A SCATTERING PROBLEM FOR A LOCAL PERTURBATION OF AN OPEN PERIODIC WAVEGUIDE

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ABSTRACT. In this paper we consider the propagation of waves in an open waveguide in $\mathbb{R}^2$ where the index of refraction is a local perturbation of a function which is periodic along the axis of the waveguide (which we chose to be the x_1-axis) and equal to one for $|x_2| > h_0$ for some $h_0 > 0$. Motivated by the limiting absorption principle (proven in [13]) for the case of an open waveguide in the half space $\mathbb{R} \times (0, \infty)$) we formulate a radiation condition which allows the existence of propagating modes and prove uniqueness, existence, and stability of a solution. In the last part we investigate the decay properties of the radiating part in the direction of periodicity and orthogonal to it.

1. INTRODUCTION

Let $k > 0$ be the wavenumber which is fixed throughout the paper and $n \in L^\infty(\mathbb{R}^2)$ the real valued index of refraction which is assumed to be $2\pi-$periodic with respect to x_1 and equals to 1 for $|x_2| > h_0$ for some $h_0 > 0$. Furthermore, let $q \in L^\infty(\mathbb{R}^2)$ and $f \in L^2(\mathbb{R}^2)$ have compact support $Q = (0, 2\pi) \times (-h_0, h_0)$. We assume that $n(x) + q(x) \geq n_0$ in $\mathbb{R}^2$ for some $n_0 > 0$. It is the aim to solve

$$\Delta u + k^2 n(1 + q)\, u \;=\; -f \quad \text{in } \mathbb{R}^2 \tag{1}$$

subject to a suitable radiating condition stated below.

The solution of (1) is understood in the variational sense; that is,

$$\int_{\mathbb{R}^2} \big[\nabla u \cdot \nabla \psi - k^2 n(1 + q)\, u\, \psi\big]\, dx \;=\; \int_Q f\, \psi\, dx \tag{2}$$

for all $\psi \in H^1(\mathbb{R}^2)$ with compact support. By standard regularity theorems it is known that $u \in H^2_{loc}(\mathbb{R}^2)$ and $\Delta u + k^2 n(1 + q)u = -f$ almost everywhere. For $|x_2| > h_0$ the solution u is a classical solution of the Helmholtz equation and thus analytic.

As mentioned above, a further condition is needed to assure uniqueness (see Definition 1.6 below). In contrast to the closed waveguide; that is, where $\mathbb{R}^2$ is replaced by $\mathbb{R} \times (a_-, a_+)$ and boundary conditions for $x_2 = a_\pm$ are added, not only a radiation condition in the direction of periodicity; that is, x_1, is needed but also one in direction of x_2. The radiation condition should be in accordance with the limiting absorption principle; that is, the solution u should be the limit (as $\varepsilon > 0$ tends to zero) of the solutions $u_\varepsilon \in H^1(\mathbb{R}^2)$ corresponding to wave numbers $k + i\varepsilon$ instead of k. Candidates are the Sommerfeld radiation condition (see, e.g., [8] for bounded media or [1] for periodic open waveguides) or the "upward propagating radiation condition" which is popular for scattering problems

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by rough surfaces (see, e.g. [4]). While the first excludes the existence of propagating modes (see Definition 1.2); that is, is two restrictive, the second is not sufficient for uniqueness; that is, is not restrictive enough. For layered media; that is media where n is constant with respect to x_1; radiation conditions have been developped in, e.g., [16] or [5, 6].

In this paper we first investigate uniqueness, existence and continuous dependence on f of equation (1) complemented by the radiation condition which has been introduced in [13, 14, 11, 12]. For closed waveguides this radiation condition is equivalent to the condition based on the dispersion curves (see, e.g., [9]). For the proof of uniqueness we were inspired by [10]. We had, however, to modify his proof considerably because of the full space waveguide instead of the half-space waveguide considered in [10].

Second, we investigate the asymptotic behaviour of the solution in the direction of the waveguide and orthogonal to it. While for closed waveguides the solution is (for $x_1 \to +\infty$ and $x_1 \to -\infty$) a finite sum of propagating modes and a function which decays exponentially (evanescent mode) we will show that the decaying part for open waveguides behaves only as $\mathcal{O}(|x_1^{-3/2}|)$ in the direction of the waveguide and as $\mathcal{O}(|x_1^{-1/2}|)$ orthogonal to it.

First, we make the assumption that k^2 does not belong to the point spectrum of $-\frac{1}{n(1+q)}\,\Delta$; that is,

Assumption 1.1.
There does not exist a nontrivial $u \in H^1(\mathbb{R}^2)$ with $\Delta u + k^2 n(1+q)u = 0$ in $\mathbb{R}^2$.

Even for the unperturbed case q it is in general not known whether this assumption is needed or if it is automatically satisfied.

Definition 1.2. $\alpha \in (-1/2, 1/2]$ *is called an exceptional value (or Floquet spectral value) if there exists a non-trivial $u \in H^1_{\alpha,loc}(\mathbb{R}^2) = \big\{u \in H^1_{loc}(\mathbb{R}^2) : u(\cdot, x_2)$ is $\alpha-$quasi-periodic$\big\}$ such that*

$$\Delta u + k^2 n\, u \;=\; 0 \quad in\ \mathbb{R}^2\,, \tag{3a}$$

$$u(x) \;=\; \sum_{\ell\in\mathbb{Z}} u_\ell^\pm\, e^{i(\ell+\alpha)x_1}\, e^{i\sqrt{k^2-(\ell+\alpha)^2}(\pm x_2 - h_0)} \quad for\ \pm x_2 > h_0 \tag{3b}$$

for some $u_\ell^\pm \in \mathbb{C}$ where the convergence is uniformly for $|x_2| \geq h_0 + \varepsilon$ for every $\varepsilon > 0$. We recall that a function $u(\cdot, x_2)$ is $\alpha-$quasi-periodic if $u(x_1 + 2\pi, x_2) = e^{2\pi\alpha i}u(x_1, x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$. The functions u are called propagating (or guided) modes.

It is not difficult to see that α is an exceptional value if, and only if, there exists a nontrivial $u \in H^1_\alpha(Q) = \{u \in H^1(Q) : u(\cdot, x_2)$ is $\alpha-$quasi-periodic$\}$ with

$$\int\limits_Q [\nabla u \cdot \nabla\overline{\psi} - k^2 n\, u\, \overline{\psi}]\, dx \;-\; \int\limits_\Gamma (\Lambda_\alpha u)\, \overline{\psi}\, ds \;=\; 0 \quad \text{for all } \psi \in H^1_\alpha(Q) \tag{4}$$

where $\Gamma = (\mathbb{R}\times\{h_0\}) \cup (\mathbb{R}\times\{-h_0\})$ and $\Lambda_\alpha : H^{1/2}_\alpha(\Gamma) \to H^{-1/2}_\alpha(\Gamma)$ is the $\alpha-$quasi-periodic Dirichlet-to-Neumann operator given by

$$(\Lambda_\alpha\phi)(x_1, \pm h_0) \;=\; \frac{i}{\sqrt{2\pi}} \sum_{\ell\in\mathbb{Z}} \sqrt{k^2 - (\ell+\alpha)^2}\, \phi_\ell(\pm h_0)\, e^{i(\ell+\alpha)x_1}\,, \quad x_1 \in \mathbb{R}\,, \tag{5}$$

for $\phi \in H^{1/2}(\Gamma)$. Here, $\phi_\ell(\pm h_0) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \phi(x_1, \pm h_0)\, \exp(-i(\ell+\alpha)x_1)\, dx_1$ are the Fourier coefficients of $\phi(\cdot, \pm h_0)$. If we set $\psi = u$ in (4) and take the imaginary part we observe directly that $u_\ell(\pm h_0) = 0$ for all $\ell \in \mathbb{Z}$ with $|\ell + \alpha| < k$. Therefore, if α is an exceptional value with corresponding eigenfunction u and α is not a cut-off value; that is, $|\alpha + \ell| \neq k$ for all $\ell \in \mathbb{Z}$, then u is evanescent; that is, exponentially decaying as $|x_2|$ tends to infinity; that is, satisfies $|u(x)| \leq c\, e^{-\delta|x_2|}$ for $|x_2| \geq h_0$ and some $c, \delta > 0$ which are independent of x. We formulate the latter condition as an additional assumption

Assumption 1.3. *Let $|\ell + \alpha| \neq k$ for all exceptional values α and all $\ell \in \mathbb{Z}$; that is, the cut-off values are not exceptional values.*

Under Assumptions 1.1 and 1.3 it can be shown (see, e.g. [13]) that at most a finite number of exceptional values exist. Furthermore, if α is an exceptional value with eigenfunction u then $-\alpha$ is an exceptional value with eigenfunction $\overline{u}$. Therefore, we can numerate the exceptional values such they are given by $\{\alpha_j : j \in J\}$ where $J \subset \mathbb{Z}$ is symmetric with respect to 0 and $\alpha_{-j} = -\alpha_j$ for $j \in J$. Furthermore, it is known that every eigenspace

$$X_j \;=\; \big\{ u \in H^1_{\alpha_j, loc}(\mathbb{R}^2) : u \text{ satisfies (3a) and (3b)} \big\} \tag{6}$$

is finite dimensional with some dimension $m_j > 0$. We construct a special orthonormal basis in X_j by considering the following finite dimensional self-adjoint eigenvalue problem in X_j.

Let $j \in J$ be fixed. Determine $\lambda_{\ell,j} \in \mathbb{R}$, $\ell = 1, \ldots, m_j$, and non-trivial $\hat{\phi}_{\ell,j} \in X_j$ such that

$$-i \int\limits_{Q^\infty} \frac{\partial \hat{\phi}_{\ell,j}}{\partial x_1}\, \overline{\psi}\, dx \;=\; \lambda_{\ell,j}\, k \int\limits_{Q^\infty} n\, \hat{\phi}_{\ell,j}\, \overline{\psi}\, dx \quad \text{for all } \psi \in X_j\,, \tag{7a}$$

where $Q^\infty = (0, 2\pi) \times \mathbb{R}$. Let the eigenfunctions be normalized such that

$$2k \int\limits_{Q^\infty} n\, \hat{\phi}_{\ell,j}(x)\, \overline{\hat{\phi}_{\ell',j}(x)}\, dx \;=\; \delta_{\ell,\ell'}\,, \quad \ell, \ell' = 1, \ldots, m_j\,. \tag{7b}$$

We note that $\hat{\phi}_{\ell,j} \in H^2(Q^\infty)$ and even analytic for $|x_2| > h_0$. We make a further assumption.

Assumption 1.4. *Let $\lambda_{\ell,j} \neq 0$ for all $\ell = 1, \ldots, m_j$ and $j \in J$; that is, there is no non-trivial $\phi \in X_j$ with $\int_{Q^\infty} \frac{\partial \phi}{\partial x_1}\, \overline{\psi}\, dx = 0$ for all $\psi \in X_j$.*

Remark 1.5. *This condition is equivalent to the requirement that the group velocity does not vanish. Indeed, assume that for all α there exists eigenvalues $\mu_\nu(\alpha) \in \mathbb{R}$ and corresponding eigenfunctions $u_\nu(\alpha) \in H^1_\alpha(Q^\infty)$ that satisfy $\Delta u_\nu(\alpha) + \mu_\nu(\alpha)\, n\, u_\nu(\alpha) = 0$ in Q^∞. Then $\hat{\alpha}$ is exceptional if $\mu_\nu(\hat{\alpha}) = k^2$ for some ν. We transform u_ν to its periodic form by setting $\tilde{u}_\nu(x) = e^{-i\alpha x_1} u_\nu(x)$. Then $\tilde{u}_\nu(\alpha)$ is $2\pi-$periodic with respect to x_1 and satisfies $\Delta \tilde{u}_\nu(\alpha) + 2i\alpha\, \partial \tilde{u}_\nu(\alpha)/\partial x_1 + (\mu_\nu(\alpha)\, n - \alpha^2)\tilde{u}_\nu(\alpha) = 0$ in Q^∞. Assuming that $\tilde{u}_\nu(\alpha)$ is differentiable with respect to α we differentiate this equation and set $\alpha = \hat{\alpha}$. This yields*

$$\Delta \tilde{u}'_\nu(\hat{\alpha}) \;+\; 2i\, \hat{\alpha}\, \frac{\partial \tilde{u}'_\nu(\hat{\alpha})}{\partial x_1} \;+\; (k^2\, n - \hat{\alpha}^2)\, \tilde{u}'_\nu(\hat{\alpha}) \;=\; -2i\, \frac{\partial \tilde{u}_\nu(\hat{\alpha})}{\partial x_1} \;+\; [2\hat{\alpha} - \mu'_\nu(\hat{\alpha})\, n]\, \tilde{u}_\nu(\hat{\alpha})$$

in Q^∞. We multiply this equation by $\overline{\tilde{u}_\nu(\hat{\alpha})}$, integrate over Q^∞, and use Green's second theorem. This yields

$$2i \int\limits_{Q^\infty} \overline{\tilde{u}_\nu(\hat{\alpha})} \left[\frac{\partial \tilde{u}_\nu(\hat{\alpha})}{\partial x_1} + i\hat{\alpha} |\tilde{u}_\nu(\hat{\alpha})|^2 \right] dx \; + \; \mu_\nu'(\hat{\alpha}) \int\limits_{Q^\infty} n \, |\tilde{u}_\nu(\hat{\alpha})|^2 \, dx \; = \; 0 \, .$$

Formulated with u_ν instead of $\tilde{u}_\nu$ this yields

$$2i \int\limits_{Q^\infty} \overline{u_\nu(\hat{\alpha})} \, \frac{\partial u_\nu(\hat{\alpha})}{\partial x_1} \, dx \; + \; \mu_\nu'(\hat{\alpha}) \int\limits_{Q^\infty} n \, |u_\nu(\hat{\alpha})|^2 \, dx \; = \; 0 \, .$$

Therefore, the condition of Assumption 1.4 (for $m_j = 1$) is equivalent to $\mu_\nu'(\hat{\alpha}) \neq 0$.

Now we are able to formulate the radiation condition. In all of the paper we make Assumptions 1.1, 1.3, and 1.4 without mentioning this always.

Definition 1.6. *Let $\psi_+, \psi_- \in C^\infty(\mathbb{R})$ be any functions with $\psi_\pm(x_1) = 1$ for $\pm x_1 \geq \sigma_0$ (for some $\sigma_0 > 2\pi + 1$) and $\psi_\pm(x_1) = 0$ for $\pm x_1 \leq \sigma_0 - 1$.*
A solution $u \in H^1_{loc}(\mathbb{R}^2)$ of (1); that is,

$$\Delta u + k^2 n (1+q) \, u \; = \; -f \quad \text{in } \mathbb{R}^2 \, , \tag{8}$$

satisfies the open waveguide radiation condition if

(a) *u has a decomposition in the form $u = u^{(1)} + u^{(2)}$ where*

$$u^{(2)}(x) \; = \; \sum_{j \in J} \left[\psi_+(x_1) \sum_{\lambda_{\ell,j} > 0} a_{\ell,j} \, \hat{\phi}_{\ell,j}(x) \; + \; \psi_-(x_1) \sum_{\lambda_{\ell,j} < 0} a_{\ell,j} \, \hat{\phi}_{\ell,j}(x) \right] \tag{9}$$

for $x \in \mathbb{R}^2$ and some $a_{\ell,j} \in \mathbb{C}$ where $u^{(1)} \in H^1(W_h)$ for all $h > h_0$ and where $W_h = \mathbb{R} \times (-h, h) \subset \mathbb{R}^2$,

(b) *the Fourier transform $(\mathcal{F}u^{(1)})(\cdot, x_2)$ of $u^{(1)}(\cdot, x_2)$ with respect to x_1 satisfies the generalized Sommerfeld radiation condition*

$$\int\limits_{-\infty}^{\infty} \left| (\operatorname{sign} x_2) \, \frac{\partial (\mathcal{F}u^{(1)})(\omega, x_2)}{\partial x_2} - i\sqrt{k^2 - \omega^2} \, (\mathcal{F}u^{(1)})(\omega, x_2) \right|^2 d\omega \; \longrightarrow \; 0 \, , \quad |x_2| \to \infty \, . \tag{10}$$

Here we define the Fourier transform as

$$(\mathcal{F}\phi)(t) \; = \; \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \phi(s) \, e^{-ist} \, ds \, , \quad t \in \mathbb{R} \, .$$

It has been shown in [13] for the case of a half plane problem that this radiation condition is a consequence of the limiting absorption principle. A second motivation is the following result on the direction of the energy flow which plays a central role in the proof of uniqueness.

Lemma 1.7. *Let $u^{(2)}$ be given by (9). With $I_r = \{r\} \times \mathbb{R}$ for $|r| \geq \sigma_0$ we have*

$$4\pi \operatorname{Im} \int\limits_{I_r} \overline{u^{(2)}} \, \frac{\partial u^{(2)}}{\partial x_1} \, ds \; = \; 2 \operatorname{Im} \int\limits_{Q^\infty} \overline{u^{(2)}} \, \frac{\partial u^{(2)}}{\partial x_1} \, dx \; = \; \begin{cases} \displaystyle\sum_{j \in J} \sum_{\lambda_{\ell,j} > 0} \lambda_{\ell,j} \, |a_{\ell,j}|^2 \, , & r > \sigma_0 \, , \\ \displaystyle\sum_{j \in J} \sum_{\lambda_{\ell,j} < 0} \lambda_{\ell,j} \, |a_{\ell,j}|^2 \, , & r < -\sigma_0 \, . \end{cases}$$

Proof: We only consider $r > \sigma_0$. Then $u^{(2)}(x) = \sum_{j\in J}\sum_{\lambda_{\ell,j}>0} a_{\ell,j}\,\hat\phi_{\ell,j}(x)$ for $x_1 > \sigma_0$. First we fix $j \in J$ and define $u_j^+(x) = \sum_{\lambda_{\ell,j}>0} a_{\ell,j}\,\hat\phi_{\ell,j}$. Since we fix j in the first part we drop the index j and write u^+ for u_j^+. Furthermore, we define $v(x) = (x_1 - r)u^+(x)$. Then $\frac{\partial v}{\partial x_1} = u^+ + (x_1 - r)\frac{\partial u^+}{\partial x_1}$ and $\Delta v + k^2 n v = 2\frac{\partial u^+}{\partial x_1}$. Therefore, with $r + Q^\infty = (r, r+2\pi)\times\mathbb{R} \subset \mathbb{R}^2$,

$$
\begin{aligned}
2\int\limits_{Q^\infty} \overline{u^+}\,\frac{\partial u^+}{\partial x_1}\,dx &= 2\int\limits_{r+Q^\infty} \overline{u^+}\,\frac{\partial u^+}{\partial x_1}\,dx \;=\; \int\limits_{r+Q^\infty} \overline{u^+}\,\big(\Delta v + k^2 n v\big)\,dx \\
&= \int\limits_{r+Q^\infty} v\,\big(\Delta\overline{u^+} + k^2 n\overline{u^+}\big)\,dx \;+\; \int\limits_{r+\partial Q^\infty}\left(\overline{u^+}\,\frac{\partial v}{\partial \nu} - v\,\frac{\partial \overline{u^+}}{\partial \nu}\right) ds \\
&= -\int\limits_{I_r} |u^+|^2 ds + \int\limits_{I_{r+2\pi}}\left[\overline{u^+}\left(u^+ + 2\pi\,\frac{\partial u^+}{\partial x_1}\right) - 2\pi u^+\,\frac{\partial \overline{u^+}}{\partial x_1}\right] ds \\
&= 2\pi\int\limits_{I_r}\left(\overline{u^+}\,\frac{\partial u^+}{\partial x_1} - u^+\,\frac{\partial\overline{u^+}}{\partial x_1}\right) ds \;=\; 4\pi\, i\,\mathrm{Im}\int\limits_{I_r}\overline{u^+}\,\frac{\partial u^+}{\partial x_1}\,ds
\end{aligned}
$$

Furthermore, with $L_j^+ = \{\ell : \lambda_{\ell,j} > 0\}$,

$$
\begin{aligned}
\int\limits_{Q^\infty}\overline{u^+}\,\frac{\partial u^+}{\partial x_1}\,dx &= \sum_{\ell,\ell'\in L_j^+}\overline{a_{\ell,j}}\,a_{\ell',j}\int\limits_{Q^\infty}\overline{\hat\phi_{\ell,j}}\,\frac{\partial\hat\phi_{\ell',j}}{\partial x_1}\,dx \\
&= ik\sum_{\ell,\ell'\in L_j^+}\overline{a_{\ell,j}}\,a_{\ell',j}\,\lambda_{\ell',j}\int\limits_{Q^\infty} n\,\overline{\hat\phi_{\ell,j}}\,\hat\phi_{\ell',j}\,dx \;=\; \frac{i}{2}\sum_{\ell\in L_j^+}\lambda_{\ell,j}\,|a_{\ell,j}|^2
\end{aligned}
$$

by the orthonormalization of $\hat\phi_{\ell,j}$. Therefore, we have shown

$$
4\pi\,\mathrm{Im}\int\limits_{I_r}\overline{u_j^+}\,\frac{\partial u_j^+}{\partial x_1}\,ds \;=\; 2\,\mathrm{Im}\int\limits_{Q^\infty}\overline{u_j^+}\,\frac{\partial u_j^+}{\partial x_1}\,dx \;=\; \sum_{\ell\in L_j^+}\lambda_{\ell,j}\,|a_{\ell,j}|^2
$$

where we indicated the dependence on j. In the second part we take $j, j' \in J$, apply Green' theorem in $r + Q^\infty$, and use the quasi-periodicities of u_j^+ and $u_{j'}^+$.

$$
\begin{aligned}
0 &= \int\limits_{r+\partial Q^\infty}\left(\overline{u_j^+}\,\frac{\partial u_{j'}^+}{\partial \nu} - u_{j'}^+\,\frac{\partial\overline{u_j^+}}{\partial\nu}\right) ds \\
&= -\int\limits_{I_r}\left(\overline{u_j^+}\,\frac{\partial u_{j'}^+}{\partial x_1} - u_{j'}^+\,\frac{\partial\overline{u_j^+}}{\partial x_1}\right) ds \;+\; \int\limits_{I_{r+2\pi}}\left(\overline{u_j^+}\,\frac{\partial u_{j'}^+}{\partial x_1} - u_{j'}^+\,\frac{\partial\overline{u_j^+}}{\partial x_1}\right) ds \\
&= \left(e^{i(\hat\alpha_{j'}-\hat\alpha_j)2\pi} - 1\right)\int\limits_{I_r}\left(\overline{u_j^+}\,\frac{\partial u_{j'}^+}{\partial x_1} - u_{j'}^+\,\frac{\partial\overline{u_j^+}}{\partial x_1}\right) ds\,.
\end{aligned}
$$

Therefore, the last integral vanishes for $j \neq j'$. Thus we have

$$
\begin{aligned}
& 4\pi i \,\mathrm{Im} \int\limits_{I_r} \overline{u^{(2)}}\, \frac{\partial u^{(2)}}{\partial x_1}\, ds \\
= \;& 2\pi \int\limits_{I_r} \left[\overline{u^{(2)}}\, \frac{\partial u^{(2)}}{\partial x_1} - u^{(2)}\, \frac{\partial \overline{u^{(2)}}}{\partial x_1} \right] ds \;=\; 2\pi \sum_{j\in J} \int\limits_{I_r} \left[\overline{u_j^+}\, \frac{\partial u_j^+}{\partial x_1} - u_j^+\, \frac{\partial \overline{u_j^+}}{\partial x_1} \right] ds \\
= \;& 4\pi i \sum_{j\in J} \mathrm{Im} \int\limits_{I_r} \overline{u_j^+}\, \frac{\partial u_j^+}{\partial x_1}\, ds \;=\; i \sum_{j\in J} \sum_{\ell\in L_j^+} \lambda_{\ell,j}\, |a_{\ell,j}|^2 \,.
\end{aligned}
$$

□

We note that all of the three terms in the representation of $(Fu^{(1)})(\cdot,\alpha)$ appear only in the cases $\kappa = 0$ or $\kappa = 1/2$; that is, if $k \in \frac{1}{2}\mathbb{N}$. If $0 < |\kappa| < 1/2$ then only the first or the second term appears, depending on the sign of κ.

Because $q\psi_\pm$ vanishes identically by our choice of $\psi_\pm$ we observe that the part $u^{(1)}$ satisfies

$$
\Delta u^{(1)} \;+\; k^2 n(1+q)\, u^{(1)} \;=\; -f \;-\; \sum_{j\in J} \sum_{\ell=1}^{m_j} a_{\ell,j} \varphi_{\ell,j} \quad \text{in } \mathbb{R}^2 \tag{11a}
$$

where

$$
\varphi_{\ell,j}(x) \;=\; \begin{cases} 2\,\psi_+'(x_1)\, \frac{\partial \hat\phi_{\ell,j}(x)}{\partial x_1} + \psi_+''(x_1)\, \hat\phi_{\ell,j}(x)\,, & \lambda_{\ell,j} > 0\,, \\ 2\,\psi_-'(x_1)\, \frac{\partial \hat\phi_{\ell,j}(x)}{\partial x_1} + \psi_-''(x_1)\, \hat\phi_{\ell,j}(x)\,, & \lambda_{\ell,j} < 0\,. \end{cases} \tag{11b}
$$

We note that f has compact support in Q and $\varphi_{\ell,j}$ vanish for $|x_1| \geq \sigma_0$, and are evanescent; that is, there exist $c, \delta > 0$ with $|\varphi_{\ell,j}(x)| \leq c \exp(-\delta |x_2|)$.

Therefore, we decompose $u^{(1)}$ in the upper and lower half planes $x_2 > h_0$ and $x_2 < -h_0$, respectively, as a sum of solutions with homogeneous boundary conditions and one with a homogeneous Helmholtz equation.

Lemma 1.8. *Let Assumptions 1.1, 1.3, and 1.4 hold, and let $u \in H_{loc}(\mathbb{R}^2)$ be a solution of (8) satisfying the radiation condition of Definition 1.6.*

(a) *Then the part $u^{(1)}$ has a decomposition in the half planes $x_2 > h_0$ and $x_2 < -h_0$, respectively, in the forms*

$$
u^{(1)}(x) \;=\; u_0^\pm(x) \;+\; \sum_{\ell,j} a_{\ell,j} v_{\ell,j}^\pm(x) \quad \textit{for } \pm x_2 > h_0\,,
$$

where $v_{\ell,j}^\pm$ are the unique solutions of $\Delta v_{\ell,j}^\pm + k^2 v_{\ell,j}^\pm = -\varphi_{\ell,j}$ for $\pm x_2 > h_0$ and $v_{\ell,j}^\pm = 0$ for $x_2 = \pm h_0$ satisfying the generalized Sommerfeld radiation condition (10), and $u_0^\pm$ is the unique radiating solution of $\Delta u_0^\pm + k^2 u_0^\pm = 0$ for $\pm x_2 > h_0$ and $u_0^\pm = u^{(1)}$ for $x_2 = \pm h_0$.

(b) *There exists $c > 0$ such that $|\nabla v_{\ell,j}^\pm(x)| + |v_{\ell,j}^\pm(x)| \leq c\, \frac{|x_2|}{1+|x|^{3/2}}$ for all $x \in \mathbb{R}^2$ with $\pm x_2 > h_0$ where $c > 0$ is independent of x and ℓ, j.*

(c) *There exists $c > 0$ with*

$$\left|u^{(1)}(x)\right| \;+\; \left|\nabla u^{(1)}(x)\right| \;\leq\; c\,|x_2|\,\rho(x_1) \tag{12}$$

for all $x \in \mathbb{R}^2$ with $|x_2| \geq h_0 + 1$, where $\rho \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is given by

$$\rho(x_1) \;=\; \sum_{\sigma\in\{+1,-1\}} \int_{\mathbb{R}} \frac{|u^{(1)}(y_1, \sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy \;+\; \frac{1}{1+|x_1|^{3/2}}\,, \quad x_1 \in \mathbb{R}\,. \tag{13}$$

Proof: We only consider the upper half plane $x_2 > h_0$.
(a) We show that $v^+_{\ell,j}$ and u^+_0 are given by

$$v^+_{\ell,j}(x) \;=\; \int_{h_0}^{\infty}\int_{-\sigma_0}^{\sigma_0} G^+(x,y)\,\varphi_{\ell,j}(y)\,dy_1 dy_2\,, \quad x_2 > h_0\,, \tag{14a}$$

$$u^+_0(x) \;=\; 2\int_{-\infty}^{\infty} u^{(1)}(y_1,h_0)\,\frac{\partial}{\partial y_2}\Phi(x_1,x_2,y_1,\pm h_0)\,dy_1\,, \quad x_2 > h_0\,, \tag{14b}$$

respectively, with the fundamental solution $\Phi(x,y) = \frac{i}{4}\,H^{(1)}_0(k|x-y|)$ of the Helmholtz equation and the Green's function $G^+(x,y) = \Phi(x,y) - \Phi(x,y^*)$, $x,y \in \mathbb{R}^2$, $x_2, y_2 > h_0$ where $y^* = (y_1, 2h_0 - y_2)^\top$ is the reflection of y at the line $y_2 = h_0$.
First we show that $v^+_{\ell,j}$ is a solution of the inhomogeneous differential equation. For any $x \in W^{h_0,R} := \{x \in \mathbb{R}^2_+ : h_0 < x_2 < R\}$ we decompose $v^+_{\ell,j}(x)$ as

$$v^+_{\ell,j}(x) \;=\; \int_{h_0}^{R+1}\int_{-\sigma_0}^{\sigma_0} G^+(x,y)\,\varphi_{\ell,j}(y)\,dy_1 dy_2 \;+\; \int_{R+1}^{\infty}\int_{-\sigma_0}^{\sigma_0} G^+(x,y)\,\varphi_{\ell,j}(y)\,dy_1 dy_2\,.$$

The first integral ist just the volume integral over a bounded region. This term satisfies the inhomogeneous differential equation for $x \in W^{h_0,R}$ and the homogeneous boundary condition for $x_2 = h_0$. The integrand of the second integral is regular for $x \in W^{h_0,R}$ and, therefore, satisfies the homogeneous differential equation and the boundary condition. Also, u^+_0 satisfies the homogeneous Helmholtz equation. It remains to show the generalized Sommerfeld condition for $v^+_{\ell,j}$ and u^+_0.

Taking the Fourier transforms of u_0^+ and $v_{\ell,j}$ with respect to x_1 and noting that the integral with respect to y_1 is a convolution yields

$$\begin{aligned}
(\mathcal{F}u_0^+)(\omega,x_2) &= (\mathcal{F}u^{(1)})(\omega,h_0)\, e^{i\sqrt{k^2-\omega^2}(x_2-h_0)} \quad \text{and} \\
(\mathcal{F}v_{\ell,j}^+)(\omega,x_2) &= \frac{i}{2\sqrt{k^2-\omega^2}} \int_{h_0}^{\infty} (\mathcal{F}\varphi_{\ell,j})(\omega,y_2)\left[e^{i\sqrt{k^2-\omega^2}|x_2-y_2|} - e^{i\sqrt{k^2-\omega^2}(x_2+y_2-2h_0)}\right] dy_2 \\
&= \frac{i}{2\sqrt{k^2-\omega^2}} \int_{h_0}^{x_2} (\mathcal{F}\varphi_{\ell,j})(\omega,y_2)\, e^{i\sqrt{k^2-\omega^2}(x_2-y_2)}\, dy_2 \\
&\quad + \frac{i}{2\sqrt{k^2-\omega^2}} \int_{x_2}^{\infty} (\mathcal{F}\varphi_{\ell,j})(\omega,y_2)\, e^{i\sqrt{k^2-\omega^2}(y_2-x_2)}\, dy_2 \\
&\quad - \frac{i}{2\sqrt{k^2-\omega^2}} \int_{h_0}^{\infty} (\mathcal{F}\varphi_{\ell,j})(\omega,y_2)\, e^{i\sqrt{k^2-\omega^2}(x_2+y_2-2h_0)}\, dy_2\,.
\end{aligned}$$

The first term $(\mathcal{F}u_0^+)(\omega,x_2)$ satisfies the radiation condition (10) trivially. For the second term we have

$$\frac{\partial(\mathcal{F}v_{\ell,j}^+)(\omega,x_2)}{\partial x_2} - i\sqrt{k^2-\omega^2}\,(\mathcal{F}v_{\ell,j}^+)(\omega,x_2) \;=\; \int_{x_2}^{\infty} (\mathcal{F}\varphi_{\ell,j})(\omega,y_2)\, e^{i\sqrt{k^2-\omega^2}(y_2-x_2)}\, dy_2\,.$$

For $|\omega| < k$ we just estimate

$$\left|\frac{\partial(\mathcal{F}v_{\ell,j}^+)(\omega,x_2)}{\partial x_2} - i\sqrt{k^2-\omega^2}\,(\mathcal{F}v_{\ell,j}^+)(\omega,x_2)\right| \;\le\; c\int_{x_2}^{\infty} e^{-\delta y_2}\, dy_2 \;=\; \frac{c}{\delta}\, e^{-\delta x_2}\,.$$

For $|\omega| > k$ we estimate

$$\begin{aligned}
&\left|\frac{\partial(\mathcal{F}v_{\ell,j}^+)(\omega,x_2)}{\partial x_2} - i\sqrt{k^2-\omega^2}\,(\mathcal{F}v_{\ell,j}^+)(\omega,x_2)\right| \\
&\quad\le\; c\int_{x_2}^{\infty} e^{-\delta y_2 - \sqrt{\omega^2-k^2}(y_2-x_2)}\, dy_2 \;\le\; \frac{c}{\delta+\sqrt{\omega^2-k^2}}\, e^{-\delta x_2}\,.
\end{aligned}$$

Together we have the existence of $c > 0$ such that

$$\left|\frac{\partial(\mathcal{F}v_{\ell,j}^+)(\omega,x_2)}{\partial x_2} - i\sqrt{k^2-\omega^2}\,(\mathcal{F}v_{\ell,j}^+)(\omega,x_2)\right| \le \frac{c}{\delta+\sqrt{|\omega^2-k^2|}}\, e^{-\delta x_2}$$

for all $\omega \in \mathbb{R}$ and $x_2 > h_0$. Squaring and integrating with respect to ω yields the radiation condition.

To show uniqueness let v satisfy $\Delta v + k^2 v = 0$ for $x_2 > h_0$, $v = 0$ for $x_2 = h_0$, and also (10). Taking the Fourier transform and solving the resulting ordinary differential equation yields $(\mathcal{F}v)(\omega,x_2) = a(\omega)e^{i\sqrt{k^2-\omega^2}x_2} + b(\omega)e^{-i\sqrt{k^2-\omega^2}x_2}$ for $x_2 > h_0$, thus

$$(\mathcal{F}v)'(\omega,x_2) - i\sqrt{k^2-\omega^2}\,(\mathcal{F}v)(\omega,x_2) \;=\; -2i\sqrt{k^2-\omega^2}\, b(\omega)e^{-i\sqrt{k^2-\omega^2}x_2}\,,$$

and therefore

$$\left|(\mathcal{F}v)'(\omega,x_2) - i\sqrt{k^2-\omega^2}\,(\mathcal{F}v)(\omega,x_2)\right|^2 \;=\; \begin{cases} 4(k^2-\omega^2)\,|b(\omega)|^2\,, & |\omega|<k\,,\\ 4(\omega^2-k^2)\,|b(\omega)|^2 e^{2\sqrt{\omega^2-k^2}x_2}\,, & |\omega|>k\,. \end{cases}$$

Integrating with respect to ω und using (10) yields $b(\omega)=0$ for all ω. The initial condition yields $a(\omega)=0$.

(b) We know from [3] that there exists $c>0$ with

$$\begin{aligned} \left|G^+(x,y)\right| \;+\; \left|\nabla_x G^+(x,y)\right| \;&\le\; c\,\frac{x_2\,y_2}{|x-y|^{3/2}} \quad \text{for all } x,y\in\mathbb{R}^2_{h_0} \text{ with } |x-y|\ge 1, \text{ and}\\ \left|G^+(x,y)\right| \;&\le\; c\,\bigl|\ln|x-y|\bigr| \quad \text{for all } x,y\in\mathbb{R}^2_{h_0} \text{ with } |x-y|\le 1\,,\\ \left|\nabla_x G^+(x,y)\right| \;&\le\; \frac{c}{|x-y|} \quad \text{for all } x,y\in\mathbb{R}^2_{h_0} \text{ with } |x-y|\le 1 \end{aligned}$$

where we have set $\mathbb{R}^2_{h_0}=\mathbb{R}\times(h_0,\infty)$. First we consider $|x_1|\le 2\sigma_0$. We split the region of integration with respect to y_2 into $\{y_2 : |y_2-x_2|<1\}\cup\{y_2 : 1<|y_2-x_2|<x_2/2\}\cup\{y_2 : |y_2-x_2|>x_2/2\}$ and use the estimates of G^+ is each of the regions. (Note that $|y_1|\le\sigma_0$). Therefore,

$$\begin{aligned} |v^+_{\ell,j}(x)| \;&\le\; c \int\limits_{|x_2-y_2|<1}\int\limits_{-\sigma_0}^{\sigma_0} e^{-\delta y_2}\,\bigl|\ln|x-y|\bigr|\,dy_1\,dy_2\\ &\quad + c\,x_2 \int\limits_{1<|x_2-y_2|<x_2/2}\int\limits_{-\sigma_0}^{\sigma_0} e^{-\delta y_2}\,\frac{y_2}{|x_2-y_2|^{3/2}}\,dy_1\,dy_2\\ &\quad + c\,x_2 \int\limits_{|x_2-y_2|>x_2/2}\int\limits_{-\sigma_0}^{\sigma_0} e^{-\delta y_2}\,\frac{y_2}{|x_2-y_2|^{3/2}}\,dy_1\,dy_2\\ &\le\; c\,e^{-\delta(x_2-1)}\int\limits_{-1}^{1}\int\limits_{-3\sigma_0}^{3\sigma_0}\bigl|\ln|z|\bigr|\,dz_1\,dz_2 \;+\; c_1\,x_2^3\,e^{-\delta x_2/2}\\ &\quad + \frac{c_2}{\sqrt{x_2}}\int\limits_{h_0}^{\infty} y_2\,e^{-\delta y_2}\,dy_2 \;=\; \frac{c_2}{\sqrt{x_2}}\left(\frac{h_0}{\delta}+\frac{1}{\delta^2}\right)e^{-\delta h_0} \end{aligned} \tag{16}$$

for all $x_2\ge h_0$ and $|x_1|\le 2\sigma_0$ where $c_2>0$ is independent of x. We indicated the dependence on h_0 in (16) (and (17), (18)) for later use. This proves the desired estimate for $|x_1|\le 2\sigma_0$. Now we consider $|x_1|>2\sigma_0$. Then $|y_1-x_1|\ge|x_1|-\sigma_0>|x_1|/2$ and thus

$$|v^+_{\ell,j}(x)| \;\le\; c\,x_2\int\limits_{h_0}^{\infty} y_2\,e^{-\delta y_2}\int\limits_{-\sigma_0}^{\sigma_0}\frac{dy_1}{|x-y|^{3/2}}\,dy_2\,.$$

We split the integral with respect to y_2 into $|y_2 - x_2| > x_2/2$ and $|y_2 - x_2| < x_2/2$. Then

$$x_2 \int\limits_{|y_2-x_2|>x_2/2} y_2\, e^{-\delta y_2} \int\limits_{-\sigma_0}^{\sigma_0} \frac{dy_1}{|x-y|^{3/2}}\, dy_2$$

$$(17) \quad \leq \; x_2 \int\limits_{|y_2-x_2|>x_2/2} y_2\, e^{-\delta y_2}\, dy_2 \, \frac{2\sigma_0}{(|x|/2)^{3/2}} \;\leq\; \frac{c\, x_2}{|x|^{3/2}} \int\limits_{h_0}^{\infty} y_2\, e^{-\delta y_2}\, dy_2 \;\leq\; \frac{c'\, x_2}{|x|^{3/2}}\, e^{-\delta h_0}$$

because $|x-y|^2 = (x_1-y_1)^2+(x_2-y_2)^2 \geq \frac{1}{4}(x_1^2+x_2^2)$. Finally, since $|y_2| \geq |x_2|-|y_2-x_2| \geq x_2/2$ for $|y_2 - x_2| < x_2/2$ we have by splitting $e^{-\delta y_2} = e^{-\delta y_2/2} e^{-\delta y_2/2}$

$$x_2 \int\limits_{|y_2-x_2|<x_2/2} y_2\, e^{-\delta y_2} \int\limits_{-\sigma_0}^{\sigma_0} \frac{dy_1}{|x-y|^{3/2}}\, dy_2 \;\leq\; x_2\, e^{-\delta x_2/4} \int\limits_{h_0}^{\infty} y_2\, e^{-\delta y_2/2}\, dy_2 \int\limits_{-\sigma_0}^{\sigma_0} \frac{dy_1}{|x_1-y_1|^{3/2}}$$

$$(18) \qquad \leq \; c\, x_2\, e^{-\delta x_2/4} \frac{1}{|x_1|^{3/2}} \int\limits_{h_0}^{\infty} y_2\, e^{-\delta y_2/2}\, dy_2 \;\leq\; \frac{c\, x_2}{|x|^{3/2}}\, e^{-\delta h_0}$$

because $|x_1| e^{\delta x_2/6} \geq c|x|$ for some $c > 0$ (note that $|x_1| > 2\sigma_0$ and $x_2 \geq h_0$).
The proof for the derivatives follow exactly the same lines. (Only the intgral over $\ln |x-y|$ has to be replaced by the the integral over $1/|x-y|$.)
(c) We know from the asymptotic behavior of the Hankel functions that for all $a > 0$ there exists $c = c(a) > 0$ with

$$(19) \qquad \left| \frac{\partial}{\partial y_2} \Phi(x,y) \right| \;\leq\; c\, \frac{x_2\, y_2}{|x-y|^{3/2}}$$

for all $x, y \in \mathbb{R}^2$ with $|x-y| \geq a$. Therefore, $\rho \in L^2(\mathbb{R})$ because the first term can be expressed as the convolution of the L^2-function $|u^{(1)}(y_1, h_0)|$ and the L^1-function $y_1 \mapsto (1+|y_1|)^{-3/2}$. It is also bounded by the inequality of Cauchy-Schwarz.
Using (19) and the form (14a) we estimate for $x_2 > h_0 + 1$

$$|u_0^+(x)| \;\leq c\, x_2\, h_0 \int\limits_{-\infty}^{\infty} \frac{|u^{(1)}(y_1, h_0)|}{[(x_1-y_1)^2+1]^{3/4}}\, dy_1$$

which proves the desired estimate in combination with part (b). $\square$

During the proof we have shown the following sharper version of the radiation condition.

Corollary 1.9. *Let Assumptions 1.1, 1.3, and 1.4 hold, and let $u \in H_{loc}(\mathbb{R}^2)$ be a solution of (8) satisfying the radiation condition of Definition 1.6 and let $\sigma \in \{-1, +1\}$ be fixed.*

(a) *For $\sigma x_2 > h_0$ the Fourier transform $(\mathcal{F}u^{(1)})(\omega, x_2)$ of $u^{(1)}(\cdot, x_2)$ has the form*

$$(20) \qquad (\mathcal{F}u^{(1)})(\omega, x_2) \;=\; (\mathcal{F}u^{(1)})(\omega, \sigma h_0)\, e^{i\sqrt{k^2-\omega^2}(\sigma x_2 - h)}$$
$$+ \frac{i}{2\sqrt{k^2-\omega^2}} \int\limits_{h_0}^{\infty} (\mathcal{F}g)(\omega, \sigma t) \left[e^{i\sqrt{k^2-\omega^2}|x_2-\sigma t|} - e^{i\sqrt{k^2-\omega^2}(x_2+\sigma(t-2h_0))} \right] dt$$

for $\omega \in \mathbb{R}$ *where* $(\mathcal{F}g)(\omega, x_2)$ *is the Fourier transform of* $g = \Delta u^{(2)} + k^2 u^{(2)} = \sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j}\varphi_{\ell,j}$ *for* $|x_2| > h_0$.

(b) *We have the following stronger form of the radiation condition (10):*

$$\left|\sigma\,(\mathcal{F}u^{(1)})'(\omega,x_2) - i\sqrt{k^2-\omega^2}\,(\mathcal{F}u^{(1)})(\omega,x_2)\right| \;\leq\; \frac{c}{\delta + \sqrt{|\omega^2-k^2|}}\, e^{-\delta|x_2|} \tag{21}$$

for all $\omega \in \mathbb{R}$ *and* $|x_2| > h_0$. *Note that the right hand side is the product of a* $L^2(\mathbb{R})-$*function and the exponential function* $\exp(-\delta|x_2|)$.

2. Uniqueness

In this section we follow the proof of uniqueness giben by T. Furuya in [10] for the half-plane case. We had to to modify his appoach, however, because the free space Green's function; that is, the fundamental solution, does not decay as fast as the Green's function for the half-plane as $|x_1|$ tends to infinity. Therefore, we can't use his integral representations.

We begin with the following technical result.

Lemma 2.1. *Let Assumptions 1.1, 1.3, and 1.4 hold, and let* $u \in H_{loc}(\mathbb{R})$ *be a solution of (1) satisfying the radiation condition of Definition 1.6. Analogously to* $\rho(x_1)$ *of (13) (see Lemma 1.8) we define*

$$\rho_N(x_1) \;=\; \sum_{\sigma\in\{+1,-1\}} \int_{-N}^{N} \frac{|u^{(1)}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 \;+\; \frac{1}{1+|x_1|^{3/2}}\,, \quad x_1\in\mathbb{R}\,,\ \ \in\mathbb{N}\,. \tag{22}$$

Then there exists $c > 0$ *and a sequence* (N_m) *in* $\mathbb{N}$ *converging to infinity such that*

$$\int\limits_{|x_1|>N_m} \rho_{N_m}(x_1)^2\,dx_1 \;\leq\; \frac{c}{\sqrt{N_m}}\,, \qquad \int\limits_{|x_1|<N_m} |\rho(x_1)-\rho_{N_m}(x_1)|^2\,dx_1 \;\leq\; \frac{c}{\sqrt{N_m}}\,,$$

and

$$\int\limits_{N_m<|x_1|<N_m+1} \rho(x_1)^2\,dx_1 \;\leq\; \frac{c}{\sqrt{N_m}}$$

for all $j \in \mathbb{N}$.

Proof: We define the sets $J_N = (-N-\sqrt{N}, -N+\sqrt{N}) \,\cup\, (N-\sqrt{N}, N+\sqrt{N})$. As in [2] we first note that for every $m \in \mathbb{N}$ there exists $N_m \geq m$ with $\|u^{(1)}(\cdot,h_0)\|_{L^2(J_{N_m})} + \|u^{(1)}(\cdot,-h_0)\|_{L^2(J_{N_m})} \leq \frac{1}{N_m^{1/4}}$. Indeed, otherwise there exists $m \in \mathbb{N}$ such that $\|u^{(1)}(\cdot,h_0)\|_{L^2(J_N)} + \|u^{(1)}(\cdot,-h_0)\|_{L^2(J_N)} \geq \frac{1}{N^{1/4}}$ for all $N \geq m$. Since $J_{N^2} \cap J_{M^2}$ for $N \neq M$ we would have

$$\begin{aligned}
\sum_{\sigma\in\{-1,+1\}} \int\limits_{|x_1|>N_m^2-N_m} |u^{(1)}(\cdot,\sigma h_0)|^2\,dx_1 \;&\geq\; \sum_{\sigma\in\{-1,+1\}}\sum_{N=m}^{\infty}\int\limits_{J_{N^2}} |u^{(1)}(\cdot,\sigma h_0)|^2\,dx_1 \\
&\geq\; \sum_{N=m}^{\infty}\frac{1}{N} \;=\; \infty\,,
\end{aligned}$$

a contradiction to $u^{(1)}(\cdot,\pm h_0) \in L^2(\mathbb{R})$.

We set $N_m^- = N_m - \sqrt{N_m}$ for abbreviation and estimate for $|x_1| > N_m$:

$$\begin{aligned}
& \int\limits_{|y_1|<N_m} \frac{|u^{(1)}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 \\
= & \int\limits_{|y_1|<N_m^-} \frac{|u^{(1)}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 \;+\; \int\limits_{N_m^-<|y_1|<N_m} \frac{|u^{(1)}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 \\
\leq & \;\|u^{(1)}\|_{L^2(\mathbb{R})}\sqrt{\int\limits_{|y_1|<N_m^-} \frac{dy_1}{(1+|x_1|-|y_1|)^3}} \;+\; \|u^{(1)}\|_{L^2(J_{N_m})}\sqrt{\int\limits_{N_m^-<|y_1|<N_m} \frac{dy_1}{(1+|x_1|-|y_1|)^3}} \\
\leq & \;\frac{c}{1+|x_1|-N_m^-} \;+\; \frac{1}{N_m^{1/4}}\,\frac{c}{1+|x_1|-N_m}
\end{aligned}$$

and thus

$$\begin{aligned}
\int\limits_{|x_1|>N_m} \rho_{N_m}(x_1)^2\,dx_1 \;\leq\; & \frac{8}{(1+N_m)^2} \;+\; c\int\limits_{|x_1|>N_m} \frac{dx_1}{(1+|x_1|-N_m^-)^2} \\
& + \frac{c}{\sqrt{N_m}}\int\limits_{|x_1|>N_m} \frac{dx_1}{(1+|x_1|-N_m)^2} \\
\leq\; & \frac{8}{(1+N_m)^2} \;+\; \frac{c}{1+\sqrt{N_m}} \;+\; \frac{c}{\sqrt{N_m}}\,.
\end{aligned}$$

Analogously, with $N_m^+ = N_m + \sqrt{N_m}$, we estimate for $|x_1| < N_m$:

$$\begin{aligned}
\rho(x_1) - \rho_{N_m}(x_1) \;=\; & \int\limits_{|y_1|>N_m} \frac{|u^{(1)}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 \\
= & \int\limits_{|y_1|>N_m^+} \frac{|u^{(1)}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 \;+\; \int\limits_{N_m<|y_1|<N_m^+} \frac{|u^{(1)}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 \\
\leq & \;\|u^{(1)}\|_{L^2(\mathbb{R})}\sqrt{\int\limits_{|y_1|>N_m^+} \frac{dy_1}{(1+|y_1|-|x_1|)^3}} \;+\; \|u^{(1)}\|_{L^2(J_{N_m})}\sqrt{\int\limits_{N_m<|y_1|<N_m^+} \frac{dy_1}{(1+|y_1|-|x_1|)^3}} \\
\leq & \;\frac{c}{1+N_m^+-|x_1|} \;+\; \frac{1}{N_m^{1/4}}\,\frac{c}{1+N_m-|x_1|}
\end{aligned}$$

and thus $\int_{|x_1|<N_m} |\rho(x_1) - \rho_{N_m}(x_1)|^2 \mathrm{d}x_1 \le c/\sqrt{N_m}$ as before. Finally, for $N_m < |x_1| < N_m + 1$ we estimate

$$
\begin{aligned}
\rho(x_1) \;&=\; \int\limits_{J_{N_m}} \frac{|u^{(1)}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 \\
&\quad + \int\limits_{|y_1|<N_m^-} \frac{|u^{(1)}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 \;+\; \int\limits_{|y_1|>N_m^+} \frac{|u^{(1)}(y_1,\sigma h_0)|}{(1+|x_1-y_1|)^{3/2}}\,dy_1 \\
&\le\; c\,\|u^{(1)}\|_{L^2(J_{N_m})} \;+\; \|u^{(1)}\|_{L^2(\mathbb{R})} \sqrt{\int\limits_{|y_1|<N_m^-} \frac{dy_1}{(1+|x_1|-|y_1|)^3}} \\
&\quad + \|u^{(1)}\|_{L^2(\mathbb{R})} \sqrt{\int\limits_{|y_1|>N_m^+} \frac{dy_1}{(1+|y_1|-|x_1|)^3}} \\
&\le\; \frac{c}{N_m^{1/4}} \;+\; \frac{c}{1+|x_1|-N_m^-} \;+\; \frac{c}{1+N_m^+-|x_1|} \;\le\; \frac{c'}{N_m^{1/4}}\,.
\end{aligned}
$$

Integration with respect to x_1 yields the last assertion. $\square$

After these preparations we turn to the proof of uniqueness.

Theorem 2.2. *Let Assumptions 1.1, 1.3, and 1.4 hold, and let u solve the problem (1) for $f = 0$ and the radiation condition of Definition 1.6. Then u vanishes.*

Proof: The proof is lengthy, and we try to structure it. In part (A) we show that the coefficients $a_{\ell,j}$ vanish, and in part (B) we show that $u^{(1)}$ vanishes under a smoothness assumption on its Fourier transform. The latter property is shown in Part (C).
Part (A): Choose $\psi_N \in C^\infty(\mathbb{R})$ with $\psi_N(x_1) = 1$ for $|x_1| \le N$ and $\psi_N(x_1) = 0$ for $|x_1| \ge N+1$. We define the regions $D_{N,H} = (-N,N)\times(-H,H)$ and $W_{N,H}^- = (-N-1,-N)\times(-H,H)$ and $W_{N,H}^+ = (N,N+1)\times(-H,H)$ and the vertical and horizontal segments $I_{\pm N,H} = \{\pm N\}\times(-H,H)$ and $\Gamma_{N,\pm H} = (-N,N)\times\{\pm H\}$ for any $H > h_0+1$ and $N > \sigma_0 + 1$. We apply Green's theorem in $D_{N+1,H}$ to $v(x) = \psi_N(x_1)\,u(x)$:

$$
\begin{aligned}
(23)\qquad & \sum_{\sigma\in\{+1,-1\}} \sigma \int\limits_{\Gamma_{N+1,\sigma H}} \psi_N^2\,\overline{u}\,\frac{\partial u}{\partial x_2}\,ds \\
&= \sum_{\sigma\in\{+1,-1\}} \sigma \int\limits_{\Gamma_{N+1,\sigma H}} \overline{v}\,\frac{\partial v}{\partial x_2}\,ds \;=\; \int\limits_{D_{N+1,H}} \left[\left|\nabla v\right|^2 + \overline{v}\,\Delta v\right] dx \\
&= \int\limits_{D_{N,H}} \left[\left|\nabla u\right|^2 + \overline{u}\,\Delta u\right] dx \;+\; \int\limits_{W_{N,H}^+} \left[\left|\nabla v\right|^2 + \overline{v}\,\Delta v\right] dx \;+\; \int\limits_{W_{N,H}^-} \left[\left|\nabla v\right|^2 + \overline{v}\,\Delta v\right] dx\,.
\end{aligned}
$$

We note that $\Delta u = -k^2 n(1+q)u$ and $\Delta v = -\psi_N k^2 n(1+q)u + 2\psi_N' \frac{\partial u}{\partial x_1} + \psi_N''\,u$ and $\nabla v = \psi_N \nabla u + u\,\psi_N' e^{(1)}$ with $e^{(1)} = (1,0)^\top$. The decomposition $u = u^{(1)} + u^{(2)}$ yields 4 terms on the left hand side of (23) and also the corresponding terms on the right hand side.

(A1) First, we look at the terms on the right hand side of (23). Note that the first integral on the right hand side is real valued. We define $v^{(j)} = \psi_N u^{(j)}$ for $j = 1,2$ and estimate the terms

$$a^{\pm}_{N,H}(j,\ell) \;:=\; \int\limits_{W^{\pm}_{N,H}} \left[\nabla\overline{v^{(j)}}\cdot\nabla v^{(\ell)} + \overline{v^{(j)}}\,\Delta v^{(\ell)}\right] dx$$

for $j,\ell \in \{1,2\}$. Then, with (12),

$$\begin{aligned}
|a^{+}_{N,H}(1,1)| \;&\le\; c\,\|u^{(1)}\|^2_{H^1(W^+_{N,h_0+1})} \;+\; c\,\|u^{(1)}\|^2_{H^1(W^+_{N,H}\setminus W^+_{N,h_0+1})} \\
&\le\; c\|u^{(1)}\|^2_{H^1(W^+_{N,h_0+1})} \;+\; c\int\limits_N^{N+1}\;\int\limits_{h_0+1<|x_2|<H} x_2^2\,\rho(x_1)^2\,dx_2\,dx_1 \\
&\le\; c\,\gamma_{N,H} \qquad \text{with} \qquad (24)
\end{aligned}$$

$$\gamma_{N,H} \;=\; \|u^{(1)}\|^2_{H^1(Q_N)} \;+\; H^3 \int\limits_{N<|x_1|<N+1} \rho(x_1)^2\,dx_1 \qquad (25)$$

and $Q_N = W^+_{N,h_0+1} \cup W^-_{N,h_0+1} = \{x \in \mathbb{R}^2 : N < |x_1| < N+1,\ |x_2| < h_0+1\}$. Analogously, since $\|u^{(2)}\|_{H^1(W^+_{N,H})}$ and $\|\nabla u^{(2)}\|_{H^1(W^+_{N,H})}$ are bounded with respect to N and H,

$$|a^{+}_{N,H}(1,2)|+|a^{+}_{N,H}(2,1)| \;\le\; c\left[\|u^{(1)}\|^2_{H^1(W^+_{N,h_0+1})} + \|u^{(1)}\|^2_{H^1(W^+_{N,H}\setminus W^+_{N,h_0+1})}\right]^{1/2} \;\le\; c\sqrt{\gamma_{N,H}}\,.$$

For $a^{+}_{N,H}(2,2)$ we apply Green's theorem:

$$\begin{aligned}
a^{+}_{N,H}(2,2) \;&=\; -\int\limits_{I_{N,H}} \overline{u^{(2)}}\,\frac{\partial u^{(2)}}{\partial x_1}\,ds \;+\; \sum_{\sigma\in\{+1,-1\}} \sigma \int\limits_{\substack{N<x_1<N+1\\ x_2=\sigma H}} \psi_N^2\,\overline{u^{(2)}}\,\frac{\partial u^{(2)}}{\partial x_2}\,ds \\
&=\; -\int\limits_{I_N} \overline{u^{(2)}}\,\frac{\partial u^{(2)}}{\partial x_1}\,ds \;+\; \beta^{+}_{N,H}
\end{aligned}$$

with $I_N = \{N\}\times\mathbb{R}$ and

$$|\beta^{+}_{N,H}| \;\le\; \sum_{\sigma\in\{+1,-1\}} \left|\int\limits_{\substack{N<x_1<N+1\\ x_2=\sigma H}} \psi_N^2\,\overline{u^{(2)}}\,\frac{\partial u^{(2)}}{\partial x_2}\,ds\right| \;+\; \left|\int\limits_{I_N\setminus I_{N,H}} \overline{u^{(2)}}\,\frac{\partial u^{(2)}}{\partial x_1}\,ds\right| \;\le\; c\,e^{-2\delta H}\,.$$

The same estimates hold for $a^{-}_{N,H}(j,\ell)$; that is, the integrals over $W^{-}_{N,H}$. Therefore, taking the imaginary part of (23) and using Lemma 1.7 we have shown that

$$\begin{aligned}
&\sum_{\sigma\in\{+1,-1\}} \sigma\,\mathrm{Im} \int\limits_{\Gamma_{N+1,\sigma H}} \psi_N^2\,\overline{u}\,\frac{\partial u}{\partial x_2}\,ds \qquad (26)\\
&\le\; -\,\mathrm{Im}\int\limits_{I_N} \overline{u^{(2)}}\,\frac{\partial u^{(2)}}{\partial x_1}\,ds \;+\; \mathrm{Im}\int\limits_{I_{-N}} \overline{u^{(2)}}\,\frac{\partial u^{(2)}}{\partial x_1}\,ds \;+\; c\,e^{-2\delta H} \;+\; c\left[\gamma_{N,H} + \sqrt{\gamma_{N,H}}\right] \\
&\le\; -\frac{1}{4\pi}\sum_{j\in J}\sum_{\lambda_{\ell,j}>0} \lambda_{\ell,j}\,|a_{\ell,j}|^2 \;+\; \frac{1}{4\pi}\sum_{j\in J}\sum_{\lambda_{\ell,j}<0} \lambda_{\ell,j}\,|a_{\ell,j}|^2 \;+\; c\,e^{-2\delta H} \;+\; c\left[\gamma_{N,H} + \sqrt{\gamma_{N,H}}\right].
\end{aligned}$$

(A2) Now we look at the left hand side of (23); that is, of (26), and decompose again u into $u = u^{(1)} + u^{(2)}$. Using Cauchy-Schwarz and (12) we estimate for $\sigma \in \{-1, 1\}$

$$
\begin{aligned}
& \int\limits_{\Gamma_{N+1,\sigma H}} \psi_N^2 \left| \overline{u^{(1)}} \frac{\partial u^{(2)}}{\partial x_2} + \overline{u^{(2)}} \frac{\partial u^{(1)}}{\partial x_2} + \overline{u^{(2)}} \frac{\partial u^{(2)}}{\partial x_2} \right| ds \\
\leq \; & c \, \|u^{(1)}\|_{L^2(\Gamma_{N+1,\sigma H})} \Big\| \frac{\partial u^{(2)}}{\partial x_2} \Big\|_{L^2(\Gamma_{N+1,\sigma H})} + c \, \|u^{(2)}\|_{L^2(\Gamma_{N+1,\sigma H})} \Big\| \frac{\partial u^{(1)}}{\partial x_2} \Big\|_{L^2(\Gamma_{N+1,\sigma H})} \\
& + c \, \|u^{(2)}\|_{L^2(\Gamma_{N+1,\sigma H})} \Big\| \frac{\partial u^{(2)}}{\partial x_2} \Big\|_{L^2(\Gamma_{N+1,\sigma H})} \;\leq\; c \left[H \, \|\rho\|_{L^2(\mathbb{R})} \sqrt{N} + N \right] e^{-\delta H} .
\end{aligned}
$$

Finally, we consider $\int_{\Gamma_{N+1,\pm H}} \psi_N^2 \, \overline{u^{(1)}} \, \frac{\partial u^{(1)}}{\partial x_2} \, ds$. We restrict ourselves to the line integral over $\Gamma_{N+1,H}$ and approximate $u^{(1)}$ by functions $u^{(1)}_{N,H}$ which satisfy the homogeneous Helmholtz equation for $x_2 > H$. To do this we set $u^{(1)}_{N,H} = u_N^+ + v_H^+$ for $x_2 > h_0$. Here, u_N^+ is the unique radiating solution of $\Delta u_N^+ + k^2 u_N^+ = 0$ for $x_2 > h_0$ and $u_N^+(x_1, h_0) = u^{(1)}(x_1, h_0)$ for $|x_1| < N$ and $u_N^+(x_1, h_0) = 0$ for $|x_1| > N$, and the function v_H^+ is defined as the unique radiating solution of

$$
\Delta v_H^+ \;+\; k^2 v_H^+ \;=\; \begin{cases} \sum_{j \in J} \sum_{\ell=1}^{m_j} a_{\ell,j} \, \varphi_{\ell,j} & \text{for } h_0 < x_2 < H \,, \\ 0 & \text{for } x_2 > H \,, \end{cases}
$$

and $v_H^+ = 0$ for $x_2 = h_0$. Then u_N^+ and v_H^+ are given by (compare with (14a), (14b))

$$
\begin{aligned}
u_N^+(x) &= 2 \int\limits_{-N}^{N} u^{(1)}(y_1, h_0) \, \frac{\partial}{\partial y_2} \Phi(x_1, x_2, y_1, h_0) \, dy_1 \,, \quad x_2 > h_0 \,, \\
v_H^+(x) &= \sum_{j \in J} \sum_{\ell=1}^{m_j} a_{\ell,j} \int\limits_{h_0}^{H} \int\limits_{-\sigma_0}^{\sigma_0} G^+(x,y) \, \varphi_{\ell,j}(y) \, dy_1 dy_2 \,, \quad x_2 > h_0 \,,
\end{aligned}
$$

and it is easy to show by modifying the proof of Lemma 1.8 that

$$
\begin{aligned}
\big| u^{(1)}_{N,H}(x) \big| \;+\; \big| \nabla u^{(1)}_{N,H}(x) \big| \;&\leq\; c \, |x_2| \, \rho_N(x_1) \,, \\
\big| u^{(1)}(x) - u^{(1)}_{N,H}(x) \big| + \big| \nabla \big(u^{(1)}(x) - u^{(1)}_{N,H}(x) \big) \big| \;&\leq\; c \, x_2 \left[\rho_N(x_1) - \rho(x_1) \right] + \frac{c \, x_2}{|x|^{3/2}} \, e^{-\delta H} \,,
\end{aligned}
$$

for all $x \in \mathbb{R}^2$ with $x_2 \geq h_0 + 1$, where $\rho, \rho_N \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ are given by (13) and (22), respectively.[1] With $\Gamma_{\infty,\pm H} = \mathbb{R} \times \{\pm H\}$ we decompose

$$
\begin{aligned}
& \int\limits_{\Gamma_{N+1,H}} \psi_N^2 \, \overline{u^{(1)}} \, \frac{\partial u^{(1)}}{\partial x_2} \, ds \\
= & \int\limits_{\Gamma_{\infty,H}} \overline{u^{(1)}_{N,H}} \, \frac{\partial u^{(1)}_{N,H}}{\partial x_2} \, ds \; + \int\limits_{\Gamma_{N+1,H}} \psi_N^2 \left[\overline{u^{(1)}} \, \frac{\partial u^{(1)}}{\partial x_2} - \overline{u^{(1)}_{N,H}} \, \frac{\partial u^{(1)}_{N,H}}{\partial x_2} \right] ds \\
& - \int\limits_{\Gamma_{\infty,H} \setminus \Gamma_{N+1,H}} \overline{u^{(1)}_{N,H}} \, \frac{\partial u^{(1)}_{N,H}}{\partial x_2} \, ds \; + \int\limits_{\Gamma_{N+1,H} \setminus \Gamma_{N,H}} (\psi_N^2 - 1) \, \overline{u^{(1)}_{N,H}} \, \frac{\partial u^{(1)}_{N,H}}{\partial x_2} \, ds \\
= & \int\limits_{\Gamma_{\infty,H}} \overline{u^{(1)}_{N,H}} \, \frac{\partial u^{(1)}_{N,H}}{\partial x_2} \, ds \; + \; \eta_{N,H}
\end{aligned}
$$

where

$$
\begin{aligned}
|\eta_{N,H}| \; \leq \; & c \, \|u^{(1)} - u^{(1)}_{N,H}\|_{L^2(\Gamma_{N+1,H})} \left\| \frac{\partial u^{(1)}}{\partial x_2} \right\|_{L^2(\Gamma_{N+1,H})} \\
& + c \, \|u^{(1)}_{N,H}\|_{L^2(\Gamma_{N+1,H})} \left\| \frac{\partial u^{(1)}}{\partial x_2} - \frac{\partial u^{(1)}_{N,H}}{\partial x_2} \right\|_{L^2(\Gamma_{N+1,H})} \\
& + c \, \|u^{(1)}_{N,H}\|_{L^2(\Gamma_{\infty,H} \setminus \Gamma_{N,H})} \left\| \frac{\partial u^{(1)}_{N,H}}{\partial x_2} \right\|_{L^2(\Gamma_{\infty,H} \setminus \Gamma_{N,H})} \\
\leq \; & c \, H^2 \, \|\rho\|_{L^2(\mathbb{R})} \sqrt{ \int\limits_{|x_1| < N} |\rho(x_1) - \rho_N(x_1)|^2 \, dx_1 } \; + \; c \, H^2 \int\limits_{|x_1| > N} \rho_N(x_1)^2 \, dx_1 \, . \qquad (27)
\end{aligned}
$$

The same estimates hold also for $\int_{\Gamma_{N+1,-H}} \psi_N^2 \, \overline{u^{(1)}} \, \frac{\partial u^{(1)}}{\partial x_2} \, ds$. Substituting this into the left hand side of (26) yields

$$
\begin{aligned}
\sum_{\sigma \in \{-1,+1\}} \sigma \, \mathrm{Im} \int\limits_{\Gamma_{\infty,\sigma H}} \overline{u^{(1)}_{N,H}} \, \frac{\partial u^{(1)}_{N,H}}{\partial x_2} \, ds \; \leq \; & -\frac{1}{4\pi} \sum_{j \in J} \sum_{\lambda_{\ell,j} > 0} \lambda_{\ell,j} \, |a_{\ell,j}|^2 + \frac{1}{4\pi} \sum_{j \in J} \sum_{\lambda_{\ell,j} < 0} \lambda_{\ell,j} \, |a_{\ell,j}|^2 \\
& + c \, e^{-2\delta H} \; + \; c \left[\gamma_{N,H} + \sqrt{\gamma_{N,H}} \right] \\
& + c \left[H \sqrt{N} + N \right] e^{-\delta H} \; + \; |\eta_{N,H}| \, .
\end{aligned}
$$

At this point we set $N = N_m$ where (N_m) is the sequence from Lemma 2.1. Then from (25) and (27) in combination with the estimates of Lemma 2.1 we conclude that $\gamma_{N_m,H} \; \leq \; c \, \|u^{(1)}\|^2_{H^1(Q_{N_m})} + c \, \frac{H^3}{\sqrt{N_m}}$ and $|\eta_{N_m,H}| \; \leq \; c \, \frac{H^2}{N_m^{1/4}}$. We choose H_m such that the

[1]For the estimate with v_H^+ replace h_0 in (16), (17), (18) by H!

reminders converge to zero, for example, $H_m = N_m^{1/10}$. Then

$$\sum_{\sigma\in\{-1,+1\}} \sigma \limsup_{m\to\infty} \ \mathrm{Im} \int\limits_{\Gamma_{\infty,\sigma H_m}} \overline{u^{(1)}_{N_m,H_m}} \frac{\partial u^{(1)}_{N_m,H_m}}{\partial x_2}\, ds \tag{28}$$

$$\leq \ -\frac{1}{4\pi}\sum_{j\in J}\sum_{\lambda_{\ell,j}>0} \lambda_{\ell,j}\, |a_{\ell,j}|^2 \ + \ \frac{1}{4\pi}\sum_{j\in J}\sum_{\lambda_{\ell,j}<0} \lambda_{\ell,j}\, |a_{\ell,j}|^2 \ \leq \ 0\,.$$

Now we show that the left hand side is non-negative. Indeed, we fix m, write $N = N_m$ and $H = H_m$ for short and take the Fourier transform $\hat u_{N,H}(\omega, x_2) = (\mathcal{F}u^{(1)}_{N,H})(\omega, x_2)$ for $x_2 > H$. Then, for $\sigma \in \{-1,+1\}$,

$$\int\limits_{\Gamma_{\infty,\sigma H}} \overline{u^{(1)}_{N,H}} \frac{\partial u^{(1)}_{N,H}}{\partial x_2}\, ds \ = \ \int\limits_{-\infty}^{\infty} \overline{\hat u_{N,H}(\omega,\sigma H)}\, \hat u'_{N,H}(\omega,\sigma H)\, d\omega\,. \tag{29}$$

Furthermore, $\hat u''_{N,H}(\omega, x_2) + (k^2-\omega^2)\,\hat u_{N,H} = 0$ for $|x_2| > H$. Therefore, $\hat u_{N,H}$ has the form (compare with (20))

$$\hat u_{N,H}(\omega, x_2) \ = \ \begin{cases} \hat u_{N,H}(\omega, H)\, e^{i\sqrt{k^2-\omega^2}(x_2-H)} & \text{for } x_2 > H\,, \\ \hat u_{N,H}(\omega, -H)\, e^{i\sqrt{k^2-\omega^2}(-x_2-H)} & \text{for } x_2 < -H\,, \end{cases}$$

and thus $\sigma\, \overline{\hat u_{N,H}(\omega,\sigma H)}\, \hat u'_{N,H}(\omega,\sigma H) = i\, |\hat u_{N,H}(\omega,\sigma H)|^2 \sqrt{k^2-\omega^2}$ and thus $\sigma\ \mathrm{Im}\big[\overline{\hat u_{N,H}(\omega,\sigma H)}\, \hat u'_{N,H}(\omega,\sigma H)\big] \geq 0$. Therefore, the left hand side of (28) is non-negative which implies that all $a_{\ell,j}$ vanish; that is, $u^{(2)} = 0$. This ends the proof of Part (A).

Part (B): Now $u = u^{(1)} \in H^1(W_h)$ for all $h > h_0$ where again $W_h = \mathbb{R}\times(-h,h)$. From (23) we conclude for $N\to\infty$ and $H := h_0 + 1$ that

$$\sum_{\sigma\in\{+1,-1\}} \sigma \int\limits_{-\infty}^{\infty} \overline{u(x_1,\sigma H)} \frac{\partial u(x_1,\sigma H)}{\partial x_2}\, dx_1 \ = \ \int\limits_{W_H} \big[|\nabla u|^2 - k^2 n(1+q)\, |u|^2\big]\, dx\,.$$

The imaginary part of this expression vanishes again. Transforming this equation to the Fourier space we observe just as in (29) that $(\mathcal{F}u)(\omega, \pm H)$ vanishes for all $|\omega| < k$. Therefore, from (20) we conclude that

$$(\mathcal{F}u)(\omega, x_2) \ = \ (\mathcal{F}u)(\omega, \pm H)\, e^{-\sqrt{\omega^2-k^2}(\pm x_2 - H)} \quad \text{for } \pm x_2 > H \text{ and } |\omega| > k$$

and thus for $|\omega| > k$:

$$\int\limits_H^\infty |(\mathcal{F}u)(\omega,x_2)|^2\, dx_2 \ = \ |(\mathcal{F}u)(\omega,H)|^2 \int\limits_H^\infty e^{-2\sqrt{\omega^2-k^2}(x_2-H)}\, dx_2 \ = \ \frac{|(\mathcal{F}u)(\omega,H)|^2}{2\sqrt{\omega^2-k^2}}\,.$$

The integal vanishes for $|\omega| < k$. The analogous formula holds for the integral $\int_{-\infty}^{-H} |(\mathcal{F}u)(\omega,x_2)|^2\, dx_2$. If $(\mathcal{F}u)(\cdot, \pm H)$ would be continuous in a neighborhood of $\omega = \pm k$ then this integral would be integrable with respect to $\omega \in \mathbb{R}$ and, by Parseval's theorem, $u \in L^2(\mathbb{R}^2)$. This would imply that u vanishes because k^2 is not in the point spectrum of $-\frac{1}{n(1+q)}\Delta$ by Assumption 1.1.

Part (C): Therefore, it remains to prove continuity of $(\mathcal{F}u)(\cdot, \pm H)$ in neighborhoods of $\omega = \pm k$. Let

$$(Fu)(x_1, x_2, \alpha) \;=\; \tilde{u}(x_1, x_2, \alpha) \;=\; \sum_{\ell \in \mathbb{Z}} u(x_1 + 2\pi\ell, x_2)\, e^{-i2\pi\ell\alpha} \tag{30}$$

for $x \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ denote the Floquet-Bloch transform of $u(\cdot, x_2)$ with respect to x_1. Since u satisfies the Helmholtz equation $\Delta u + k^2 n u = -k^2 nqu$ in $\mathbb{R}^2$ standard regularity results yield $u \in H^2(W_h)$ for all $h > h_0$. Then $\tilde{u} \in L^2\big((-1/2, 1/2), H^2_{qp}(Q^h)\big)$ for all $h > h_0$ by well known mapping properties of the Floquet-Bloch transform (see [15]). Here, $Q^h = (0, 2\pi) \times (-h, h)$. Therefore, $\tilde{u}(\cdot, \alpha)$ satisfies the equation

$$\Delta \tilde{u}(\cdot, \alpha) \;+\; k^2\, n\, \tilde{u}(\cdot, \alpha) \;=\; -k^2 F(nqu)(\cdot, \alpha) \quad \text{in } Q^\infty$$

for almost all $\alpha \in [-1/2, 1/2]$. We consider this equation as an equation for $\tilde{u}(\cdot, \alpha)$ for fixed right hand side $\tilde{g} := k^2 F(nqu)$. As in the case of the modes (see (4)) this equation is equivalent to

$$\int_Q [\nabla \tilde{u}(\cdot, \alpha) \cdot \nabla \overline{\psi} - k^2 n\, \tilde{u}(\cdot, \alpha)\, \overline{\psi}]\, dx \;-\; \int_\Gamma (\Lambda_\alpha \tilde{u}(\cdot, \alpha))\, \overline{\psi}\, ds \;=\; \int_Q \tilde{g}(\cdot, \alpha)\, \overline{\psi}\, dx$$

for all $\psi \in H^1_\alpha(Q)$ and almost all $\alpha \in (-1/2, 1/2)$. Here, Λ_α is the $\alpha-$quasi-periodic Dirichlet-to-Neumann operator given by (5). Transforming the $\alpha-$dependence from the solution space to the equation we set $v(x, \alpha) = e^{-i\alpha x_1} \tilde{u}(x, \alpha)$ and $(\tilde{\Lambda}_\alpha \phi)(x) = e^{-i\alpha x_1} (\Lambda_\alpha \phi)(x)$ and arrive at the periodic variational equation

$$\begin{aligned} &\int_Q [\nabla v(\cdot, \alpha) \cdot \nabla \overline{\psi} - 2i\alpha \frac{\partial v(\cdot, \alpha)}{\partial x_1}\, \overline{\psi} - (k^2 n - \alpha^2)\, v(\cdot, \alpha)\, \overline{\psi}]\, dx \;-\; \int_\Gamma (\tilde{\Lambda}_\alpha v(\cdot, \alpha))\, \overline{\psi}\, ds \\ = &\int_Q e^{-i\alpha x_1} \tilde{g}(x, \alpha)\, \overline{\psi(x)}\, dx \quad \text{for all } \psi \in H^1_{per}(Q)\,. \end{aligned}$$

We note that the right hand side depends analytically on α (because nqu has compact support in Q) and the coefficients of the left hand side depends continuously (because of the square-root term in Λ_α) on α. Furthermore, since this equation is of Fredholm type with index zero, α is not exceptional if, and only if, this equation is uniquely solvable for all right hand sides.

We decompose k again as $k = \hat{\ell} + \kappa$ with $\hat{\ell} \in \mathbb{N}_0$ and $\kappa \in (-1/2, 1/2]$. Then $\pm\kappa$ are the cut-off values. By Assumption 1.3 these are not exceptional values. Therefore, for α in some neighborhoods of $\pm\kappa$ the variational equation is uniquely solvable, and $\alpha \mapsto \tilde{u}(\cdot, \alpha)$ is continuous from $[-1/2, 1/2]$ to $H^1(Q^H)$ in neighborhoods of $\pm\kappa$. This implies that also the Fourier coefficients $\tilde{u}_\ell(\alpha, x_2) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \tilde{u}(x_1, x_2, \alpha)\, e^{-i(\ell+\alpha)x_1}\, dx_1$ of the $\alpha-$quasi-periodic function $\tilde{u}(\cdot, x_2, \alpha)$ are continuous from $[-1/2, 1/2]$ to $\mathbb{C}$ in neighborhoods of $\pm\kappa$ for every $x_2 \in \mathbb{R}$.
Finally, we show the following relationship between the Fourier transform and the Fourier

coefficients of the Floquet-Bloch transform. For $\alpha \in [-1/2, 1/2]$ and $\ell \in \mathbb{Z}$ we compute

$$
\begin{aligned}
(\mathcal{F}u^{(1)})(\ell+\alpha, x_2) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^{(1)}(x_1, x_2)\, e^{-i(\ell+\alpha)x_1}\, dx_1 \\
&= \frac{1}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}} \int_0^{2\pi} u^{(1)}(x_1 + 2\pi m, x_2)\, e^{-i(\ell+\alpha)(x_1+2\pi m)}\, dx_1 \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \tilde{u}(x_1, x_2, \alpha)\, e^{-i(\ell+\alpha)x_1}\, dx_1 \;=\; \tilde{u}_\ell(x, \alpha) \qquad (31)
\end{aligned}
$$

which shows continuity of $\alpha \mapsto (\mathcal{F}u^{(1)})(\ell+\alpha, x_2)$ in neighborhoods of $\pm\kappa$. In particular, $(\mathcal{F}u^{(1)})(\cdot, \pm H)$ is continuous in neighborhoods of $\pm k$. This ends the proof. $\square$

3. Existence

In this section we will prove existence of a solution under the Assumptions 1.1, 1.3, and 1.4. The main part deals with the unperturbed case $q = 0$. The general case follows by a compactness argument. Therefore, for given $f \in L^2(Q)$ we consider first the problem to determine $u \in H^1_{loc}(\mathbb{R}^2)$ which satisfies

$$\Delta u + k^2 n\, u \;=\; -f \quad \text{in } \mathbb{R}^2 \qquad (32)$$

and the radiation condition of Definition 1.6. With the exceptional values $\hat{\alpha}_j$ for $j \in J$ and their eigenfunctions $\hat{\phi}_{\ell,j}$, $\ell = 1, \ldots, m_j$, $j \in J$, determined in (7a), (7b), we define the coefficients $a_{\ell,j} \in \mathbb{C}$ as

$$a_{\ell,j} \;:=\; \frac{2\pi\, i}{|\lambda_{\ell,j}|} \int_Q f(x)\, \overline{\hat{\phi}_{\ell,j}(x)}\, dx\,, \quad \ell = 1, \ldots, m_j,\ j \in J\,. \qquad (33)$$

Therefore, we have to solve the equation (11a) for $q = 0$; that is,

$$\Delta u^{(1)} \;+\; k^2 n\, u^{(1)} \;=\; -g \quad \text{in } \mathbb{R}^2 \quad \text{with} \quad g = f + \sum_{j\in J} \sum_{\ell=1}^{m_j} a_{\ell,j} \varphi_{\ell,j}\,, \qquad (34)$$

where $\varphi_{\ell,j}$ are given by (11b). Furthermore, $u^{(1)}$ has to satisfy the generalized Sommerfeld radiation condition (10). The plan is to take the Floquet-Bloch transform of this equation, show solvability for all $\alpha \in [-1/2, 1/2]$ (without exception) and continuity with respect to α and apply the inverse transform.

We note that the right hand side g of (34) is in $L^2(\mathbb{R}^2)$ (and has even compact support with respect to x_1). Therefore, for every $\alpha \in [-1/2, 1/2]$ we try to solve the Floquet-Bloch transformed (with respect to x_1) equation; that is, find $w_\alpha \in H^1_{\alpha,loc}(Q^\infty)$ with

$$\Delta w_\alpha + k^2 n\, w_\alpha \;=\; -(Fg)(\cdot, \alpha) \quad \text{in } Q^\infty = (0, 2\pi) \times \mathbb{R} \qquad (35a)$$

satisfying the radiating condition

$$(\operatorname{sign} x_2)\, w'_\ell(\alpha, x_2) - i\sqrt{k^2 - (\ell+\alpha)^2}\, w_\ell(\alpha, x_2) \;\longrightarrow 0\,, \quad |x_2| \to \infty\,, \qquad (35b)$$

for the Fourier coefficients $w_\ell(\alpha, x_2)$ of $w_\alpha(\cdot, x_2)$. Here, Fg denotes the Flochet-Bloch transform of g, defined in (30). Fg is analytic with respect to α because the right hand

side of (34) has compact support with respect to x_1.
The decomposition of $u^{(1)}$ in the half spaces $x_2 > h_0$ and $x_2 < -h_0$ of Lemma 1.8 carries over to the quasi-periodic case; that is w_α has a decomposition in the strips $Q_{h_0}^+ = (0,2\pi)\times(h_0,\infty)$ and $Q_{h_0}^- = (0,2\pi)\times(-\infty,-h_0)$, respectively, in the forms

$$w_\alpha \;=\; w_{\alpha,0}^\pm \;+\; w_{\alpha,g}^\pm \quad \text{in } Q_{h_0}^\pm\,,$$

where $w_{\alpha,g}^\pm$ is the unique $\alpha-$quasi-periodic solution of $\Delta w_{\alpha,g}^\pm + k^2 w_{\alpha,g}^\pm \;=\; -Fg$ in $Q_{h_0}^\pm$ and $w_{\alpha,g}^\pm = 0$ for $x_2 = \pm h_0$ satisfying radiation condition (35b), and $w_{\alpha,0}^\pm$ is the unique radiating solution of $\Delta w_{\alpha,0}^\pm + k^2 w_{\alpha,0}^\pm = 0$ in $Q_{h_0}^\pm$ and $w_{\alpha,0}^\pm = w_\alpha$ for $x_2 = \pm h_0$. This can be seen by taking the Floquet-Bloch transform of $u_0^\pm$ and $v_{\ell,j}^\pm$ or directly by solving the upper and lower half plane problems by means of the $\alpha-$quasi-periodic Green's functions

$$\begin{aligned} G_\alpha^\pm(x,y) \;&=\; \frac{i}{4\pi}\sum_{\ell\in\mathbb{Z}} \frac{1}{\sqrt{k^2-(\ell+\alpha)^2}}\Big[e^{i\sqrt{k^2-(\ell+\alpha)^2}|x_2-y_2|} \\ &\qquad -\; e^{i\sqrt{k^2-(\ell+\alpha)^2}(\pm(x_2+y_2)-2h_0)}\Big]e^{i(\ell+\alpha)(x_1-y_1)} \end{aligned}$$

for $x,y\in Q_{h_0}^\pm$, $x\neq y$. Then $w_{\alpha,g}^\pm$ is given by

$$w_{\alpha,g}^\pm(x) \;=\; \int\limits_{Q_{h_0}^\pm} (Fg)(y,\alpha)\,G_\alpha^\pm(x,y)\,dy\,, \quad x\in Q_{h_0}^\pm\,. \tag{36}$$

The proof of the following result is standard and omitted.

Lemma 3.1. *Let $\alpha\in[-1/2,1/2]$ be fixed.*

(a) *Let $w_\alpha \in H^1_{\alpha,loc}(Q^\infty)$ solve (35a) and (35b). Then $w_\alpha|_Q \in H^1_\alpha(Q)$ satisfies*

$$\begin{aligned} &\int\limits_Q [\nabla w_\alpha\cdot\nabla\overline{\psi} - k^2 n\, w_\alpha\,\overline{\psi}]\,dx \;-\; \int\limits_\Gamma (\Lambda_\alpha w\alpha)\,\overline{\psi}\,ds \\ &=\; \int\limits_Q (Fg)\,\overline{\psi}\,dx \;+\; \int\limits_\Gamma \frac{\partial w_{\alpha,g}}{\partial\nu}\,\overline{\psi}\,ds \quad \textit{for all } \psi\in H^1_\alpha(Q) \end{aligned} \tag{37}$$

where again $\Gamma = (\mathbb{R}\times\{h_0\})\cup(\mathbb{R}\times\{-h_0\})$ and $\Lambda_\alpha : H_\alpha^{1/2}(\Gamma)\to H_\alpha^{-1/2}(\Gamma)$ is the $\alpha-$quasi-periodic Dirichlet-to-Neumann operator given by (5). Here $\partial w_{\alpha,g}/\partial\nu = \pm\partial w_{\alpha,g}^\pm/\partial x_2$ for $x_2 = \pm h_0$.

(b) *Let $w_\alpha\in H^1_\alpha(Q)$ satisfy (37). Extend w_α by $w_\alpha = w_{\alpha,0}^\pm + w_{\alpha,g}^\pm$ into $Q_{h_0}^\pm$. Then w_α satisfies (35a) and (35b).*

By the representation theorem of Riesz we can write the variational equation (37) as

$$A_\alpha w_\alpha \;=\; r_\alpha \quad \text{in } H^1_\alpha(Q)\,, \tag{38}$$

where $r_\alpha \in H^1_\alpha(Q)$ and the linear and bounded operator A_α from $H^1_\alpha(Q)$ into itself are defined as

$$\begin{aligned}
\big(A_\alpha w,\psi\big)_{H^1(Q)} &= \int_Q [\nabla w\cdot\nabla\overline{\psi} - k^2 n\, w\,\overline{\psi}]\,dx \;-\; \int_\Gamma (\Lambda_\alpha w)\,\overline{\psi}\,ds \\
\big(r_\alpha,\psi\big)_{H^1(Q)} &= \int_Q (Fg)\,\overline{\psi}\,dx \;+\; \int_\Gamma \frac{\partial w_{\alpha,g}}{\partial\nu}\,\overline{\psi}\,ds
\end{aligned}$$

for all $w,\psi \in H^1_\alpha(Q)$. Then A_α is Fredholm with index zero. The equivalence implies that α is an exceptional value if, and only if, A_α fails to be invertible. This form (37) allows the application of Fredholm's theorem; that is, $A_\alpha w_\alpha = r_\alpha$ is solvable if, and only if r_α is orthogonal to the null space of the adjoint A^*_α of A_α. This is indeed the case for this particular form of the right hand side. Before we prove this we show the following properties of the operators A_α and the right hand side r_α.

Lemma 3.2. *Let Assumptions 1.1, 1.3, and 1.4 hold, and let $\alpha = \hat{\alpha}_j$ for some $j \in J$ be an exceptional value.*

(a) *The null spaces $\mathcal{N}(A_{\hat{\alpha}_j})$ and $\mathcal{N}(A^*_{\hat{\alpha}_j})$ of $A_{\hat{\alpha}_j}$ and $A^*_{\hat{\alpha}_j}$, respectively, coincide and are given by the restrictions of the functions in X_j to Q.*

(b) *The Riesz number of $A_{\hat{\alpha}_j}$ is one; that is, the geometric and algebraic multiplicities of the eigenvalue zero coincide.*

(c) *For all $\hat{\phi} \in X_j$ we have*

$$\big(r_{\hat{\alpha}_j},\hat{\phi}\big)_{H^1(Q)} \;=\; \int_{Q^\infty} (Fg)(x,\hat{\alpha}_j)\,\overline{\hat{\phi}(x)}\,dx$$

where $(Fg)(\cdot,\hat{\alpha}_j)$ is again the right hand side of (35a).

Proof: (a) $A^*_\alpha\phi = 0$ is equivalent to $(A_\alpha\psi,\phi)_{H^1(Q)} = 0$ for all ψ; that is,

$$\int_Q [\nabla\psi\cdot\nabla\overline{\phi} - k^2 n\,\psi\,\overline{\phi}]\,dx \;-\; \int_\Gamma (\Lambda_\alpha\psi)\,\overline{\phi}\,ds \;=\; 0\,;\quad \text{that is,}$$

$$\int_Q [\nabla\psi\cdot\nabla\overline{\phi} - k^2 n\,\psi\,\overline{\phi}]\,dx \;-\; i\sum_{\sigma\in\{-1,+1\}}\sum_{\ell\in\mathbb{Z}} \sqrt{k^2-(\ell+\alpha)^2}\,\psi_\ell(\sigma h_0)\,\overline{\phi_\ell(\sigma h_0)} \;=\; 0$$

for all $\psi \in H^1_\alpha(Q)$. If $\alpha = \hat{\alpha}_j$ this yields, by taking $\psi = \phi$ and the imaginary part, that $\phi_\ell(\pm h_0) = 0$ for $|\ell + \hat{\alpha}_j| < k$; that is, $\phi \in X_j$.

(b) Let ϕ with $A^2_{\hat{\alpha}_j}\phi = 0$. Then $w = A_{\hat{\alpha}_j}\phi \in \mathcal{N}(A_{\hat{\alpha}_j}) = \mathcal{N}(A^*_{\hat{\alpha}_j})$ and thus $\|w\|^2_{H^1(Q)} = \big(w, A_{\hat{\alpha}_j}\phi\big)_{H^1(Q)} = \big(A^*_{\hat{\alpha}_j}w,\phi\big)_{H^1(Q)} = 0$; that is, $w = 0$.

(c) We compute (note that $w_{\alpha,g} = w_{\alpha,g}(\cdot, \hat\alpha_j$ vanishes on Γ):

$$
\begin{aligned}
\big(r_{\hat\alpha_j}, \hat\phi\big)_{H^1(Q)} &= \int\limits_Q (Fg)(\cdot,\hat\alpha_j)\,\overline{\hat\phi}\,dx \;+\; \int\limits_\Gamma \left[\frac{\partial w_{\alpha,g}}{\partial\nu}\,\overline{\hat\phi} - \frac{\partial\overline{\hat\phi}}{\partial\nu}\,w_{\alpha,g}\right] ds \\
&= \int\limits_Q (Fg)(\cdot,\hat\alpha_j)\,\overline{\hat\phi}\,dx \;-\; \int\limits_{Q^\infty\setminus Q} \big[\overline{\hat\phi}\,\Delta w_{\alpha,g} - w_{\alpha,g}\,\Delta\overline{\hat\phi}\big]\,dx \\
&= \int\limits_Q (Fg)(\cdot,\hat\alpha_j)\,\overline{\hat\phi}\,dx \;-\; \int\limits_{Q^\infty\setminus Q} \overline{\hat\phi}\,\big[\Delta w_{\alpha,g} + k^2 w_{\alpha,g}\big]\,dx \\
&= \int\limits_{Q^\infty} (Fg)(\cdot,\hat\alpha_j)\,\overline{\hat\phi}\,dx\,.
\end{aligned}
$$

This ends the proof. $\square$

Lemma 3.3. *For every exceptional value $\alpha = \hat\alpha_{j_0}$ the right hand side $(Fg)(\cdot,\hat\alpha_{j_0})$ of (35a) is orthogonal to the eigenspace X_{j_0} (see (6)) in $L^2(Q^\infty)$. Therefore, by the previous lemmata, the variational equation (37) and the equations (35a), (35b) are solvable for all $\alpha \in [-1/2, 1/2]$ without exception.*

Proof: Recall the definition of g and thus $Fg = Ff + \sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j} F\varphi_{\ell,j}$ where φ_{ℓ_j} are defined in (11b).
Since $\hat\phi_{\ell,j}$ is $\hat\alpha_j-$quasi-periodic it follows easily from the properties of the Floquet-Bloch transform that

$$
(F\varphi_{\ell,j})(x,\alpha) \;=\; 2\,(F\chi^\pm)(x_1,\alpha-\hat\alpha_j)\,\frac{\partial\hat\phi_{\ell,j}(x)}{\partial x_1} \;+\; (F\chi^\pm)'(x_1,\alpha-\hat\alpha_j)\,\hat\phi_{\ell,j}(x)
$$

where $\chi^\pm = \psi'_\pm$ for $\lambda_{\ell,j} \gtrless 0$. (Note that $\psi'_\pm \in L^2(\mathbb{R})$ in contrast to $\psi_\pm$ itself.) Since $(F\chi^\pm)(\cdot,\beta)$ is $\beta-$quasi-periodic its Fourier series is given by

$$
(F\chi^\pm)(x_1,\beta) \;=\; \frac{1}{\sqrt{2\pi}}\sum_{\ell\in\mathbb{Z}} \chi^\pm_\ell(\beta)\,e^{i(\ell+\beta)x_1} \;=\; \frac{1}{\sqrt{2\pi}}\sum_{\ell\in\mathbb{Z}} (\mathcal{F}\chi^\pm)(\ell+\beta)\,e^{i(\ell+\beta)x_1}
$$

where we used (31) for the relationship between the Fourier transform $\mathcal{F}\chi^\pm$ and the Fourier coefficients $\chi^\pm_\ell(\beta)$ of the Floquet-Bloch transform $(F\chi\pm)(\cdot,\beta)$. With $(\mathcal{F}\chi^\pm)(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty (\psi_\pm)'(t)\,dt = \pm\frac{1}{\sqrt{2\pi}}$ we can write

$$
(F\chi^\pm)(x_1,\beta) \;=\; \begin{cases} \frac{1}{\sqrt{2\pi}}\frac{d}{dx_1}\sum\limits_{\ell\in\mathbb{Z}} \frac{(\mathcal{F}\chi^\pm)(\ell+\beta)}{i(\ell+\beta)}\,e^{i(\ell+\beta)x_1}\,, & \beta\notin\mathbb{Z}\,, \\ \pm\frac{1}{2\pi} + \frac{1}{\sqrt{2\pi}}\frac{d}{dx_1}\sum\limits_{\ell\neq 0} \frac{(\mathcal{F}\chi^\pm)(\ell)}{i\ell}\,e^{i\ell x_1}\,, & \beta\in\mathbb{Z}\,, \end{cases}
$$

which we abbreviate as $(F\chi^\pm)(x_1,\beta) = \pm\frac{1}{2\pi}\,\delta_\beta + \frac{d}{dx_1}\rho^\pm(x_1,\beta)$ where $\delta_\beta = 0$ for $\beta \notin \mathbb{Z}$ and $\delta_\beta = 1$ for $\beta \in \mathbb{Z}$. This allows us to write

$$
\begin{aligned}
(F\varphi_{\ell,j})(x,\alpha) \;&=\; \pm\frac{1}{\pi}\,\frac{\partial\hat\phi_{\ell,j}(x)}{\partial x_1}\,\delta_{\alpha-\hat\alpha_j} \\
&\quad + 2\,\frac{d}{dx_1}\rho^\pm(x_1,\alpha-\hat\alpha_j)\,\frac{\partial\hat\phi_{\ell,j}(x)}{\partial x_1} \;+\; \frac{d^2}{dx_1^2}\rho^\pm(x_1,\alpha-\hat\alpha_j)\,\hat\phi_{\ell,j}(x) \\
&=\; \pm\frac{1}{\pi}\,\frac{\partial\hat\phi_{\ell,j}(x)}{\partial x_1}\,\delta_{\alpha-\hat\alpha_j} \;+\; \Delta\tilde v^\pm_{\ell,j}(x,\alpha) \;+\; k^2 n(x)\tilde v^\pm_{\ell,j}(x,\alpha)
\end{aligned}
$$

for $\lambda_{\ell,j} \gtrless 0$ where $\tilde v^\pm_{\ell,j}(x,\alpha) = \rho^\pm(x_1,\alpha-\hat\alpha_j)\hat\phi_{\ell,j}(x)$. Substituting this into (35a) we obtain

$$
\begin{aligned}
&\Delta w_\alpha + k^2 n\, w_\alpha \;=\; -(Ff)(\cdot,\alpha) \\
&-\frac{1}{\pi}\sum_{j\in J}\delta_{\alpha-\hat\alpha_j}\sum_{\ell=1}^{m_j}(\operatorname{sign}\lambda_{\ell,j})\,a_{\ell,j}\,\frac{\partial\hat\phi_{\ell,j}}{\partial x_1} \;-\; \big[\Delta v(\cdot,\alpha) \;+\; k^2 n\, v(\cdot,\alpha)\big]
\end{aligned}
$$

with an obvious meaning of v.

Now the proof of orthogonality is not difficult anymore. Let $\alpha = \hat\alpha_{j_0}$ for some $j_0 \in J$ and $\hat\phi_{\ell_0,j_0} \in X_{j_0}$. Then $\int_{Q^\infty}\big[\Delta v(\cdot,\hat\alpha_{j_0}) + k^2 n\, v(\cdot,\hat\alpha_{j_0})\big]\,\overline{\hat\phi_{\ell_0,j_0}}\,dx$ vanishes by Green's second theorem. Furthermore,

$$
\begin{aligned}
&\frac{1}{\pi}\sum_{\ell=1}^{m_{j_0}}(\operatorname{sign}\lambda_{\ell,j_0})\,a_{\ell,j_0}\int\limits_{Q^\infty}\frac{\partial\hat\phi_{\ell,j_0}}{\partial x_1}\,\overline{\hat\phi_{\ell_0,j_0}}\,dx \\
=\;& \frac{1}{\pi}\,(\operatorname{sign}\lambda_{\ell_0,j_0})\,a_{\ell_0,j_0}\,\frac{i\lambda_{\ell_0,j_0}}{2} \;=\; -\int\limits_{Q} f(x)\,\overline{\hat\phi_{\ell_0,j_0}(x)}\,dx \\
=\;& -\int\limits_{Q^\infty}(Ff)(x,\hat\alpha_{j_0})\,\overline{\hat\phi_{\ell_0,j_0}(x)}\,dx
\end{aligned}
$$

by the properties of $\hat\phi_{\ell,j}$ from (7a), (7b) and the definition (33) of $a_{\ell,j}$. The last equation holds because f has support in Q. This ends the proof. $\square$

Therefore we have shown that the source problem (35a), ((35b) in the Floquet-Bloch space is solvable for all $\alpha \in [-1/2, 1/2]$. In order to apply the inverse Floquet-Bloch transform we have to show that the mapping $\alpha \mapsto w_\alpha$ from $[-1/2, 1/2]$ to $H^1(Q^H)$ is square integrable. This is not obvious. We show that it is even continuous. As in Remark 1.5 we transform again the dependence on α from the space to the differential equation. The operator $(T_\alpha u)(x) = e^{-i\alpha x_1}u(x)$ transforms $H^1_\alpha(Q)$ into $H^1_{per}(Q)$ (and also $H^1_\alpha(Q^\infty)$ into $H^1_{per}(Q^\infty)$) and the variational equation (37) into the task to determine

$\tilde w_\alpha \in H^1_{per}(Q)$ with

$$\int_Q [\nabla \tilde w_\alpha \cdot \nabla \overline{\psi} - 2i\,\alpha\,\frac{\partial \tilde w_\alpha}{\partial x_1}\,\overline{\psi} - (k^2 n - \alpha^2)\,\tilde w_\alpha\,\overline{\psi}]\,dx \;-\; \int_\Gamma \tilde\Lambda_\alpha(\tilde w_\alpha)\,\overline{\psi}\,ds \tag{39}$$

$$= \int_Q (T_\alpha F g)\,\overline{\psi}\,dx \;+\; \int_\Gamma \frac{\partial (T_\alpha w_{\alpha,g})}{\partial \nu}\,\overline{\psi}\,ds \quad \text{for all } \psi \in H^1_{per}(Q)\,,$$

where $\tilde\Lambda_\alpha = T_\alpha \Lambda_\alpha T_\alpha^{-1}$ is the periodic form of Λ_α (as in Part (C) of the proof of Theorem 2.2). We write this as $\tilde A_\alpha \tilde w_\alpha = \tilde r_\alpha$ in $H^1_{per}(Q)$, analogously to above. Let $\hat\alpha$ be some fixed exceptional value. (We drop the index.) In order to apply Theorem 5.1 of the Appendix we have to show that $\tilde r_\alpha$ and $\tilde A_\alpha$ are differentiable at $\hat\alpha$ with respect to α and that $P\frac{\partial \tilde A_{\hat\alpha}}{\partial \alpha}|_{\mathcal N(\tilde A_{\hat\alpha})}$ is bijective from the null space $\mathcal N(\tilde A_{\hat\alpha})$ of $\tilde A_{\hat\alpha}$ onto itself. Here, $P : H^1_{per}(Q) \to \mathcal N(\tilde A_{\hat\alpha})$ is the projection along the direct sum $H^1_{per}(Q) = \mathcal N(\tilde A_{\hat\alpha}) \oplus \mathcal R(\tilde A_{\hat\alpha})$. Differentiability is seen directly from the definitions because $\hat\alpha$ is not a cut-off value; that is, $|\ell + \hat\alpha| \neq k$ for all $\ell \in \mathbb Z$.

Let $w, \psi \in \mathcal N(\tilde A_{\hat\alpha})$. Then $T_{\hat\alpha} w, T_{\hat\alpha}\psi \in \mathcal N(A_{\hat\alpha})$, and w, ψ have therefore expansions in the forms

$$w(x) \;=\; \frac{1}{\sqrt{2\pi}} \sum_{|\ell+\hat\alpha|>k} w_\ell^\pm(\pm h_0)\, e^{-\sqrt{(\ell+\hat\alpha)^2-k^2}(\pm x_2 - h_0)}\, e^{i\ell x_1}\,,$$

$$\psi(x) \;=\; \frac{1}{\sqrt{2\pi}} \sum_{|\ell+\hat\alpha|>k} \psi_\ell^\pm(\pm h_0)\, e^{-\sqrt{(\ell+\hat\alpha)^2-k^2}(\pm x_2 - h_0)}\, e^{i\ell x_1}$$

for $\pm x_2 > h_0$. Therefore,

$$\int_\Gamma (\tilde\Lambda_\alpha w)\,\overline{\psi}\,ds \;=\; -\sum_{\sigma\in\{+1,-1\}} \sum_{|\ell+\hat\alpha|>k} \sqrt{(\ell+\hat\alpha)^2 - k^2}\, w_\ell^\pm(\sigma h_0)\,\overline{\psi_\ell^\pm(\sigma h_0)}\,.$$

From this form we observe that

$$\left(\frac{\partial \tilde A_{\hat\alpha}}{\partial \alpha} w\,,\,\psi\right)_{H^1_{per}(Q)} \;=\; 2\int_Q \left[-i\,\frac{\partial w}{\partial x_1} + \hat\alpha\, w\right]\overline{\psi}\,dx$$

$$+ \sum_{\sigma\in\{+1,-1\}} \sum_{|\ell+\hat\alpha|>k} \frac{\ell+\hat\alpha}{\sqrt{(\ell+\hat\alpha)^2-k^2}}\, w_\ell^\sigma(\sigma h_0)\,\overline{\psi_\ell^\sigma(\sigma h_0)}$$

Using the expansions of w and ψ we compute

$$2\int_{Q_{h_0}^+} \left[-i\,\frac{\partial w}{\partial x_1} + \hat\alpha\, w\right]\overline{\psi}\,dx \;=\; 2\sum_{|\ell+\hat\alpha|>k} (\ell+\hat\alpha)\, w_\ell^+(h_0)\,\overline{\psi_\ell^+(h_0)} \int_{h_0}^\infty e^{-2\sqrt{(\ell+\hat\alpha)^2-k^2}(x_2-h_0)} dx_2$$

$$= \sum_{|\ell+\hat\alpha|>k} \frac{\ell+\hat\alpha}{\sqrt{(\ell+\hat\alpha)^2-k^2}}\, w_\ell^+(h_0)\,\overline{\psi_\ell^+(h_0)}$$

and analogously for the integral over $Q_{h_0}^-$. Therefore,

$$\left(\frac{\partial \tilde{A}_{\hat\alpha}}{\partial\alpha} w\,,\,\psi\right)_{H^1_{per}(Q)} \;=\; 2\int\limits_{Q^\infty}\left[-i\,\frac{\partial w}{\partial x_1}+\alpha\,w\right]\overline{\psi}\,dx \;=\; -2i\int\limits_{Q^\infty}\frac{\partial}{\partial x_1}[e^{i\hat\alpha x_1}w]\,\overline{[e^{i\hat\alpha x_1}\psi]}\,dx$$

and $P\frac{\partial \tilde{A}_{\hat\alpha}}{\partial\alpha}w$ vanishes if, and only if, the last integral vanishes for all $\psi\in\mathcal{N}(\tilde{A}_{\hat\alpha})$. This implies that w vanishes by Assumption 1.4.

Therefore, all assumption of Theorem 5.1 of the Appendix are satisfied which yields that the unique solution $\tilde{w}_\alpha\in H^1_{per}(Q)$ for $\alpha\neq\hat\alpha$ can be continuously extended into $\hat\alpha$. Therefore, the mapping $\alpha\mapsto w_\alpha$ is continuous from $[-1/2,1/2]$ into $H^1(Q^H)$ (for every $H>h_0$) which implies that the inverse Floquet-Bloch transform $u^{(1)}=F^{-1}w_\alpha\in H^1(W_H)$ for every $H>h_0$. We have therefore shown the following result for the special case $q=0$:

Theorem 3.4. *Let Assumptions 1.1, 1.3, and 1.4 hold. Then there exists a unique solution $u\in H^1_{loc}(\mathbb{R}^2)$ of the source problem (8) satisfying the radiation condition of Definition 1.6. Furthermore, for every $H>h_0$ the mapping $f\mapsto u$ is bounded from $L^2(Q)$ into $H^1(W_H)$.*

Proof: It remains to study the case of a general q. Let $L:L^2(Q)\to H^1(Q)$ be the linear and bounded operator which maps $f\in L^2(Q)$ into $u|_Q$ where where u solves (8) for $q=0$ and the radiation condition. For arbitrary q the solution of (8) is equivalent to the fixpoint equation $u=L(f+k^2nqu)$ for $u\in L^2(Q)$. Since L is compact from $L^2(Q)$ into itself the uniqueness result of Section 2 yields existence. $\square$

4. The Asymptotic Behaviour of the Solution

It is well known (see, e.g. [9]) that for closed waveguides the solution decays exponentially as $|x_1|$ tends to infinity. This follows also from Theorem 5.1. Indeed, in this case the variational equation (39) holds in the space $\{\psi\in H^1_{per}(Q):\psi=0\text{ on }\Gamma\}$ (assuming Dirichlet boundary conditions on Γ) without the boundary terms involving the Dirichlet-to-Neumann map. This shows even analytic dependence of $\tilde{A}_\alpha$ and $\tilde{r}_\alpha$ on α. Therefore, from well known properties of analytic functions, continuity of $\alpha\mapsto w_\alpha$ at an exceptional value implies even analyticity which shows that the Floquet-Bloch transform is analytic in $[-1/2,1/2]$ which implies that $u^{(1)}$ itself decays exponentially. The situation is different in the case of an open waveguide because of the existence of cut-off values. They destroy the analytic dependence on α but, as we show next, allow a special form of Hölder continuity.

Let now $\hat\alpha\in[-1/2,1/2]$ be a cut-off value; that is, the set $L=\{\ell\in\mathbb{Z}:|\ell-\hat\alpha|=k\}$ is not empty. L can consist of one or two elements. Indeed, if we decompose k as $k=\hat\ell+\kappa$ with $\hat\ell\in\mathbb{N}_0$ and $\kappa\in(-1/2,1/2]$ then $L=\{(\operatorname{sign}\kappa)\hat\ell\}$ if $0<|\kappa|<\frac{1}{2}$ and $L=\{\hat\ell,-\hat\ell\}$ if $\hat\alpha=\kappa=0$ (then $\hat\ell\geq 1$) and $L=\{\hat\ell,-\hat\ell-1\}$ if $\hat\alpha=\kappa=\frac{1}{2}$ and $L=\{-\hat\ell,\hat\ell+1\}$ if $\hat\alpha=-\kappa=-\frac{1}{2}$.

Lemma 4.1. *Let Assumptions 1.1, 1.3, and 1.4 hold. Let $k=\hat\ell+\kappa$ for some $\hat\ell\in\mathbb{N}_0$ and $\kappa\in(-1/2,1/2]$. Then for every $\hat\alpha\in[-1/2,1/2]$ which is not a cut-off value and every $H>h_0$ there exists a neighborhood of $\hat\alpha$ in which the mapping $\alpha\mapsto(Fu^{(1)})(\cdot,\alpha)$ from $\mathbb{R}$ into $H^1(Q^H)$ is analytic.*[2]

[2] Here, $Q^H=(0,2\pi)\times(-H,H)$.

Furthermore, in a neighborhood $(\hat\alpha-\delta,\hat\alpha+\delta)$ of a cut-off value $\hat\alpha\in[-1/2,1/2]$ the transform $Fu^{(1)}$ has the form

$$(Fu^{(1)})(\cdot,\alpha) \;=\; \tilde u_0^{(1)}(\cdot,\alpha)+\sqrt{\hat\alpha-\alpha}\,\tilde u_+^{(1)}(\cdot,\alpha)+\sqrt{\alpha-\hat\alpha}\,\tilde u_-^{(1)}(\cdot,\alpha)+|\alpha-\hat\alpha|\,\tilde u_1^{(1)}(\cdot,\alpha)$$

with smooth (with respect to α) functions $\tilde u_0^{(1)}(\cdot,\alpha)$, $\tilde u_1^{(1)}(\cdot,\alpha)$, and $\tilde u_\pm^{(1)}(\cdot,\alpha)$ in a neighborhood of $\hat\alpha$.

Proof: We look again at the variational form (39) for the periodic transform $\tilde w_\alpha(x)=e^{-i\alpha x_1}(Fu^{(1)})(x,\alpha)$ which we write in the form

$$\begin{aligned}(40)\qquad &\int\limits_Q[\nabla\tilde w_\alpha\cdot\nabla\overline{\psi}-2i\,\alpha\,\frac{\partial\tilde w_\alpha}{\partial x_1}\,\overline{\psi}-(k^2n-\alpha^2)\,\tilde w_\alpha\,\overline{\psi}]\,dx\\ &-i\sum_{\sigma\in\{+1,-1\}}\sum_{\ell\in\mathbb{Z}}\sqrt{k^2-(\ell+\alpha)^2}\,w_\ell^\sigma(\sigma h_0)\,\overline{\psi_\ell^\sigma(\sigma h_0)}\\ =\;&\int\limits_Q e^{-i\alpha x_1}(Fg)(x,\alpha)\,\overline{\psi}(x)\,dx\;+\;\int\limits_\Gamma e^{-i\alpha x_1}\frac{\partial w_{\alpha,g}(x)}{\partial\nu}\,\overline{\psi(x)}\,ds\quad\text{for all }\psi\in H^1_{per}(Q)\,,\end{aligned}$$

where again $\frac{\partial w_{\alpha,g}}{\partial\nu}=\pm\frac{\partial w^\pm_{\alpha,g}}{\partial x_2}$ for $x_2=\pm h_0$ and $w^\pm_{\alpha,g}$ is given by (36) and $w_\ell^\pm(\pm h_0)$ and $\psi_\ell^\pm(\pm h_0)$ are Fourier coefficients of $\tilde w_\alpha$ and ψ, respectively, for $x_2=\pm h_0$.
Again, with the representation theorem of Riesz we write this $\tilde A_\alpha\tilde w_\alpha=\tilde r_\alpha$ in $H^1_{per}(Q)$.

Let first $\hat\alpha$ be not a cut-off value. Since the square root function is holomorphic in $\mathbb{C}\setminus i\mathbb{R}_{\le 0}$ we note that the function $\tilde r_\alpha$ and the operator $\tilde A_\alpha$ depend analytically on α in a neighborhood of $\hat\alpha$. We note that α is exceptional if, and only if, $\tilde A_\alpha$ fails to be invertible. Therefore, if $\hat\alpha$ is not exceptional then the (unique) solution $\tilde w_\alpha$ depends analytically on α in a neighborhood of $\hat\alpha$. If $\hat\alpha$ is exceptional then the assumptions of Theorem 5.1 of the Appendix are satisfied (as we have shown above) and, therefore, $\tilde w_\alpha$ has a continuous – and thus analytic – extension into $\hat\alpha$.

Now we study the case that α is in a neighborhood of a cut-off value $\hat\alpha\in[-1/2,1/2]$. We set $t_+(\alpha)=\sqrt{\hat\alpha-\alpha}$ and $t_-(\alpha)=\sqrt{\alpha-\hat\alpha}$. In order to treat all cases of the sets L simultaneously we set $\hat\ell_\pm=\pm\hat\ell$ in the case that $|\kappa|<1/2$ and $\hat\ell_+=\hat\ell$, $\hat\ell_-=-\hat\ell-1$ in the case that $\kappa=1/2$ and $\hat\alpha=1/2$ and $\hat\ell_+=\hat\ell+1$, $\hat\ell_-=-\hat\ell$ in the case that $\kappa=1/2$ and $\hat\alpha=-1/2$. In all cases we have that $\sqrt{k^2-(\hat\ell_\pm+\alpha)^2}=t_\pm(\alpha)\,\rho_\pm(\alpha)+\eta_\pm(\alpha)$ where $\rho_\pm$ and $\eta_\pm$ are analytic in a neighborhood of $\hat\alpha$.

First we consider the right hand side $\tilde r_\alpha$; that is, the right hand side of (40). The first term is analytic with respect to α. The second term involves $w^\pm_{\alpha,g}$ from (36). We split the Green's function $G^\pm_\alpha(x,y)$ into the series over $\ell\notin\{\hat\ell_+,\hat\ell_-\}$ and the terms $\ell=\hat\ell_+$ and $\ell=\hat\ell_-$. We consider G^+. The Taylor expansion yields

$$\begin{aligned}&\frac{e^{i\sqrt{k^2-(\hat\ell_\pm+\alpha)^2}(|x_2-y_2|}-e^{i\sqrt{k^2-(\hat\ell_\pm+\alpha)^2}(x_2+y_2-2h_0)}}{\sqrt{k^2-(\hat\ell_\pm+\alpha)^2}}\\ =\;&\tilde a_\pm(x_2,y_2,\alpha)+\sqrt{k^2-(\hat\ell_\pm+\alpha)^2}\,\tilde b_\pm(x_2,y_2,\alpha)\;=\;a_\pm(x_2,y_2,\alpha)+t_\pm(\alpha)\,b_\pm(x_2,y_2,\alpha)\end{aligned}$$

with functions $a_\pm$ and $b_\pm$ which depend analytically on α in a neighborhood of $\hat\alpha$. Substituting this decomposition (also for the lower boundary with G^-) into the form of $w_{\alpha,g}$ yields the right hand side $\tilde r_\alpha$ has the form $\tilde r_\alpha = \tilde r_{\alpha,0} + t_+(\alpha)\,\tilde r_{\alpha,+} + t_-(\alpha)\,\tilde r_{\alpha,-}$ where $\tilde r_{\alpha,0}$ and $\tilde r_{\alpha,\pm}$ depend analytically on α in a neighborhood of $\hat\alpha$. The same decomposition is done with the (periodic) Dirichlet-to-Neumann map $T_\alpha\Lambda_\alpha T_\alpha^{-1}$. We note that this operator consists of two components, which acts on $H^{1/2}(\Gamma_+)$ and $H^{1/2}(\Gamma_-)$ where $\Gamma_\pm = (0,2\pi)\times\{\pm h_0\}$. We write it as

$$\begin{aligned}
(T_\alpha\Lambda_\alpha T_\alpha^{-1}\phi)(x_1,\pm h_0) \;&=\; \frac{i}{\sqrt{2\pi}}\sum_{\ell\in\mathbb{Z}}\sqrt{k^2-(\ell+\alpha)^2}\,\phi_\ell(\pm h_0)\,e^{i\ell x_1}\\
=\;(\tilde\Lambda_\alpha^{\pm}\phi)(x_1,\pm h_0) \;+\; t_+(\alpha)\,&\frac{i\rho_+(\alpha)}{\sqrt{2\pi}}\,\phi_{\hat\ell_+}(\pm h_0)\,e^{i\hat\ell_+x_1} \;+\; t_-(\alpha)\,\frac{i\rho_-(\alpha)}{\sqrt{2\pi}}\,\phi_{\hat\ell_-}(\pm h_0)\,e^{i\hat\ell_-x_1}
\end{aligned}$$

This decomposition defines two two-dimensional operators $E_+(\alpha)$ and $E_-(\alpha)$ from $H^1_{per}(Q)$ into itself by

$$\big(E_\pm(\alpha)\phi,\psi\big)_{H^1(Q)} \;=\; i\rho_\pm(\alpha)\left[\phi_{\hat\ell_\pm}(h_0)\,\overline{\psi_{\hat\ell_\pm}(h_0)}+\phi_{\hat\ell_\pm}(-h_0)\,\overline{\psi_{\hat\ell_\pm}(-h_0)}\right],\ \ \phi,\psi\in H^1_{per}(Q)\,.$$

Then the operator $\tilde A_\alpha$ has a decomposition in the form

$$\tilde A_\alpha \;=\; B_\alpha \;+\; t_+(\alpha)\,E_+(\alpha) \;+\; t_-(\alpha)\,E_-(\alpha)$$

where B_α depends analtically on α. Then (37) is equivalent to

$$[B_\alpha + t_+(\alpha)\,E_+(\alpha) + t_-(\alpha)\,E_-(\alpha)]\,\tilde w_\alpha \;=\; \tilde r_{\alpha,0} + t_+(\alpha)\,\tilde r_{\alpha,+} + t_-(\alpha)\,\tilde r_{\alpha,-} \tag{41}$$

Since the cut-off value $\hat\alpha$ is not exceptional by Assumption 1.3 we conclude that $\tilde A_{\hat\alpha}$ is invertible and thus also B_α in a neighborhood of $\hat\alpha$. Since the operator on the left hand side of (41) is a small perturbation of $B_{\hat\alpha}$ the solution is given by the Neumann series as

$$\tilde w_\alpha = \sum_{m=0}^{\infty}(-1)^m[t_+(\alpha)\,B_\alpha^{-1}E_+(\alpha)+t_-(\alpha)\,B_\alpha^{-1}E_-(\alpha)]^m B_\alpha^{-1}[\tilde r_{\alpha,0}+t_+(\alpha)\,\tilde r_{\alpha,+}+t_-(\alpha)\,\tilde r_{\alpha,-}]\,.$$

Therefore, noting that $t_+(\alpha)^2$ and $t_-(\alpha)^2$ are analytic functions, the assertion follows by sorting this expression for $t_+(\alpha)$, $t_-(\alpha)$, and $t_+(\alpha)t_-(\alpha) = i|\alpha-\hat\alpha|$. $\square$

The following lemma will be needed:

Lemma 4.2. *For every $a>0$ and $\sigma\in\{+1,-1\}$*

$$\lim_{\sigma T\to\infty}\left[\sqrt{|T|}\int_0^a\frac{1}{\sqrt\alpha}\,e^{-iT\alpha}\,d\alpha\right] \;=\; (1-i\sigma)\sqrt{\frac{\pi}{2}}\,,$$

$$\lim_{\sigma T\to\infty}\left[\sqrt{|T|}\int_{-a}^a\frac{1}{\sqrt\alpha}\,e^{-iT\alpha}\,d\alpha\right] \;=\; \begin{cases}(1-i)\sqrt{2\pi}\,, & \sigma=1\,,\\ 0\,, & \sigma=-1\,.\end{cases}$$

Proof: Using the substitution $t=|T|\alpha=\sigma T\alpha$ the first formula follows from

$$\int_0^a\frac{1}{\sqrt\alpha}\,e^{-iT\alpha}\,d\alpha \;=\; \frac{1}{\sqrt{|T|}}\int_0^{a|T|}\frac{1}{\sqrt t}\,e^{-i\sigma t}\,dt \;=\; \frac{1}{\sqrt{|T|}}\int_0^{a|T|}\frac{\cos t}{\sqrt t}\,dt \;-\; i\sigma\frac{1}{\sqrt{|T|}}\int_0^{a|T|}\frac{\sin t}{\sqrt t}\,dt$$

and

$$\lim_{T\to\infty}\int_0^T \frac{\cos t}{\sqrt{t}}\,dt \;=\; \lim_{T\to\infty}\int_0^T \frac{\sin t}{\sqrt{t}}\,dt \;=\; \sqrt{\frac{\pi}{2}}\,.$$

For the second formula we note that

$$\begin{aligned}\int_{-a}^{a} \frac{1}{\sqrt{\alpha}}\,e^{-iT\alpha}\,d\alpha \;&=\; \int_0^a \frac{1}{\sqrt{\alpha}}e^{-iT\alpha}\,d\alpha \;+\; \frac{1}{i}\int_0^a \frac{1}{\sqrt{\alpha}}e^{iT\alpha}\,d\alpha \\ &=\; (1-i)\,\mathrm{Re}\int_0^a \frac{1}{\sqrt{\alpha}}e^{-iT\alpha}\,d\alpha \;-\; (1-i)\,\mathrm{Im}\int_0^a \frac{1}{\sqrt{\alpha}}e^{-iT\alpha}\,d\alpha\end{aligned}$$

which yields the second assertion. □

Theorem 4.3. *Let Assumptions 1.1, 1.3, and 1.4 hold. For all $H > h_0$ there exists $c > 0$ such that $\|u^{(1)}\|_{H^1(Q_\ell^H)} \le \frac{c}{|\ell|^{3/2}}$ for all $\ell \ge 1$. Here, $Q_\ell^H = (2\pi\ell, 2\pi(\ell+1)) \times (-H, H)$.*

Proof: For the different cases of exceptional values we define open sets I_1, I_2, and/or I_3 and corresponding functions $\psi_1, \psi_2, \psi_3 \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\psi_j \subset I_j$ as follows.
Case I: If $|\kappa| < \frac{1}{2}$ we define $I_1 = (-1/2-\varepsilon, 1/2+\varepsilon) \setminus \{\pm\kappa\}$, $I_2 = (\kappa-\varepsilon, \kappa+\varepsilon)$, and $I_3 = (-\kappa-\varepsilon, -\kappa+\varepsilon)$. (The latter only if $\kappa \neq 0$.) The functions ψ_j are chosen such that $\sum_j \psi_j(\alpha) = 1$ for all $\alpha \in [-1/2, 1/2]$ (partition of unity).
Case II: If $\kappa = 1/2$ we define $I_1 = (-\varepsilon, 1+\varepsilon) \setminus \{1/2\}$ and $I_2 = (1/2-\varepsilon, 1/2+\varepsilon)$. The functions ψ_1, ψ_2 are chosen such that $\psi_1(\alpha) + \psi_2(\alpha) = 1$ for all $\alpha \in [0, 1]$.
The inverse Floquet-Bloch transform is given by

$$u^{(1)}(x_1 + 2\pi\ell, x_2) \;=\; \int_I (Fu^{(1)})(x,\alpha)\,e^{i2\pi\ell\alpha}d\alpha \;=\; \sum_j \int_I \psi_j(\alpha)\,(Fu^{(1)})(x,\alpha)\,e^{i2\pi\ell\alpha}d\alpha$$

for $x \in Q^H$ and $\ell \in \mathbb{Z}$. Because of periodicity we can choose the interval of integration as $I = (-1/2, 1/2)$ which we do in the first case or $I = (0, 1)$ which we do in the case of $\kappa = 1/2$. In the following, however, we restrict ourselves to the first case. The second case is treated as the case $\kappa = 0$.
The integrand of the term containing ψ_1 vanishes in a neighborhood of the cut-off values $\pm\kappa$ and is therefore smooth. Furthermore, since $\psi_1 = 1$ in neighborhoods of $\pm 1/2$ and since $(Fu^{(1)})(x, \cdot)$ is $1-$periodic, partial integration (two times) yields

$$\left\| \int_{-1/2}^{1/2} \psi_1(\alpha)\,(Fu^{(1)})(\cdot,\alpha)\,e^{i2\pi\ell\alpha}\,d\alpha \right\|_{H^1(Q^H)} \;\le\; \frac{c}{\ell^2}\,.$$

Next, we consider the case containing ψ_j for $j \in \{2,3\}$; that is by Lemma 4.1,

$$\begin{aligned}
&\int_{-1/2}^{1/2} \psi_j(\alpha)\,(Fu^{(1)})(x,\alpha)\,e^{i2\pi\ell\alpha}d\alpha \\
&= \int_{-1/2}^{1/2} \psi_j(\alpha)\,\tilde u_0^{(1)}(x,\alpha)\,e^{i2\pi\ell\alpha}d\alpha \;+\; \int_{-1/2}^{1/2} \psi_j(\alpha)\,|\alpha-\hat\alpha|\,\tilde u_1^{(1)}(\cdot,\alpha)\,e^{i2\pi\ell\alpha}d\alpha \\
&\quad + \int_{-1/2}^{1/2} \psi_j(\alpha)\left[\sqrt{\hat\alpha-\alpha}\,\tilde u_+^{(1)}(x,\alpha)+\sqrt{\alpha-\hat\alpha}\,\tilde u_-^{(1)}(x,\alpha)\right]e^{i2\pi\ell\alpha}d\alpha
\end{aligned}$$

where $\hat\alpha=\kappa$ or $\hat\alpha=-\kappa$ if $j=2$ or $j=3$, respectively. Two times partial integration of the first term gives $\mathcal{O}(1/\ell^2)$ (note that ψ_j vanishes near $\pm 1/2$). Also the second term can be partially integrated twice and gives $\mathcal{O}(1/\ell^2)$. Partial integration of the third term yields

$$\begin{aligned}
&\int_{-1/2}^{1/2} \psi_j(\alpha)\left[\sqrt{\hat\alpha-\alpha}\,\tilde u_+^{(1)}(x,\alpha)+\sqrt{\alpha-\hat\alpha}\,\tilde u_-^{(1)}(x,\alpha)\right]e^{i2\pi\ell\alpha}d\alpha \\
&= -\frac{1}{i2\pi\ell}\int_{-1/2}^{1/2} \frac{\partial}{\partial\alpha}\left[\sqrt{\hat\alpha-\alpha}\,\psi_j(\alpha)\,\tilde u_+^{(1)}(x,\alpha)+\sqrt{\alpha-\hat\alpha}\,\psi_j(\alpha)\,\tilde u_-^{(1)}(x,\alpha)\right]e^{i2\pi\ell\alpha}d\alpha \\
&= \frac{1}{i4\pi\ell}\int_{\hat\alpha-\varepsilon}^{\hat\alpha+\varepsilon}\left(\frac{1}{\sqrt{\hat\alpha-\alpha}}\,\psi_j(\alpha)\,\tilde u_+^{(1)}(x,\alpha)-\frac{1}{\sqrt{\alpha-\hat\alpha}}\,\psi_j(\alpha)\,\tilde u_-^{(1)}(x,\alpha)\right)e^{i2\pi\ell\alpha}d\alpha \\
&\quad -\frac{1}{i2\pi\ell}\int_{\hat\alpha-\varepsilon}^{\hat\alpha+\varepsilon}\left(\sqrt{\hat\alpha-\alpha}\,\frac{\partial}{\partial\alpha}\left[\psi_j(\alpha)\,\tilde u_+^{(1)}(x,\alpha)+\sqrt{\alpha-\hat\alpha}\,\frac{\partial}{\partial\alpha}\left[\psi_j(\alpha)\,\tilde u_-^{(1)}(x,\alpha)\right]\right)e^{i2\pi\ell\alpha}d\alpha\,.
\end{aligned}$$

The second integral on the right hand side is again of order $\mathcal{O}(1/\ell^2)$. For the first integral we write

$$\begin{aligned}
&\int_{\hat\alpha-\varepsilon}^{\hat\alpha+\varepsilon}\left(\frac{1}{\sqrt{\hat\alpha-\alpha}}\,\psi_j(\alpha)\,\tilde u_+^{(1)}(x,\alpha)-\frac{1}{\sqrt{\alpha-\hat\alpha}}\,\psi_j(\alpha)\,\tilde u_-^{(1)}(x,\alpha)\right)e^{i2\pi\ell\alpha}d\alpha \\
&= \tilde u_+^{(1)}(x,\hat\alpha)\int_{\hat\alpha-\varepsilon}^{\hat\alpha+\varepsilon}\frac{1}{\sqrt{\hat\alpha-\alpha}}\,e^{i2\pi\ell\alpha}d\alpha \;-\; \tilde u_-^{(1)}(x,\hat\alpha)\int_{\hat\alpha-\varepsilon}^{\hat\alpha+\varepsilon}\frac{1}{\sqrt{\alpha-\hat\alpha}}\,e^{i2\pi\ell\alpha}d\alpha \;+\; \int_{\hat\alpha-\varepsilon}^{\hat\alpha+\varepsilon} v(x,\alpha)\,e^{i2\pi\ell\alpha}d\alpha
\end{aligned}$$

with

$$v(x,\alpha)=\frac{1}{\sqrt{\hat\alpha-\alpha}}\left[\psi_j(\alpha)\,\tilde u_+^{(1)}(x,\alpha)-\tilde u_+^{(1)}(x,\hat\alpha)\right]\;-\;\frac{1}{\sqrt{\alpha-\hat\alpha}}\left[\psi_j(\alpha)\,\tilde u_-^{(1)}(x,\alpha)-\tilde u_-^{(1)}(x,\hat\alpha)\right].$$

We show that $v \in W^{1,1}\big((-1/2,1/2), H^1(Q^H)\big)$. Indeed, for the first term, which we denote by $v_1(x,\alpha)$ we compute

$$\begin{aligned}\frac{\partial v_1(x,\alpha)}{\partial\alpha} &= \frac{1}{2(\hat\alpha-\alpha)^{3/2}}\left[\psi_j(\alpha)\,\tilde u_+^{(1)}(x,\alpha) - \tilde u_+^{(1)}(x,\hat\alpha)\right] \\ &\quad + \frac{1}{\sqrt{\hat\alpha-\alpha}}\,\frac{\partial}{\partial\alpha}\left[\psi_j(\alpha)\,\tilde u_+^{(1)}(x,\alpha)\right].\end{aligned}$$

We estimate (note that $\psi_j(\hat\alpha)=1$)

$$\begin{aligned}&\frac{1}{|\alpha-\hat\alpha|}\left\|\psi_j(\alpha)\,\tilde u_+^{(1)}(\cdot,\alpha) - \tilde u_+^{(1)}(\cdot,\hat\alpha)\right\|_{H^1(Q^H)} \\ = \;& \frac{1}{|\alpha-\hat\alpha|}\left\|\int_{\hat\alpha}^{\alpha}\frac{\partial}{\partial\beta}\left[\psi_j(\beta)\,\tilde u_+^{(1)}(\cdot,\beta)\right]d\beta\right\|_{H^1(Q^H)} \;\le\; \max_\beta\left\|\frac{\partial}{\partial\beta}\left[\psi_j(\beta)\,\tilde u_+^{(1)}(\cdot,\beta)\right]\right\|_{H^1(Q^H)}.\end{aligned}$$

This shows that $\partial v_1/\partial\alpha$ satisfies an estimate of the form $\|\partial v_1(\cdot,\alpha)/\partial\alpha\|_{H^1(Q^H)} \le c/\sqrt{|\alpha-\hat\alpha|}$. The second integral is estimated in the same way. Therefore, also the integral $\int_{-1/2}^{1/2} v(x,\alpha)\,e^{i2\pi\ell\alpha}d\alpha$ is of order $\mathcal{O}(1/|\ell|)$ by partial integration. Finally, we compute

$$\begin{aligned}&e^{-i2\pi\ell\hat\alpha}\sqrt{2\pi|\ell|}\int_{\hat\alpha-\varepsilon}^{\hat\alpha+\varepsilon}\frac{1}{\sqrt{\hat\alpha-\alpha}}\,e^{i2\pi\ell\alpha}d\alpha \\ = \;& \sqrt{2\pi|\ell|}\int_{-\varepsilon}^{\varepsilon}\frac{1}{\sqrt{\alpha}}\,e^{-i2\pi\ell\alpha}d\alpha \;\longrightarrow\; \begin{cases}(1-i)\sqrt{2\pi}\,, & \ell\to\infty\,,\\ 0\,, & \ell\to-\infty\,,\end{cases}\end{aligned}$$

and analogously

$$e^{-i2\pi\ell\hat\alpha}\sqrt{2\pi|\ell|}\int_{\hat\alpha-\varepsilon}^{\hat\alpha+\varepsilon}\frac{1}{\sqrt{\alpha-\hat\alpha}}\,e^{i2\pi\ell\alpha}d\alpha \;\longrightarrow\; \begin{cases}0\,, & \ell\to\infty\,,\\ (1-i)\sqrt{2\pi}\,, & \ell\to-\infty\,.\end{cases}$$

Therefore, we conclude that

$$\lim_{\ell\to\pm\infty}\left[|\ell|^{3/2}e^{-i2\pi\ell\hat\alpha}\int_{-1/2}^{1/2}\psi_j(\alpha)\,(Fu^{(1)})(\cdot,\alpha)\,e^{i2\pi\ell\alpha}d\alpha\right] \;=\; -\frac{1+i}{4\pi}\,e^{i2\pi\hat\alpha}\,\tilde u_\pm^{(1)}(\cdot,\hat\alpha)$$

in $H^1(Q^H)$. □

Theorem 4.4. *Let Assumptions 1.1, 1.3, and 1.4 hold. For every $R>0$ we have that*

$$u^{(1)}(x) \;=\; \mathcal{O}(1/\sqrt{|x_2|}) \quad \text{as } |x_2|\to\infty$$

uniformly with respect to $|x_1|\le R$.

Proof: Let again $k=\hat\ell+\kappa$ with $\hat\ell\in\mathbb{N}_0$ and $\kappa\in(-1/2,1/2]$. As in Lemma 1.8 we decompose $u^{(1)}$ in the half planes $x_2>h_0$ and $x_2<-h_0$, respectively, in the forms $u^{(1)}=u_0^\pm+\sum_{\ell,j}a_{\ell,j}v_{\ell,j}^\pm$ where $u_0^\pm$ solves $\Delta u_0^\pm+k^2u_0^\pm=0$ for $\pm x_2>h_0$ and $u_0^\pm=u^{(1)}$ for $x_2=\pm h_0$ and $v_{\ell,j}^+$ is given by (14a) ($v_{\ell,j}^-$ is defined analogously). The functions $v_{\ell,j}^\pm$ decay as $\mathcal{O}(1/\sqrt{|x_2|})$ by Lemma 1.8. It remains to consider the part $u_0^\pm$. We restrict ourselves to the upper half plane and write u_0^+ as

$$u_0^+(x_1+2\pi m, x_2) \;=\; \int\limits_{-1/2}^{1/2} (Fu_0^{(1)})(x,\alpha)\, e^{i2\pi m\alpha} d\alpha\,, \quad m\in\mathbb{Z}\,,\ x_2>h_0\,,$$

where $Fu_0^{(1)}$ denotes again the Floquet-Bloch transform of u_0^+. Lemma 4.1 implies that the trace $Fu^{(1)}|_\Gamma \in C^1\big([-1/2,1/2]\setminus\{\kappa,-\kappa\}, H^{1/2}(\Gamma)\big)\cap W^{1,1}\big((-1/2,1/2), H^{1/2}(\Gamma)\big)$. Therefore, also $\tilde u_0^+ \in C^1\big([-1/2,1/2]\setminus\{\kappa,-\kappa\}, H^1(Q^H\setminus Q)\big)\cap W^{1,1}\big((-1/2,1/2), H^1(Q^H\setminus Q)\big)$ for every $H>h_0$. Therefore, partial integration with respect to α is possible. For $x_2>h_0$ the solution $Fu_0^{(1)}$ is given by the Rayleigh expansion as

$$(Fu_0^{(1)})(x,\alpha) \;=\; \frac{1}{\sqrt{2\pi}}\sum_{\ell\in\mathbb{Z}} \tilde u_\ell(\alpha)\, e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2-h_0)+i(\ell+\alpha)x_1}$$

where $\tilde u_\ell(\alpha) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi}(Fu_0^{(1)})(x_1,h_0,\alpha)\exp(-i(\ell+\alpha)x_1)\,dx_1$ are the Fourier coefficients of $(Fu_0^{(1)})(\cdot,h_0,\alpha)$. Therefore,

$$u_0^+(x_1+2\pi m, x_2) \;=\; \sum_{\ell\in\mathbb{Z}}\int\limits_{-1/2}^{1/2} \tilde u_\ell(\alpha)\, e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2-h_0)+i(\ell+\alpha)x_1}\, e^{i2\pi m\alpha}\, d\alpha\,.$$

Let $m,\ell\in\mathbb{Z}$ and $x\in Q^H\setminus Q$. We set $\phi(\alpha,x_1)=\tilde u_\ell(\alpha)\,e^{i(\ell+\alpha)x_1}e^{i2\pi m\alpha}$ for abbreviation, and consider several cases.

Case I: $0<|\ell|\le\hat\ell+2$ and $\ell\ne\hat\ell$. Then either $|\ell+\alpha|<k$ for all $|\alpha|\le\frac12$ or $|\ell+\alpha|>k$ for all $|\alpha|\le\frac12$. Both cases are treated in the same way. Let $|\ell+\alpha|<k$ for all $|\alpha|\le\frac12$. Then we make the transform $t=t(\alpha)=\sqrt{k^2-(\ell+\alpha)^2}$. It is smooth and $t'(\alpha)=-\frac{\ell+\alpha}{\sqrt{k^2-(\ell+\alpha)^2}}$ which does not vanish because $\ell\ne 0$. Therefore, the transform is regular and $\alpha=\sqrt{k^2-t^2}-\ell$ and $d\alpha=-\frac{t}{\sqrt{k^2-t^2}}\,dt$. We get

$$\int\limits_{-1/2}^{1/2}\phi(\alpha,\cdot)\,e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2-h_0)}\,d\alpha \;=\; \int\limits_{\sqrt{k^2-(\ell+1/2)^2}}^{\sqrt{k^2-(\ell-1/2)^2}} \frac{\phi(\sqrt{k^2-t^2}-\ell,\cdot)\,t}{\sqrt{k^2-t^2}}\,e^{it(x_2-h_0)}\,dt\,.$$

As mentioned above we can apply partial integration and get an estimate of the form $\mathcal{O}(1/x_2)$ uniformly with respect to $x_1\in[0,2\pi]$ and m from bounded sets and $|\ell|\le\hat\ell+2$. This finishes the part $|\ell+\alpha|<k$ of Case I. If $|\ell+\alpha|>k$ for all $|\alpha|\le\frac12$ then one uses the transform $t=\sqrt{(\ell+\alpha)^2-k^2}$ and procceeds as before.

Case II: $\ell = 0$. Then, with the same substitution and $\alpha_0 = \min\{k, 1/2\}$,

$$\int\limits_{\alpha_0}^{\alpha_0} \phi(\alpha,\cdot)\, e^{i\sqrt{k^2-\alpha^2}(x_2-h_0)}\, d\alpha$$

$$= \int\limits_{0}^{\alpha_0} [\phi(\alpha,\cdot)+\phi(-\alpha,\cdot)]\, e^{i\sqrt{k^2-\alpha^2}(x_2-h_0)}\, d\alpha \;=\; \int\limits_{\sqrt{k^2-\alpha_0^2}}^{k} \frac{\tilde{\phi}(\sqrt{k^2-t^2},\cdot)\, t}{\sqrt{k^2-t^2}}\, e^{it(x_2-h_0)}\, dt$$

$$= \frac{\tilde{\phi}(0,\cdot)\, k}{\sqrt{2k}} \int\limits_{\sqrt{k^2-\alpha_0^2}}^{k} \frac{1}{\sqrt{k-t}}\, e^{it(x_2-h_0)}\, dt$$

$$+ \int\limits_{\sqrt{k^2-\alpha_0^2}}^{k} \frac{1}{\sqrt{k-t}} \left[\frac{\tilde{\phi}(\sqrt{k^2-t^2},\cdot)\, t}{\sqrt{k+t}} - \frac{\tilde{\phi}(0,\cdot)\, k}{\sqrt{2k}} \right] e^{it(x_2-h_0)}\, dt$$

where we have set $\tilde{\phi}(\alpha,\cdot) = \phi(\alpha,\cdot)+\phi(-\alpha,\cdot)$. Since $\tilde{\phi}$ is smooth at $\alpha = 0$ we observe that the term in the brackets $[\cdots]$ is smooth as well and vanishes at $t = k$. Therefore, partial integration implies that the second term behaves as $\mathcal{O}(1/x_2)$ as $x_2 \to \infty$ uniformly with respect to x_1 and m from bounded sets. By Lemma 4.2 the first integral behaves as

$$\sqrt{x_2}\, e^{-ik(x_2-h_0)} \int\limits_{\sqrt{k^2-\alpha_0^2}}^{k} \frac{1}{\sqrt{k-t}}\, e^{it(x_2-h_0)}\, dt \;=\; \sqrt{x_2} \int\limits_{0}^{k-\sqrt{k^2-\alpha_0^2}} \frac{1}{\sqrt{s}}\, e^{-is(x_2-h_0)}\, dt \;\to\; (1-i)\sqrt{\frac{\pi}{2}}$$

as $x_2 \to \infty$. Together one gets an estimate of $\mathcal{O}(1/\sqrt{x_2})$. Analogously, if $\alpha_0 = k < 1/2$,

$$\int\limits_{|\alpha|>k} \phi(\alpha,\cdot)\, e^{i\sqrt{k^2-\alpha^2}(x_2-h_0)}\, d\alpha$$

$$= \int\limits_{k}^{1/2} [\phi(\alpha,\cdot)+\phi(-\alpha,\cdot)]\, e^{-\sqrt{\alpha^2-k^2}(x_2-h_0)}\, d\alpha \;=\; \int\limits_{0}^{\sqrt{1/4-k^2}} \frac{\tilde{\phi}(\sqrt{k^2+t^2},\cdot)\, t}{\sqrt{k^2+t^2}}\, e^{-t(x_2-h_0)}\, dt$$

which behaves as $\mathcal{O}(1/x_2)$.

Case III: $\ell = \hat{\ell} \geq 1$. Then we have to distinguish between $\alpha \leq \kappa$ and $\alpha > \kappa$ (since $|\ell+\alpha| = \hat{\ell}+\alpha$). With $t = \sqrt{k^2-(\hat{\ell}+\alpha)^2}$ the first case is transfomed to

$$\int\limits_{-1/2}^{\kappa} \phi(\alpha,\cdot)\, e^{i\sqrt{k^2-(\hat{\ell}+\alpha)^2}(x_2-h_0)}\, d\alpha \;=\; \int\limits_{0}^{\sqrt{k^2-(\hat{\ell}-1/2)^2}} \frac{\phi(\sqrt{k^2-t^2}-\hat{\ell},\cdot)\, t}{\sqrt{k^2-t^2}}\, e^{it(x_2-h_0)}\, dt$$

while with $t = \sqrt{(\hat\ell+\alpha)^2 - k^2}$ the second case is transfomed to

$$\int\limits_\kappa^{-1/2} \phi(\alpha,\cdot)\, e^{-\sqrt{(\hat\ell+\alpha)^2-k^2}(x_2-h_0)}\,d\alpha \;=\; \int\limits_0^{\sqrt{(\hat\ell-1/2)^2-k^2}} \frac{\phi(\sqrt{k^2+t^2}-\hat\ell,\cdot)\,t}{\sqrt{k^2+t^2}}\, e^{-t(x_2-h_0)}\,dt\,.$$

Both terms decay as $\mathcal{O}(1/x_2)$.

Case IV: $|\ell| \geq \hat\ell + 2$. Then $|\ell+\alpha|^2 - k^2 = (|\ell+\alpha|-k)(|\ell+\alpha|+k) \geq [|\ell| - (k+1/2)]^2 \geq [|\ell| - (\hat\ell+1)]^2$ and thus

$$\begin{aligned} &\sum_{|\ell|\geq\hat\ell+2} \Bigg\| \int\limits_{-1/2}^{1/2} \phi(\alpha,\cdot)\, e^{-\sqrt{(\hat\ell+\alpha)^2-k^2}(x_2-h_0)}\,d\alpha \Bigg\|_\infty \\ &\leq\; 2\sum_{\ell=\hat\ell+2}^{\infty} \int\limits_{-1/2}^{1/2} \|\phi(\alpha,\cdot)\|_\infty\, e^{-[\ell-(\hat\ell+1)](x_2-h_0)}\,dt \;\leq\; c \max_{|\alpha|\leq 1/2} \|\phi(\alpha,\cdot)\|_\infty\, e^{-(x_2-h_0)}\,. \end{aligned}$$

This ends the proof because the cases I–III appear only finitely often. $\square$

5. Appendix

The following result is a special case of a slightly more general result of Colton and Kress (see Section 1.4 in [7]).

Theorem 5.1. *Let X be a Banach space, $I \subset \mathbb{R}$ an open interval and $r(\alpha,\cdot) \in X$ and $K(\alpha,\cdot) \in \mathcal{K}(X)$ for $\alpha \in I$ a family of linear and compact operators*[3] *such that $r \in C^1(\hat I, X)$ and $K \in C^1\big(\hat I, \mathcal{K}(X)\big)$, respectively, where $\hat I = (\hat\alpha - \varepsilon, \hat\alpha + \varepsilon) \subset I$ for some $\hat\alpha \in I$ and $\varepsilon > 0$.*

Let $I - K(\alpha,\cdot)$ be bijective for $\alpha \neq \hat\alpha$ but $\mathcal{N}\big(I - K(\hat\alpha,\cdot)\big) \neq \{0\}$. Let the Riesz number of $I - K(\hat\alpha,\cdot)$ be one; that is, $\mathcal{N}\big((I - K(\hat\alpha,\cdot))^2\big) = \mathcal{N}\big(I - K(\hat\alpha,\cdot)\big)$ and $P : X \to \mathcal{N} := \mathcal{N}\big(I - K(\hat\alpha,\cdot)\big)$ be the projection operator onto the null space with respect to the direct sum $X = \mathcal{N} \oplus \mathcal{R}\big(I - K(\hat\alpha,\cdot)\big)$. Assume, furthermore, that $P\frac{\partial}{\partial\alpha}K(\hat\alpha,\cdot)\big|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$ is one-to-one and $r(\hat\alpha,\cdot) \in \mathcal{R}\big(I - K(\hat\alpha,\cdot)\big)$.

Then the unique solution $u(\alpha,\cdot) \in X$ of $\big(I - K(\alpha,\cdot)\big)u(\alpha,\cdot) = r(\alpha,\cdot)$ for $\alpha \neq \hat\alpha$ converges to a solution $u(\hat\alpha,\cdot)$ of $\big(I - K(\hat\alpha,\cdot)\big)u(\hat\alpha,\cdot) = r(\hat\alpha,\cdot)$. In other words, $u \in C(\hat I, X)$ where $\varepsilon > 0$ in the definition of $\hat I$ is possibly smaller than the original one. Furthermore, the mapping $r \mapsto u$ is bounded from $C^1(\hat I, X)$ into $C(\hat I, X)$.

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