Generalization of Kim's Estimates In Terms of the Trace of Divergencefree Symmetric Tensor

Serhan EKER

Ağrı İbrahım Çeçen University, Department of Mathematics, Ağrı, TURKEY.

Abstract

In this paper, we generalized E.C. Kim' s estimates by taking in to account the trace of the divergencefree symmetric tensor non−zero. We have also shown that E.C. Kim's estimates still valid in case of the trace of the divergencefree symmetric tensor vanished identically. In the equality case, we characterized eta−Killing spinor with Killing pair over the Sasakian spin manifolds.

Keywords: Spin and Spin^c geometry, Dirac operator, Estimation of eigenvalues
2010 MSC: 53C25, 53C27, 34L40

1. Introduction

The Dirac operator is one of the fundemantal tool of the geometry and topology for mathematicians [1, 19, 7]. The Dirac operator owes its fame to the fact that Witten made basic proof of the positive mass theorem on the basis of this operator [19]. The self−adjoining of the Dirac operator makes it possible to study its eigenvalues [4, 5, 7, 9, 10, 11]. One of these studies is the estimating lower bound for the eigenvalues of the Dirac operator [2, 3, 13, 14, 15, 18]. The lower bound estimates corresponding to the square of the first eigenvalue of the Dirac operator defined on the closed Riemannian spin manifolds have been studied intensively in terms of the scalar curvature, Energy-Momentum tensor and divergencefree tensor [5, 8, 12, 17]. In this direction, the first estimates is given in 1963 by A. Lichnerowicz [17]. The author is used an integration of the Schrödinger−Lichnerowicz formula defined on the closed Riemannian spin manifold to get the following lower bound:

$$\lambda^2 \geq \frac{Scal}{4}, \tag{1.1}$$

where $Scal$ is the scalar curvature of the manifold. Of course, this result is interesting when the scalar curvature is positive, but the estimate can be improved. Accordingly, the first sharp estimate is obtained in 1980 by T. Friedrich

Email address: serhane@agri.edu.tr (Serhan EKER)

as follows [6]:

$$\lambda^2 \geq \frac{m}{4(m-1)} Scal. \tag{1.2}$$

This proof is based on a modification of the spinorial Levi−Civita connection. The equality case is characterized by a non−trivial real−Killing spinor with positive scalar curvature.

In 1986, O. Hijazi obtained the following optimal estimates for $m \geq 3-$dimensional in terms of the Yamabe operator [10]

$$\lambda^2 \geq \frac{m}{4(m-1)} \mu_1, \tag{1.3}$$

where μ_1 is the eigenvalue of the Yamabe operator. Later on, C. Bar obtained an estimates in term of the Euler characteristic of M denoted by $\chi(M)$ [4] as follows:

$$\lambda^2 \geq \frac{2\Pi\chi(M)}{Area(M)}. \tag{1.4}$$

After this point, mathematicians choose to use different geometric invariants such as Energy-momentum tensor and divergencefree symmetric tensor to bring an optimal lower bound for the eigenvalues of the Dirac operator. In this paper we optimized E.C.Kim estimates in terms of the trace of divergencefree symmetric tensor.

Before giving an optimal lower bound, let's give some basic information about the $\beta-$twist Dirac operator defined as follows:

Consider an $m-$dimensional closed Riemannian spin manifold with a spinor bundle $\mathbb{S}$ over (M, g) and a spinorial Levi−Civita connection denoted ∇ lifted to the spinor bundle by using the Levi−Civita connection given on M. Also, the Levi−Civita connection given on M is denoted by ∇. With respect to the spinorial Levi−Civita conndection ∇, Dirac operator is locally expressed as:

$$D\Phi = \sum_{i=1}^{m} e_i \cdot \nabla_{e_i} \Phi, \tag{1.5}$$

where $\{e_1, ..., e_m\}$ orthonormal frame field on M, " $\cdot$ " denotes the Clifford multiplication and $\Phi \in \Gamma(\mathbb{S})$. On the spinor bundle $\mathbb{S}$ Hermitian inner product is defined by [16]:

$$\big(V \cdot \Phi, V \cdot \Psi\big) = |v|^2 \big(\Phi, \Psi\big), \tag{1.6}$$

where $v \in \Gamma(TM)$ and $\Phi, \Psi \in \Gamma(\mathbb{S})$. The Spinorial Levi−Civita connection is satisfied the following properties for all vector fields $V, W \in \Gamma(TM)$ and spinor fields $\Phi, \Psi \in \Gamma(S)$,

$$\begin{aligned} V\big(\Phi, \Psi\big) &= \big(\nabla_V \Phi, \Psi\big) + \big(\Phi, \nabla_V \Psi\big) \\ \nabla_V\big(W \cdot \Phi\big) &= \nabla_V W \cdot \Phi + W \cdot \nabla_V \Phi. \end{aligned}$$

In 2009, E.C. Kim is obtained two estimates on an $m-$dimensional closed Riemannian spin manifold for the eigenvalues of the Dirac operator in terms of the eigenvalue of the $B-$twist Dirac operator define as [5]:

$$D_B\Phi = \sum_{i=1}^{m} B^{-1}(e_i)\cdot\nabla_{e_i}\Phi = \sum_{i=1}^{m} e_i\cdot\nabla_{B^{-1}(e_i)}\Phi, \tag{1.7}$$

where B is a nondegenerate symmetric $(0,2)$ tensor field on (M,g) identified with the induced $(1,1)-$tensor field B via $B(V,W)=g(V,B(W))$. E.C Kim's estimate depends on the divergencefree symmetric tensors defined by:

$$div(B^{-1}) = \sum_{i=1}^{m}(\nabla_{e_i}B)(e_i) = 0.$$

E.C. Kim estimate obtained in [[5], Theorem 2.1] is describing with $tr(B^{-1}) = 0$ and $div(B^{-1}) = 0$. Accordingly, it is impossible to comment on the geometry of the manifold in case $tr(B^{-1})$ is non$-$zero. In this case by expanding E.C.Kim' s twistor$-$like operator

$$T_X(\Phi) = \nabla_X\Psi + pX\cdot\Psi + qB^{-1}(X)\cdot D_B\Phi \tag{1.8}$$

to

$$T_X(\Phi) = \nabla_X\Psi + pX\cdot\Psi + qB^{-1}(X)\cdot D_B\Phi + rtr(B^{-1})X\cdot D_B\Phi \tag{1.9}$$

we generalized E.C. Kim' s estimates as follows:

$$\lambda_1^2 \;\geq\; \sup_{p}\,\inf_{M}\left(\frac{1}{(mp^2-2p+1)}\Big(\frac{Scal}{4} + \frac{\overline{\lambda}_1^2}{\alpha} + \frac{\Delta\alpha}{2\alpha}\Big)\right), \tag{1.10}$$

where $\overline{\lambda}_1 \in \mathbb{R}$ is the smallest nonzero eigenvalue of the D_B, $p < \frac{1}{2m}$ and $\alpha : M \longrightarrow \mathbb{R}$ is a positive real$-$valued function defined as

$$\alpha = \frac{\Big((1-pm)|B^{-1}|^2 + p|tr(B^{-1})|^2\Big)^2}{|B^{-1}|^2(1-pm)^2 - mp^2|tr(B^{-1})|^2}. \tag{1.11}$$

Recall that, if $tr(B^{-1}) = 0$, twistor$-$like operator given in (1.9) induces to (1.8) which is given by E.C. Kim in [5]. estimates still valid. Finally, in case $tr(B^{-1}) \neq 0$, we apply our estimates to the Sasakian spin manifolds which lead us to describe eta$-$Killing spinor.

Theorem 1.1. *On an $m-$dimensional closed Riemannian manifold (M,g), assume that β is a nondegenerate symmetric tensor on M such that $div(B^{-1}) = 0$.*

Then, the first nonzero eigenvalue $\lambda_1 \in \mathbb{R}$ of D associated with the eigenspinor Φ_1, satisfied

$$\lambda_1^2 \quad \geq \quad \sup_p \inf_M \left(\frac{1}{(mp^2 - 2p + 1)} \Big(\frac{Scal}{4} + \frac{\widetilde{\lambda}_1^2}{\alpha} + \frac{\Delta\alpha}{2\alpha} \Big) \right), \tag{1.12}$$

where $\widetilde{\lambda}_1 \in \mathbb{R}$ is the smallest nonzero eigenvalue of the D_B, $p < \frac{1}{2m}$ and $\alpha : M \longrightarrow \mathbb{R}$ is a positive real−valued function defined as

$$\alpha = \frac{\Big((1 - pm)|B^{-1}|^2 + p|tr(B^{-1})|^2 \Big)^2}{|B^{-1}|^2(1 - pm)^2 - mp^2|tr(B^{-1})|^2}. \tag{1.13}$$

The equality case of (1.12) is satisfied if and only if $\Delta\alpha = 0$ vanishes identically and if the spinorial Levi−Civita connection satisfies

$$\nabla_V \Phi_1 = -p\lambda V \cdot \Phi_1 - q\widetilde{\lambda} B^{-1}(V) \cdot \Phi_1 - rtr(B^{-1}) V \cdot \Phi_1 \tag{1.14}$$

for some constants $\lambda, \widetilde{\lambda} \in \mathbb{R}$ and for all vector fields V. Here p, q, r are real−valued functions and Φ_1 is the first eigenspinor field of both D and D_B.

Proof. For any real−valued functions p, q, r and spinor field $\Phi \in \Gamma(\mathbb{S})$, define the following modified spinorial Levi−Civita connection $\widehat{\nabla}$ on $\Gamma(\mathbb{S})$ by

$$T_i\Phi \quad = \quad \nabla_i\Phi + pe_i \cdot D\Phi + qB^{-1}(e_i) \cdot D_B\Phi + rtr(B^{-1})e_i \cdot D_B\Phi.$$

One can easily compute,

$$\begin{aligned} \sum_{i=1}^m \big(T_i\Phi, T_i\Phi\big) \quad = \quad & div\big[\sum_{i=1}^m \big(\Phi, e_i \cdot D\Phi + \nabla_{e_i}\Phi\big)e_i \big] + \big(mp^2 - 2p + 1\big)|D\Phi|^2 \\ & + \big(q^2|B^{-1}|^2 - 2q + (2qr + mr^2)|tr(B^{-1})|^2\big)|D_B\Phi|^2 \\ & - \frac{Scal}{4}|\Phi|^2 + \big(2pq - 2r + 2prm\big)tr(B^{-1})\big(D\Phi, D_B\Phi\big). \end{aligned} \tag{1.15}$$

Remember that, the real−valued function α with the eigenspinor Φ_1 corresponding to the eigenvalue λ_1 of D satisfies the following equation [5]:

$$\int_M \alpha div\big[\sum_{i=1}^m \big(\Phi_1, e_i \cdot D\Phi_1 + \nabla_{e_i}\Phi_1\big)e_i \big]\mu \quad = \quad -\frac{1}{2}\int_M \Delta(\alpha)|\Phi_1|^2\mu. \tag{1.16}$$

To vanishing the last term of equation (1.15), we set r as

$$r = \frac{pq}{1 - pm}. \tag{1.17}$$

We still use the parameter r for ease of the calculation obtained below. Then, taking integral of (1.15) over M by considering equation (1.16), we get

$$\begin{aligned}\int_M \alpha|T\Phi_1|^2\mu &= -\frac{1}{2}\int_M \Big(\Delta(\alpha)|\Phi_1|^2 + \big(mp^2-2p+1\big)\alpha\lambda_1^2|\Phi_1|^2 \\ &\quad +\big(q^2|B^{-1}|^2-2q+(2qr+mr^2)|tr(B^{-1})|^2\big)\alpha|D_B\Phi_1|^2 \\ &\quad -\alpha\frac{Scal}{4}|\Phi_1|^2\Big)\mu. \end{aligned} \tag{1.18}$$

Let's define the P_1 positive function as below to get an optimal result:

$$\begin{aligned} P_1 : &= \int_M \Big[\big(D_B\Phi_1, D_B\Phi_1\big) - \widetilde{\lambda}_1^2\big(\Phi_1,\Phi_1\big)\Big]\mu + \int_M \Big[\alpha\sum_{i=1}^m \big(T_i\Phi_1, T_i\Phi_1\big)\Big]\mu \\ &= \int_M \Big(\big(mp^2-2p+1\big)\alpha\lambda_1^2|\Phi_1|^2\Big)\mu - \frac{1}{2}\int_M \Delta(\alpha)|\Phi_1|^2\mu - \int_M \widetilde{\lambda}_1^2|\Phi_1|^2\mu \\ &\quad + \int_M \Big[\Big(\big(q^2|B^{-1}|^2-2q+(2qr+mr^2)|tr(B^{-1})|^2\big)\alpha+1\Big)|D_B\Phi_1|^2 \\ &\quad -\alpha\frac{Scal}{4}|\Phi_1|^2\Big]\mu \geq 0. \end{aligned} \tag{1.19}$$

Set the free parameters q and α as:

$$q = \tfrac{1-r|tr(B^{-1})|^2}{|B^{-1}|^2} \text{ and } \alpha = \tfrac{\left((1-pm)|B^{-1}|^2+p|tr(B^{-1})|^2\right)^2}{|B^{-1}|^2(1-pm)^2-mp^2|tr(B^{-1})|^2}.$$

Taking into acoount (1.17) one can obtained the following equality:

$$q = \frac{1-r|tr(B^{-1})|^2}{|B^{-1}|^2} = \frac{r(1-pm)}{p}. \tag{1.20}$$

Solving (1.20) we get $r = \frac{p}{(1-pm)|B^{-1}|^2+p|trB^{-1}|^2}$. Then, in terms of the parameter p, P_1 can be rewritten as

$$\begin{aligned} P_1 &= \int_M \Big(\big(mp^2-2p+1\big)\alpha\lambda_1^2|\Phi_1|^2\Big)\mu - \frac{1}{2}\int_M \Delta(\alpha)|\Phi_1|^2\mu - \int_M \widetilde{\lambda}_1^2|\Phi_1|^2\mu \\ &\quad + \int_M \Big[\Big(\Big(\frac{mp^2|tr(B^{-1})|^2-(1-pm)^2|B^{-1}|^2}{|B^{-1}|^2\big(|B^{-1}|^2(1-pm)^2+p|tr(B-1)|^2\big)^2}\Big)\alpha+1\Big)|D_B\Phi_1|^2 \\ &\quad -\alpha\frac{Scal}{4}|\Phi_1|^2\Big]\mu \geq 0. \end{aligned} \tag{1.21}$$

Finally, if $|B^{-1}|^2 \geq \frac{|tr(B^{-1})|^2}{m}$ and $\frac{1}{2m} > p$ are choosen to make α positive, then one can get the desired inequality given in (1.12). $\square$

Note that, if $tr(B^{-1})$ vanishes identically, the spinorial Levi−Civita connection (1.9) reduces to (1.8). In this case all results are same with Kim' s estimations.

Consider the ratio of $\lambda_1 \neq 0$ and $\widetilde{\lambda}_1$ which are first eigenvalues of D and D_B, respectively and denote this ration by $\kappa_1 := \frac{\widetilde{\lambda}_1}{\lambda_1}$. Then under the same contitions as in Theorem 1.1, the equation (1.21) can be rewritten as follows:

$$P_2 : \quad = \quad \int_M \Big(\alpha\lambda_1^2\big(mp^2 - 2p + 1 - \frac{\kappa_1^2}{\alpha}\big)|\Phi_1|^2 - \frac{1}{2}\Delta(\alpha)|\Phi_1|^2 - \alpha\frac{Scal}{4}|\Phi_1|^2\Big)\mu \geq 0. \tag{1.22}$$

This give us the following corollary:

Corollary 1.2. *Under the same contitions as in Theorem* 1.1*, one has*

$$\lambda_1^2 \quad \geq \quad \sup_{\zeta} \inf_{M_*}\Big(\frac{Scal}{4\zeta} + \frac{\Delta\alpha}{2\alpha\zeta}\Big). \tag{1.23}$$

where $\zeta = mp^2 - 2p + 1 - \frac{\kappa_1^2}{\alpha}$ *and* $\widetilde{M} \subset M$ *is given by* $\widetilde{M} := \{x \in M : \zeta(x) > 0\}$.

Corollary 1.3. *Under the same conditions as in Theorem* 1.1*, if real−valued functions* $p = \frac{2}{m}$ *and* $r = -\frac{2q}{m}$ *are taken in the equation* (1.19)*, one gets*

$$\lambda_1^2 \quad \geq \quad \inf_M\Big(\frac{Scal}{4} + \frac{\widetilde{\lambda}_1^2}{|B^{-1}|^2} + \frac{\Delta\alpha}{2\alpha}\Big), \tag{1.24}$$

where $\alpha : M \longrightarrow \mathbb{R}$ *is a real−valued function defined by* $\alpha = |B^{-1}|^2$*.*

Proof. Inserting $p = \frac{2}{m}$ into the equation (1.20), one gets $r = -\frac{2q}{m}$. This means that the term $(2qr + mr^2)|tr(B^{-1})|^2$ given in equation (1.19) is vanished. Then P_1 induces to:

$$\begin{aligned} P_1 : \quad = \quad & \int_M \Big(\alpha\lambda_1^2|\Phi_1|^2 - \frac{1}{2}\Delta(\alpha)|\Phi_1|^2 - \widetilde{\lambda}_1^2|\Phi_1|^2 \\ & + \big((q^2|B^{-1}|^2 - 2q)\alpha + 1\big)|D_B\Phi_1|^2 - \alpha\frac{Scal}{4}|\Phi_1|^2\Big)\mu \geq 0. \end{aligned} \tag{1.25}$$

If the free parameters q and α is taken as follows

$$q = \frac{1}{|B^{-1}|^2} \text{ and } \alpha = |B^{-1}|^2. \tag{1.26}$$

one gets the desired result given in (1.24). $\square$

As in Corollary 1.2, consider the ratio of $\lambda_1 \neq 0$ and $\widetilde{\lambda}_1$ which are the first eigenvalues of D and D_B, respectively and denotes this ration by $\kappa_1 := \frac{\widetilde{\lambda}_1}{\lambda_1}$. Then under the same contitions as in Corollary 1.3, the equation (1.25) can be rewritten as follows:

$$\int_M \Big(\alpha\lambda_1^2\big(1-\frac{\kappa_1^2}{|B^{-1}|^2}\big)|\Phi_1|^2 - \frac{1}{2}\Delta(\alpha)|\Phi_1|^2 - \alpha\frac{Scal}{4}|\Phi_1|^2\Big)\mu \geq 0, \tag{1.27}$$

This proves:

Corollary 1.4. *In the notations of Corallary* 1.3*, we have*

$$\lambda_1^2 \quad \geq \quad \underset{\zeta}{sup}\ \underset{M_*}{inf}\Big(\frac{Scal}{4\zeta} + \frac{\Delta\alpha}{2\alpha\zeta}\Big). \tag{1.28}$$

where $\zeta = 1 - \frac{\kappa_1^2}{|B^{-1}|^2}$ *and* $\widetilde{M} \subset M$ *is given by* $\widetilde{M} := \{x \in M : \zeta(x) > 0\}$.

In the next theorem, we construct a spinorial Levi$-$Civita connection with respect to the nondegenerate symmetric tensor and its trace to give a lower bound esimate.

Theorem 1.5. *Assume that β is a nondegenerate symmetric tensor defined on an $m-$dimensional closed Riemannian manifold (M,g), such that $div(B^{-1}) = 0$. Let κ_1 be the ratio of $\widetilde{\lambda}_1 \in \mathbb{R}$, $\lambda_1 \neq 0 \in \mathbb{R}$ which are the first eigenvalues of D and D_B, respectively. Then, for any real$-$valued functions $q, r : M \longrightarrow \mathbb{R}$ satisfying both*

$$\frac{1}{2|B^{-1}|^2} > q > 0 \ \text{and}\ q(1+q|B^{-1}|^2) > (2qr + mr^2)|tr(B^{-1})|^2 \tag{1.29}$$

we have

$$\lambda_1{}^2 \geq \underset{\kappa(q,r,\kappa_1)}{sup}\ \underset{\widetilde{M}}{inf}\left(\frac{Scal}{4\kappa(q,r,\kappa_1)} + \frac{\Delta(\alpha)}{2\kappa(q,r,\kappa_1)}\right), \tag{1.30}$$

where $\kappa(q,r,\kappa_1)$ and α are real$-$valued functions defined on M by

$$
\begin{aligned}
\kappa(q,r,\kappa_1) \quad = \quad & \Bigg(\frac{(m-1)\big(q|B^{-1}|^2(2m+mq|B^{-1}|^2+2q|tr(B^{-1})|^2)\big)}{\big(q-2q|B^{-1}|^2\big)\big(m+mq|B^{-1}|^2+q|tr(B^{-1})|^2\big)^2} \\
& + \frac{(m-1)\big(q|tr(B^{-1})|^2\big(1-m|tr(B^{-1})|^2\big)-m\big)}{\big(q-2q|B^{-1}|^2\big)\big(m+mq|B^{-1}|^2+q|tr(B^{-1})|^2\big)^2} \\
& + \frac{\big(mr^2+2q^2\big)|tr(B^{-1})|^4+\big(q^3-m^2qr^2-2mq^2r\big)|tr(B^{-1})|^4|B^{-1}|^2}{\big(q-2q|B^{-1}|^2\big)\big(m+mq|B^{-1}|^2+q|tr(B^{-1})|^2\big)^2} \\
& + \frac{+mq^3|tr(B^{-1})|^2|B^{-1}|^4-\big(mqr^2+2q^2r\big)|tr(B^{-1})|^6}{\big(q-2q|B^{-1}|^2\big)\big(m+mq|B^{-1}|^2+q|tr(B^{-1})|^2\big)^2} - \kappa_1^2 \Bigg) \\
\alpha \quad = \quad & \frac{1}{q-2q^2|B^{-1}|^2}
\end{aligned}
\tag{1.31}
$$

respectively, and $\widetilde{M} = \{x \in M | \kappa(q,r,\kappa_1)(x) > 0\}$.

Egualité case of (1.30) *is satisfied if and only if the spinorial Levi−Civita connection satisfies*

$$
\nabla_V \Phi_1 = -p\lambda V \cdot \Phi_1 - q\widetilde{\lambda} B^{-1}(V) \cdot \Phi_1 - rtr(B^{-1}) V \cdot \Phi_1 \tag{1.32}
$$

for some constant $\lambda, \widetilde{\lambda} \in \mathbb{R}$, $\widetilde{\lambda} \neq 0$ *and for all vector fields* V. *Here* p, q, r *are real−valued functions given in* (1.45), (1.47) *and* Φ_1 *is the first eigenspinor of both* D *and* D_B.

Proof. Considering the following modified spinorial Levi−Civita connection T defined on $\Gamma(\mathbb{S})$ by

$$
T_V \Phi \quad = \quad \nabla_V \Phi + pV \cdot D\Phi + qB^{-1}(V) \cdot D_B \Phi + rtr(B^{-1}) V \cdot D_B \Phi, \tag{1.33}
$$

where p, q, r are real−valued functions defined on M. Assume that λ_1 is the first eigenvalue of D asssociated with the eigenspinor Φ_1 and $\widetilde{\lambda}_1$ is the first eigenvalue of D_B such that $\widetilde{\lambda}_1 = \kappa_1 \lambda_1$. Using (1.15) − (1.16) with positive real−valued function α and free functions $\gamma, \eta : M \longrightarrow \mathbb{R}$ to define the following posive real−valued function P_5 as follows:

$$
\begin{aligned}
P_5 \quad := \quad & \int_M \Big[\big(D_B\Phi_1, D_B\Phi_1\big) - \kappa_1^2 \lambda_1^2 \big(\Phi_1, \Phi_1\big) \Big] \mu + \int_M \alpha \sum_{i=1}^{m} |T\Phi_1|^2 \\
& + \gamma^2 \big(D_B\Phi_1 - \eta D\Phi_1, D_B\Phi_1 - \eta D\Phi_1\big) \Big] \mu \\
= \quad & \int_M \Big[\big(mp^2 - 2p + 1\big)\alpha + \gamma^2\eta^2 - \kappa_1^2 \big) \lambda_1^2 |\Phi_1|^2 - \frac{Scal}{4} \alpha |\Phi_1|^2 - \frac{\Delta\alpha}{2} |\Phi_1|^2 \\
& + 2\Big((pq - r + prn) tr(B^{-1}) \alpha - \gamma^2 \eta \Big) \lambda_1 \big(D_B\Phi_1, \Phi_1\big) \\
& + \Big(\big(q^2|B^{-1}|^2 - 2q + (2qr + mr^2)|tr(B^{-1})|^2\big)\alpha + \gamma^2 + 1 \Big) |D_B\Phi_1|^2 \geq 0.
\end{aligned}
\tag{1.34}
$$

Using the relation $D_B\Phi_1 = \kappa D\Phi_1$, then multiplying modified spinorial Levi−Civita connection defined in 1.33 with e_i and $B^{-1}(e_i)$ and then summing over $i = 1, ..., m$ gives equations 1.35 and 1.36, respectively. To vanish the last two lines of (1.34) the real−valued functions $p, q, r, \alpha, \gamma, \kappa$ have to satify the following four equations:

$$(1-mp) - (q+mr)tr(B^{-1})\kappa = 0 \qquad (1.35)$$

$$\big(1 + q|B^{-1}|^2 - r\big(tr(B^{-1})\big)^2\big)\kappa - ptr(B^{-1}) = 0 \qquad (1.36)$$

$$\big(pq - r + prm\big)tr(B^{-1})\alpha - \gamma^2\kappa = 0 \qquad (1.37)$$

$$\big(q^2|B^{-1}|^2 - 2q + (2qr + mr^2)|tr(B^{-1})|^2\big)\alpha + \gamma^2 + 1 = 0 \qquad (1.38)$$

Solving equation (1.36), one has

$$p = \frac{\big(1 + q|B^{-1}|^2 - r|tr(B^{-1})|^2\big)\kappa}{tr(B^{-1})}. \qquad (1.39)$$

Inserting (1.35) into equation (1.37), we have:

$$\big(pq - r(1-pm)\big)tr(B^{-1})\alpha - \gamma^2\kappa = \big(pq - r\kappa(q+mr)tr(B^{-1})\big)tr(B^{-1})\alpha - \gamma^2\kappa \qquad (1.40)$$

Inserting p into equation (1.40), we get

$$\gamma^2 = \alpha\big(q(1 + q|B^{-1}|^2) - (2qr + mr^2)|tr(B^{-1})|^2\big). \qquad (1.41)$$

Putting γ^2 into equation (1.38), we obtain

$$\alpha = \frac{1}{q(1 - 2q|B^{-1}|^2)}. \qquad (1.42)$$

Note that equation (1.35) and (1.36) together give

$$\kappa = \frac{1-mp}{(q+mr)tr(B^{-1})} = \frac{ptr(B^{-1})}{(1 + q|B^{-1}|^2 - r|tr(B^{-1})|^2)}. \qquad (1.43)$$

Also solving equations (1.35) and (1.36) we have,

$$\kappa = \frac{tr(B^{-1})}{(m + mq|B^{-1}|^2 + q|tr(B^{-1})|^2)}. \qquad (1.44)$$

This means

$$p = \frac{(1 + q|B^{-1}|^2 - r|tr(B^{-1})|^2)}{(m + mq|B^{-1}|^2 + q|tr(B^{-1})|^2)}. \qquad (1.45)$$

So the relations

$$\begin{aligned}
\alpha &= \frac{1}{q(1-2q|B^{-1}|^2)} > 0, \gamma^2 = \frac{\big(q(1 + q|B^{-1}|^2) - (2qr + mr^2)|tr(B^{-1})|^2\big)}{q(1-2q|B^{-1}|^2)} > 0.\\
\kappa^2 &= \frac{p(1-mp)}{(q+mr)(1 + q|B^{-1}|^2 - r|tr(B^{-1})|^2)}\\
&= \frac{(1-mp)}{(q+mr)(m + mq|B^{-1}|^2 + q|tr(B^{-1})|^2)} > 0
\end{aligned} \qquad (1.46)$$

imply the restriction

$$\frac{1}{2|B^{-1}|^2} > q > 0 \text{ and } q(1+q|B^{-1}|^2) > (2qr+mr^2)|tr(B^{-1})|^2. \tag{1.47}$$

Inserting (1.45) and (1.46) into (1.34) gives the desired estimates given in (1.30). The limiting case of (1.30) are easy to check. □

2. Estimating over Sasakian Spin Manifolds

In this section, we get an eigenvalue estimate with the aid of the divergence-free symmetric nondegenerate tensor $B^{-1} = \frac{I}{n} - \frac{1}{n}\xi \otimes \kappa$ by applying Theorem 1.1 and Theorem 1.5 to the Sasakian spin manifolds. Then, by using the relations between the eta−Killing pair (u, v) corresponding to the eta−Killing spinor and the decomposition property of the spinor bundle built over Sasakian spin manifold we investigate the geometric properties of the equality case.

On a $2n+1$ dimensional manifold M, Sasakian structure is defined by an almost contact metric structure of M. An almost contact metric structure is expressed by (ϕ, ξ, η, g). Here ϕ is a $(1,1)-$ tensor field, ξ is a vector field , η is a $1-$form and g is denoted the metric. An almost contact metric structure satisfies

$$\eta(\xi) = 1,\ \Theta^2 = -V + \eta(V)\xi,\ g(\phi V, \phi W) = g(V, W) - \eta(V)\eta(W). \tag{2.1}$$

On an almost contact metric manifold a fundamental $2-$form Θ is defined as

$$\Theta(V, W) = g\big(V, \phi(W)\big),$$

where V, W is a vector fields. In addition, if the following condition is satisfied

$$\big(\nabla_V \phi)W = g(V, W) - \eta(W)V \tag{2.2}$$

for all vector fields V, W, then an almost contact metric structure is called Sasakian structure and with this structure manifold is called Sasakian. Moreover, if the Ricci curvature tensor Ric defined on the Sasakian manifold $(M, \phi, \xi, \kappa, g)$ satisfies

$$Ric = ug + v\eta \otimes \eta \tag{2.3}$$

for some constants $u, v \in \mathbb{R}$ with $u+v = 2n$, then Sasakian manifold $(M, \phi, \xi, \eta, g)$is called eta−Einstein. T.Friedrich and E.C.Kim showed that any Spinor bundle connsructed on an almost contact metric manifold is splits under the action of the fundemantal $2-$form. For more information see Θ [8].

Definition. *On a Sasakian manifold (M, ϕ, ξ, η, g) endowed with spin structure, eta−Killing spinor with Killing pair (u, v) is characterized by a nontrivial spinor field Φ which is satisfied*

$$\nabla_V \Phi = \alpha V \cdot \Phi + B\kappa(V)\xi \cdot \Phi \tag{2.4}$$

for some real numbers $u, v \in \mathbb{R}, u \neq 0$ and for all vector fields V.

Since eta−Killing spinor with Killing pair (α, B) is an eigenspinor of the Dirac operator with eigenvalue $\lambda = -(2m+1)\alpha - B$, Killing pair (α, B) can be expressed in terms of the scalar curvature by considering some special decomposition of the spinor bundle [8].

Theorem 2.1. *Let (M, ϕ, ξ, κ) be a $2m+1(m \geq 1)-$dimensional a closed Sasakian spin manifold. Let B^{-1} be a nondegenerate symmetric tensor field on M defined by $B^{-1} = \frac{I}{n} - \frac{1}{n}\xi \otimes \kappa$, $div(B^{-1}) = 0$. Let $\lambda_1 \in \mathbb{R}$ and $\widetilde{\lambda}_1 \in \mathbb{R}$ be the first eigenvalue of D and D_B, respectively. Then we have*

$$\lambda_1^2 \geq \sup_{p,\kappa(p)} \inf_M \frac{1}{(np^2 - 2p + 1)} \Big(\frac{Scal}{4} + \frac{n^2(np^2 - 2pn + 1)\widetilde{\lambda}_1^2}{(n-1)(p-1)^2} \Big), \tag{2.5}$$

where $p < \frac{1}{2n}$. In case that $\lambda_1 \neq 0$, inequality (2.5) can be rewritten as

$$\lambda_1^2 \geq \sup_{p} \inf_M \frac{1}{(np^2 - 2p + 1)} \Big(\frac{Scal}{4\kappa(p)} \Big), p < \frac{1}{2n}, \tag{2.6}$$

where $\kappa(p)$ and μ_1 are real−valued functions on M defined by

$$\begin{aligned} \kappa(p) &= \frac{(n-1)(p-1)^2(np^2 - 2p + 1) - n^2(np^2 - 2pn + 1)\mu_1^2}{(n-1)(p-1)^2(np^2 - 2p + 1)}, \\ \mu_1 &= \frac{\widetilde{\lambda}_1}{\lambda_1}. \end{aligned} \tag{2.7}$$

The limiting case of (2.5) occurs, in case

1. *$n \geq 5$, if and only if there exist an eta−Killing spinor Φ_1 with Killing pair*

$$\Big(\frac{1}{2}, -\frac{n}{4} + \frac{Scal}{4(n-1)}\Big), \ \Big(-\frac{1}{2}, \frac{n}{4} - \frac{Scal}{4(n-1)}\Big) \tag{2.8}$$

   such that Φ_1 is a first eigenspinor of D_B.

2. *$n = 3$ if if and only if there exist an eta−Killing spinor Φ_1 with killing pair*

$$\Big(\frac{-2 + \sqrt{4 + 2Scal}}{4}, \frac{4 - \sqrt{4 + 2Scal}}{4}\Big), \tag{2.9}$$

   such that Φ_1 is a first eigenspinor of D_B.

Proof. Let's define the nondegenerate symmetric tensor field B^{-1} on M by $B^{-1} = \frac{I}{n} - \frac{1}{n}\xi \otimes \kappa$. So, the positive function given in (1.13) can be rewritten as follows:

$$\alpha = \frac{(n-1)(p-1)^2}{n^2(np^2 - 2pn + 1)}. \tag{2.10}$$

Inserting the function α into (1.12), one gets ineguality (2.5) and (2.6), respectively. Considering $B^{-1} = \frac{I}{n} - \frac{1}{n}\xi \otimes \kappa$, we get

$$tr(B^{-1}) = \frac{n-1}{n},\ |B^{-1}|^2 = \frac{n-1}{n^2}. \tag{2.11}$$

In limiting case, by taking into account (1.14), one can rewrite spinorial Levi$-$Civita connection as follows:

$$\begin{aligned}
\nabla_{e_i}\Phi_1 &= -\Big(p\lambda_1 + \frac{np}{1-p}\widetilde{\lambda}_1\Big)e_i \cdot \Phi_1 - \frac{(1-pn)n^2}{(n-1)(1-p)}\widetilde{\lambda}_1 B^{-1}(e_i)\cdot\Phi_1 \\
&= -\Big(p\lambda_1 + \frac{np}{1-p}\widetilde{\lambda}_1\Big)e_i \cdot \Phi_1 - \frac{(1-pn)n^2}{(n-1)(1-p)}\widetilde{\lambda}_1 \big(\frac{e_i}{n} - \frac{\kappa(e_i)\xi}{n}\big)\cdot\Phi_1 \\
&= -\Big(p\lambda_1 + \frac{n}{n-1}\widetilde{\lambda}_1\Big)e_i \cdot \Phi_1 - \frac{n(1-pn)}{(n-1)(1-p)}\widetilde{\lambda}_1 \kappa(e_i)\xi\cdot\Phi_1. \qquad (2.12)
\end{aligned}$$

Accordingly, in limiting case, eta$-$Killing spinor with Killing pair is described as:

$$\big(\alpha_1, B_1\big) \quad = \quad \Big(\frac{p(n-1)\lambda_1 + n\widetilde{\lambda_1}}{n-1}, \frac{n(1-pn)\widetilde{\lambda}_1}{(n-1)(1-p)}\Big). \tag{2.13}$$

In the case $m \geq 2$, eta$-$Killing spinor is characterized with eta$-$Killing pair given in (2.8). Let M be a 3$-$dimensional Sasakian spin manifold. Then, by using Proposition 3.2 and Proposition 3.3 given in [5], eta$-$Killing pair can be rewritten in terms of the scalar curvature of M as follows:

$$(\alpha_1, B_1) \quad = \quad \Big(\frac{-2+\sqrt{4+2Scal}}{4}, \frac{4-\sqrt{4+2Scal}}{4}\Big). \tag{2.14}$$

This means,

$$\Big(\frac{p(n-1)\lambda_1 + np\lambda_1}{n-1}, \frac{n(1-pn)\widetilde{\lambda}_1}{(n-1)(1-p)}\Big) \quad = \quad \Big(\frac{-2+\sqrt{4+2Scal}}{4}, \frac{4-\sqrt{4+2Scal}}{4}\Big). \tag{2.15}$$

Accordingly, the first eigenvalue of D and D_B can be desribed as follows;

$$\begin{aligned}
\lambda_1 &= \frac{(3p-1)\big(-2+\sqrt{4+2Scal}\big) - p(1-p)\big(4-\sqrt{4+2Scal}\big)}{4p(3p-1)}, \\
\widetilde{\lambda}_1 &= \frac{(1-p)\big(4-\sqrt{4+2Scal}\big)}{6(3p-1)}, \qquad (2.16)
\end{aligned}$$

where $p < \frac{1}{2n}$. □

In the next theorem to consider application of Theorem 1.5 to the Sasakian spin manifolds, we use same divergencefree symmetric tensor given in the proof ot Theorem 1.1.

Theorem 2.2. *Let (M,ϕ,ξ,κ) be a $2m+1(m \geq 1)-$dimensional a closed Sasakian spin manifold. Let B^{-1} be a nondegenerate symmetric tensor field on M defined by $B^{-1} = \frac{I}{n} - \frac{1}{n}\xi \otimes \kappa$, $div(B^{-1}) = 0$. Let $\lambda_1 \in \mathbb{R}$ and $\widetilde{\lambda}_1 \in \mathbb{R}$ be the first eigenvalue of D and D_B, respectively and denote $\kappa_1 = \frac{\widetilde{\lambda}_1}{\lambda_1}$. Then, for any real$-$valued functions $q, r : M \longrightarrow \mathbb{R}$ satisfying*

$$\frac{n^2}{2(n-1)} > q > 0 \ \text{and} \ \frac{q\big(n^2 + q(n-1)\big)}{(n-1)^2} > (2qr + nr^2) \tag{2.17}$$

we have

$$\lambda_1{}^2 \geq \sup_{\kappa(q,r,\kappa_1)} \inf_M \Big(\frac{Scal}{4\kappa(q,r,\kappa_1)}\Big), \tag{2.18}$$

where $\kappa(q,r,\kappa_1)$ and α are real$-$valued functions on M defined by

$$\begin{aligned} \kappa(q,r,\kappa_1) \quad = \quad & \left(\frac{(n-1)^2(n^2q - 2q^2(n-1))}{n^4\big(n^3 + q(n-1)(2n-1)\big)^2}\Big((n-1)^3\big(nq^3 - 2n^3qr\right. \\ & -4n^2q^2r + n^2qr^2 + 2nq^2r - q^3 + 2n^2qr + 4nq^2r - nqr^2 \\ & -2q^2r + n^3r^2 + 2qn^2 + nq^3\big) + 3n^4q - qn^3 + 3n^4q^2 \\ & \left. -6n^3 + 2n^2\Big) - \kappa_1^2\right). \end{aligned} \tag{2.19}$$

The limiting case of (2.18) occurs, in case

1. *$n \geq 5$, if and only if there exist an eta$-$Killing spinor Φ_1 with killing pair*

$$\Big(\frac{1}{2}, -\frac{n}{4} + \frac{Scal}{4(n-1)}\Big), \ \Big(-\frac{1}{2}, \frac{n}{4} - \frac{Scal}{4(n-1)}\Big) \tag{2.20}$$

   such that Φ_1 is a first eigenspinor of D_B.

2. *$n = 3$ if and only if there exist an eta$-$Killing spinor Φ_1 with killing pair*

$$\Big(\frac{-2 + \sqrt{4 + 2Scal}}{4}, \frac{4 - \sqrt{4 + 2Scal}}{4}\Big), \tag{2.21}$$

   such that Φ_1 is a first eigenspinor of D_B.

Proof. Let's define the nondegenerate symmetric tensor field B^{-1} on M by $B^{-1} = \frac{I}{n} - \frac{1}{n}\xi \otimes \kappa$. So, the positive functions $\kappa(q,r,\kappa_1)$ can be rewritten as in (2.19). These give us desired inegualty (2.18).

In limiting case, with respect to real functions p, q, r spinorial Levi−Civita connection can be writtes as:

$$\begin{aligned}\nabla_{e_i}\Phi_1 &= -p\lambda_1 e_i\cdot\Phi_1 - q\widetilde{\lambda}_1 B^{-1}(e_i)\cdot\Phi_1 - r\Big(\frac{n-1}{n}\Big)\widetilde{\lambda}_1 e_i\cdot\Phi_1\\ &= -p\lambda_1 e_i\cdot\Phi_1 - q\widetilde{\lambda}_1\Big(\frac{e_i}{n}-\frac{\kappa(e_i)\xi}{n}\Big)\cdot\Phi_1 - r\Big(\frac{n-1}{n}\Big)\widetilde{\lambda}_1 e_i\cdot\Phi_1\\ &= -\frac{1}{n}\Big(np\lambda_1+q\widetilde{\lambda}_1+r(n-1)\widetilde{\lambda}_1\Big)e_i\cdot\Phi_1+\frac{1}{n}q\widetilde{\lambda}_1\kappa(e_i)\xi\cdot\Phi_1,\end{aligned} \tag{2.22}$$

where real−valued functions p is given by

$$p=\frac{n(n-1)\big(q-r(n-1)\big)}{n^3+(n-1)q(2n-1)}. \tag{2.23}$$

Accordingly, in limiting case eta−Killing spinor with Killing pair, is described by:

$$\big(\alpha_1,B_1\big) = \Bigg(-\frac{1}{n}\Big(np\lambda_1+q\widetilde{\lambda}_1+r(n-1)\widetilde{\lambda}_1\Big),\frac{1}{n}q\widetilde{\lambda}_1\Bigg) \tag{2.24}$$

In the case $m\geq 2$, eta−Killing spinor is characterized with Killing pair given in (2.20).

Let M be 3−dimensional Sasakian spin manifold. By using Proposition 3.2 and Proposition 3.3 given in [5], (α_1, B_1) can be characterized with Killing pair given in (2.21). □

Acknowledgements: This study was supported by TUBITAK The Scientific and Technological Research Council of Turkey (Project Number: 120F109)

References

[1] Eker, S., *Seiberg-Witten-like equations on the strictly pseudoconvex CR−3 manifolds*,Bull. Korean Math. Soc. **56** (2019), no.6, 1551–1567.

[2] Eker, S., *Lower Bound Eigenvalue Problems of the Compact Riemannian Spin-Submanifold Dirac Operator*, Erzincan Üniversitesi Fen Bilimleri Enstitüsü Dergisi, **13** (ÖZEL SAYI I) (2020), 56–62.

[3] Eker, S., *Lower Bounds for the Eigenvalues of the Dirac Operator on Spin^c Manifolds*, Iranian Journal of Science and Technology Transaction A-science, **44** (2020), 251–257.

[4] Bär, C., *Lower eigenvalue estimates for Dirac operators*, Math. Ann. **239** (1992), 39–46.

[5] Kim, E.C., *Dirac eigenvalues estimates in terms of divergencefree symmetric tensors*, Bull.Korean Math. Soc. **46**(2009), no.5, 949–966.

[6] Friedrich, T., *Der erste Eigenwert des Dirac-Operators einer kompakten, Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung.* Math. Nach. **97** (1980), 117–146.

[7] T. Friedrich, *Dirac operators in Riemannian geometry*, American Mathematical Society, **25**, 2000.

[8] Friedrich, T. and Kim, E.C., *Some remarks on the Hijazi inequality and generalizations of the Killing equation for spinors*, Journal of Geometry and Physics, **37**(2001), no.1-2, 1–14.

[9] Habib, G., *Energy-Momentum tensor on foliations*, J. Geom. Phys. **57** (2007), 2234–2248.

[10] O. Hijazi, *A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors*, Comm. Math. Phys. **104** (1986), no. 1, 151–162.

[11] O. Hijazi, *Première valeur propre de l'opérateur de Dirac et nombre de Yamabe*, Comptes rendus de l'Académie des sciences. Série 1, Mathématique, **313** (1991), no. 12, 865–868.

[12] O. Hijazi, *Lower bounds for the eigenvalues of the Dirac operator*, J. Geom. Phys, **16** (1995), no. 1, 27–38.

[13] O. Hijazi and X. Zhang, *Lower bounds for the eigenvalues of the Dirac operator: part I. The hypersurface Dirac operator*, Ann. Global Anal. Geom. **19** (2001), no. 4, 355–376.

[14] O. Hijazi and X. Zhang, *Lower bounds for the eigenvalues of the Dirac operator: Part II. The submanifold Dirac operator*, Ann. Global Anal. Geom. **20** (2001), no. 2, 163–181.

[15] O. Hijazi, S. Montiel and X. Zhang, *Eigenvalues of the Dirac operator on manifolds with boundary*, Comm. Math. Phys. **221** (2001), no.2, 255–265.

[16] H.B. Lawson, M.L. Michelsohn, Spin geometry, Princeton university press, 1989.

[17] A. Lichnerowicz, *Spineurs harmoniques*, C.R. Acad. Sci. Paris Ser. A–B, **257** (1963), 7–9.

[18] X.Zhang, *Lower bounds for eigenvalues of hypersurface Dirac operators*, Math. Res. Lett. **5** (1998), no.2, 199–210.

[19] Witten, E., *A simple proof of the positive energy theorem*, Commun. Math. Phys. **80** (1931), 381–402.