\frac{\partial}{\partial w_x}+w_{xx}\frac{\partial}{\partial w_{xx}}+.... \tag{13}
$$

The extended infinitesimal function of θ-th order ($\eta^{\theta,t}$) concerned to RL fractional derivative is describedby

$$
\eta^{\theta,t}=D_t^{\theta}(\eta)+\xi.D_t^{\theta}(u_x)-D_t^{\theta}(\xi u_x)+D_t^{\theta}(D_t(\tau)u)-D_t^{\theta+1}(\tau u)+\tau D_t^{\theta+1}(u). \tag{14}
$$

Also $D_t^{\theta+1}(f(t))=D_t^{\theta}(D_t(f(t)),$ then above expression simplified to

$$
\eta^{\theta,t}=D_t^{\theta}(\eta)+\xi.D_t^{\theta}(u_x)-D_t^{\theta}(\xi u_x)+\tau D_t^{\theta}(u)-D_t^{\theta}(\tau u_t). \tag{15}
$$

Applying the generalized Liebnitz rule on (15), we obtain

$$
\eta^{\theta,t}=D_t^{\theta}(\eta)-\theta D_t^{\theta}(\tau)\frac{\partial^{\theta}u}{\partial t^{\theta}}-\sum_{\lambda=1}^{\infty}\binom{\theta}{n}D_t^{\lambda}(\xi)D_t^{\theta-\lambda}(u_x)-\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda+1}D_t^{\lambda+1}(\tau)D_t^{\theta-\lambda}(u). \tag{16}
$$

Using the generalized Liebnitz rule and chain rule (5) the term $D_t^{\theta}(\eta)$ in (16) can be defined as

$$
\begin{aligned}
D_t^{\theta}(\eta)=&\frac{\partial^{\theta}\eta}{\partial t^{\theta}}+\left(\eta_u\frac{\partial^{\theta}u}{\partial t^{\theta}}-u\frac{\partial^{\theta}(\eta_u)}{\partial t^{\theta}}\right)+\left(\eta_v\frac{\partial^{\theta}v}{\partial t^{\theta}}-v\frac{\partial^{\theta}(\eta_v)}{\partial t^{\theta}}\right)+\left(\eta_w\frac{\partial^{\theta}w}{\partial t^{\theta}}-v\frac{\partial^{\theta}(\eta_w)}{\partial t^{\theta}}\right)\\
&+\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^{n}(\eta_u)}{\partial t^{\lambda}}D_t^{\theta-\lambda}(u)+\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^{\lambda}(\eta_v)}{\partial t^{\lambda}}D_t^{\theta-\lambda}(v)+\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^{\lambda}(\eta_w)}{\partial t^{\lambda}}D_t^{\theta-\lambda}(w)+\sigma_1+\sigma_2+\sigma_3
\end{aligned}
\tag{17}
$$

where
$$\begin{cases} \sigma_1 = \sum_{\lambda=2}^{\infty}\sum_{m=2}^{\lambda}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\theta}{\lambda}\binom{\lambda}{m}\binom{k}{r}\frac{t^{\lambda-\theta}}{k!\Gamma(\lambda+1-\theta)}(-u)^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{\lambda-m+k}\eta}{\partial t^{\lambda-m}\partial u^k}, \\ \sigma_2 = \sum_{\lambda=2}^{\infty}\sum_{m=2}^{\lambda}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\theta}{\lambda}\binom{\lambda}{m}\binom{k}{r}\frac{t^{\lambda-\theta}}{k!\Gamma(\lambda+1-\theta)}(-v)^r\frac{\partial^m}{\partial t^m}(v^{k-r})\frac{\partial^{\lambda-m+k}\eta}{\partial t^{\lambda-m}\partial v^k}, \\ \sigma_3 = \sum_{\lambda=2}^{\infty}\sum_{m=2}^{\lambda}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\theta}{\lambda}\binom{\lambda}{m}\binom{k}{r}\frac{t^{\lambda-\theta}}{k!\Gamma(\lambda+1-\theta)}(-w)^r\frac{\partial^m}{\partial t^m}(w^{k-r})\frac{\partial^{\lambda-m+k}\eta}{\partial t^{\lambda-m}\partial w^k}. \end{cases} \tag{18}$$

Finally, the expression for θ-th order extended infinitesimal $\eta^{\theta,t}$ of the form

$$\begin{aligned} \eta^{\theta,t} &= \frac{\partial^\theta \eta}{\partial t^\theta} + \left(\eta_u - \theta D_t(\tau)\right)\frac{\partial^\theta u}{\partial t^\theta} - u\frac{\partial^\theta(\eta_u)}{\partial t^\theta} + \left(\eta_v\frac{\partial^\theta v}{\partial t^\theta} - v\frac{\partial^\theta(\eta_v)}{\partial t^\theta}\right) + \left(\eta_w\frac{\partial^\theta w}{\partial t^\theta} - v\frac{\partial^\theta(\eta_w)}{\partial t^\theta}\right) \\ &+ \sum_{\lambda=1}^{\infty}\left[\binom{\theta}{\lambda}\frac{\partial^n \eta_u}{\partial t^n} - \binom{\theta}{\lambda+1}D_t^{\lambda+1}(\tau)\right]D_t^{\theta-\lambda}(u) + \sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\eta_v)}{\partial t^\lambda}D_t^{\theta-\lambda}(v) + \sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\eta_w)}{\partial t^\lambda}D_t^{\theta-\lambda}(w) \\ &- \sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}D_t^\lambda(\xi)D_t^{\theta-\lambda}(u_x) + \sigma_1 + \sigma_2 + \sigma_3 \end{aligned} \tag{19}$$

Similarly, expressions for $\varphi^{\theta,t}$ and $\mu^{\theta,t}$ also obtained.

$$\begin{aligned} \phi^{\theta,t} &= \frac{\partial^\theta \phi}{\partial t^\theta} + \left(\phi_v - \theta D_t(\tau)\right)\frac{\partial^\theta v}{\partial t^\theta} - v\frac{\partial^\theta(\phi_v)}{\partial t^\theta} + \left(\phi_u\frac{\partial^\theta u}{\partial t^\theta} - u\frac{\partial^\theta(\phi_u)}{\partial t^\theta}\right) + \left(\phi_w\frac{\partial^\theta w}{\partial t^\theta} - v\frac{\partial^\theta(\phi_w)}{\partial t^\theta}\right) \\ &+ \sum_{\lambda=1}^{\infty}\left[\binom{\theta}{\lambda}\frac{\partial^\lambda \phi_v}{\partial t^\lambda} - \binom{\theta}{\lambda+1}D_t^{\lambda+1}(\tau)\right]D_t^{\theta-\lambda}(v) + \sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda \phi_u}{\partial t^\lambda}D_t^{\theta-\lambda}(u) + \sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda \phi_w}{\partial t^\lambda}D_t^{\theta-\lambda}(w) \\ &+ -\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}D_t^\lambda(\xi)D_t^{\theta-\lambda}(v_x) + \sigma_4 + \sigma_5 + \sigma_6 \end{aligned} \tag{20}$$

$$\begin{aligned} \mu^{\theta,t} &= \frac{\partial^\theta \mu}{\partial t^\theta} + \left(\mu_v - \theta D_t(\tau)\right)\frac{\partial^\theta w}{\partial t^\theta} - w\frac{\partial^\theta(\mu_w)}{\partial t^\theta} + \left(\phi_u\frac{\partial^\theta u}{\partial t^\theta} - u\frac{\partial^\theta(\mu_u)}{\partial t^\theta}\right) + \left(\phi_v\frac{\partial^\theta v}{\partial t^\theta} - v\frac{\partial^\theta(\mu_v)}{\partial t^\theta}\right) \\ &+ \sum_{\lambda=1}^{\infty}\left[\binom{\theta}{\lambda}\frac{\partial^\lambda(\mu_w)}{\partial t^\lambda} - \binom{\theta}{\lambda+1}D_t^{\lambda+1}(\tau)\right]D_t^{\theta-\lambda}(w) + \sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda(\mu_u)}{\partial t^\lambda}D_t^{\theta-\lambda}(u) + \sum_{n=1}^{\infty}\binom{\theta}{\lambda}\frac{\partial^\lambda \mu_v}{\partial t^\lambda}D_t^{\theta-\lambda}(v) \\ &+ -\sum_{\lambda=1}^{\infty}\binom{\theta}{\lambda}D_t^\lambda(\xi)D_t^{\theta-\lambda}(w_x) + \sigma_7 + \sigma_8 + \sigma_9 \end{aligned} \tag{21}$$

where

$$\begin{cases}\sigma_4=\sum_{\lambda=2}^{\infty}\sum_{m=2}^{\lambda}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\theta}{\lambda}\binom{\lambda}{m}\binom{k}{r}\frac{t^{\lambda-\theta}}{k!\Gamma(\lambda+1-\theta)}(-u)^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{\lambda-m+k}\phi}{\partial t^{\lambda-m}\partial u^k}\\ \sigma_5=\sum_{\lambda=2}^{\infty}\sum_{m=2}^{\lambda}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\theta}{\lambda}\binom{\lambda}{m}\binom{k}{r}\frac{t^{\lambda-\theta}}{k!\Gamma(\lambda+1-\theta)}(-v)^r\frac{\partial^m}{\partial t^m}(v^{k-r})\frac{\partial^{\lambda-m+k}\phi}{\partial t^{\lambda-m}\partial v^k}\\ \sigma_6=\sum_{\lambda=2}^{\infty}\sum_{m=2}^{\lambda}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\theta}{\lambda}\binom{\lambda}{m}\binom{k}{r}\frac{t^{\lambda-\theta}}{k!\Gamma(\lambda+1-\theta)}(-w)^r\frac{\partial^m}{\partial t^m}(w^{k-r})\frac{\partial^{\lambda-m+k}\phi}{\partial t^{\lambda-m}\partial w^k}\\ \sigma_7=\sum_{\lambda=2}^{\infty}\sum_{m=2}^{\lambda}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\theta}{\lambda}\binom{\lambda}{m}\binom{k}{r}\frac{t^{\lambda-\theta}}{k!\Gamma(\lambda+1-\theta)}(-u)^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{\lambda-m+k}\mu}{\partial t^{\lambda-m}\partial u^k}\\ \sigma_8=\sum_{\lambda=2}^{\infty}\sum_{m=2}^{\lambda}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\theta}{\lambda}\binom{\lambda}{m}\binom{k}{r}\frac{t^{\lambda-\theta}}{k!\Gamma(\lambda+1-\theta)}(-v)^r\frac{\partial^m}{\partial t^m}(v^{k-r})\frac{\partial^{\lambda-m+k}\mu}{\partial t^{\lambda-m}\partial v^k}\\ \sigma_9=\sum_{\lambda=2}^{\infty}\sum_{m=2}^{\lambda}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\theta}{\lambda}\binom{\lambda}{m}\binom{k}{r}\frac{t^{\lambda-\theta}}{k!\Gamma(\lambda+1-\theta)}(-w)^r\frac{\partial^m}{\partial t^m}(w^{k-r})\frac{\partial^{\lambda-m+k}\mu}{\partial t^{\lambda-m}\partial w^k}\end{cases}\tag{22}$$

Whenever η, φ and μ are the linear functions of u, v and w respectively expressions σ_i ; $i=1, 2, 3\ldots 9$ given by (18) and (22) vanishes.

4. Invariance Analysis of Time Fractional generalized Hirota-Satsuma coupled KdVsystem:

Applying prolongation on system of equations (1) the invariance criterion obtained

$$\begin{aligned}&\eta^{\theta,t}-\frac{1}{4}\eta^{xxx}-3u\eta^{x}-3\eta u_x-3\mu^x+6\varphi v_x+6v\varphi^x=0,\\ &\varphi^{\theta,t}+\frac{1}{2}\varphi^{xxx}+3u\varphi^x+3\eta v_x=0,\\ &\mu^{\theta,t}+\frac{1}{2}\mu^{xxx}+3u\mu^x+3\eta w_x=0.\end{aligned}\tag{23}$$

Using (16-22) in (23) and takingall the coefficients of 'u' and its derivatives to zero, the set of determining equations for $0<\theta<1$ can be obtained. On solving PDEs and FDEs, the infinitesimal symmetry generators [24] are given below.

$$\xi=px+q;\tau=\frac{3pt}{\theta};\ \eta=-2pu;\ \varphi=-2pv\ ;\ \mu=-4pw+r\,t^{\theta-1},\tag{24}$$

where p, q and r are arbitrary constants.

The symmetry generators to form a lie algebra of Eq. (24) are found as:

$$\begin{aligned}&X_1=x\partial_x+\frac{3t}{\theta}\partial_t-2u\partial_u-2v\partial_v-4w\partial_w,\\ &X_2=\partial_x\ ;\ \ X_3=t^{\theta-1}\partial_w.\end{aligned}\tag{25}$$

Now characteristic equations formed with respect to the vector field X_1 are as follows:

$$\frac{dx}{x}=\frac{\theta\,dt}{3t}=\frac{du}{-2u}=\frac{dv}{-2v}=\frac{dw}{-4w}. \tag{26}$$

The similarity transformations with similarity variable ‘ ζ ’ formed from (26) is

$$u=t^{-\frac{2\theta}{3}}F(\zeta);\ v=t^{-\frac{2\theta}{3}}G(\zeta);\ w=t^{-\frac{4\theta}{3}}H(\zeta);\ \zeta=xt^{-\frac{\theta}{3}}. \tag{27}$$

In this part, we carried (27) along with HSc-KdV system (1) and EK differ-integral operators (6-7) with formal calculations to convert the system (1) into FODEs.

The similarity transformations are

$$\zeta=x\,t^{-\theta/3}\ and\ u=t^{-2\theta/3}F(\zeta). \tag{28}$$

RL definition of time fractional order treatment for similarity transformation is

$$D_t^{\theta}u=D_t^{\lambda}\left(\frac{1}{\Gamma(\lambda-\theta)}\int_0^t(t-s)^{\lambda-\theta-1}s^{-2\theta/3}F(xs^{-\theta/3})ds\right). \tag{29}$$

Substituting $s=t\gamma^{-1}$ in (29), we get

$$D_t^{\theta}u=D_t^{\lambda}\left(\frac{1}{\Gamma(\lambda-\theta)}\int_1^{\infty}(t-\frac{t}{\gamma})^{\lambda-\theta-1}\left(\frac{t}{\gamma}\right)^{-2\theta/3}F(x(t/\gamma)^{-\theta/3})\frac{t}{\gamma^2}d\gamma\right) \tag{30}$$

$$=D_t^{\lambda}\left(\frac{t^{\lambda-\frac{5\theta}{3}}}{\Gamma(\lambda-\theta)}\int_1^{\infty}(-1)^{\lambda-\theta-1}\gamma^{-(\lambda+1-5\theta/3)}F(\zeta\gamma^{\theta/3})d\gamma\right) \tag{31}$$

The definition of EK integral operator reduced (31) into

$$D_t^{\theta}u=D_t^{\lambda}\left(t^{\lambda-\frac{5\theta}{3}}\left[\left(K_{3/\theta}^{1-\frac{2\theta}{3},\lambda-\theta}F\right)(\zeta)\right]\right) \tag{32}$$

$$as\ \zeta=xt^{-\theta/3},\ F\in C'(0,\infty)\text{ then }t\,D_tF(\zeta)=tx\left(-\frac{\theta}{3}\right)t^{\frac{-\theta}{3}-1}D_{\zeta}F(\zeta)=\frac{-\theta}{3}\zeta D_{\zeta}F(\zeta) \tag{33}$$

$$Now\ D_t^{\lambda}\left(t^{\lambda-\frac{5\theta}{3}}\left(K_{\frac{3}{\theta}}^{1-\frac{2\theta}{3},\lambda-\theta}F\right)\zeta\right)=D_t^{\lambda-1}\left(D_t\left(t^{\lambda-\frac{5\theta}{3}}\left(K_{\frac{3}{\theta}}^{1-\frac{2\theta}{3},\lambda-\theta}F\right)\zeta\right)\right) \tag{34}$$

$$=D_t^{\lambda-1}\left(t^{\lambda-1-\frac{5\theta}{3}}\left((\lambda-\frac{5\theta}{3}-\frac{\theta}{3}\zeta\,D_{\zeta})\left(K_{\frac{3}{\theta}}^{1-\frac{2\theta}{3},\lambda-\theta}F\right)\zeta\right)\right) \tag{35}$$

Repeating above arguments (λ-1) times, to generate

$$D_t^{\lambda}\left(t^{\lambda-\frac{5\theta}{3}}\left(K_{\frac{3}{\theta}}^{1-\frac{2\theta}{3},\lambda-\theta}F\right)\zeta\right)=t^{-4\theta/3}\prod_{j=0}^{\lambda-1}\left(1+j-\frac{5\theta}{3}-\frac{\theta}{3}\zeta D_{\zeta}\right)\left(K_{3/\theta}^{1-\frac{2\theta}{3},\lambda-\theta}F\right)(\zeta)=t^{-\frac{5\theta}{3}}\left(E_{3/\theta}^{1-\frac{5\theta}{3},\theta}F\right)(\zeta) \quad (36)$$

$$\therefore D_t^{\theta}u=t^{-\frac{5\theta}{3}}\left(E_{3/\theta}^{1-\frac{5\theta}{3},\theta}F\right)(\zeta) \quad (37)$$

Similarly, proceeding above steps (29-37) on distinct transformations.

From similarity transformations $\zeta = x.t^{-\theta/3}$ *and* $u=t^{-2\theta/3}G(\zeta)$, we obtain

$$D_t^{\theta}v=t^{-\frac{5\theta}{3}}\left(E_{3/\theta}^{1-\frac{5\theta}{3},\theta}G\right)(\zeta) \quad (38)$$

From similarity transformations $\zeta = x.t^{-\theta/3}$ *and* $u=t^{-4\theta/3}H(\zeta)$, we obtain

$$D_t^{\theta}w=t^{-\frac{7\theta}{3}}\left(E_{3/\theta}^{1-\frac{7\theta}{3},\theta}G\right)(\zeta) \quad (39)$$

Using above obtained similarity transformations and (37-39), we formed a set of nonlinear system of HSc-KdVFODEs.

$$\begin{cases}\left(E_{3/\alpha}^{1-\frac{5\theta}{3},\theta}F\right)(\zeta)=\frac{1}{4}F'''(\zeta)+3F(\zeta).F'(\zeta)-6G(\zeta)G'(\zeta)+3H'(\zeta)\\ \left(E_{3/\alpha}^{1-\frac{5\theta}{3},\theta}G\right)(\zeta)=-\frac{1}{2}G'''(\zeta)-3F(\zeta).G'(\zeta)\\ \left(E_{3/\alpha}^{1-\frac{7\theta}{3},\theta}G\right)(\zeta)=-\frac{1}{2}H'''(\zeta)-3F(\zeta).H'(\zeta)\end{cases} \quad (40)$$

5. Construction of explicit solution

Now, we shall obtain explicit solutions for coupled time fractional HSC-KdV system (1), by applying the power series expansion technique on set of equations (40), we set

$$F(\zeta)=\sum_{n=0}^{\infty}a_n\zeta^n\,;G(\zeta)=\sum_{n=0}^{\infty}b_n\zeta^n \text{ and } H(\zeta)=\sum_{n=0}^{\infty}c_n\zeta^n \quad (41)$$

where a_n, b_n *and* c_n are constants to be find later after necessary calculations. Now substitute (41) in the set of equations (40), we get

$$\sum_{n=0}^{\infty}\frac{\Gamma(2-\frac{2\theta}{3}+\frac{n\theta}{3})}{\Gamma(2-\frac{5\theta}{3}+\frac{n\theta}{3})}a_n\zeta^n=\frac{1}{4}\left(\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)a_{n+3}\zeta^n\right)+3\left(\sum_{n=0}^{\infty}a_n\zeta^n\right)\left(\sum_{n=0}^{\infty}(n+1)a_{n+1}\zeta^n\right)$$
$$-6\left(\sum_{n=0}^{\infty}b_n\zeta^n\right)\left(\sum_{n=0}^{\infty}(n+1)b_{n+1}\zeta^n\right)+3\left(\sum_{n=0}^{\infty}(n+1)c_{n+1}\zeta^n\right)$$
$$\sum_{n=0}^{\infty}\frac{\Gamma(2-\frac{2\theta}{3}+\frac{n\theta}{3})}{\Gamma(2-\frac{5\theta}{3}+\frac{n\theta}{3})}b_n\zeta^n=-\frac{1}{2}\left(\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)b_{n+3}\zeta^n\right)-3\left(\sum_{n=0}^{\infty}a_n\zeta^n\right)\left(\sum_{n=0}^{\infty}(n+1)b_{n+1}\zeta^n\right) \tag{42}$$
$$\sum_{n=0}^{\infty}\frac{\Gamma(2-\frac{4\theta}{3}+\frac{n\theta}{3})}{\Gamma(2-\frac{7\theta}{3}+\frac{n\theta}{3})}b_n\zeta^n=-\frac{1}{2}\left(\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)c_{n+3}\zeta^n\right)-3\left(\sum_{n=0}^{\infty}a_n\zeta^n\right)\left(\sum_{n=0}^{\infty}(n+1)c_{n+1}\zeta^n\right)$$

Putting *n=0* in (42), we obtain

$$a_3=\frac{2}{3}\frac{\Gamma(2-\frac{2\theta}{3})}{\Gamma(2-\frac{2\theta}{3})}a_0-2a_0a_1+4b_0b_1-2c_1$$
$$b_3=-\frac{1}{3}\frac{\Gamma(2-\frac{2\theta}{3})}{\Gamma(2-\frac{2\theta}{3})}b_0-a_0b_1 \tag{43}$$
$$c_3=-\frac{1}{3}\frac{\Gamma(2-\frac{4\theta}{3})}{\Gamma(2-\frac{7\theta}{3})}b_0-a_0c_1$$

And comparing the coefficients of ζ^n in (42), to get

$$a_{n+3}=\frac{4}{(n+1)(n+2)(n+3)}\left[\sum_{n=0}^{\infty}\frac{\Gamma(2-\frac{2\theta}{3}+\frac{n\theta}{3})}{\Gamma(2-\frac{5\theta}{3}+\frac{n\theta}{3})}a_n-3\left(\sum_{k=0}^{\infty}(n+1-k)a_ka_{n+1-k}\right)\right.$$
$$\left.+6\left(\sum_{k=0}^{\infty}(n+1-k)b_kb_{n+1-k}\right)-3\left(\sum_{n=0}^{\infty}(n+1)c_{n+1}\right)\right]$$
$$b_{n+3}=\frac{-2}{(n+1)(n+2)(n+3)}\left[\sum_{n=0}^{\infty}\frac{\Gamma(2-\frac{2\theta}{3}+\frac{n\theta}{3})}{\Gamma(2-\frac{5\theta}{3}+\frac{n\theta}{3})}b_n+3\left(\sum_{k=0}^{\infty}(n+1-k)a_kb_{n+1-k}\right)\right] \tag{44}$$
$$c_{n+3}=\frac{-2}{(n+1)(n+2)(n+3)}\left[\sum_{n=0}^{\infty}\frac{\Gamma(2-\frac{4\theta}{3}+\frac{n\theta}{3})}{\Gamma(2-\frac{7\theta}{3}+\frac{n\theta}{3})}c_n+3\left(\sum_{k=0}^{\infty}(n+1-k)a_kc_{n+1-k}\right)\right]$$

The exact explicit solution by using above set of equations (59-62) simultaneously

$$F(\zeta)=a_0+a_1(xt^{-\theta/3})+a_2(xt^{-\theta/3})^2+\left[\frac{2}{3}\frac{\Gamma(2-\frac{2\theta}{3})}{\Gamma(2-\frac{2\theta}{3})}a_0-2a_0a_1+4b_0b_1-2c\right](xt^{-\theta/3})^3+$$

$$+\sum_{n=1}^{\infty}\frac{4}{(n+1)(n+2)(n+3)}\left[\sum_{n=0}^{\infty}\frac{\Gamma(2-\frac{2\theta}{3}+\frac{nv}{3})}{\Gamma(2-\frac{5\theta}{3}+\frac{n\theta}{3})}a_n-3\left(\sum_{k=0}^{\infty}(n+1-k)a_ka_{n+1-k}\right)+\right. \tag{45}$$

$$\left.6\left(\sum_{k=0}^{\infty}(n+1-k)b_kb_{n+1-k}\right)-3\left(\sum_{n=0}^{\infty}(n+1)c_{n+1}\right)\right](xt^{-\theta/3})^{n+3}$$

$$G(\zeta)=b_0+b_1(xt^{-\theta/3})+b_2(xt^{-\theta/3})^2+\left[-\frac{1}{3}\frac{\Gamma(2-\frac{2\theta}{3})}{\Gamma(2-\frac{2\theta}{3})}b_0-a_0b_1\right](xt^{-\theta/3})^3+$$

$$+\sum_{n=1}^{\infty}\frac{-2}{(n+1)(n+2)(n+3)}\left[\sum_{n=0}^{\infty}\frac{\Gamma(2-\frac{2\theta}{3}+\frac{n\theta}{3})}{\Gamma(2-\frac{5\theta}{3}+\frac{n\theta}{3})}b_n+3\left(\sum_{k=0}^{\infty}(n+1-k)a_kb_{n+1-k}\right)\right](xt^{-\theta/3})^{n+3} \tag{46}$$

$$H(\zeta)=c_0+c_1(xt^{-\theta/3})+c_2(xt^{-\theta/3})^2+\left[-\frac{1}{3}\frac{\Gamma(2-\frac{4\theta}{3})}{\Gamma(2-\frac{7\theta}{3})}c_0-a_0c_1\right](xt^{-\theta/3})^3+$$

$$+\sum_{n=1}^{\infty}\frac{-2}{(n+1)(n+2)(n+3)}\left[\sum_{n=0}^{\infty}\frac{\Gamma(2-\frac{4\theta}{3}+\frac{n\theta}{3})}{\Gamma(2-\frac{7\theta}{3}+\frac{n\theta}{3})}c_n+3\left(\sum_{k=0}^{\infty}(n+1-k)a_kc_{n+1-k}\right)\right](xt^{-\theta/3})^{n+3} \tag{47}$$

6. Convergence Analysis

In this part, we will discuss the convergence of obtained power series solution. Consider equations (44-47)

$$
\begin{aligned}
|a_{n+3}| &\le \left[\left|\frac{\Gamma(2-\frac{2\theta}{3}+\frac{n\theta}{3})}{\Gamma(2-\frac{5\theta}{3}+\frac{n\theta}{3})}\right||a_n|+3\left(\sum_{k=0}^{\infty}|a_k||a_{n+1-k}|\right)+6\left(\sum_{k=0}^{\infty}|b_k||b_{n+1-k}|\right)+3\left(\sum_{n=0}^{\infty}|c_{n+1}|\right)\right]\\
|b_{n+3}| &\le \left[\left|\frac{\Gamma(2-\frac{2\theta}{3}+\frac{n\theta}{3})}{\Gamma(2-\frac{5\theta}{3}+\frac{n\theta}{3})}\right||b_n|+3\left(\sum_{k=0}^{\infty}|a_k||b_{n+1-k}|\right)\right]\\
|c_{n+3}| &\le \left[\left|\frac{\Gamma(2-\frac{4\theta}{3}+\frac{n\theta}{3})}{\Gamma(2-\frac{7\theta}{3}+\frac{n\theta}{3})}\right||c_n|+3\left(\sum_{k=0}^{\infty}|a_k||c_{n+1-k}|\right)\right]
\end{aligned} \tag{48}
$$

It is well known that expressions like $\left|\frac{\Gamma(n)}{\Gamma(m)}\right| < 1$ for arbitrary real values of m and n. Thus (48) becomes

$$
\begin{aligned}
|a_{n+3}| &\leq M_1\left[|a_n| + \left(\sum_{k=0}^{\infty}|a_k||a_{n+1-k}|\right) + \left(\sum_{k=0}^{\infty}|b_k||b_{n+1-k}|\right) + \left(\sum_{n=0}^{\infty}|c_{n+1}|\right)\right] \\
|b_{n+3}| &\leq M_2\left[|b_n| + \left(\sum_{k=0}^{\infty}|a_k||b_{n+1-k}|\right)\right] \\
|c_{n+3}| &\leq M_3\left[|c_n| + \left(\sum_{k=0}^{\infty}|a_k||c_{n+1-k}|\right)\right]
\end{aligned}
\tag{49}
$$

where M_1,M_2 and M_3 are maximums of the arbitrary coefficients involved in set of equations (49). Now we introduce some another power series

$$P(\chi) = \sum_{n=0}^{\infty} p_n \chi^n ; Q(\chi) = \sum_{n=0}^{\infty} q_n \chi^n \text{ and } R(\chi) = \sum_{n=0}^{\infty} r_n \chi^n \tag{50}$$

where $p_i=|a_i|$, $q_i =|b_i|$, $r_i= |c_i|$, i = 1, 2, 3, …then we can have

$$
\begin{aligned}
p_{n+3} &\leq M_1\left[p_n + \sum_{k=0}^{n} p_n p_{n+1-k} + \sum_{k=0}^{n} p_n q_{n+1-k} + \sum_{n=0}^{n} r_{n+1}\right] \\
q_{n+3} &\leq M_2\left[q_n + \sum_{k=0}^{n} p_n q_{n+1-k}\right] \\
r_{n+3} &\leq M_3\left[r_n + \sum_{k=0}^{n} p_n r_{n+1-k}\right]
\end{aligned}
\tag{51}
$$

It is easily seen that $|a_i| \leq p_i,\ |b_i| \leq q_i,\ |c_i| \leq r_i, i = 1,2,3...$ then

$$
\begin{aligned}
P(\chi) &= p_0 + p_1\chi + p_2\chi^2 + p_3\chi^3 + \sum_{n=1}^{\infty} M_1\left[p_n + \sum_{k=0}^{n} p_n p_{n+1-k} + \sum_{k=0}^{n} p_n q_{n+1-k} + \sum_{n=0}^{n} r_{n+1}\right]\chi^{n+3} \\
Q(\chi) &= q_0 + q_1\chi + q_2\chi^2 + q_3\chi^3 + \sum_{n=1}^{\infty} M_2\left[q_n + \sum_{k=0}^{n} p_n q_{n+1-k}\right]\chi^{n+3} \\
R(\chi) &= r_0 + r_1\chi + r_2\chi^2 + r_3\chi^3 + \sum_{n=1}^{\infty} M_3\left[r_n + \sum_{k=0}^{n} p_n r_{n+1-k}\right]\chi^{n+3}
\end{aligned}
\tag{52}
$$

Assuming the implicit function system with independent variable χ

$$
\begin{aligned}
P_1(\chi, P) &= P(\chi) - p_0 - p_1\chi - p_2\chi^2 - p_3\chi^3 - \sum_{n=1}^{\infty} M_1\left[p_n + \sum_{k=0}^{n} p_n p_{n+1-k} + \sum_{k=0}^{n} p_n q_{n+1-k} + \sum_{n=0}^{n} r_{n+1}\right]\chi^{n+3} \\
Q_1(\chi, Q) &= Q(\chi) - q_0 - q_1\chi - q_2\chi^2 - q_3\chi^3 - \sum_{n=1}^{\infty} M_2\left[q_n + \sum_{k=0}^{n} p_n q_{n+1-k}\right]\chi^{n+3} \\
R_1(\chi, R) &= R(\chi) - r_0 - r_1\chi - r_2\chi^2 - r_3\chi^3 - \sum_{n=1}^{\infty} M_3\left[r_n + \sum_{k=0}^{n} p_n r_{n+1-k}\right]\chi^{n+3}
\end{aligned}
\tag{53}
$$

Here P_1, Q_1 and R_1 are analytic in a neighborhood of *(0, p_0), (0, q_0) and (0, r_0)* respectively, where $P_1(0, p_0) = 0$; $Q_1(0,q_0)= 0$ and $R_1(0, r_0) = 0$ and $\frac{\partial}{\partial P}(P_1(0,p_0)) \neq 0; \frac{\partial}{\partial Q}(Q_1(0,q_0)) \neq 0 \; and \; \frac{\partial}{\partial R}(R_1(0,r_0)) \neq 0$ then by Implicit function theorem[40], we reached at convergence of power series solution.

7. Conservation laws

In mathematical and physical point of view, conservation laws play most important role in analysis of existence, stability and uniqueness of solutions of fractional and classical PDEs. To obtain the conservation laws of system of FPDEs, generalization of Noether's theorem suggested by Ibragimov [38]. The conservation laws in fractional system is almost similar to classical order system. These conservation laws can extend to fractional system as explained in [35-37] for convenience of readers.

Let us define the conserved vector for coupled HSC-KdV system of fractional PDEs

$$\lambda = (\lambda^t, \lambda^x), \tag{54}$$

with λ^x and λ^t are components of vector, known as conserved flux and density functions of variables *x, t, u, v, w* and partial derivatives of *u, v, w*, which satisfy continuity equation

$$D_t(\lambda^t) + D_x(\lambda^x) = 0, \tag{55}$$

where $D_t$and D_x are total derivatives with variables*t* and *x*.

Let us define the formal Lagrangian of system (1) with *A, B, C* as new dependent variables of *x* and *t.*

$$\ell = A(\partial_t^\theta u - \frac{1}{4}u_{xxx} - 3uu_x + 6vv_x - 3w_x) + B(\partial_t^\theta v + \frac{1}{2}v_{xxx} - 3uv_x) + C(\partial_t^\theta w + \frac{1}{2}w_{xxx} + 3uw_x) \tag{56}$$

The adjoint system of (1) given as

$$\frac{\delta \ell}{\delta u} = 0, \; \frac{\delta \ell}{\delta v} = 0, \; \frac{\delta \ell}{\delta w} = 0, \tag{57}$$

with Euler-Lagrange operators for *u, v, w* is given by

$$\frac{\delta}{\delta u^i} = \frac{\partial}{\partial u^i} + (D_t^\theta)^* \frac{\partial}{\partial (D_t^\theta u^i)} - D_x \frac{\partial}{\partial u_x^{\;i}} + D_{xx} \frac{\partial}{\partial u_{xx}^{\;i}} - D_{xxx} \frac{\partial}{\partial u_{xxx}^{\;i}}, \tag{58}$$

where $(D_t^\theta)^*$ represents the adjoint operator to D_t^θ, which is defined in right-sided Caputotime-fractional derivative of order $'\theta'$ as

$$(D_t^\theta)^* = {}_t^C D_T^\theta u = \frac{(-1)^n}{\Gamma(n-\theta)} \int_t^T (\upsilon - t)^{n-1-\theta} D_\upsilon^\theta u(\upsilon, x) d\upsilon; \;\; n = [\theta] + 1 \tag{59}$$

Using equations (54-59), we obtained the adjoint system of equations of system (1)

$$(D_t^\theta)^* A+3uA_x+3Bv_x+3Cw_x+\frac{1}{4}A_{xxx}=0,$$
$$(D_t^\theta)^* B+6vA_x-3uB_x-3Bu_x-\frac{1}{2}B_{xxx}=0, \quad (60)$$
$$(D_t^\theta)^* C-3u_xC-3uC_x+3A_x-\frac{1}{2}C_{xxx}=0.$$

Now, the components of conserved vector are given by the following expressions

$$\lambda^x=\xi\ell+W_j\left[\frac{\partial\ell}{\partial u_x^j}-D_x\left(\frac{\partial\ell}{\partial u_{xx}^j}\right)+D_x^2\left(\frac{\partial\ell}{\partial u_{xxx}^j}\right)\right]+D_x(W_j)\left[\frac{\partial\ell}{\partial u_{xx}^j}-D_x\left(\frac{\partial\ell}{\partial u_{xxx}^j}\right)\right]+D_x^2(W_j)\left[\frac{\partial\ell}{\partial u_{xxx}^j}\right],$$
$$\lambda^t=\tau\ell+D_t^{\theta-1}(W_j)\frac{\partial\ell}{\partial D_t^\theta u^j}+I\left(W_j,\ D_t\frac{\partial\ell}{\partial D_t^\theta u^j}\right), \quad (61)$$

here $W_j=\eta_j-\xi_ju_x-\tau_ju_t$, ℓ is defined above in (61) and I is integral defined as

$$I(f,g)=\frac{1}{\Gamma(1-\theta)}\int_0^t\int_t^T\frac{f(s,x)g(\mu,x)}{(\mu-s)^\theta}d\mu ds \quad (62)$$

For symmetry generator X_2 we have $W_1=-u_x$, $W_2=-v_x$ and $W_3=-w_x$ and calculated components of the conserved vector 'λ' with the help of (61)

$$\lambda^x=(A\partial_t^\theta u+B\partial_t^\theta v+C\partial_t^\theta w)+\frac{u_xA_{xx}}{4}-\frac{u_{xx}A_x}{4}+\frac{v_{xx}B_x}{2}-\frac{v_xB_{xx}}{2}-\frac{w_xC_{xx}}{2}+\frac{w_{xx}C_x}{2}$$
$$\lambda^t=AD_t^{\theta-1}(-u_x)+I(-u_x,A_t)+BD_t^{\theta-1}(-v_x)+I(-v_x,B_t)+CD_t^{\theta-1}(-w_x)+I(-w_x,C_t) \quad (63)$$

For symmetry generator X_1, we have $W_1=-2u-xu_x-\frac{3t}{\theta}u_t, W_2=-2v-xv_x-\frac{3t}{\theta}v_t, W_3=-4w-xw_x-\frac{3t}{\theta}w_t$ as characteristic functions and components of the conserved vector 'λ' with the help of (61-62)

$$\begin{aligned}\lambda^x=&\,x(A\partial_t^\theta u+B\partial_t^\theta v+C\partial_t^\theta w)+6Au^2+\frac{uA_{xx}}{2}+Au_{xx}+\frac{xu_xA_{xx}}{4}+\frac{9tuu_tA}{\theta}-\frac{3u_xA_x}{4}-\frac{xu_{xx}A_x}{4}-\\&\frac{3tu_{tx}A_x}{4\theta}+\frac{3tu_{txx}A}{4\theta}-12v^2A-6uvB-vB_{xx}-\frac{xv_xB_{xx}}{2}-\frac{18tuv_tA}{\theta}-\frac{9tuv_tB}{\theta}-\frac{3tv_tB_{xx}}{2\theta}+\frac{3B_xv_x}{2}+\\&\frac{xv_{xx}B_x}{2}+\frac{3tB_xv_{xt}}{2\theta}-\frac{3Btv_{txx}}{2\theta}+12wA-12uwC-\frac{4wC_{xx}}{2}-\frac{xw_xC_{xx}}{2}-\frac{9Atw_t}{\theta}-\frac{9Cutw_t}{\theta}-\\&\frac{3tw_tC_{xx}}{2\theta}+\frac{4w_xC_x}{2}+\frac{xw_{xx}C_x}{2}+\frac{3tw_{tx}C_x}{2\theta}-2w_{xx}C-\frac{3tw_{txx}C}{2\theta}.\end{aligned} \quad (64)$$

$$
\begin{aligned}
\lambda^{t} = \frac{3t}{\theta}\Bigg[& (A\partial_t^{\theta} u + B\partial_t^{\theta} v + C\partial_t^{\theta} w) - \frac{Au_{xxx}}{4} + \frac{Bv_{xxx}}{2} + \frac{Cw_{xxx}}{2} - 3Auu_x + 6Avv_x - 3Aw_x + 3Buv_x + 3Cuw_x \Bigg] \\
& + AD_t^{\theta-1}(-2u - xu_x - \frac{3t}{\theta}u_t) + I\left(-2u - xu_x - \frac{3t}{\theta}u_t, A_t\right) + BD_t^{\theta-1}(-2v - xv_x - \frac{3t}{\theta}v_t) \\
& + I\left(-2v - xv_x - \frac{3t}{\theta}v_t, B_t\right) + CD_t^{\theta-1}(-4w - xw_x - \frac{3t}{\theta}w_t) + I\left(-4w - xw_x - \frac{3t}{\alpha}w_t, C_t\right)
\end{aligned} \tag{65}
$$

8. Conclusions

In this work, we performed the application of the fractional Lie symmetry reduction analysis to the time fractional HSC-KdV system. By symmetry, we determined the vector field corresponding to the system of equations and reduced it into FODEs. Further, we have treated the system of reduced FODEs with EK differential and integral operators and found the explicit solution by using the power expansion technique. Sequentially, the convergence of the power series solution is analyzed and concluded that the combination of the two techniques has achieved better results and could be applied to fractional fluid dynamical problems. Finally, pointed out the importance of conservation laws of time fractional HSC-KdV system with the use of Noether's theorem.

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