Hypercube embeddings and Cayley graphs generated by transpositions

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Abstract

A graph is called a *partial cube* if it can be embedded into a hypercube isometrically. In this paper, we study a class of Cayley graphs —*Cayley graphs generated by transpositions* and show that a Cayley graph Γ generated by transpositions is a partial cube if and only if Γ is a bubble sort graph. This result enhances a result of Alahmadi et al. [Math. Meth. Appl. Sci. 39 (2016), 4856–4865]: BS_n is a partial cube. As a corrollary, we give the analytical expressions of the Wiener indices of bubble sort graphs.

Keywords: Cayley graph; Partial cube; Isometric embedding; Bubble sort graph.

1 Introduction

Graphs that can be isometrically embedded into hypercubes are called *partial cubes*, introduced by Graham and Pollak [13] as a model for interconnection networks and later for other applications, for examples see [3, 9, 20]. For research on partial cubes, we refer the readers to two books [7, 14] and a survey [25].

One of the most challenging open problems in the area is to classify regular partial cubes, in particular, vertex-transitive partial cubes. Wiechsel in 1992 [26] considered distance-regular partial cubes and classified all distance-regular partial cubes based on their girth: hypercubes are the only ones with girth 4, the 6-cycle and the middle-level graphs are the only ones with girth 6, and even cycles of length at least 8 are the only ones with higher girths. Koolen [21] generalized this result to a certain broader metrical hierarchy. Restricting the cubic case, Marc [22] classified all cubic, vertex-transitive partial cubes. On the other hand, B. Brešar et al. [4] introduced a new family of graphs: mirror graphs and proved that mirror graphs are a subfamily of vertex-transitive partial cubes and, classified all mirror graphs that are obtained by cubic inflation (thus are cubic graphs). And furthermore, Marc [23] proved that the mirror graphs are equivalent to the Cayley graphs of a finite Coxeter group with canonical generators and the tope graphs of a reflection arrangement.

A motivation for the study of partial cubes with high minimum degree comes from the theory of oriented matroids [11]. Besides hypercubes, we know a little about partial cubes with

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high minimum degree. On the other hand, Cayley graphs are vertex-transitive. In view of these, in this paper, we would like to form a natural connection between the study of vertex-transitive partial cubes and the study of Cayley graphs with high degree, and consider an important class of graphs Γ in networks theory—Cayley graphs on the symmetric group S_n generated by transpositions, and obtain the main theorem: Γ is a partial cube if and only if Γ is a bubble sort graph BS_n, i.e., Theorem 3.3. Notice that BS_n is an $(n-1)$-regular mirror graph.

This paper is organized as follows. In Section 2, we provide some definitions and lemmas. Next, we prove Theorem 3.3 and, as one of its applications, calculate the Wiener index of the bubble sort graph BS_n in Section 3.

2 Preliminaries

In this paper a graph Γ considered is simple, undirected with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For $u, v \in V(\Gamma)$, the *distance* $d_\Gamma(u, v)$ (without causing confusion, $d(u, v)$ for short) is the length of a shortest path between u and v in Γ. A subgraph H of a graph Γ is called *isometric* if $d_H(u, v) = d_\Gamma(u, v)$ for all $u, v \in V(H)$.

An *embedding* from graph Γ to graph Γ' is an injective mapping $\pi : V(\Gamma) \to V(\Gamma')$ satisfying $uv \in E(\Gamma) \implies \pi(u)\pi(v) \in E(\Gamma')$ for $u, v \in V(\Gamma)$. If π additionally satisfies $d_\Gamma(u, v) = d_{\Gamma'}(\pi(u), \pi(v))$ for any $u, v \in V(\Gamma)$, then π is called *isometric*. Notice that if Γ can be embedded into Γ' isometrically, denoted by $\Gamma \hookrightarrow \Gamma'$, then the subgraph H of Γ' induced by the set $\pi(V(\Gamma)) = \{\pi(u)|u \in V(\Gamma)\}$ is isometric and $H \cong \Gamma$. It's easy to see that the relation '$\hookrightarrow$' is transitive.

Observation 2.1. *Let Γ, Γ', Γ'' be graphs. If $\Gamma \hookrightarrow \Gamma'$ and $\Gamma' \hookrightarrow \Gamma''$, then $\Gamma \hookrightarrow \Gamma''$.*

Proof. Since $\Gamma \hookrightarrow \Gamma'$, $\Gamma' \hookrightarrow \Gamma''$, there exist isometric embeddings $\pi : V(\Gamma) \to V(\Gamma')$ and $\tau : V(\Gamma') \to V(\Gamma'')$, respectively. Then, for any $u, v \in V(\Gamma)$,

$$d_\Gamma(u, v) = d_{\Gamma'}(\pi(u), \pi(v)) = d_{\Gamma''}(\tau(\pi(u)), \tau(\pi(v))) =: d_{\Gamma''}(\tau\pi(u), \tau\pi(v)),$$

where $\tau\pi$ is the composition of π and τ from Γ to Γ'' defined as $\tau\pi(v) = \tau(\pi(v))$ for any $v \in V(\Gamma)$. Then $\tau\pi$ is an isometric embedding from Γ to Γ''. Therefore, $\Gamma \hookrightarrow \Gamma''$ □

Let X be a finite or infinite set. A *hypercube* $\mathcal{H}(X)$ on X is a graph whose vertex set is the family of all subsets of X, denoted by $\mathcal{P}(X)$, two vertices are adjacent if and only if they differ by a singleton. If X is finite, the size of X is called the *dimension* of $\mathcal{H}(X)$. In this case, we can also denote $\mathcal{H}(X)$ as Q_n if $|X| = n$. Let X_1, X_2 be finite subsets of X. The distance between X_1 and X_2 on the hypercube $\mathcal{H}(X)$ is called the *Hamming distance*, which evidently equals $|X_1 \triangle X_2|$, where $X_1 \triangle X_2$ is the symmetric difference of X_1 and X_2. We say a graph is a *partial cube* on the set X, if it can be embedded into $\mathcal{H}(X)$ isometrically.

2.1 Djoković-Winkler relation

In this subsection, we only consider connected graphs.

Definition 2.2 (Djoković-Winkler relation [8, 29])**.** Let Γ be a graph, $e = uv$, $f = xy$ two edges of Γ. The *Djoković-Winkler relation* Θ on $E(\Gamma)$ is defined by: $e\,\Theta\, f \iff d(u, x) + d(v, y) \neq d(u, y) + d(v, x)$.

The following properties can be easily proved by simple calculation, but they are useful for our proof in the next section.

Observation 2.3. *Let Γ be a bipartite graph and $e, f \in E(\Gamma)$. If $e\,\Theta\,f$, then e and f are disjoint.*

Observation 2.4. *Let Γ be a graph and C an isometric even cycle in it. Then each pair of antipodal edges of C are in relation Θ.*

It is obvious that Djoković-Winkler relation Θ is reflexive and symmetric, but need not be transitive. By means of transitivity of the relation Θ, a characterization of partial cubes by means of relation Θ was obtained as follows:

Lemma 2.5. [29] *Let Γ be a connected graph. Γ is a partial cube if and only if Γ is bipartite and Θ is a transitive relation on $E(\Gamma)$.*

In other words, Γ is a partial cube if and only if Γ is bipartite and Θ is an equivalence relation on $E(\Gamma)$. In a partial cube Γ with edge set $E(\Gamma)$, an *equivalence class* is a subset of $E(\Gamma)$ in which edges are related to each other by the equivalence relation Θ. Obviously the equivalence classes of $E(\Gamma)$ form a partition of $E(\Gamma)$.

2.2 Cayley graphs generated by transpositions

Let's denote the set $\{1, 2, \cdots, n\}$ by $[n]$ and refer to its elements as *points*. A bijection of $[n]$ onto itself is called a *permutation* of $[n]$. Denote by ι the identity permutation. There are two common ways in which permutations are written. First of all, a permutation $\mathbf{u}$ can be written as a linear order $u_1u_2\cdots u_n$ of different points in $[n]$ for which $\mathbf{u}(i) = u_i$. This notation is called the *one-line notation* of permutations. A permutation $\mathbf{u}$ is called an *r-cycle* ($r = 1, 2, \cdots$) if for r distinct points $u_1, u_2, \cdots, u_r$ of $[n]$, $\mathbf{u}$ maps u_i onto u_{i+1} ($i = 1, \cdots, r-1$), maps u_r onto u_1, and leaves all other points fixed; denote by $\mathbf{u} = (u_1u_2\cdots u_r)$. In particular, 2-cycle $\mathbf{u} = (u_1u_2)$ is called a *transposition*. The second common way to specify a permutation $\mathbf{u}$ is to write $\mathbf{u}$ as a product of disjoint cycles, i.e., compositions of permutations represented by cycles, in which 1-cycles are omitted, and call it a *cycle notation* of $\mathbf{u}$. For example, a permutation $\mathbf{u} = 62837154$ in the one-line notation can be written as $\mathbf{u} = (16)(384)(57)$ in the cycle notation. In the present paper, we mostly use the one-line notation to represent a permutation, but for convenience, we use the cycle notation in Claim 3 of proof of Theorem 3.3.

Let $\mathbf{u}$ be a permutation of $[n]$ with $\mathbf{u}(i) = i$ for some $i \in [n]$, i.e., i is in a 1-cycle of $\mathbf{u}$. Then we call i a *fixed point* of $\mathbf{u}$. The set of all fixed points of $\mathbf{u}$ is denoted by $\mathrm{fix}(\mathbf{u})$. If i is not a fixed point of $\mathbf{u}$, we say that it's a *support point* and denote the set of all support points by $\mathrm{supp}(\mathbf{u})$. In other words, $\mathrm{supp}(\mathbf{u}) = [n] \setminus \mathrm{fix}(\mathbf{u})$. The *inversion set* $\mathrm{Inv}(\mathbf{u})$ of $\mathbf{u}$ is defined as: $\mathrm{Inv}(\mathbf{u}) = \{\{i, j\} | (i-j)(\mathbf{u}^{-1}(i) - \mathbf{u}^{-1}(j)) < 0\}$. The cardinality of $\mathrm{Inv}(\mathbf{u})$ is called the *inverse number* of $\mathbf{u}$, denoted by $\mathrm{inv}(\mathbf{u})$.

Let G be a group with the group operation denoted by '$\cdot$' and S a subset of G satisfying: (i) $x \in S \Longleftrightarrow x^{-1} \in S$, (ii) the identity element is not in S. The *Cayley graph* Γ associated with (G, S), denoted by $\mathrm{Cay}(G, S)$, is a simple graph with vertex set G and $u, v \in G$ are adjacent if and only if $u^{-1} \cdot v \in S$. Notice that all Cayley graphs are vertex-transitive because, for any $g \in G$, the mapping l_g on G: $x \mapsto g \cdot x$ for $x \in G$, is an automorphism of $\mathrm{Cay}(G, S)$.

We denote $\mathfrak{S}_n$ as the symmetric group on n letters, most often points of $[n]$, i.e., the set of all permutations with function compositions as the group operation. If S is a subset of transpositions on $[n]$, then the Cayley graph $\mathrm{Cay}(\mathfrak{S}_n, S)$ on $\mathfrak{S}_n$ is called the *Cayley graph generated by transpositions* S. by the definition of Cayley graphs, permutation $\mathbf{u} = u_1u_2\cdots u_n$ is adjacent to permutation $\mathbf{v} = v_1v_2\cdots v_n$ in $\mathrm{Cay}(\mathfrak{S}_n, S)$ if and only if for $(ij) \in S$, $u_i = v_j$, $u_j = v_i$ and $u_k = v_k$ for $k \neq i, j$. In this case, we say that the edge $e = \mathbf{uv}$ is an *(ij)-edge* and denote $l(e) = (ij)$. Clearly, $\mathrm{Cay}(\mathfrak{S}_n, S)$ is bipartite with one part consisting of permutations of odd inverse number and the other part consisting of permutations of even inverse number. Let S be a set of transpositions on $[n]$. The *transposition generating graph* $T(S)$ of S is the graph with vertex set $[n]$ and two vertices i and j are adjacent if and only if $(ij) \in S$. Among Cayley graphs generated by transpositions, there are three special graphs as follows: the *bubble sort graph* BS_n, the *modified bubble sort graph* MB_n and the *star graph* ST_n, satisfying their transposition generating graphs $T(S)$ of order n are the path P_n, the cycle C_n and the star $K_{1,n-1}$, respectively. For example, $MB_3 = Cay(\mathfrak{S}_3, \{(12), (23), (13)\})$, which is isomorphic to the complete bipartite graph $K_{3,3}$.

We have the following properties about the Cayley graphs generated by transpositions:

Lemma 2.6. [12] *Let* $\Gamma = \mathrm{Cay}(\mathfrak{S}_n, S)$ *be a Cayley graph generated by transpositions. Then* Γ *is connected if and only if* $T(S)$ *is connected.*

Lemma 2.7. [6] *Let* S *and* S' *be two sets of transpositions on* $[n]$*. Then* $\mathrm{Cay}(\mathfrak{S}_n, S)$ *and* $\mathrm{Cay}(\mathfrak{S}_n, S')$ *are isomorphic if and only if* $T(S)$ *and* $T(S')$ *are isomorphic.*

Lemma 2.8. *Let* n', n $(n' \leqslant n)$ *be two positive integers. Let* S' *be a transposition set on* $[n']$, S *a transposition set on* $[n]$*. If* $T(S')$ *can be embedded into* $T(S)$ *as an induced subgraph, then* $\mathrm{Cay}(\mathfrak{S}_{n'}, S')$ *can also be embedded into* $\mathrm{Cay}(\mathfrak{S}_n, S)$ *as an induced subgraph.*

Proof. Let $\pi : [n'] \to [n]$ be the embedding from $T(S')$ to $T(S)$ as an induced subgraph. Then by the definition of transposition generating graphs, for any $i, j \in [n']$, $(ij) \in S'$ if and only if $(\pi(i)\pi(j)) \in S$.

Define an injection τ from $\mathfrak{S}_{n'}$ to $\mathfrak{S}_n$: $\mathbf{u}' = u'_1 u'_2 \cdots u'_{n'} \mapsto \mathbf{u} = u_1 u_2 \cdots u_n$ satisfying $u_{\pi(i)} = \pi(u'_i)$ for $i \in [n']$, otherwise $u_j = j$. Denote $A = \tau(\mathfrak{S}_{n'}) = \{\mathbf{u} \in \mathfrak{S}_n | \mathbf{u} \text{ is fixed on } [n] \setminus \{\pi(1), \pi(2), \cdots, \pi(n')\}\}$ Let $\mathbf{u}' = u'_1 u'_2 \cdots u'_{n'}$ and $\mathbf{v}' = v'_1 v'_2 \cdots v'_{n'}$ be vertices in $\mathrm{Cay}(\mathfrak{S}_{n'}, S')$, and $\mathbf{u} = \tau(\mathbf{u}') = u_1 u_2 \cdots u_n$, $\mathbf{v} = \tau(\mathbf{v}') = v_1 v_2 \cdots v_n$ vertices in A. Now, we prove that $\mathbf{u}'$, $\mathbf{v}'$ are adjacent in $\mathrm{Cay}(\mathfrak{S}_{n'}, S')$ if and only if $\mathbf{u}$, $\mathbf{v}$ are adjacent in $\mathrm{Cay}(\mathfrak{S}_n, S)$.

If $\mathbf{u}'$, $\mathbf{v}'$ are adjacent, then $\mathbf{v}' = \mathbf{u}' \cdot (ij)$ for some $(ij) \in S'$ and $i, j \in [n']$, that is, $u'_i = v'_j$, $u'_j = v'_i$ and $u'_k = v'_k$ for $k \in [n'] \setminus \{i, j\}$. By the definition of τ, $v_{\pi(i)} = \pi(v'_i) = \pi(u'_j) = u_{\pi(j)}$, $v_{\pi(j)} = \pi(v'_j) = \pi(u'_i) = u_{\pi(i)}$ and $v_k = u_k$ for $k \in [n]$ and $k \neq \pi(i), \pi(j)$. Then $\mathbf{u} = \mathbf{v} \cdot (\pi(i)\pi(j))$, that is, $\mathbf{u}$, $\mathbf{v}$ are adjacent in $\mathrm{Cay}(\mathfrak{S}_n, S)$. On the other hand, Assume that $\mathbf{u}$, $\mathbf{v}$ are adjacent in $\mathrm{Cay}(\mathfrak{S}_n, S)$, that is, for some $(st) \in S$, $u_s = v_t$, $u_t = v_s$ and $u_k = v_k$ for $k \neq s, t$. By the definition of τ, there exist i, j satisfying $s = \pi(i)$, $t = \pi(j)$ and $(ij) \in S'$. Then $u'_i = \pi^{-1}(u_{\pi(i)}) = \pi^{-1}(u_s) = \pi^{-1}(v_t) = \pi^{-1}(v_{\pi(j)}) = v'_j$. Similarly, we can prove $u'_j = v'_i$ and $u'_k = v'_k$ for $k \in [n'] \setminus \{i, j\}$. Thus $\mathbf{v}' = \mathbf{u}' \cdot (ij)$ and, further $\mathbf{u}'$ and $\mathbf{v}'$ are adjacent in $\mathrm{Cay}(\mathfrak{S}_{n'}, S')$.

Thus $\mathrm{Cay}(\mathfrak{S}_{n'}, S')$ is isomorphic to the subgraph of $\mathrm{Cay}(\mathfrak{S}_n, S)$ induced by A. Therefore, $\mathrm{Cay}(\mathfrak{S}_{n'}, S')$ can also be embedded into $\mathrm{Cay}(\mathfrak{S}_n, S)$ as an induced subgraph. □

2.3 Sorting a permutation by a sequence of cyclically adjacent transpositions

In this subsection, we consider the distance between any pair of vertices $\mathbf{u}$ and $\mathbf{v}$ in the modified bubble sort graph MB_n, which is used for the proof of the main theorem in the next section. In fact, since $l_{\mathbf{u}^{-1}} : \mathbf{w} \to \mathbf{u}^{-1} \cdot \mathbf{w}$ $(\mathbf{w} \in \mathfrak{S}_n)$ is an automorphism of MB_n, $d(\mathbf{u}, \mathbf{v}) = d(\iota, \mathbf{u}^{-1} \cdot \mathbf{v})$. So we only consider the distance between the identity element and the others. Jerrum [16] gave a polynomial-time algorithm for computing the distance between the identity element ι and any permutation in the modified bubble sort graph MB_n (i.e., Lemma 2.9). Before that, we first introduce relevant notations in [16].

We call a transposition (st) on $[n]$ is a *cyclically adjacent transposition* (shortly, a *cat*) if $t \equiv s + 1 \pmod{n}$, that is, $(st) \in \{(12), (23), (34), \ldots, ((n-1)n), (1n)\} =: \mathit{\Pi}_{\mathrm{cat}}$. For a permutation π of length n, A vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ is called a *displacement vector* (or just *dvec*) of π if (i) $x_i \equiv \pi^{-1}(i) - i \pmod{n}$ for $i \in [n]$; (ii) $\sum_{i=1}^{n} x_i = 0$. In particular, if let $x_i = \pi^{-1}(i) - i$ for $i \in [n]$, then $\mathbf{x}$ is obviously a dvec of π, which is called the *initial displacement vector* (or *initial dvec*) of π.

Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ be an n-dimensional vector in $\mathbb{Z}^n$. Let $T_{ij} : \mathbb{Z}^n \to \mathbb{Z}^n$ $(i \neq j)$ be a *transformation* defined as follows: if T_{ij} is applied on $\mathbf{x}$, the result $T_{ij}(\mathbf{x}) = (x'_1, x'_2, \cdots, x'_n)$ is given by

$$x'_k = \begin{cases} x_k, & k \neq i \text{ or } j; \\ x_i - n, & k = i; \\ x_j + n, & k = j. \end{cases}$$

It's easy to see that T_{ij} map a dvec to another dvec of π. For a devc $\mathbf{x}$ of a permutation π of length n, we say that T_{ij} *strictly contracts* $\mathbf{x}$ if and only if $\max(\mathbf{x}) = x_i$, $\min(\mathbf{x}) = x_j$ and $x_i - x_j > n$, where $\max(\mathbf{x}) = \max\limits_{1 \leqslant k \leqslant n} x_k$, $\min(\mathbf{x}) = \min\limits_{1 \leqslant k \leqslant n} x_k$. For all values of i and j, if $x_i - x_j \leqslant n$, then we say that $\mathbf{x}$ admits *no strictly contracting transformation.* We call the dvec of a permutation π, which is obtained after applying a series of transformations (probably empty) on its initial dvec until it admits no strictly contracting transformation, the *optimal displacement vector* (shortly, *optimal dvec*).

Suppose $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{Z}^n$, $i, j \in [n]$ and $i \neq j$, and let $r = i - j$ and $s = (i + x_i) - (j + x_j)$. Define the *crossing number* (shortly *cnum*), $c_{ij}(\mathbf{x})$, of i and j with respect to $\mathbf{x}$ by

$$c_{ij}(\mathbf{x}) = \begin{cases} |\{r \leq k \leq s | k \equiv 0 \pmod{n}\}|, & \text{if } r \leqslant s \text{ (i.e., } x_i \geqslant x_j); \\ -|\{s \leq k \leq r | k \equiv 0 \pmod{n}\}|, & \text{otherwise.} \end{cases}$$

Notice that, for any $i, j \in [n]$, $c_{ij}(\mathbf{x}) = -c_{ji}(\mathbf{x})$ and $x_i = \sum_{k=1}^{n} c_{ik}(\mathbf{x})$. We define $i_c(\mathbf{x})$ by

$$i_c(\mathbf{x}) = \frac{1}{2} \sum_{\substack{i \neq j \\ i,j \in [n]}} |c_{ij}(\mathbf{x})| = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |c_{ij}(\mathbf{x})|.$$

Jerrum stated that for all optimal dvecs, the values of $i_c(\mathbf{x})$ are same, and, specially, proved the following lemma.

Lemma 2.9. [10, 16] *Let π be a permutation, $\mathbf{x}$ its displacement vector. Then $L(\pi, \mathit{\Pi}_{\text{cat}}) = \min i_c(\mathbf{x})$ and $i_c(\mathbf{x})$ will be minimized when $\mathbf{x}$ is one of its optimal dvecs.*

Combined Lemma 2.9 with the definition of optimal dvecs, for a permutation π, an algorithm for computing $L(\pi, \mathit{\Pi}_{\text{cat}})$ is presented as Algorithm 1.

If $\mathbf{x}$ is the initial dvec of π, then $r = i - j$, $s = \pi^{-1}(i) - \pi^{-1}(j)$ for $i, j \in [n]$. So the cnum $c_{ij}(\mathbf{x}) = 1$ or -1 if and only if $\{i, j\}$ is an inversion of π. Further, $i_c(\mathbf{x}) = \text{inv}(\pi)$. In a sense $i_c(\mathbf{x})$ is a generalization of inverse number $\text{inv}(\pi)$. Combined with Lemma 2.9, we obtain

Corollary 2.10. *Let π be a permutation. If its initial displacement vector $\mathbf{x}$ admits no strictly contracting transformation, then $L(\pi, \mathit{\Pi}_{\text{cat}}) = \text{inv}(\pi)$.*

Let us go back the problem we consider at the beginning of this subsection: the distance between the identity element and the others π in the modified bubble sort graph MB_n. If $\pi \cdot \pi_1 \cdot \pi_2 \cdot \cdots \cdot \pi_l = \iota$ $(\pi_1, \pi_2, \ldots, \pi_l \in \mathit{\Pi}_{\text{cat}})$ is a sequence that sorts π into ι, then each π_i is corresponding to an edge of MB_n. So it's easy to see that $d_{MB_n}(\iota, \pi) = L(\pi, \mathit{\Pi}_{\text{cat}})$. For any pair of vertices $\mathbf{u}$ and $\mathbf{v}$, since $d(\mathbf{u}, \mathbf{v}) = d(\iota, \mathbf{u}^{-1} \cdot \mathbf{v})$, we can compute the distance between $\mathbf{u}$ and $\mathbf{v}$ by running Algorithm 1 when we set $\pi = \mathbf{u}^{-1} \cdot \mathbf{v}$.

Algorithm 1 [16] Computing the length of a minimum-length sequence of cat

Require: a permutation π on $[n]$.
Ensure: the length of a minimum-length sequence of cat to sort π, $L(\pi, \Pi_{\text{cat}})$.

$\mathbf{x} := (x_1, x_2, \cdots, x_n)$;
for $i = 1$ to n **do**
  $x_i := \pi^{-1}(i) - i$;
end for
while $\max(\mathbf{x}) - \min(\mathbf{x}) > n$ **do**
  choose i, j with $x_i = \max(\mathbf{x})$, $x_j = \min(\mathbf{x})$;
  $x_i \leftarrow x_i - n$;
  $x_j \leftarrow x_j + n$;
end while
$i_c(\mathbf{x}) \leftarrow 0$;
for $i = 1$ to $n - 1$ **do**
  for $j = i + 1$ to n **do**
    $r \leftarrow i - j$; $s \leftarrow (i + x_i) - (j + x_j)$;
    for $k = r$ to s **do**
      if $k \equiv 0 \pmod{n}$ **then**
        $i_c(\mathbf{x}) \leftarrow i_c(\mathbf{x}) + 1$;
      end if
    end for
  end for
end for
return $L(\pi, \Pi_{\text{cat}}) = i_c(\mathbf{x})$;

3 Main result

In this section, we give our main result: characterizing the partial cubes in Cayley graphs generated by transpositions. First, we present a lemma due to Alahmadi et al. [2] and give its detail proof here.

Lemma 3.1. *[2] The bubble sort graph BS_n is a partial cube.*

Proof. Let X be the family of 2-element subsets of $[n]$. Then $\mathcal{H}(X) = Q_{\binom{n}{2}}$. Let $\mathrm{Inv} : \mathbf{u} \mapsto \mathrm{Inv}(\mathbf{u})$ be an injective map from $\mathfrak{S}_n$ to $\mathcal{P}(X)$, where $\mathrm{Inv}(\mathbf{u})$ is the inversion set of $\mathbf{u}$. We know that the distance of two permutations $\mathbf{u}, \mathbf{v}$ in BS_n is *Kendall τ distance* (see [17, 18]): $d_{BS_n}(\mathbf{u}, \mathbf{v}) = |\{\{i,j\} | (\mathbf{u}^{-1}(i) - \mathbf{u}^{-1}(j))(\mathbf{v}^{-1}(i) - \mathbf{v}^{-1}(j)) < 0\}|$. It deduces that $d_{BS_n}(\mathbf{u}, \mathbf{v}) = |\mathrm{Inv}(\mathbf{u}) \triangle \mathrm{Inv}(\mathbf{v})|$, which is the Hamming distance in $Q_{\binom{n}{2}}$. Therefore, Inv is an isometric embedding from BS_n to $Q_{\binom{n}{2}}$, i.e., The bubble sort graph BS_n is a partial cube. $\square$

Before giving the main result, we would like to introduce the characterization of cubic vertex-transitive partial cubes, due to Marc [22].

Let's denote $\Gamma_1 \square \Gamma_2$ as the *Cartesian product* of graphs Γ_1 and Γ_2, which is with vertex set $V(\Gamma_1) \times V(\Gamma_2)$. And two vertices (u, v) and (u', v') are adjacent in $\Gamma_1 \square \Gamma_2$ if and only if either $u = u'$ and $vv' \in E(\Gamma_2)$ or $uu' \in E(\Gamma_1)$ and $v = v'$. Marc [22] proved there are only five kinds of cubic vertex-transitive graphs which are partial cubes.

Theorem 3.2. [22] *Let Γ be cubic vertex-transitive partial cubes. Then Γ is isomorphic to one of the following graphs: $K_2 \square C_{2n}$, the generalized Petersen graph $G(10, 3)$, the cubic permutahedron ($= BS_4$), the truncated cuboctahedron, or the truncated icosidodecahedron (see Fig. 1).*

Now we give our main result as follows.

Theorem 3.3. *Let Γ be a connected Cayley graph generated by transpositions. Then Γ is a partial cube if and only if $\Gamma \cong BS_n$.*

Proof. The sufficiency is from Lemma 3.1. Now we prove the necessity.

Let $\Gamma = Cay(\mathfrak{S}_n, S)$ be a connected Cayley graph generated by a transposition set S such that Γ is a partial cube. By Lemma 2.7, it is sufficient to prove that $T(S)$ is isomorphic to the

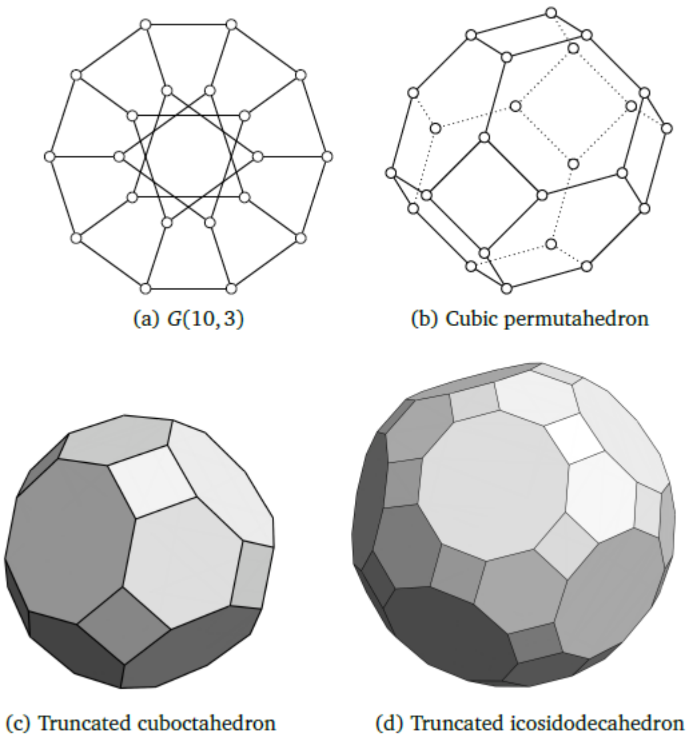

Fig. 1: The four sporadic examples of cubic vertex-transitive partial cubes. (The picture is from [22].)

path P_n. In what follows we prove it by three claims. Evidently, $T(S)$ is connected by Lemma 2.6.

Claim 1. $T(S)$ contains no triangles.

By contradiction. By Lemma 2.7, without loss of generality, we can assume that $\{(12),(23),(13)\} \subseteq S$ constructs a triangle in $T(S)$. By Lemma 2.8, $\mathrm{Cay}(\mathfrak{S}_3, \{(12),(23),(13)\})$ isomorphic to $K_{3,3}$ can be embedded into Γ as an induced subgraph, say, H. Then there exist two isometric cycles C_1 and C_2 of length 4 in H sharing a path P_3, say, one of two edges e. Then the antipodal edge of e in C_1 and the one of e in C_2 are adjacent and are in relation Θ by Observation 2.4 and Lemma 2.5, a contradiction to Observation 2.3.

Now, we show that the maximum degree of $T(S)$ is at most 2. If the maximum degree of $T(S)$ is at least 3, then $T(S)$ contains a claw $K_{1,3}$ as an induced subgraph by Claim 1.

Claim 2. $T(S)$ contains no $K_{1,3}$'s as induced subgraphs.

We assume to the contrary that $T(S)$ contains an induced subgraph $K_{1,3}$, by Lemma 2.7, say induced by $\{1,2,3,4\}$ with center 1. Then $(12),(13),(14) \in S$ and $(23),(34),(24) \notin S$. By Lemma 2.8, Then we obtain that $ST_4 = \mathrm{Cay}(\mathfrak{S}_3, \{(12),(13),(14)\})$ is isomorphic to the subgraph H' of Γ induced by the vertex set $A = \{\mathbf{u} \in \mathfrak{S}_n | 5,6,\cdots,n \text{ are the fixed points of } \mathbf{u}\}$. Now, we show that H' is isometric in Γ.

If H' is not isometric in Γ, then there exist two vertices $\mathbf{u},\mathbf{v} \in A$ satisfying $d_\Gamma(\mathbf{u},\mathbf{v}) < d_{H'}(\mathbf{u},\mathbf{v})$, that is, there exists a path from $\mathbf{u}$ to $\mathbf{v}$ not in H' which is shorter than any paths in H'. Since Γ is bipartite and the diameter of ST_4 is 4 (see [1]), we can deduce that $d_\Gamma(\mathbf{u},\mathbf{v}) = 2$ and $d_{H'}(\mathbf{u},\mathbf{v}) = 4$. By vertex-transitivity of Γ, we assume $\mathbf{u} = \iota$. Let $P = \mathbf{uwv}$ be a shortest $\mathbf{u}$-$\mathbf{v}$ path and say, $e_1 = \mathbf{uw}$, $e_2 = \mathbf{wv}$. Then $\mathbf{v} \in A$ and $\mathbf{w} \notin A$. Set $l(e_1) = (ij)$, $l(e_2) = (st)$, then $(ij) \neq (st)$ and $(ij),(st) \notin \{(12),(13),(14)\}$, otherwise $\mathbf{w} \in A$ by Lemma 2.8. Since $\mathbf{w} \notin A$, $\{i,j\} \nsubseteq \{1,2,3,4\}$, $\{s,t\} \nsubseteq \{1,2,3,4\}$ by the definition of A. Assume $i,s \notin \{1,2,3,4\}$ (maybe $i = s$). From $\mathbf{v} = \mathbf{u}(ij)(st) = (ij)(st)$, i and s must be in $\mathrm{supp}(\mathbf{v})$, contradicting to the fact that $\mathbf{v} \in A$.

Thus, $ST_4 \hookrightarrow \Gamma$ and $\Gamma \hookrightarrow \mathcal{H}(X)$ for some set X. We deduce that $ST_4 \hookrightarrow \mathcal{H}(X)$ by Observation 2.1. Combined with the obvious fact that ST_4 is a cubic vertex-transitive graph with Theorem 3.2, we obtain a contradiction.

Thus, we obtain that the maximum degree of $T(S)$ is at most 2. Since $T(S)$ is connected and is of order n, $T(S)$ is isomorphic either to P_n or to C_n.

Claim 3. $T(S) \ncong C_n$ $(n \geqslant 4)$.

For convenience, let's use the cycle notation to represent permutations.

We assume to the contrary that $T(S) \cong C_n$ and, say $S = \{(12), (23), (34), \ldots, ((n-1)n), (1n)\}$ by Lemma 2.7. So $\Gamma \cong MB_n$. We construct a cycle of length $2n-2$ in MB_n. Denote

$$\mathbf{v_i} = \begin{cases} \Big(12\cdots\big(\lceil\frac{i}{2}\rceil+1\big)\Big)\Big(n(n-1)\cdots\big(n-\lfloor\frac{i}{2}\rfloor\big)\Big), & \text{for } 0 \leqslant i \leqslant n-2; \\ \Big(12\cdots\big(n-\lceil\frac{i}{2}\rceil\big)\, n(n-1)(n-2)\cdots\big(\lfloor\frac{i}{2}\rfloor+2\big)\Big), & \text{for } n-1 \leqslant i \leqslant 2n-3. \end{cases}$$

Then $C = \mathbf{v}_0\mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_{2n-3}$ is a $(2n-2)$-cycle. Now we prove that C is an isometric subgraph in MB_n, that is, to prove $d_\Gamma(\mathbf{v}_{k_1}, \mathbf{v}_{k_2}) = d_C(\mathbf{v}_{k_1}, \mathbf{v}_{k_2})$ for any $0 \leqslant k_1 < k_2 \leqslant 2n-3$. We divide three cases to discuss.

Case 1. $0 \leqslant k_1 < k_2 \leqslant n-2$.

Since $k_2 - k_1 < \frac{2n-2}{2}$, $d_C(\mathbf{v}_{k_1}, \mathbf{v}_{k_2}) = k_2 - k_1$. In this case, $\mathbf{v}_{k_1} = \Big(123\cdots\big(\lceil\frac{k_1}{2}\rceil+1\big)\Big)\Big(n(n-1)(n-2)\cdots\big(n-\lfloor\frac{k_1}{2}\rfloor\big)\Big)$, $\mathbf{v}_{k_2} = \big(123\cdots(\lceil\frac{k_2}{2}\rceil+1)\big)\big(n(n-1)(n-2)\cdots(n-\lfloor\frac{k_2}{2}\rfloor)\big)$. $\mathbf{v}_{k_1}^{-1} \cdot \mathbf{v}_{k_2} = \Big(\big(\lceil\frac{k_1}{2}\rceil+1\big)\big(\lceil\frac{k_1}{2}\rceil+2\big)\cdots\big(\lceil\frac{k_2}{2}\rceil+1\big)\Big)\Big(\big(n-\lfloor\frac{k_1}{2}\rfloor\big)\big(n-\lfloor\frac{k_1}{2}\rfloor-1\big)\cdots\big(n-\lfloor\frac{k_2}{2}\rfloor\big)\Big)$. Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ be the initial dvec of $\mathbf{v}_{k_1}^{-1} \cdot \mathbf{v}_{k_2}$. Then

$$x_i = \left(\mathbf{v}_{k_1}^{-1} \cdot \mathbf{v}_{k_2}\right)^{-1}(i) - i = \begin{cases} \lceil\frac{k_2}{2}\rceil - \lceil\frac{k_1}{2}\rceil, & \text{if } i = \lceil\frac{k_1}{2}\rceil + 1; \\ \lfloor\frac{k_1}{2}\rfloor - \lfloor\frac{k_2}{2}\rfloor, & \text{if } i = n - \lfloor\frac{k_1}{2}\rfloor; \\ -1, & \text{if } \lceil\frac{k_1}{2}\rceil + 2 \leqslant i \leqslant \lceil\frac{k_2}{2}\rceil + 1; \\ 1, & \text{if } n - \lfloor\frac{k_2}{2}\rfloor \leqslant i \leqslant n - \lfloor\frac{k_1}{2}\rfloor - 1; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\max(\mathbf{x}) - \min(\mathbf{x}) = k_2 - k_1 < n$, $\mathbf{x}$ admits no strictly contracting transformation. Then, by Corollary 2.10, $d_\Gamma(\mathbf{v}_{k_1}, \mathbf{v}_{k_2}) = L(\mathbf{v}_{k_1}^{-1} \cdot \mathbf{v}_{k_2}, \mathit{\Pi}_{\text{cat}}) = \text{inv}(\mathbf{v}_{k_1}^{-1} \cdot \mathbf{v}_{k_2}) = k_2 - k_1$. So $d_\Gamma(\mathbf{v}_{k_1}, \mathbf{v}_{k_2}) = d_C(\mathbf{v}_{k_1}, \mathbf{v}_{k_2})$.

Case 2. $0 \leqslant k_1 \leqslant n-2 < n-1 \leqslant k_2 \leqslant 2n-3$.

In this case, $\mathbf{v}_{k_1} = \Big(123\cdots\big(\lceil\frac{k_1}{2}\rceil+1\big)\Big)\Big(n(n-1)(n-2)\cdots\big(n-\lfloor\frac{k_1}{2}\rfloor\big)\Big)$, $\mathbf{v}_{k_2} = \Big(12\cdots\big(n-\lceil\frac{k_2}{2}\rceil\big)n(n-1)(n-2)\cdots\big(\lfloor\frac{k_2}{2}\rfloor+2\big)\Big)$. In order to compute $\mathbf{v}_{k_1}^{-1} \cdot \mathbf{v}_{k_2}$, we discuss it by two subcases.

Subcase 2.1. $n-1-k_1 \geqslant k_2 - n + 2$.

In this subcase, $\mathbf{v}_{k_1}^{-1} \cdot \mathbf{v}_{k_2} = \Big(\big(\lceil\frac{k_1}{2}\rceil+1\big)\big(\lceil\frac{k_1}{2}\rceil+2\big)\cdots\big(n-\lceil\frac{k_2}{2}\rceil\big)\big(n-\lfloor\frac{k_1}{2}\rfloor\big)\big(n-\lfloor\frac{k_1}{2}\rfloor-1\big)$ $\big(n-\lfloor\frac{k_1}{2}\rfloor-2\big)\cdots\big(\lfloor\frac{k_2}{2}\rfloor+2\big)\Big)$. Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ be the initial dvec of $\mathbf{v}_{k_1}^{-1} \cdot \mathbf{v}_{k_2}$. Then

$$x_i = \left(\mathbf{v}_{k_1}^{-1} \cdot \mathbf{v}_{k_2}\right)^{-1}(i) - i = \begin{cases} \lfloor\frac{k_1}{2}\rfloor - \lceil\frac{k_2}{2}\rceil, & \text{if } i = n - \lfloor\frac{k_1}{2}\rfloor; \\ \lfloor\frac{k_2}{2}\rfloor - \lceil\frac{k_1}{2}\rceil + 1, & \text{if } i = \lceil\frac{k_1}{2}\rceil + 1; \\ -1, & \text{if } \lceil\frac{k_1}{2}\rceil + 2 \leqslant i \leqslant n - \lceil\frac{k_2}{2}\rceil; \\ 1, & \text{if } \lfloor\frac{k_2}{2}\rfloor + 2 \leqslant i \leqslant n - \lfloor\frac{k_1}{2}\rfloor - 1; \\ 0, & \text{otherwise.} \end{cases}$$

We obtain that $\max(\mathbf{x})-\min(\mathbf{x}) = k_2-k_1+1$. When $k_2-k_1 \geqslant n$, $d_C(\mathbf{v}_{k_1}, \mathbf{v}_{k_2}) = 2n+k_1-k_2-2$ and $\mathbf{x}$ can be contracted. By running Algorithm 1, we obtain $d_\Gamma(\mathbf{v}_{k_1}, \mathbf{v}_{k_2}) = L(\mathbf{v}_{k_1}^{-1}\cdot\mathbf{v}_{k_2}, \mathit{\Pi}_{\mathrm{cat}}) = 2n+k_1-k_2-2$; When $k_2-k_1 \leqslant n-1$, $d_C(\mathbf{v}_{k_1}, \mathbf{v}_{k_2}) = k_2-k_1$ and $\mathbf{x}$ admits no strictly contracting transformation. Then, by Corollary 2.10, $d_\Gamma(\mathbf{v}_{k_1}, \mathbf{v}_{k_2}) = L(\mathbf{v}_{k_1}^{-1}\cdot\mathbf{v}_{k_2}, \mathit{\Pi}_{\mathrm{cat}}) = \mathrm{inv}(\mathbf{v}_{k_1}^{-1}\cdot\mathbf{v}_{k_2}) = k_2-k_1$.

Subcase 2.2. $n-1-k_1 < k_2-n+2$.

In this subcase, $\mathbf{v}_{k_1}^{-1}\cdot\mathbf{v}_{k_2} = \Big(\left(n-\left\lfloor\frac{k_1}{2}\right\rfloor\right)\left(n-\left\lfloor\frac{k_1}{2}\right\rfloor+1\right)\left(n-\left\lfloor\frac{k_1}{2}\right\rfloor+2\right)\cdots\left(\left\lfloor\frac{k_2}{2}\right\rfloor+2\right)\left(\left\lceil\frac{k_1}{2}\right\rceil+1\right)\left\lceil\frac{k_1}{2}\right\rceil\left(\left\lceil\frac{k_1}{2}\right\rceil-1\right)\cdots\left(n-\left\lceil\frac{k_2}{2}\right\rceil\right)\Big)$. Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ be the initial dvec of $\mathbf{v}_{k_1}^{-1}\cdot\mathbf{v}_{k_2}$. Then

$$x_i = \left(\mathbf{v}_{k_1}^{-1}\cdot\mathbf{v}_{k_2}\right)^{-1}(i) - i = \begin{cases} \left\lfloor\frac{k_1}{2}\right\rfloor-\left\lceil\frac{k_2}{2}\right\rceil, & \text{if } i = n-\left\lfloor\frac{k_1}{2}\right\rfloor; \\ \left\lfloor\frac{k_2}{2}\right\rfloor-\left\lceil\frac{k_1}{2}\right\rceil+1, & \text{if } i = \left\lceil\frac{k_1}{2}\right\rceil+1; \\ -1, & \text{if } n-\left\lfloor\frac{k_1}{2}\right\rfloor+1 \leqslant i \leqslant \left\lfloor\frac{k_2}{2}\right\rfloor+2; \\ 1, & \text{if } n-\left\lceil\frac{k_2}{2}\right\rceil \leqslant i \leqslant \left\lceil\frac{k_1}{2}\right\rceil; \\ 0, & \text{otherwise.} \end{cases}$$

We obtain that $\max(\mathbf{x})-\min(\mathbf{x}) = k_2-k_1+1$. Similar to Subcase 2.1, we obtain $d_\Gamma(\mathbf{v}_{k_1}, \mathbf{v}_{k_2}) = d_C(\mathbf{v}_{k_1}, \mathbf{v}_{k_2}) = 2n+k_1-k_2-2$ when $k_2-k_1 \geqslant n$, and $d_\Gamma(\mathbf{v}_{k_1}, \mathbf{v}_{k_2}) = d_C(\mathbf{v}_{k_1}, \mathbf{v}_{k_2}) = k_2-k_1$ when $k_2-k_1 \leqslant n-1$.

Case 3. $n-1 \leqslant k_1 < k_2 \leqslant 2n-3$.

In this case, $k_2-k_1 < \frac{2n-2}{2}$, so $d_C(\mathbf{v}_{k_1}, \mathbf{v}_{k_2}) = k_2-k_1$. $\mathbf{v}_{k_1} = \Big(12\cdots\left(n-\left\lceil\frac{k_1}{2}\right\rceil\right)n(n-1)(n-2)\cdots\left(\left\lfloor\frac{k_1}{2}\right\rfloor+2\right)\Big)$, $\mathbf{v}_{k_2} = \Big(12\cdots\left(n-\left\lceil\frac{k_2}{2}\right\rceil\right)n(n-1)(n-2)\cdots\left(\left\lfloor\frac{k_2}{2}\right\rfloor+2\right)\Big)$. Then $\mathbf{v}_{k_1}^{-1}\cdot\mathbf{v}_{k_2} = \Big(\left(n-\left\lceil\frac{k_1}{2}\right\rceil\right)\left(n-\left\lceil\frac{k_1}{2}\right\rceil-1\right)\cdots\left(n-\left\lceil\frac{k_2}{2}\right\rceil\right)\Big)\Big(\left(\left\lfloor\frac{k_1}{2}\right\rfloor+2\right)\left(\left\lfloor\frac{k_1}{2}\right\rfloor+3\right)\cdots\left(\left\lfloor\frac{k_2}{2}\right\rfloor+2\right)\Big)$. Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ be the initial dvec of $\mathbf{v}_{k_1}^{-1}\cdot\mathbf{v}_{k_2}$. Then

$$x_i = \left(\mathbf{v}_{k_1}^{-1}\cdot\mathbf{v}_{k_2}\right)^{-1}(i) - i = \begin{cases} \left\lceil\frac{k_1}{2}\right\rceil-\left\lceil\frac{k_2}{2}\right\rceil, & \text{if } i = n-\left\lceil\frac{k_1}{2}\right\rceil; \\ \left\lfloor\frac{k_2}{2}\right\rfloor-\left\lfloor\frac{k_1}{2}\right\rfloor, & \text{if } i = \left\lfloor\frac{k_1}{2}\right\rfloor+2; \\ -1, & \text{if } \left\lfloor\frac{k_1}{2}\right\rfloor+3 \leqslant i \leqslant \left\lfloor\frac{k_2}{2}\right\rfloor+2; \\ 1, & \text{if } n-\left\lceil\frac{k_2}{2}\right\rceil \leqslant i \leqslant n-\left\lceil\frac{k_1}{2}\right\rceil-1; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\max(\mathbf{x})-\min(\mathbf{x}) = k_2-k_1 < n$, $\mathbf{x}$ admits no strictly contracting transformation. Then $d_\Gamma(\mathbf{v}_{k_1}, \mathbf{v}_{k_2}) = L(\mathbf{v}_{k_1}^{-1}\cdot\mathbf{v}_{k_2}, \mathit{\Pi}_{\mathrm{cat}}) = \mathrm{inv}(\mathbf{v}_{k_1}^{-1}\cdot\mathbf{v}_{k_2}) = k_2-k_1$.

Thus, the cycle C is isometric in MB_n. Let's denote $\mathbf{v}'_{2n-4} = (1n(n-1))$. Then $C' = \mathbf{v}_0\mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_{2n-5}\mathbf{v}'_{2n-4}\mathbf{v}_{2n-3}$ is another $(2n-2)$-cycle. It can be proven that C' is also isometric in MB_n by the similar discussion. Let $e = \mathbf{v}_{n-3}\mathbf{v}_{n-2}$. Then e is both the antipodal edge of $\mathbf{v}_{2n-4}\mathbf{v}_{2n-3}$ in C and the antipodal edge of $\mathbf{v}'_{2n-4}\mathbf{v}_{2n-3}$ in C'. By Observation 2.4, $e\,\Theta\,\mathbf{v}_{2n-4}\mathbf{v}_{2n-3}$ and $e\,\Theta\,\mathbf{v}'_{2n-4}\mathbf{v}_{2n-3}$. However, $\mathbf{v}_{2n-4}\mathbf{v}_{2n-3}$ and $\mathbf{v}'_{2n-4}\mathbf{v}_{2n-3}$ are not in Djoković-Winkler relation by Observation 2.3, that is, the Djoković-Winkler relation Θ is not transitive on $E(\Gamma)$. Therefore, by Lemma 2.5, Γ is not a partial cube, a contradiction. $\square$

In what follows, we compute the Wiener index of BS_n by Theorem 3.3. First we give some concepts we need.

For a connected graph Γ, the *Wiener index* $W(\Gamma)$ of Γ is defined as

$$W(\Gamma) = \sum_{\{u,v\}\subseteq V(\Gamma)} d_\Gamma(u, v),$$

which was introduced by H. Wiener in 1947 [27].

Definition 3.4. Let Γ be a graph and a, b a pair of adjacent vertices. Let W_{ab} be the set of vertices closer to a than to b, i.e.,

$$W_{ab} = \{v \in V(\Gamma) | d(v,a) < d(v,b)\}.$$

We call W_{ab} a *semicube* of the graph Γ and the semicubes W_{ab} and W_{ba} a pair of *opposite semicubes.*

The formula for computing the Wiener index of partial cubes is presented in the following lemma:

Lemma 3.5. [15, 19] *Let Γ be a partial cube of order n and $E_1, E_2, \ldots, E_k$ its Θ equivalent classes. For $i = 1, 2, \ldots, k$, let $u_iv_i \in E_i$ be a representative element of E_i and $n_i = |W_{u_iv_i}|$. Then*

$$W(\Gamma) = \sum_{i=1}^{k} |W_{u_iv_i}||W_{v_iu_i}| = \sum_{i=1}^{k} n_i(n - n_i). \tag{1}$$

Now, the Wiener index of BS_n is obtained:

Theorem 3.6. $W(BS_n) = \dfrac{(n!)^2}{4}\dbinom{n}{2}$.

Proof. Let i, j be any two integers where $1 \leqslant i < j \leqslant n$. Denote $V_{ij} = \{\mathbf{v} \in \mathfrak{S}_n | \mathbf{v}^{-1}(i) < \mathbf{v}^{-1}(j)\}$, $V_{ji} = \{\mathbf{v} \in \mathfrak{S}_n | \mathbf{v}^{-1}(i) > \mathbf{v}^{-1}(j)\}$, then $V_{ij} \cap V_{ji} = \emptyset$ and $V_{ij} \cup V_{ji} = \mathfrak{S}_n$. We prove that V_{ij} and V_{ji} are a pair of opposite semicubes for all pairs of i, j. Let $\mathbf{u} \in V_{ij}, \mathbf{v} \in V_{ji}$ be two adjacent vertices in BS_n (for example, $\mathbf{u} = ij12\cdots(i-1)(i+1)\cdots(j-1)(j+1)\cdots n$, $\mathbf{v} = ji12\cdots(i-1)(i+1)\cdots(j-1)(j+1)\cdots n$ if $j \neq i+1$; Or $\mathbf{u} = \iota$, $\mathbf{v} = \big(i(i+1)\big)$ if $j = i+1$). Then $\mathrm{Inv}(\mathbf{u}) = \mathrm{Inv}(\mathbf{v}) \setminus \{\{i,j\}\}$. Let $\mathbf{w}$ be a vertex in BS_n. If $\mathbf{w} \in V_{ij}$, we can see that $\mathrm{Inv}(\mathbf{w}) \triangle \mathrm{Inv}(\mathbf{v}) = (\mathrm{Inv}(\mathbf{w}) \triangle \mathrm{Inv}(\mathbf{u})) \cup \{\{i,j\}\}$, that is, $d_{BS_n}(\mathbf{w},\mathbf{v}) = d_{BS_n}(\mathbf{w},\mathbf{u}) + 1$. Thus, $V_{ij} \subseteq W_{\mathbf{uv}}$. Otherwise, $\mathbf{w} \in V_{ji}$. We can see that $\mathrm{Inv}(\mathbf{w}) \triangle \mathrm{Inv}(\mathbf{u}) = (\mathrm{Inv}(\mathbf{w}) \triangle \mathrm{Inv}(\mathbf{v})) \cup \{\{i,j\}\}$, that is, $d_{BS_n}(\mathbf{w},\mathbf{u}) = d_{BS_n}(\mathbf{w},\mathbf{v}) + 1$. Thus, $V_{ji} \subseteq W_{\mathbf{vu}}$. Since both $(W_{\mathbf{uv}}, W_{\mathbf{vu}})$ and (V_{ij}, V_{ji}) are partitions of vertex set of BS_n, $W_{\mathbf{uv}} = V_{ij}$ and $W_{\mathbf{vu}} = V_{ji}$. Therefore, all the edges linked between V_{ij} and V_{ji} are in a Θ equivalent class for each pair of i, j. Since $|W_{\mathbf{uv}}| = |W_{\mathbf{vu}}| = |V_{ij}| = |V_{ji}| = \frac{n!}{2}$, by Eq. (1), we obtain that

$$W(BS_n) = \frac{(n!)^2}{4}\binom{n}{2}.$$

□

Now, we give the definition of mirror graphs, which is introduced by B. Brešar et al. [4].

Definition 3.7. Let $\Gamma = (V, E)$ be a connected graph. Call a partition $\mathcal{P} = \{E_1, E_2, \cdots, E_k\}$ of E a *mirror partition* if for every $i \in \{1, 2, \cdots, k\}$, there is an automorphism α_i of Γ such that

(M1) for every $uv \in E_i$, $\alpha_i(u) = v$, $\alpha_i(v) = u$, and

(M2) $\Gamma - E_i$ consists of two components Γ_1^i, Γ_2^i, and α_i maps Γ_1^i isomorphically onto Γ_2^i.

A connected graph is called a *mirror graph* if it admits a mirror partition.

Marc [23] proved that the mirror graphs are equivalent to the Cayley graphs of a finite Coxeter group with canonical generators. It's easy to varify that BS_n is in this class of Cayley graphs. Thus, we obtain:

Proposition 3.8. *The bubble sort graph BS_n is a mirror graph.*

4 Conclusions

Thomassen [5] provided diverse examples of vertex-transitive subgraphs of hypercubes. Marc [22] characterized all cubic vertex-transive partial cubes in 2017. But they were far from classifying vertex-transitive partial cubes with high degree. In this paper we consider a class of Cayley graphs with high degree—Cayley graphs Γ on the symmetric group generated by transportations, and classify that Γ is a partial cube if and only if Γ is the bubble sort graph BS_n.

For bipartite vertex-transitive graphs, Mulder [24] proved that hypercubes are the only regular—and so the only vertex-transitive—median graphs. Cayley graphs Γ on the symmetric group generated by transportations we considered are also bipartite vertex-transitive graphs. So one question in the future is to decide which class of bipartite Cayley graphs, except the mirror graphs, are partial cubes.

References

[1] S.B. Akers, B. Krishnamurthy, A group-theoretic model for symmetric interconnection networks, IEEE Trans. Comput. 38 (4) (1989) 555–566.

[2] A. Alahmadi, H. Alhazmi, S. Ali, M. Deza, M.D. Sikirić, P. Solé, Hypercube emulation of interconnection networks topologies, Mathematical Methods in the Applied Sciences, 39.16 (2016) 4856–4865.

[3] H.-J. Bandelt, Evolutionary Networks, in: eLS, John Wiley & Sons Ltd., Chichester, 2006.

[4] B. Brešar, S. Klavžar, A. Lipovec, and B. Mohar, Cubic inflation, mirror graphs, regular maps, and partial cubes, European J. Combin. 25 (2004) 55–64.

[5] A.E. Brouwer, I.J. Dejter, and C. Thomassen, Highly symmetric subgraphs of hypercubes, J. Algebraic Combin. 2 (1993) 25–29.

[6] C. Delorme, Isomorphism of transposition graphs, Tech. rep., L.R.I., Orsay, 1997.

[7] M. Deza, M. Laurent, Geometry of cuts and metrics, in: Algorithms and Combinatorics, vol. 15, Springer-Verlag, Berlin, 1997.

[8] D. Ž. Djoković, Distance preserving subgraphs of hypercubes, J. Combin. Theory Ser. B 14 (1973) 263–267.

[9] D. Eppstein, J.-C. Falmagne, S. Ovchinnikov, Media Theory, Springer-Verlag, Berlin, 2008.

[10] X. Feng, B. Chitturi, H. Sudborough, Sorting circular permutations by bounded transpositions, in: Advances in Computational Biology, in: Adv. Exp. Med. Biol., vol. 680, 2010, pp. 725–736.

[11] K. Fukuda and K. Handa, Antipodal graphs and oriented matroids, Discrete Math. 111 (1993) 245–256.

[12] C. Godsil, G. Royle, Algebraic Graph Theory, New York: Springer-Verlag, 2001, vol. 207, Graduate Texts in Mathematics.

[13] R.L. Graham, H. Pollak, On the addressing problem for loop switching, Bell System Tech. J. 50 (1971) 2495–2519.

[14] R. Hammack, W. Imrich, S. Klavžar, Handbook of product graphs, CRC press, 2011.

[15] W. Imrich, Klavžar, Product graphs: structure and recognition, Wiley, 2000.

[16] M. Jerrum, The complexity of finding minimum-length generator sequences, Theor. Comput. Sci. 36 (1985) 265–289.

[17] M. G. Kendall, A new measure of rank correlation, Biometrika, 30(1-2) (1938) 81–93.

[18] M. G. Kendall, Rank correlation methods, Brit. J. Psych. 25(1) (1948) 86–91.

[19] S. Klavžar, I. Gutman and B. Mohar, Labeling of benzenoid systems which reflects the vertex-distance relations, J. Chem. Inf. Comput. Sci. 35 (1995) 590–593.

[20] S. Klavžar, M.J. Nadjafi-Arani, Cut method: update on recent developments and equivalence of independent approaches, Curr. Org. Chem. 19 (2015) 348–358.

[21] J.H. Koolen, S.V. Shpectorov, Distance-regular graphs the distance matrix of which has only one positive eigenvalue, European J. Combin. 15 (1994) 269–275.

[22] T. Marc, Classification of vertex-transitive cubic partial cubes, J. Graph Theory, 86 (2017) 406–421.

[23] T. Marc, Mirror graphs: graph theoretical characterization of reflection arrangements and finite coxeter groups, European J. Combin., 63 (2017), pp. 115–123.

[24] H.M. Mulder, n-cubes and median graphs, J. Graph Theory, 4 (1980) 107–110.

[25] S. Ovchinnikov, Partial cubes: structures, characterizations, and constructions. Discrete Math. 308 (2008) 5597–5621.

[26] P.M. Weichsel, Distance regular subgraphs of a cube, Discrete Math. 109 (1992) 297–306.

[27] H. Wiener, Structural determination of paraffin boiling points, J. Amer. Chem. Soc. 69 (1947) 17–20.

[28] E. Wilkeit, Isometric embeddings in Hamming graphs, J. Combin. Theory Ser. B 50 (1990) 179–197.

[29] P. M. Winkler, Isometric embedding in products of complete graphs, Discrete Appl. Math. 7 (1984) 221–225.