Statistical Solutions for a Nonautonomous Modified Swift-Hohenberg Equation

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Abstract

We consider the nonautonomous modified Swift-Hohenberg equation

$$u_t + \Delta^2 u + 2\Delta u + au + b|\nabla u|^2 + u^3 = g(t,x)$$

on a bounded smooth domain $\Omega \subset \mathbb{R}^n$ with $n \leqslant 3$. We show that, if $|b| < 4$ and the external force g satisfies some appropriate assumption, then the associated process has a unique pullback attractor in the Sobolev space $H_0^2(\Omega)$. Based on this existence, we further prove the existence of a family of invariant Borel probability measures and a statistical solution for this equation.

Keywords: Nonautonomous modified Swift-Hohenberg equation; Invariant Borel probability measure; Statistical solution; Pullback attractor; Pullback flattening.

AMS Subject Classification: 76D06, 37L40, 35Q35, 35B41.

1 Introduction

In this article, we study the existence of invariant Borel probability measures and statistical solutions for the following nonautonomous problem

$$\begin{cases} u_t + \Delta^2 u + 2\Delta u + au + b|\nabla u|^2 + u^3 = g(t,x), \quad x \in \Omega,\, t > \tau, & (1.1) \\ u = \dfrac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega,\, t \geq \tau, & (1.2) \\ u(x,\tau) = \phi(x), \quad x \in \Omega, & (1.3) \end{cases}$$

where Ω is an open connected bounded smooth domain in $\mathbb{R}^n$, $n \leqslant 3$, a and b are arbitrary real constants, $u_t = \frac{\partial u}{\partial t}$, g is the forcing satisfying $g \in L^2_{\text{loc}}(\mathbb{R}, L^2(\Omega))$, ν is the external normal vector on the boundary of Ω and ϕ is the initial datum. The equation (1.1) is known in the literature as the modified Swift-Hohenberg equation, and when $b = 0$, the equation (1.1) is known as the Swift-Hohenberg equation. The modified term $b|\nabla u|^2$, reminiscent of the Kuramoto-Sivashinsky equation, comes from the study of various pattern formation phenomena involving some kind of phase turbulence or phase transition ([1, 2]), which prevents the symmetry $u \to -u$.

The Swift-Hohenberg type equation was introduced in 1977 by Swift and Hohenberg ([3]) in the research of Rayleigh-Bénard's convective hydrodynamics (see also [4]), arising in geophysical fluid flows in the atmosphere, oceans and the earth's mantle. It is closely contacted with nonlinear Navier-Stokes equations coupled with the temperature equation. Later, it has also played a valuable role extensively in the study of plasma confinement in toroidal devices ([5]), viscous film flow, lasers ([6]) and pattern formation ([7]).

In the previous work, most attention was paid to the existence of attractors (global attractor [8, 9], uniform attractor [10], pullback attractor [11, 12] and random attractor [12–14]), bifurcations (dynamical bifurcations [15, 16], nontrivial-solution bifurcations [17]) and optimal control ([18–21]) of different types of modified Swift-Hohenberg equations. Wang, Yang and Duan presented in [22] a lower number of recurrent solutions for the nonautonomous case by topological methods (see more in [23–25]). Nevertheless, up to our knowledge, invariant measures and statistical solutions of (1.1) have been barely discussed until now.

The motivation of the current article is to investigate the existence of invariant Borel probability measures and statistical solutions for the nonautonomous modified Swift-Hohenberg equation. These two concepts are very useful in the research of turbulence (see [26]), an important research target in fluid dynamics. This is mainly due to the fact that some time-average quantities essentially measure several important aspects of turbulent flows. Later, the invariant measures and statistical properties of dissipative systems were studied

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in a series of references. For instance, Wang considered the upper semi-continuity of stationary statistical properties for dissipative systems in [27]. Łukaszewicz, Real and Robinson in [28] constructed the invariant measures for general continuous dynamical systems on metric spaces by using the generalized Banach limit. For a much wider class of dissipative semigroups, Chekroun and Glatt-Holtz [29] also applied the generalized Banach limit to constructing the invariant measures, but they generalized and simplified the proofs of [27,28].

Recently, a series of works developed some techniques to provide a construction of invariant measures for nonautonomous systems with minimal assumptions on the underlying dynamical process (see Foias *et al.* [26], Wang [27] and Łukaszewicz *et al.* [28, 30, 31]). Nowadays, these theories have been employed to establish the existence of invariant measures and (trajectory) statistical solutions for some evolution equations, (see e.g. [32,35–47] and the references therein). However, invariant measures in these works were usually discussed in a closed subspace of $L^2(\Omega)$ (see [26,30,31,33,34,45]) or its own product space ([47]), and their regularity can be considered ([26]). In this article we directly investigate the invariant measures and statistical solutions in a more regular Sobolev space $H_0^2(\Omega)$ (denoted by V in the sequel).

The solution operator of problem (1.1)-(1.3) generates a norm-to-weak continuous process $\{U(t,\tau)\}_{t\geqslant\tau}$ on the phase space V. We will use the abstract theory for dissipative nonautonomous systems in [31, Theorem 3.1] to construct the invariant measures. Accordingly, we need first to obtain the existence of pullback attractors in V. To be frank, for the argument, we can use the procedure in [11], in which the two-dimensional modified Swift-Hohenberg equation was considered. Nevertheless, in our article, we not only extend the dimensional from two to the range $n = 1, 2, 3$, but also relax the restriction imposed on the nonautonomous term $g(t, x)$ (see the assumption $(\mathbf{A})$ in Subsection 2.1) to some extent. This is sufficient to ensure the existence of pullback attractors.

Besides the existence of pullback attractor of $\{U(t,\tau)\}_{t\geqslant\tau}$, we also require the boundedness and continuity of the mapping $\tau \mapsto U(t,\tau)\phi$ on $(-\infty, t]$ for every $t \in \mathbb{R}$ and $\phi \in V$. However, this property is not trivial (see Lemmas 3.3 and 3.2). The continuous dependence of the nonautonomous dynamical system on their initial times differs essentially from that of the autonomous one. The continuity of the V-valued function $t \mapsto U(t,\tau)\phi$ does not imply the convergence $\|U(t,\tau)\phi - \phi\|_V \to 0$ as $\tau \to t$, due to the dependence of the continuity on τ. Actually, when $\tau \to t^-$, $U(t,\tau)\phi$ also changes simultaneously with different initial times τ. This is caused naturally by the nonautonomy. We will take advantage of the structure of the nonautonomous modified Swift-Hohenberg equation to cope with this problem.

The remainder of this article is organized as follows. In Section 2, we give the assumption on g and some estimates for the solutions of the problem (1.1)-(1.3) to guarantee the existence of the pullback attractor. In the last section, we establish the existence of invariant Borel probability measures and statistical solutions.

2 Existence of Pullback Attractors

In this section we first introduce some basic notations that will be used through this article, give the existence of the weak solution, and then estimate the solutions of the problem (1.1)-(1.3) to ensure the existence of the existence of pullback attractors. We will always assume that $n = 1, 2, 3$.

2.1 Basic notations and weak solutions

For metric spaces X and Y, we conventionally denote by $\mathcal{C}(X,Y)$ ($\mathcal{C}_{\mathrm{b}}(X,Y)$) the collection of continuous (and bounded) functionals from X to Y. When $Y = \mathbb{R}$, we simply use $\mathcal{C}(X)$ ($\mathcal{C}_{\mathrm{b}}(X)$) to represent $\mathcal{C}(X,\mathbb{R})$ ($\mathcal{C}_{\mathrm{b}}(X,\mathbb{R})$). We also use the following abbreviations,

$$H = L^2(\Omega),\ V = H_0^2(\Omega),\ W = H_0^2(\Omega) \cap H^4(\Omega),\ \|\cdot\| = \|\cdot\|_2,\ \text{where } \|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$$

for each $p \geqslant 1$, and choose $\|\Delta\cdot\|$ as the norm of V. Let V' and W' be the dual spaces of V and W, respectively. At the same time, we denote by $(\cdot,\cdot)$ the inner product of $L^2(\Omega)$, by $\langle\cdot,\cdot\rangle$ the dual pairing between V' and V and by $\langle\langle\cdot,\cdot\rangle\rangle$ the dual pairing between W' and W, respectively. Obviously, we have the following dense embeddings

$$W \subset V \subset H \subset V' \subset W'. \tag{2.1}$$

Let $A := \Delta^2 : \mathscr{D}(A) = W \to H$ be the principal operator of (1.1). The operator A is positive, self-adjoint and possesses a basis of eigenfunctions $\{\omega_i\}_{i\in\mathbb{N}_+}$, which is orthonormal in H and associated with the eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}_+}$ such that

$$0 < \lambda := \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_i \leqslant \cdots \to +\infty.$$

Then $\{\omega_i\}_{i\in\mathbb{N}_+}$ can be assumed to be an orthonormal basis of H.

In the sequel, we write $g(\cdot, x)$ as $g(\cdot) : \mathbb{R} \to H$ with $g(t)x = g(t, x)$. Then in the sense of distribution $\mathcal{D}(\tau, +\infty; W')$, the problem (1.1)-(1.3) can be written as

$$u_t + Au + 2\Delta u + au + b|\nabla u|^2 + u^3 = g(t),\ t > \tau, \qquad u(\tau) = \phi \in V. \tag{2.2}$$

We now specify the definition of solutions to the problem (2.2).

Definition 2.1. *A function*

$$u \in \mathcal{C}([\tau, +\infty), V) \cap L^2(0, T; W) \quad \text{for each } T \in (\tau, +\infty), \tag{2.3}$$

is called a global **weak solution** *of* (2.2)*, provided that the generalized derivative $u'(t)$ of $u(t)$ with respect to t satisfies $u'(t) \in W'$ for all $t > \tau$, and* (2.2) *holds in the sense of distribution $\mathcal{D}(\tau, +\infty; W')$.*

In order to obtain the existence of the pullback attractor and statistical solutions for the problem (2.2), we need some assumption on the parameter b and the external force g:

(**A**) Assume that $|b| < 4$, $g \in L^2_{\text{loc}}(\mathbb{R}, H)$ and there exists a $T_0 \geqslant 0$ and an

$$\alpha \in \left[0, \left(\frac{n-4}{n-12}\right)^2 \lambda\right) \quad \text{such that} \quad \|g(t)\|^2 \leqslant \beta \mathrm{e}^{-\alpha t},$$

for all $t \leqslant -T_0$ and some $\beta > 0$.

For each $t \in \mathbb{R}$, we define

$$G(t) := \int_{-\infty}^{t} \mathrm{e}^{\lambda s}\|g(s)\|^2 \mathrm{d}s \quad \text{and} \quad \mathcal{G}(t) := \int_{-\infty}^{t} \mathrm{e}^{-\frac{8\lambda s}{4-n}} [G(s)]^{\frac{12-n}{4-n}} \mathrm{d}s.$$

With this assumption (**A**), we know that the nonautonomous term g satisfies

$$G(t),\, \mathcal{G}(t) < +\infty \quad \text{for each } t \in \mathbb{R}.$$

According to the discussions in [8, 10] and standard methods in [48, 49], under the assumption (**A**), for all $\tau \in \mathbb{R}$ and $\phi \in V$, the problem (2.2) is globally well-posed in V and the corresponding weak solution $u(t, \tau; \phi)$ satisfies (2.3).

In the following estimates, we denote c as an arbitrary positive constant, which only depends the parameters of the original problem and the assumption (**A**) (i.e., a, b, λ, n, Ω), and may be different from line to line and even in the same line.

2.2 Existence of pullback attractors

We here give the necessary estimates of the weak solution of (2.3) to certify the existence of the pullback attractors.

Lemma 2.2. *Let the assumption* (**A**) *hold. Then for each initial datum $\phi \in V$, we have the following estimate for the solution $u(t) = u(t, \tau; \phi)$, for some positive constant $C_1 = C_1(a, b, \lambda, n, \Omega)$,*

$$\begin{aligned} &\|\Delta u(t)\|^2 + \int_{\tau}^{t} \mathrm{e}^{\lambda(s-t)}\|\Delta^2 u(s)\|^2 \mathrm{d}s \\ \leqslant & C_1 \left[1 + \mathrm{e}^{\lambda(\tau-t)}\|\Delta\phi\|^{\frac{2(12-n)}{4-n}} + \mathrm{e}^{-\lambda t}[G(t) + \mathcal{G}(t)]\right]. \end{aligned} \tag{2.4}$$

Proof. Let $|b| < 4$. We first give some estimates of the solutions of (2.2) in H. Taking the inner product of (2.2) with u in H and the inequalities

$$|(|\nabla u|^2, u)| = \frac{1}{2}|(u^2, \Delta u)| \leqslant \frac{1}{4}\|\Delta u\|^2 + \frac{1}{4}\|u\|_4^4, \tag{2.5}$$

$$|(2\Delta u, u)| \leqslant \frac{4-|b|}{8}\|\Delta u\|^2 + \frac{8}{4-|b|}\|u\|^2, \quad |(g(t), u)| \leqslant \frac{1}{2}(\|u\|^2 + \|g(t)\|^2)$$

into consideration, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \lambda\|u\|^2 + \frac{4-|b|}{4}\|\Delta u\|^2\\ \leqslant&\left[\left(\lambda + 1 - 2a + \frac{16}{4-|b|}\right)\|u\|^2 - \frac{4-|b|}{2}\|u\|_4^4\right] + \|g(t)\|^2.\end{aligned}$$

Note that for every solution u, there holds

$$\left(\lambda + 1 - 2a + \frac{16}{4-|b|}\right)\|u\|^2 - \frac{4-|b|}{2}\|u\|_4^4 \leqslant 1 + \frac{2|\lambda + 1 - 2a + \frac{16}{4-|b|}|^2}{4-|b|} := N,$$

where the constant N is independent of u. Hence we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \lambda\|u\|^2 + \frac{4-|b|}{4}\|\Delta u\|^2 \leqslant N + \|g(t)\|^2. \tag{2.6}$$

Replacing the time variable t with s in (2.6), then multiplying it by $\mathrm{e}^{\lambda s}$ and integrating the resulting equality over $[\tau, t]$ with respect to s, we obtain that

$$\begin{aligned}&\|u(t)\|^2 + \frac{4-|b|}{4}\int_\tau^t \mathrm{e}^{\lambda(s-t)}\|\Delta u(s)\|^2 \mathrm{d}s\\ \leqslant&\mathrm{e}^{\lambda(\tau-t)}\|\phi\|^2 + \frac{N}{\lambda} + \int_\tau^t \mathrm{e}^{\lambda(s-t)}\|g(s)\|^2 \mathrm{d}s.\end{aligned} \tag{2.7}$$

Next we show (2.4). From the proof of [22, Lemma 4.3], we see that

$$\|\nabla u\|_4 \leqslant c\|\Delta^2 u\|^{\frac{n+4}{16}}\|u\|^{\frac{12-n}{16}} \quad \text{and} \quad \|u\|_6 \leqslant c\|\Delta^2 u\|^{\frac{n}{12}}\|u\|^{\frac{12-n}{12}}. \tag{2.8}$$

Hence by Hölder inequality and Young's inequality, we see that

$$|b(|\nabla u|^2, \Delta^2 u)| \leqslant |b|\|\nabla u\|_4^2\|\Delta^2 u\| \leqslant c\|\Delta^2 u\|^{\frac{n+12}{8}}\|u\|^{\frac{12-n}{8}} \leqslant \frac{1}{8}\|\Delta^2 u\|^2 + c\|u\|^{\frac{2(12-n)}{4-n}}, \tag{2.9}$$

$$|(u^3, \Delta^2 u)| \leqslant \|u\|_6^3\|\Delta^2 u\| \leqslant c\|\Delta^2 u\|^{\frac{n+4}{4}}\|u\|^{\frac{12-n}{4}} \leqslant \frac{1}{8}\|\Delta^2 u\|^2 + c\|u\|^{\frac{2(12-n)}{4-n}}, \tag{2.10}$$

$$|(2\Delta u, \Delta^2 u)| \leqslant \frac{1}{8}\|\Delta^2 u\|^2 + 8\|\Delta u\|^2, \quad |(g(t), \Delta^2 u)| \leqslant \frac{1}{8}\|\Delta^2 u\|^2 + 2\|g(t)\|^2. \tag{2.11}$$

Multiplying (2.2) by $\Delta^2 u$, integrating the resulting equality over Ω, and combining the inequalities (2.9)-(2.11), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2 + \lambda\|\Delta u\|^2 + \|\Delta^2 u\|^2 \leqslant c\left(\|\Delta u\|^2 + \|u\|^{\frac{2(12-n)}{4-n}}\right) + 4\|g(t)\|^2, \tag{2.12}$$

and hence

$$\begin{aligned}&\|\Delta u(t)\|^2 + \int_\tau^t \mathrm{e}^{\lambda(s-t)}\|\Delta^2 u(s)\|^2 \mathrm{d}s\\ \leqslant&\mathrm{e}^{\lambda(\tau-t)}\|\Delta\phi\|^2 + c\int_\tau^t \mathrm{e}^{\lambda(s-t)}\left(\|\Delta u(s)\|^2 + \|u(s)\|^{\frac{2(12-n)}{4-n}} + \|g(s)\|^2\right)\mathrm{d}s,\end{aligned}$$

which, together with (2.7) and (2.1), gives (2.4). This ends the proof. □

Remark 2.3. *The requirement $|b|<4$ is attributed to the estimate* (2.5)*. Originally, the ε-Young's inequality can decrease one of the coefficients of $\|\Delta u\|^2$ and $\|u\|_4^4$, but each case except the choice in* (2.5) *will make the other coefficient more than $\frac{1}{4}$, so that we have to let $|b|<\hat{b}$ for some $\hat{b}\in(0,4)$ to ensure the following estimates to go smoothly.*

To introduce the pullback attractors, we adopt the attractor theory of the norm-to-weak continuous process in [11, 50]. Here we will not repeat much of those definitions.

Let $\mathcal{P}(V)$ be the family of all nonempty subsets of V. A *nonautonomous set* $\widehat{D}=\{D(t)\}_{t\in\mathbb{R}}$ is a subset of $\mathcal{P}(V)$ indexed by time t. Given another nonautonomous set $\widehat{D}'=\{D'(t)\}_{t\in\mathbb{R}}$, we use "$\widehat{D}'\subseteq\widehat{D}$" to indicate that $D'(t)\subseteq D(t)$ for each $t\in\mathbb{R}$. Let $\mathcal{D}$ be a *universe* in $\mathcal{P}(V)$, i.e., a given collection of nonautonomous sets such that whenever $\widehat{D}\in\mathcal{D}$ and $\widehat{D}'\subseteq\widehat{D}$, we have $\widehat{D}'\in\mathcal{D}$.

The global weak solution of (2.2) generates a norm-to-weak process $\{U(t,\tau)\}_{t\geqslant\tau}$ in V by setting $U(t,\tau)\phi=u(t,\tau;\phi)$ for each $\phi\in V$. A nonautonomous set $\widehat{B}=\{B(t)\}_{t\in\mathbb{R}}$ is *pullback $\mathcal{D}$-absorbing* for $\{U(t,\tau)\}_{t\geqslant\tau}$, if for each $t\in\mathbb{R}$ and $\widehat{D}=\{D(t)\}_{t\in\mathbb{R}}\in\mathcal{D}$, there exists a $T(t,\widehat{D})\leqslant t$ such that $U(t,\tau)D(\tau)\subseteq B(t)$ for all $\tau\leqslant T(t,\widehat{D})$. What we emphasize is the concept — pullback $\mathcal{D}$-flattening, which is called (PDC) in [11, 50]. In this article we use the calling of Kloeden and Langa [51].

Definition 2.4. *The process $\{U(t,\tau)\}_{t\geqslant\tau}$ is said to be* **pullback $\mathcal{D}$-flattening**, *provided that for each $t\in\mathbb{R}$, $\widehat{D}=\{D(t)\}_{t\in\mathbb{R}}\in\mathcal{D}$ and $\varepsilon>0$, there exist $T=T(\widehat{D},t,\varepsilon)\leqslant t$ and a finite-dimensional subspace V_1 of V such that*

1. *$P(\cup_{\tau\leqslant T}U(t,\tau)D(\tau))$ is bounded, and*
2. *$\|(I-P)(\cup_{\tau\leqslant T}U(t,\tau)\phi)\|_V<\varepsilon$, for all $\phi\in D(\tau)$,*

where $P:V\to V_1$ is a bounded projection.

Now denote by $\mathcal{R}$ the set of all functions $r:\mathbb{R}\to(0,+\infty)$ such that

$$\lim_{t\to-\infty}\mathrm{e}^{\lambda t}[r(t)]^{\frac{2(12-n)}{4-n}}=0. \tag{2.13}$$

Let $\mathcal{D}_\lambda$ be the class of all families $\widehat{D}=\{D(t)\}_{t\in\mathbb{R}}\subset B(V)$ such that $D(t)$ is nonempty and $D(t)\subset\overline{\mathrm{B}}(r(t))$ for some $r\in\mathcal{R}$, where $B(V)$ denotes the set of all bounded subsets of V and $\overline{\mathrm{B}}(r(t))$ denotes the closed ball in V centered at the origin with radius $r(t)$.

We are going to prove the pullback $\mathcal{D}_\lambda$-flattening of $\{U(t,\tau)\}_{t\geqslant\tau}$. The procedure of the proof is a modification of that of [11, Theorem 4.1], according to the relaxation of the assumption (**A**) relative to that in [11]. We present it here for the reader's convenience.

Lemma 2.5. *Under the assumption (**A**), the process $\{U(t,\tau)\}_{t\geqslant\tau}$ is pullback $\mathcal{D}_\lambda$-flattening.*

Proof. Let V_k be the closed subspace of V spanned by $\{\omega_i\}_{i=1}^n$ and $P_k:V\to V_k$ be the corresponding projection with $k\geqslant 2$. We denote that

$$u=P_ku+(1-P_k)u:=u_1+u_2.$$

Fix $t\in\mathbb{R}$, $\varepsilon>0$ and $\widehat{D}=\{D(t)\}_{t\in\mathbb{R}}\in\mathcal{D}_\lambda$ and consider $\phi\in D(\tau)$ for some $\tau<t$. By the setting above, there is a function $r:\mathbb{R}\to\mathbb{R}^+$ satisfying (2.13) such that

$$\|\Delta\phi\|\leqslant r(\tau). \tag{2.14}$$

Then by (2.4), we have

$$\|\Delta u_1(t,\tau;\phi)\|^2\leqslant\|\Delta u(t)\|^2\leqslant C_1\left[1+\mathrm{e}^{\lambda(\tau-t)}[r(\tau)]^{\frac{2(12-n)}{4-n}}+\mathrm{e}^{-\lambda t}[G(t)+\mathcal{G}(t)]\right].$$

By (2.13), we have a $T=T(\widehat{D},t)\leqslant t$ such that

$$\|\Delta u_1(t)\|^2\leqslant\|\Delta u(t)\|^2\leqslant C_1\left[2+\mathrm{e}^{-\lambda t}[G(t)+\mathcal{G}(t)]\right] \tag{2.15}$$

1 for all $\tau \leqslant T$, which turns out the condition (1) of Definition 2.4.

Now we check the condition (2) of Definition 2.4. Take into consideration the scalar product of (2.2) and $\Delta^2 u_2$ in H. Similar to the treatment above (2.12) and owing to embeddings and $\lambda_k\|\Delta u_2\|^2 \leqslant \|\Delta^2 u_2\|^2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u_2\|^2 + \lambda_k\|\Delta u_2\|^2 \leqslant c\left(\|\Delta u\|^2 + \|u\|^{\frac{2(12-n)}{4-n}} + \|g(t)\|^2\right)$$

and hence

$$\begin{aligned}\|\Delta u_2(t,\tau;\phi)\|^2 \leqslant & c\mathrm{e}^{\lambda_k(\tau-t)}\|\Delta\phi\|^2 + c\int_\tau^t \mathrm{e}^{\lambda_k(s-t)}\|\Delta u(s)\|^2\mathrm{d}s \\ & + c\int_\tau^t \mathrm{e}^{\lambda_k(s-t)}\|u(s)\|^{\frac{2(12-n)}{4-n}}\mathrm{d}s + c\int_\tau^t \mathrm{e}^{\lambda_k(s-t)}\|g(s)\|^2\mathrm{d}s \\ & := I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{2.16}$$

Fix $t \in \mathbb{R}$, $\varepsilon > 0$ and $\widehat{D} \in \mathcal{D}_\lambda$ as above. Note that by (2.14)

$$I_1 \leqslant c\mathrm{e}^{\lambda(\tau-t)}\left[[r(\tau)]^{\frac{2(12-n)}{4-n}} + 1\right].$$

2 We thus deduce from (2.13) the existence of $T_1 = T_1(\widehat{D}, t, \varepsilon) \leqslant t$ such that

$$I_1 < \frac{\varepsilon}{4}, \quad \text{whenever } \tau \leqslant T_1. \tag{2.17}$$

By (2.4) and (2.14), we have

$$\begin{aligned} I_2 \leqslant & c\int_\tau^t \mathrm{e}^{\lambda_k(s-t)}\left[1 + \mathrm{e}^{\lambda(\tau-s)}[r(\tau)]^{\frac{2(12-n)}{4-n}} + \mathrm{e}^{-\lambda s}[G(s) + \mathcal{G}(s)]\right]\mathrm{d}s \\ \leqslant & c\int_\tau^t \mathrm{e}^{\lambda_k(s-t)}\mathrm{d}s + c\left[\mathrm{e}^{\lambda\tau}[r(\tau)]^{\frac{2(12-n)}{4-n}} + [G(t) + \mathcal{G}(t)]\right]\mathrm{e}^{-\lambda_k t}\int_\tau^t \mathrm{e}^{(\lambda_k-\lambda)s}\mathrm{d}s \\ \leqslant & \frac{c}{\lambda_k} + \frac{c\mathrm{e}^{-\lambda t}}{\lambda_k - \lambda}\left[\mathrm{e}^{\lambda\tau}[r(\tau)]^{\frac{2(12-n)}{4-n}} + [G(t) + \mathcal{G}(t)]\right]. \end{aligned} \tag{2.18}$$

3 By (2.13) and (2.18), we obtain a $T_2 = T_2(\widehat{D}, t, \varepsilon) \leqslant t$ and an $K_2 > 0$ such that

$$I_2 < \frac{\varepsilon}{4}, \quad \text{whenever } \tau \leqslant T_2 \text{ and } k > K_2. \tag{2.19}$$

Applying (2.7) in I_3, we have

$$\begin{aligned} I_3 \leqslant & \frac{c}{\lambda_k} + c\left[\mathrm{e}^{\lambda\tau}[r(\tau)]^{\frac{2(12-n)}{4-n}}\mathrm{e}^{\frac{8\lambda\tau}{4-n}} + [G(t)]^{\frac{12-n}{4-n}\lambda}\right]\mathrm{e}^{-\lambda_k t}\int_\tau^t \mathrm{e}^{\left(\lambda_k - \frac{12-n}{4-n}\lambda\right)s}\mathrm{d}s \\ \leqslant & \frac{c}{\lambda_k} + \frac{c(4-n)}{(4-n)\lambda_k - (12-n)\lambda}\mathrm{e}^{-\frac{12-n}{4-n}\lambda t}\left[\mathrm{e}^{\lambda\tau}[r(\tau)]^{\frac{2(12-n)}{4-n}}\mathrm{e}^{\frac{8\lambda\tau}{4-n}} + [G(t)]^{\frac{12-n}{4-n}\lambda}\right]. \end{aligned}$$

4 The existence of $T_3 \leqslant t$ and $K_3 > 0$ can be obtained with no hard, such that

$$I_3 < \frac{\varepsilon}{4}, \quad \text{when } \tau \leqslant T_3 \text{ and } k > K_3. \tag{2.20}$$

As to I_4, we know that for each $\delta > 0$,

$$\begin{aligned} I_4 \leqslant & c\int_{-\infty}^{t-\delta} \mathrm{e}^{\lambda_k(s-t)}\|g(s)\|^2\mathrm{d}s + c\int_{t-\delta}^t \mathrm{e}^{\lambda_k(s-t)}\|g(s)\|^2\mathrm{d}s \\ \leqslant & c\mathrm{e}^{-(\lambda_k-\lambda)\delta}\mathrm{e}^{-\lambda t}G(t-\delta) + c\int_{t-\delta}^t \|g(s)\|^2\mathrm{d}s. \end{aligned} \tag{2.21}$$

Since $g \in L^2_{\mathrm{loc}}(\mathbb{R}, H)$, we can find a $\delta_0 > 0$ sufficiently small such that

$$c\int_{t-\delta_0}^{t} \|g(s)\|^2 \mathrm{d}s < \frac{\varepsilon}{8}. \tag{2.22}$$

For the δ_0 above, since $-\delta_0 < 0$, we can further find from (3.1) an $K_4 > 0$ sufficiently large such that when $k > K_4$,

$$c\mathrm{e}^{-(\lambda_k-\lambda)\delta_0}\mathrm{e}^{-\lambda t}G(t-\delta_0) < \frac{\varepsilon}{8}. \tag{2.23}$$

Then it follows immediately from (2.21), (2.22) and (2.23) that

$$I_4 < \frac{\varepsilon}{4}, \quad \text{whenever } k > K_4. \tag{2.24}$$

Define

$$T = \min\{T_1, T_2, T_3\} \quad \text{and} \quad K = \max\{K_2, K_3, K_4\}.$$

One easily sees from (2.16), (2.17), (2.19), (2.20) and (2.24) that when $k > K$ and $\tau \leqslant T$,

$$\|\Delta u_2(t, \tau; \phi)\|^2 < \varepsilon \quad \text{for all } \phi \in D(\tau),$$

and the condition (2) of Definition 2.4 is satisfied, which accomplishes the proof. $\square$

Now we are ready to give the existence of the pullback attractors.

Theorem 2.6. *Under the assumption* (**A**), *the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *has a unique pullback* $\mathcal{D}_\lambda$*-attractor.*

Proof. According to [11, Theorem 2.1] (or [50]) and Lemma 2.5, it suffices to construct a pullback $\mathcal{D}_\lambda$-absorbing set $\widehat{B}$ in $\mathcal{D}_\lambda$ for $\{U(t,\tau)\}_{t\geqslant\tau}$.

Indeed, let

$$r_0^2(t) = C_1\left[2 + \mathrm{e}^{-\lambda t}(G(t) + \mathcal{G}(t))\right]$$

and $\widehat{B} = \{B(t)\}_{t\in\mathbb{R}}$ with $B(t) = \{\phi \in V : \|\Delta\phi\| \leqslant r_0(t)\}$. By the assumption (**A**), let $t < -T_0$ and we have

$$\mathrm{e}^{-\lambda t}(G(t) + \mathcal{G}(t)) \leqslant \mathrm{e}^{-\alpha t} + \mathrm{e}^{-\frac{12-n}{4-n}\alpha t},$$

which assures (2.13) by replacing r therein by r_0 and $\widehat{B} \in \mathcal{D}_\lambda$.

We now check that $\widehat{B}$ is pullback $\mathcal{D}_\lambda$-absorbing. Fix $t \in \mathbb{R}$ and $\widehat{D} = \{D(t)\}_{t\in\mathbb{R}} \in \mathcal{D}_\lambda$ with r defined as (2.13) and $D(t) \subset \overline{\mathrm{B}}(r(t))$ for all $t \in \mathbb{R}$. Then by (2.15), we have a $T' = T'(\widehat{D}, t) \leqslant t$ such that $U(t,\tau)D(\tau) \subseteq B(t)$ for all $\tau \leqslant T'$. This proves the pullback $\mathcal{D}_\lambda$-absorbing property. The proof is complete. $\square$

3 Existence of Invariant Measures and Statistical Solutions

In this section we study the existence of invariant Borel probability measures and statistical solutions for the nonautonomous modified Swift-Hohenberg problem (1.1)-(1.3). For this aim, we need first to ensure the boundedness and continuity of the weak solution $u(t,\tau;\phi)$ with respect to the initial time τ.

3.1 Bounded continuity with respect to the initial time

First, some auxiliary estimates for the weak solutions are necessary as follows for the proof of bounded continuity with respect to the initial time.

Lemma 3.1. *Let* ϕ, $\psi \in V$, $u_1(t) = u(t,\tau;\phi)$ *and* $u_2(t) = u(t,\tau;\psi)$ *for* $t \geqslant \tau$. *Then we have a positive constant* $C_2 = C_2(a, b, n, \Omega)$ *such that*

$$\begin{aligned}&\|\Delta u_1(t) - \Delta u_2(t)\|^2\\ \leqslant&\|\Delta\phi - \Delta\psi\|^2 \exp\left(C_2\int_\tau^t \left(\|\Delta u_1(s)\|^2 + 1\right)\left(\|\Delta u_2(s)\|^2 + 1\right)\mathrm{d}s\right).\end{aligned} \tag{3.1}$$

Proof. Let $v(t) = u_1(t) - u_2(t)$ for all $t \geqslant \tau$. We have

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \Delta^2 v + 2\Delta v + av + b\nabla(u_1+u_2)\cdot\nabla v + v^3 + 3u_1u_2v = 0. \tag{3.2}$$

Taking the following estimates into account

$$\begin{aligned}
&|(2\Delta v, \Delta^2 v)| \leqslant \frac{1}{3}\|\Delta^2 v\|^2 + 3\|\Delta v\|^2, \quad (v^3, \Delta^2 v) = 3(|\Delta v|^2, v^2) + 6(|\nabla v|^2 v, \Delta v),\\
&|6(|\nabla v|^2 v, \Delta v)| \leqslant 6\|\Delta v\|\|v\|_4\|\nabla v\|_8^2 \leqslant c\|\Delta v\|^4 \leqslant c(\|\Delta u_1\| + \|\Delta u_2\|)^2\|\Delta v\|^2,\\
&|b(\nabla(u_1+u_2)\cdot\nabla v, \Delta^2 v)| \leqslant \frac{1}{3}\|\Delta^2 v\|^2 + c\|\nabla(u_1+u_2)\|_4^2\|\nabla v\|_4^2\\
&\qquad\qquad\qquad\qquad\qquad \leqslant \frac{1}{3}\|\Delta^2 v\|^2 + c(\|\Delta u_1\| + \|\Delta u_2\|)^2\|\Delta v\|^2,\\
&|(3u_1u_2v, \Delta^2 v)| \leqslant \frac{1}{3}\|\Delta^2 v\|^2 + \frac{27}{4}\|u_1u_2\|_4^2\|v\|_4^2 \leqslant \frac{1}{3}\|\Delta^2 v\|^2 + c\|\Delta u_1\|^2\|\Delta u_2\|^2\|\Delta v\|^2,
\end{aligned} \tag{3.3}$$

we obtain from (3.2)

$$\frac{\mathrm{d}\|\Delta v(t)\|^2}{\mathrm{d}t} \leqslant c\left(\|\Delta u_1(t)\|^2 + 1\right)\left(\|\Delta u_2(t)\|^2 + 1\right)\|\Delta v(t)\|^2. \tag{3.4}$$

By using Gronwall's lemma to (3.4), we obtain (3.1) and complete the proof. $\square$

We next prove that the V-valued mapping $\tau \mapsto u(t,\tau;\phi)$ is continuous and bounded on $(-\infty, t]$. To this end, we define for each $t \in \mathbb{R}$,

$$M(t,\phi) := 1 + \|\Delta\phi\|^{\frac{2(12-n)}{4-n}} + \mathrm{e}^{-\lambda t}(G(t) + \mathcal{G}(t)). \tag{3.5}$$

We know from Lemma 2.2 that $C_1 M(t,\phi)$ is an upper bound of $\|\Delta u(t,\cdot;\phi)\|^2$ over $(-\infty, t]$.

Lemma 3.2. *Let the assumption* (**A**) *hold. Fix $\tau \in \mathbb{R}$ and $\phi \in V$. Then for every $\varepsilon > 0$, there exists a positive number $\delta = \delta(\varepsilon, \tau, \phi)$ (surely depending also on a, b, g, n, λ and Ω) such that*

$$\|\Delta u(\tau,\tau';\phi) - \Delta\phi\| < \varepsilon, \ \ \tau' \in (\tau - \delta, \tau). \tag{3.6}$$

Proof. Observe that

$$\begin{aligned}
\|\Delta u(\tau,\tau';\phi) - \Delta\phi\|^2 =& \|\Delta u(\tau,\tau';\phi)\|^2 - \|\Delta\phi\|^2 - 2(\Delta u(\tau,\tau';\phi) - \Delta\phi, \Delta\phi)\\
\leqslant& \int_{\tau'}^{\tau} \frac{\mathrm{d}}{\mathrm{d}s}\|\Delta u(s,\tau';\phi)\|^2 \mathrm{d}s - 2(\Delta u(\tau,\tau';\phi) - \Delta\phi, \Delta\phi).
\end{aligned} \tag{3.7}$$

For the former part of (3.7), by (2.4), (2.7) and (2.12), we have

$$\left|\int_{\tau'}^{\tau} \frac{\mathrm{d}}{\mathrm{d}s}\|\Delta u(s,\tau';\phi)\|^2 \mathrm{d}s\right| \leqslant c\left[M(\tau,\phi)(\tau - \tau') + \int_{\tau'}^{\tau}\|g(s)\|^2\mathrm{d}s\right].$$

Due to the fact $g \in L^2_{\mathrm{loc}}(\mathbb{R}, H)$, for the given ε above, we see that there is a $\delta_1 = \delta_1(\varepsilon,\tau,\phi) > 0$ such that

$$\left|\int_{\tau'}^{\tau} \frac{\mathrm{d}}{\mathrm{d}s}\|\Delta u(s,\tau';\phi)\|^2 \mathrm{d}s\right| < \frac{\varepsilon^2}{2}, \quad \tau' \in (\tau - \delta_1, \tau). \tag{3.8}$$

Now we consider the latter part of (3.7). By the density of $H_0^6(\Omega)$ in V, we have an element $\psi \in H_0^6(\Omega)$ (note that $\Delta^2\psi \in V$) with $\|\Delta(\psi - \phi)\| < \frac{\varepsilon^2}{16 C_1 M(\tau,\phi)}$ such that

$$\begin{aligned}
&|(\Delta u(\tau,\tau';\phi) - \Delta\phi, \Delta\phi)|\\
\leqslant& |(\Delta(u(\tau,\tau';\phi) - \phi), \Delta\psi)| + |(\Delta(u(\tau,\tau';\phi) - \phi), \Delta(\psi - \phi))|\\
\leqslant& |\langle u(\tau,\tau';\phi) - \phi, \Delta^2\psi\rangle| + \frac{\varepsilon^2}{8}.
\end{aligned} \tag{3.9}$$

Next we estimate the term $|\langle u(\tau,\tau';\phi)-\phi,\Delta^2\psi\rangle|$. Since

$$\begin{aligned}|\langle u(\tau,\tau';\phi)-\phi,\Delta^2\psi\rangle| =&\left|\left\langle\int_{\tau'}^{\tau}\frac{\mathrm{d}}{\mathrm{d}s}u(s,\tau';\phi)\mathrm{d}s,\Delta^2\psi\right\rangle\right|\\ \leqslant&(\tau-\tau')^{1/2}\left(\int_{\tau'}^{\tau}\left\|\frac{\mathrm{d}u(s)}{\mathrm{d}s}\right\|_{V'}^2\mathrm{d}s\right)^{1/2}\|\Delta^3\psi\|,\end{aligned}$$

as long as there exists $\delta_2=\delta_2(\tau,\phi)>0$ such that when $\tau'\in(\tau-\delta_2,\tau)$,

$$\int_{\tau'}^{\tau}\left\|\frac{\mathrm{d}u(s)}{\mathrm{d}s}\right\|_{V'}^2\mathrm{d}s\leqslant C_3,\tag{3.10}$$

for some positive constant $C_3=C_3(\tau,\phi)$, we easily find $\delta_2'=\delta_2'(\varepsilon,\tau,\phi)\in(0,\delta_2)$ such that

$$|\langle u(\tau,\tau';\phi)-\phi,\Delta^2\psi\rangle|<\frac{\varepsilon^2}{8}\quad\text{ whenever }\tau'\in(\tau-\delta_2',\tau).\tag{3.11}$$

Then we pick $\delta=\min\{\delta_1,\delta_2'\}$, combine (3.7)-(3.11) and obtain that when $\tau'\in(\tau-\delta,\tau)$, (3.6) holds true, which ends the proof.

We now only need to verify (3.10). Actually, by (2.4), (2.7) and Young's inequality, together with some computations, we have

$$\begin{aligned}&\|\Delta^2u\|_{V'}\leqslant\|\Delta u\|,\quad\|\Delta u\|_{V'}\leqslant c\|\Delta u\|,\quad\|u\|_{V'}\leqslant c\|\Delta u\|,\\ &\||\nabla u|^2\|_{V'}\leqslant c\|\nabla u\|_4^2\leqslant c\|\Delta^2u\|^{\frac{4+n}{8}}\|u\|^{\frac{12-n}{8}}\leqslant c\left(\|\Delta^2u\|+\|u\|^{\frac{12-n}{4-n}}\right),\\ &\|u^3\|_{V'}\leqslant c\|u\|_6^3\leqslant c\|\Delta^2u\|^{\frac{n}{4}}\|u\|^{\frac{12-n}{4}}\leqslant c\left(\|\Delta^2u\|+\|u\|^{\frac{12-n}{4-n}}\right).\end{aligned}\tag{3.12}$$

Note moreover that

$$\mathrm{e}^{\lambda(\tau'-\tau)}\int_{\tau'}^{\tau}\|\Delta^2u(s)\|^2\mathrm{d}s\leqslant C_1M(\tau,\phi),\quad\text{ for all }s\in[\tau',\tau].$$

By the above discussion and the estimation in the proof of Lemma 2.2, we know that

$$\begin{aligned}\int_{\tau'}^{\tau}\left\|\frac{\mathrm{d}u(s)}{\mathrm{d}s}\right\|_{V'}^2\mathrm{d}s\leqslant&c\int_{\tau'}^{\tau}(\|\Delta u(s)\|^2+\|u(s)\|^{\frac{2(12-n)}{4-n}}+\|g(s)\|^2+\|\Delta^2u\|^2)\mathrm{d}s\\ \leqslant&c\mathrm{e}^{\lambda(\tau-\tau')}\int_{\tau'}^{\tau}\mathrm{e}^{\lambda(s-\tau)}(\|\Delta u(s)\|^2+\|u(s)\|^{\frac{2(12-n)}{4-n}}\\ &+\|g(s)\|^2)\mathrm{d}s+c\int_{\tau'}^{\tau}\|\Delta^2u(s)\|^2\mathrm{d}s\\ \leqslant&c\mathrm{e}^{\lambda(\tau-\tau')}M(\tau,\phi).\end{aligned}\tag{3.13}$$

Thus (3.10) can be trivially deduced from (3.13) by choosing $\delta_2=1$. $\square$

Lemma 3.3. *Let the assumption* (**A**) *hold. Then for every $t\in\mathbb{R}$ and $\phi\in V$, the V-valued mapping $\tau\mapsto u(t,\tau;\phi)$ is bounded and continuous on $(-\infty,t]$.*

Proof. The boundedness can be easily obtained by (2.4) and (3.5). The continuity is read as follows, by fixing τ, $t\in\mathbb{R}$ and $\phi\in V$, with $t\geqslant\tau$, for every $\varepsilon>0$, there exists a positive number δ depending only on ε, τ, t, ϕ (and surely also on the parameters a, b, g, n, λ and Ω) such that

$$\|\Delta u(t,\tau';\phi)-\Delta u(t,\tau;\phi)\|<\varepsilon,\quad\text{ whenever }|\tau'-\tau|<\delta.\tag{3.14}$$

We first consider the right continuity, saying the case when $\tau'>\tau$. By Lemma 3.1, we have

$$\begin{aligned}&\|\Delta u(t,\tau';\phi)-\Delta u(t,\tau;\phi)\|^2\\ \leqslant&\|\Delta\phi-\Delta u(\tau',\tau;\phi)\|^2\exp\left(c\int_{\tau'}^{t}\left(\|\Delta u(s,\tau;\phi)\|^2+1\right)\left(\|\Delta u(s,\tau';\phi)\|^2+1\right)\mathrm{d}s\right)\\ \leqslant&\|\Delta\phi-\Delta u(\tau',\tau;\phi)\|^2\exp\left(c\int_{\tau}^{t}\left(M(s,\phi)+1\right)^2\mathrm{d}s\right).\end{aligned}$$

Then the continuity of $u(\cdot,\tau;\phi)$ over $[\tau,t]$ guarantees the existence of the positive number $\delta=\delta(\varepsilon,\tau,t,\phi)$ such that (3.14) holds for all $\tau'\in(\tau,\tau+\delta)$.

The left continuity can be similarly obtained by utilizing Lemma 3.2 and we omit the details here. $\square$

3.2 Construction of statistical solutions

We are now well prepared to combine the existence of the pullback attractor and the abstract result (see [31]) to verify the existence of invariant measures. We first recall the definition of the generalized Banach limit.

Definition 3.4 ([45, Definition 4.1]). *A* **generalized Banach limit** *is any linear functional, denoted by* $\mathop{\mathrm{LIM}}\limits_{t\to+\infty}$*, defined on the collection of all bounded real-valued functions on* $[0,+\infty)$ *and satisfying*

1. $\mathop{\mathrm{LIM}}\limits_{t\to+\infty}\zeta(t)\geqslant 0$ *for nonnegative functions* ζ *on* $[0,+\infty)$*;*
2. $\mathop{\mathrm{LIM}}\limits_{t\to+\infty}\zeta(t)=\lim\limits_{t\to+\infty}\zeta(t)$ *if the latter limit exists.*

Using Theorem 2.6, (2.3), Lemma 3.3 and [31, Theorem 3.1], we can obtain the existence of invariant measures for the process $\{U(t,\tau)\}_{t\geqslant\tau}$. The details are stated in the following theorem.

Theorem 3.5. *Let the assumption (*$\mathbf{A}$*) hold and* $\widehat{\mathcal{A}}_{\mathcal{D}_\lambda}=\{\mathcal{A}_{\mathcal{D}_\lambda}(t)\}_{t\in\mathbb{R}}$ *be the unique pullback* $\mathcal{D}_\lambda$*-attractor of* $\{U(t,\tau)\}_{t\geqslant\tau}$ *in* V *given by Theorem 2.6. Given a generalized Banach limit* $\mathop{\mathrm{LIM}}\limits_{t\to+\infty}$ *and a continuous mapping* $\xi:\mathbb{R}\to V$ *with* $\{\xi(t)\}_{t\in\mathbb{R}}\in\mathcal{D}_\lambda$*, there exists a unique family of Borel probability measures* $\{\mu_t\}_{t\in\mathbb{R}}$ *in* V *such that the support of the measures* μ_t *is contained in* $\mathcal{A}_{\mathcal{D}_\lambda}(t)$ *and for all* $\Upsilon\in\mathcal{C}(V)$*,*

$$\begin{aligned}&\mathop{\mathrm{LIM}}\limits_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t\Upsilon(U(t,s)\xi(s))\mathrm{d}s=\int_{\mathcal{A}_{\mathcal{D}_\lambda}(t)}\Upsilon(u)\mathrm{d}\mu_t(u)\\&=\int_V\Upsilon(u)\mathrm{d}\mu_t(u)=\mathop{\mathrm{LIM}}\limits_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t\int_V\Upsilon(U(t,\tau)u)\mathrm{d}\mu_t(u)\mathrm{d}s.\end{aligned}\tag{3.15}$$

Additionally, the measure μ_t *is invariant in the sense that*

$$\int_{\mathcal{A}_{\mathcal{D}_\lambda}(t)}\Upsilon(u)\mathrm{d}\mu_t(u)=\int_{\mathcal{A}_{\mathcal{D}_\lambda}(\tau)}\Upsilon(U(t,\tau)u)\mathrm{d}\mu_\tau(u),\quad t\geqslant\tau.\tag{3.16}$$

We next investigate the statistical solutions for the equation (2.2). Rewrite (1.1) as

$$\frac{\mathrm{d}u}{\mathrm{d}t}=F(u,t):=g(t)-\Delta^2u-2\Delta u-au-b|\nabla u|^2-u^3.\tag{3.17}$$

We see $F:W\times(\tau,+\infty)\to W'$. Associated to statistical solutions of (2.2), we define below the class $\mathcal{T}$ of test functions such that each function $\Psi\in\mathcal{T}$ satisfies

$$\frac{\mathrm{d}\Psi(u(t))}{\mathrm{d}t}=\langle\langle F(u(t),t),\Psi'(u(t))\rangle\rangle,\quad\text{for every global weak solution } u(t) \text{ of (2.2).}\tag{3.18}$$

Definition 3.6 ([39, Definition 3.3]). *The class* $\mathcal{T}$ *is defined to be the set of all real-valued functionals* Ψ *on* V *that are bounded on each bounded set of* V *such that*

1. *the Fréchet derivative* $\Psi'(u)$ *exists for every* $u\in W$*;*
2. $\Psi'(u)\in W$ *for all* $u\in W$ *and the mapping* $u\mapsto\Psi'(u)$ *belongs to* $\mathcal{C}_{\mathrm{b}}(W,W)$*;*
3. *for every global solution* u *of* (2.2)*, equality* (3.18) *holds true.*

We now give the definition of statistical solutions for the equation (3.17) and prove its existence.

Definition 3.7. *A family* $\{\mu_t\}_{t\in\mathbb{R}}$ *of Borel probability measures in* V *is called a* **statistical solution** *(in the phase space* V*) of* (3.17) *if the following conditions are satisfied,*

1. *the function* $t \mapsto \int_V \Gamma(u)\mathrm{d}\mu_t(u)$ *is continuous for every* $\Gamma \in \mathcal{C}_{\mathrm{b}}(V)$*;*
2. *for almost all* $t \in \mathbb{R}$*, the measure* μ_t *is carried by* V *and the function* $u \mapsto \langle\langle F(u(t),t), \phi\rangle\rangle$ *is* μ_t*-integrable for every* $\phi \in W$*, and the mapping*
$$t \mapsto \int_V \langle\langle F(u,t), \phi\rangle\rangle \mathrm{d}\mu_t(u)$$
belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$ *for every* $\phi \in W$*;*
3. *for each test function* $\Psi \in \mathcal{T}$*, it holds that*
$$\int_V \Psi(u)\mathrm{d}\mu_t(u) - \int_V \Psi(u)\mathrm{d}\mu_\tau(u) = \int_\tau^t \int_V \langle\langle F(u,s), \Psi'(u)\rangle\rangle \mathrm{d}\mu_s(u)\mathrm{d}s,$$
for all $\tau, t \in \mathbb{R}$ *with* $t \geqslant \tau$*.*

The main result of this article reads as follows.

Theorem 3.8. *Let the assumption* (**A**) *hold. Then the family of Borel probability measures* $\{\mu_t\}_{t\in\mathbb{R}}$ *guaranteed by Theorem 3.5 is a statistical solution of the equation* (3.17).

Proof. It is sufficient to check that the family of Borel probability measures $\{\mu_t\}_{t\in\mathbb{R}}$ obtained by Theorem 3.5 satisfies Definition 3.7.

Firstly, for each given $t_0 \in \mathbb{R}$, we show that for every $\Gamma \in \mathcal{C}_{\mathrm{b}}(V)$ there holds

$$\lim_{t\to t_0} \int_V \Gamma(u)\mathrm{d}\mu_t(v) = \int_V \Gamma(u)\mathrm{d}\mu_{t_0}(u). \tag{3.19}$$

Indeed, it follows from the equalities (3.15) and (3.16) that

$$\int_V \Gamma(u)\mathrm{d}\mu_t(u) - \int_V \Gamma(u)\mathrm{d}\mu_{t_0}(u) = \int_{\mathcal{A}_{\mathcal{D}_\lambda}(t_0)} \left(\Gamma(U(t,t_0)u) - \Gamma(u)\right)\mathrm{d}\mu_{t_0}(u), \quad t > t_0. \tag{3.20}$$

Since $U(t,t_0)u \to u$ in V as $t \to t_0^+$, $\Gamma \in \mathcal{C}_{\mathrm{b}}(V)$ and $\mathcal{A}_{\mathcal{D}_\lambda}(t_0)$ is compact in V, (3.20) gives (3.19) for the case when $t > t_0$. The left continuity can be similarly proved.

Secondly, for every $t \in \mathbb{R}$, we have known that μ_t is carried by $\mathcal{A}_{\mathcal{D}_\lambda}(t)$. Define for every ϕ, $u \in W$,

$$\Psi(u) = \langle\langle F(u(t),t), \phi\rangle\rangle. \tag{3.21}$$

Obviously Ψ maps W into $\mathbb{R}$. We next show that Ψ is continuous. Let $u_* \in W$ be fixed and consider $u \in W$ with $\|u - u_*\|_W \leqslant 1$. Note that F is also a function from $V \times \mathbb{R}$ into V'. By (3.12) and (3.3), we have

$$\begin{aligned}
&|\Psi(u) - \Psi(u_*)| = |\langle\langle F(u(t),t) - F(u_*(t),t), \phi\rangle\rangle| = |\langle F(u(t),t) - F(u_*(t),t), \phi\rangle| \\
\leqslant& c\|\Delta\phi\|\|\Delta(u-u_*)\| + c|(\nabla(u+u_*)\nabla(u-u_*), \phi)| \\
&+ c|((u^2 + uu_* + u_*^2)(u-u_*), \phi)| \\
\leqslant& c\|\Delta\phi\|\|\Delta(u-u_*)\| + c(\|\nabla u\|_4 + c\|\nabla u_*\|_4)\|\nabla(u-u_*)\|_4\|\phi\| \\
&+ c(\|u\|_8^2 + \|u_*\|_8^2)\|u-u_*\|_4\|\phi\| \\
\leqslant& c\|\Delta\phi\|\|\Delta(u-u_*)\| + c(\|\Delta u\| + \|\Delta u_*\|)\|\Delta(u-u_*)\|\|\phi\| \\
&+ c(\|\Delta u\|^2 + \|\Delta u_*\|^2)\|\Delta(u-u_*)\|\|\phi\| \\
\leqslant& c\left(1 + \|\Delta u\| + \|\Delta u_*\| + \|\Delta u\|^2 + \|\Delta u_*\|^2\right)\|u-u_*\|_W\|\phi\|_W,
\end{aligned}$$

where we have used the embeddings (2.1). This assures the continuity of Ψ on W. Thus (3.15) and (3.21) certifies the μ_t-integrability of the functional $u \mapsto \Psi(u)$, for every $\phi \in W$. Hence item (1) of Definition 3.7 and (3.21) indicate that

$$t \mapsto \int_V \langle\langle F(u,t), \phi\rangle\rangle \mathrm{d}\mu_t(u)$$

belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$ for every $\phi \in W$.

At last, for all $\Psi \in \mathcal{T}$ and $t, \tau \in \mathbb{R}$ with $t \geqslant \tau$, we have by (3.18) that

$$\Psi(u(t)) - \Psi(u(\tau)) = \int_\tau^t \langle\langle F(u(\theta), \theta), \Psi'(u(\theta))\rangle\rangle \mathrm{d}\theta. \tag{3.22}$$

Now for all $s < \tau$, let $u_* \in W$ and $u(\theta) = U(\theta, s)u_*$ for $\theta \geqslant s$. It follows easily from (3.22) that

$$\Psi(U(t,s)u_*) - \Psi(U(\tau,s)u_*) = \int_\tau^t \langle\langle F(U(\theta,s)u_*, \theta), \Psi'(U(\theta,s)u_*)\rangle\rangle \mathrm{d}\theta. \tag{3.23}$$

Thus by (3.15), (3.23), Fubini's theorem and some calculations, we obtain that

$$\begin{aligned}
&\int_V \Psi(u)\mathrm{d}\mu_t(u) - \int_V \Psi(u)\mathrm{d}\mu_\tau(u) \\
= &\operatorname*{LIM}_{\sigma\to-\infty} \frac{1}{\tau-\sigma}\int_\sigma^\tau \int_V \left(\Psi(U(t,s)u_*) - \Psi(U(\tau,s)u_*)\right) \mathrm{d}\mu_s(u_*)\mathrm{d}s \\
= &\operatorname*{LIM}_{\sigma\to-\infty} \frac{1}{\tau-\sigma}\int_\sigma^\tau \int_V \int_\tau^t \langle\langle F(U(\theta,s)u_*, \theta), \Psi'(U(\theta,s)u_*)\rangle\rangle \mathrm{d}\theta\mathrm{d}\mu_s(u_*)\mathrm{d}s \\
= &\operatorname*{LIM}_{\sigma\to-\infty} \frac{1}{\tau-\sigma}\int_\sigma^\tau \int_\tau^t \int_V \langle\langle F(U(\theta,s)u_*, \theta), \Psi'(U(\theta,s)u_*)\rangle\rangle \mathrm{d}\mu_s(u_*)\mathrm{d}\theta\mathrm{d}s.
\end{aligned} \tag{3.24}$$

Furthermore, by the semigroup property of the process and the property of invariant measures, we get

$$\begin{aligned}
&\int_V \langle\langle F(U(\theta,s)u_*, \theta), \Psi'(U(\theta,s)u_*)\rangle\rangle \mathrm{d}\mu_s(u_*) \\
= &\int_V \langle\langle F(U(\theta,\tau)U(\tau,s)u_*, \theta), \Psi'(U(\theta,\tau)U(\tau,s)u_*)\rangle\rangle \mathrm{d}\mu_s(u_*) \\
= &\int_V \langle\langle F(U(\theta,\tau)u_*, \theta), \Psi'(U(\theta,\tau)u_*)\rangle\rangle \mathrm{d}\mu_\tau(u_*),
\end{aligned}$$

which is independent of the variable s. We thus infer the equality

$$\begin{aligned}
(3.24) &= \int_\tau^t \int_V \langle\langle F(U(\theta,\tau)u_*, \theta), \Psi'(U(\theta,\tau)u_*)\rangle\rangle \mathrm{d}\mu_\tau(u_*)\mathrm{d}\theta \\
&= \int_\tau^t \int_V \langle\langle F(u_*, \theta), \Psi'(u_*)\rangle\rangle \mathrm{d}\mu_\theta(u_*)\mathrm{d}\theta,
\end{aligned}$$

that is rightly the item (3) of Definition 3.7. The proof is complete. $\square$

Remark 3.9. *In this article, the restriction $|b| < 4$ on b enables the nonautonomous modified Swift-Hohenberg equation to possess a unique pullback attractor, which consequently ensures the existence of invariant measures and statistical solutions. However, when $|b| \geqslant 4$, it is still an open problem whether the invariant measures and statistical solutions exist, since in this case, the existence of pullback attractors may not be guaranteed generally, which increases the difficulty and urges new methods in related area.*

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