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Global Asymptotic Synchronization of Fractional Order Multi-linked Memristive Neural Networks with Time-varying Delays via Discontinuous Control †

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Summary

In this paper, we address the global asymptotic synchronization (GAS) problem of the Master-Slave fractional order multi-linked memristive neural networks (FOMMNNs). Firstly, we propose the FOMMNNs which incorporate the fractional calculus into multi-linked memristive neural networks (MMNNs) for the rst time. Then, by utilizing the fractional dierential inclusions and set-valued mapping theories, the addressed FOMMNNs with discontinuous state switching at the right-hand side and time-varying delays are converted into the continuous FOMMNNs. Under the frameworks of fractional Caputo derivative and fractional Fillipov solution, by the way of building up appropriate Lyapunov functionals and utilizing some synchronous analysis technology, several sucient criteria ensuring that the Master-Slave FOMMNNs can realize global asymptotic synchronization (GAS) under two dierent state-feedback controllers are obtained.

KEYWORDS:
Asymptotic synchronization, fractional order, multi-linked memristive neural networks (MMNNs), adaptive controller, time-varying delays, state-feedback

1 | INTRODUCTION

Memristor, which was put forward theoretically by Chua in 1971 [1], is considered to be one of the four basic elements along with resistors, capacitors and inductors. Its resistance relies on the amount of electricity passed through. If the power is cut off, the resistance will remain unchanged until it receives the reverse current, thus it could act as a memory element. And in 2008, Strukov et al. fabricated the first memristor based on TiO2 model [2]. Since then, many kinds of memristor models are proposed by the scholars. Compared with the resistors, the memristors exibit many advantageous characteristics, such as extremely low energy consumption, excellent scalabilities, fast-speed operation, etc. Particularly, the memristor exhibits excellent memory and pinched hysteresis characteristics and has variable resistance, which bears striking resemblance to biological synapses [3]. Thus, the memristor is one of the most promising condidates for artificial synapses emulating memory and brain functions. The researchers proposed the memristive neural networks (MNNs) by rebuilding the traditional artificial NNs by utilizing the memristors rather than the common resistors. Besides, since the conductance of memristor will change owning to the change of the

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voltage pulse parameter value of the high-speed voltage flow. Therefore, the connection weights of the MNNs are discontinuous, and the MNNs system is a discontinuous system.

MNNs have attracted increasing attention from researchers in various fields because of their excellent application prospects, their dynamical behaviors, such as synchronization and stabilization, can be applied in pattern recognition [4], image processing [5,6,7], optimization [8,9], and so forth. Therefore, more and more scholars focus on investigating the synchronization and stabilization of MNNs. Many excellent results have been reported, please refer to the works [10,11,12,13,14,15,16,17,18,19] and references therein.

Fractional calculus [20] studies the integration and differentiation of arbitrary real order or even complex order, is a mathematical tool extended from ordinary calculus. Fractional differential systems can be applied widely in both science and engineering applications due to its excellent memory and hereditary characteristics. Compared with ordinary differential system, FDEs represent a more reasonable mathematical modeling framework, which can more effectively and accurately describe the memory and genetic characteristics of processes and materials, and etc. By introducing fractional order calculus into MNNs [21,22,23], the scholars constructed the fractional order memristive neural networks (FOMNNs), which not only contain the characteristics of the fractional differentital systems (FDEs), but also the characteristics of the MNNs. It is an evolutionary version of fractional order neural networks (FONNs). FOMNNs can provide an effective instrument to model some irregular dynamics simply and accurately by adopting fractional derivatives and the characteristics of the memristor. Therefore, using fractional order memristive neural networks (FOMNNs) can solve more complex problems that fractional order neural networks (FONNs) and MNNs cannot solve.

The application of various FOMNNs are mainly based on their dynamical behaviors, such as stabilization and synchronization, which have been extensively studied by the scholars recently, please refer to [24,25,26,13,27] and references therein. The researchers have proposed many kinds of control strategies [28,26,29,30,25,31,25] and various FOMNNs models. For example, complex-valued FOMNNs were studied in [32,33,34,16,35,36]. FOMNNs with reaction-diffusion terms were studied in [30,37]. Quaternion-valued FOMNNs were investigated in [38,13,39,40].

As we know, one neuron can transmit information with the next one through multiple synapses. In other words, there are many connections between two neurons in the real biological neural network s. Here, we regard one connection between two neurons as an edge and there exist mutiple edges between two neruons of biological neural networks of the brain. Multi-linked memristive neural network (MMNNs) mentioned in [41] are comprised of several single-linked MNNs, each edge indicating a kind of transmission delay. We can naturally introduce fractional derivatives into multi-linked memristive neural networks to form fractional order multi-linked memristive neural networks (FOMMNNs). Although there are many significant results concerning about the synchronization and stabilization for memristive neural network (MNNs), there are few researches concerning on FOMMNNs. For example, in [28], the synchronization of the delayed fractional order memristive BAM neural networks involving switching jumps mismatch were obtained in a finite time by adopting impulsive controller and utilizaing comparison principle. What's more, the delayed complex-valued FOMBAMNNs were investigated in [27,35]. To our best knowledge, there exist no literature concerning about the synchronization and stabilization of the FOMMNNs. Few results about the synchronization and stabilization issue of FOMNNs involving time-varying delays are obtained, most of the published reports are discussing constant delay(s) or without delay terms. For example, FOMNNs with constant delay(s) were investigated in [42,43,24,44]. FOMNNs with time-varying delay was studied in [27,14,16,28]. The global synchronization of the delayed FOMNNs involving constant delay were studied in [42]. In [13], Pratap et al. addressed the finite time stability of the impulsive quaternion-valued FOMNNs in the Mittag-Leffler sense by employing Mittage-Leffler function and Laplace transform.

Sparked by the discussions mentioned above, this article firstly incoporates fractional calculus into MMNNs and obtain the FOMMNNs, then addresses the asymptotic synchronization and stabilization of the FOMMNNs involving time-varying delays by designing state-feedback controllers under the frameworks of Caputo fractional derivative and fractional Fillipov solution.

Contributions are stated as follows

(1) We propose fractional order multi-linked memristive neural networks (FOMMNNs) with time varying delays for the first time.

(2) Two different control strategies are adopted. One is the linear control strategy, the other is the adaptive control strategy. Our proof procedure is simple to be implemented in the real applications.

(3) We design two different Lyapunov-Krasovskii functionals. Sufficient criteria guaranteeing the GAS of the Master-Slave FOMMNNs are obtained under the Filippov-framework by utilizing the theory of set-valued mapping.

The remaining of this paper is given as follows. Section 2 gives the description of the mathematical model of the Master-Slave FOMMNNs, including some necessary preliminaries, relative tools and notations. Section 3 analyzes the sufficient criteria of

GAS the Master-Slave FOMMNNs with time-varying delays, and two corollaries are obtained for the FOMMNNs involving constant delays and no-delayed terms, respectively. Section 4 gives the conclusions.

Notation. $\mathcal{R}$, $\mathcal{R}^{\mathfrak{N}_1\times\mathfrak{N}_2}$ and $\mathcal{R}^{\mathfrak{N}}$ represent the space of real numbers, the set of matrices with dimensions $\mathfrak{N}_1 \times \mathfrak{N}_2$ and $\mathfrak{N}$-dimensional vector, respectively. $\mathfrak{Q}^T$ denotes the transpose of matrix $\mathfrak{Q}$. $\mathbb{C}([-\varsigma_M, 0], \mathcal{R}^n)$ indicates the functions set Ψ : $[-\varsigma_M, 0] \to \mathcal{R}^n$ and Ψ is bounded and differential. $\overline{co}[\check{\mathcal{Q}}, \hat{\mathcal{Q}}]$ indicates the closure of convex hull generated by the real numbers set or real matrices set including $\check{\mathcal{Q}}$ and $\hat{\mathcal{Q}}$.

2 | NETWORK MODEL AND PRELIMINARIES

Throuhout the paper, we consider the GAS between the Master-Slave FOMMNNs with time-varying delays in Caputo sense with fractional order $\hat{\beta} \in (0, 1)$, denoted as ${}^{C}_{t_0}\mathcal{D}^{\hat{\beta}}_{t}$, where t_0 is the initial time. For convinience in this paper, we denote ${}^{C}_{t_0}\mathcal{D}^{\hat{\beta}}_{t}$ as $\mathcal{D}^{\hat{\beta}}_{t}$ for short.

Definition 1. (Podlubny[20]) For a continuous function $\vartheta(t)$, its Caputo derivative with fractional order $\hat{\beta} \in (p-1, p)$ is defined as

$$ {}^{C}_{t_0}\mathcal{D}^{\hat{\beta}}_{t}\vartheta(t) = \frac{1}{\Gamma(p-\hat{\beta})}\int_{t_0}^{t}(t-\lambda)^{p-\hat{\beta}-1}\vartheta^{(m)}(\lambda)d\lambda, \ t \geq t_0, $$

where $p \in \mathbb{R}^+$ and $p \in (\hat{\beta}, \hat{\beta}+1)$.

Definition 2. (Podlubny[20]) For a differential and continuous function $\vartheta(\mathcal{T})$, its Caputo fractional integral of order $\hat{\beta}$ is given as

$$ {}^{C}_{\mathcal{T}_0}\mathcal{D}^{\hat{\beta}}_{\mathcal{T}}\vartheta(\mathcal{T}) = \frac{1}{\Gamma(\hat{\beta})}\int_{\mathcal{T}_0}^{\mathcal{T}}(\mathcal{T}-\lambda)^{\hat{\beta}-1}\vartheta(\lambda)d\lambda, \ \mathcal{T} \geq \mathcal{T}_0, $$

where $\Gamma(\hat{\beta}) = \int_0^{+\infty} \lambda^{\hat{\beta}-1}e^{-\lambda}d\lambda$.

2.1 | Description of the Master-Slave FOMMNNs

2.1.1 | The Master FOMMNNs

Consider the master FOMMNNs with time-varying delays

$$ \begin{aligned} \mathcal{D}^{\hat{\beta}}_{t}\mathfrak{X}_m(t) = &-\ell_m(\mathfrak{X}_m(t))\mathfrak{X}_m(t) + \sum_{n=1}^{\Pi} \upsilon_{mn}(\mathfrak{X}_m(t))\hbar_n(\mathfrak{X}_n(t)) \\ &+ \sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} \varrho_{(\iota)mn}(\mathfrak{X}_m(t))g_n(\mathfrak{X}_n(t-\varsigma_\iota(t))) + \mathcal{I}_m(t), \ t \geq 0, \end{aligned} \tag{1} $$

where $m = 1, 2, \ldots, \Pi$, $0 < \hat{\beta} < 1$, $\mathcal{D}$ represents Caputo fractional operator, $\mathcal{D}^{\hat{\beta}}_{t}$ denotes Caputo fractional derivative with order $\hat{\beta}$ $(0 < \hat{\beta} < 1)$ between time (t_0, t), $\mathfrak{X}_m(t)$ stands for the state of the m-th neuron. $\ell_m(\cdot)$ indicates the neuron self-inhibition satisfying $\ell_m(\cdot) > 0$. $\mathcal{K}$ denotes the number of edges (links) between any two neurons of the FOMMNNs. $\hbar_n(\cdot)$ and $g_n(\cdot)$ represent the activation functions without and with time delays, $\varsigma_\iota(t)$ denotes the delay of the ι-th edge which is bounded and satisfies $0 < \varsigma_\iota(t) \leq \varsigma_M$. The initial value of FOMMNNs (1) is $\pi(s) = (\pi_1(s), \pi_2(s), \ldots, \pi_n(s))^T \in \mathbb{C}([-\varsigma_M, 0], \mathcal{R}^n)$. $\mathcal{I}_m(t)$ represents the external input. $\upsilon_{mn}(\mathfrak{X}_m(t))$ and $\varrho_{(\iota)mn}(\mathfrak{X}_m(t))$ are memristive connection weights, given by

$$ \begin{aligned} &\ell_m(\mathfrak{X}_m(t)) = \frac{1}{C_m}\left[\sum_{n=1}^{\Pi}(\widetilde{M}_{mn} + \sum_{\iota=1}^{\mathcal{K}}\widehat{M}_{(\iota)mn}) \times \varepsilon_{mn} + \frac{1}{\mathfrak{R}_m}\right], \\ &\upsilon_{mn}(\mathfrak{X}_m(t)) = \frac{\widetilde{M}_{mn}}{C_m} \times \varepsilon_{mn}, \quad \varrho_{(\iota)mn}(\mathfrak{X}_m(t)) = \frac{\widehat{M}_{(\iota)mn}}{C_m} \times \varepsilon_{mn}, \\ &\varepsilon_{mn} = \begin{cases} 1, & m \neq n, \\ -1, & m = n, \end{cases} \end{aligned} $$

where $\widetilde{M}_{mn}$, $\widehat{M}_{(\imath)mn}$ indicate the memductances associated with memristors $\widetilde{\mathfrak{R}}_{mn}$, $\widehat{\mathfrak{R}}_{(\imath)mn}$, respectively. Besides, $\widetilde{\mathfrak{R}}_{mn}$ indicate the memristor between $\mathfrak{X}_m(t)$ and $\hbar_n(\mathfrak{X}_n(t))$, and $\widehat{\mathfrak{R}}_{(\imath)mn}$ denotes the memristor between $\mathfrak{X}_m(t)$ and $g_n(\mathfrak{X}_n(t-\varsigma_\imath(t)))$. $\mathfrak{R}_m$ indicates the parallel-resistor related to C_m. According to the properties of the memristors, we set

$$\ell_m(\mathfrak{X}_m(t)) = \begin{cases} \ell_m^*, |\mathfrak{X}_m(t)| \le \mho_m, \\ \ell_m^{**}, |\mathfrak{X}_m(t)| > \mho_m, \end{cases}$$

$$\upsilon_{mn}(\mathfrak{X}_m(t)) = \begin{cases} \upsilon_{mn}^*, |\mathfrak{X}_m(t)| \le \mho_m, \\ \upsilon_{mn}^{**}, |\mathfrak{X}_m(t)| > \mho_m, \end{cases}$$

and

$$\varrho_{(\imath)mn}(\mathfrak{X}_m(t)) = \begin{cases} \varrho_{(\imath)mn}^*, |\mathfrak{X}_m(t)| \le \mho_m, \\ \varrho_{(\imath)mn}^{**}, |\mathfrak{X}_m(t)| > \mho_m, \end{cases}$$

for $m, n = 1, 2, \dots, \Pi, \imath = 1, \dots, \mathcal{K}$, and ℓ_m^*, ℓ_m^{**}, υ_{mn}^*, υ_{mn}^{**}, $\varrho_{(\imath)mn}^*$, $\varrho_{(\imath)mn}^{**}$ are the known parameters which are related to the memristors. $\mho_m$ represents the thresholds of the switching jumps.

For convenience, let

$$\begin{aligned} &\tilde{\ell}_m^u = \max\{|\ell_m^*|, |\ell_m^{**}|\},\ \breve{\ell}_m = \min\{\ell_m^*, \ell_m^{**}\},\ \hat{\ell}_m = \max\{\ell_m^*,\ \ell_m^{**}\}, \\ &\hat{\upsilon}_{mn} = \max\{\upsilon_{mn}^*, \upsilon_{mn}^{**}\},\ \breve{\upsilon}_{mn} = \min\{\upsilon_{mn}^*, \upsilon_{mn}^{**}\}, \\ &\tilde{\upsilon}_{mn}^u = \max\{|\upsilon_{mn}^*|, |\upsilon_{mn}^{**}|\}, \hat{\varrho}_{(\imath)mn} = \max\{\varrho_{(\imath)mn}^*, \varrho_{(\imath)mn}^{**}\}, \\ &\breve{\varrho}_{(\imath)mn} = min\{\varrho_{(\imath)mn}^*, \varrho_{(\imath)mn}^{**}\},\ \tilde{\varrho}_{(\imath)mn}^u = \max\{|\varrho_{(\imath)mn}^*|, |\varrho_{(\imath)mn}^{**}|\}. \end{aligned}$$

for $m, n = 1, 2, \dots, \Pi$, $\imath = 1, 2, \dots, \mathcal{K}$.

Utilizing the theories of set-valued mapping and fractional Fillipov inclusions[45], it arrives that

$$\begin{aligned} \mathcal{D}_t^{\hat{\beta}} \mathfrak{X}_m(t) \in &-\overline{co}[\ell_m(\mathfrak{X}_m(t))](\mathfrak{X}_m(t)) \\ &+ \sum_{n=1}^{\Pi} \overline{co}[\upsilon_{mn}(\mathfrak{X}_m(t))]\hbar_n(\mathfrak{X}_n(t)) \\ &+ \sum_{n=1}^{\Pi} \sum_{\imath=1}^{\mathcal{K}} \overline{co}[\varrho_{(\imath)mn}(\mathfrak{X}_m(t))] g_n(\mathfrak{X}_n(t-\varsigma_\imath(t))) + \mathcal{I}_m(t), \end{aligned} \tag{2}$$

for $t \ge 0,\ m = 1, 2, \dots, \Pi$, where

$$\overline{co}[\ell_m(\mathfrak{X}_m(t))] = \begin{cases} \ell_m^*, & |\mathfrak{X}_m(t)| < \mho_m, \\ [\breve{\ell}_m, \hat{\ell}_m], & |\mathfrak{X}_m(t)| = \mho_m, \\ \ell_m^{**}, & |\mathfrak{X}_m(t)| > \mho_m, \end{cases}$$

$$\overline{co}[\upsilon_{mn}(\mathfrak{X}_m(t))] = \begin{cases} \upsilon_{mn}^*, & |\mathfrak{X}_m(t)| < \mho_m, \\ [\breve{\upsilon}_{mn}, \hat{\upsilon}_{mn}], & |\mathfrak{X}_m(t)| = \mho_m, \\ \upsilon_{mn}^{**}, & |\mathfrak{X}_m(t)| > \mho_m, \end{cases}$$

and

$$\overline{co}[\varrho_{(\imath)mn}(\mathfrak{X}_m(t))] = \begin{cases} \varrho_{(\imath)mn}^*, & |\mathfrak{X}_m(t)| < \mho_m, \\ [\breve{\varrho}_{(\imath)mn}, \hat{\varrho}_{(\imath)mn}], & |\mathfrak{X}_m(t)| = \mho_m, \\ \varrho_{(\imath)mn}^{**}, & |\mathfrak{X}_m(t)| > \mho_m, \end{cases}$$

where $m, n = 1, 2, \dots, \Pi$. Based on the measurable selection theorem[46], there exist appropriate measurable functions $\acute{\ell}_m(t) \in \overline{co}[\ell_m(\mathfrak{X}_m(t))]$, $\acute{\upsilon}_{mn}(t) \in \overline{co}[\upsilon_{mn}(\mathfrak{X}_m(t))]$ and $\acute{\varrho}_{(\imath)mn}(t) \in \overline{co}[\varrho_{(\imath)mn}(\mathfrak{X}_m(t))]$ such that

$$\begin{aligned} \mathcal{D}_t^{\hat{\beta}} \mathfrak{X}_m(t) = &-\acute{\ell}_m(t)\mathfrak{X}_m(t) + \sum_{n=1}^{\Pi} \acute{\upsilon}_{mn}(t)\hbar_n(\mathfrak{X}_n(t)) \\ &+ \sum_{s=1}^{n} \sum_{\imath=1}^{\mathcal{K}} \acute{\varrho}_{(\imath)mn}(t) g_n(\mathfrak{X}_n(t-\varsigma_\imath(t))) + \mathcal{I}_m(t), \end{aligned} \tag{3}$$

for a.e. $t \ge 0,\ m = 1, 2, \dots, \Pi$.

Remark 1. When $\hat{\beta} = 1$, the FOMMNNs (1) degenerates into common MMNNs, which had be discussed in[47,48,49]. In other words, the FOMMNNs generalizes the usual MMNNs. The FOMMNNs (1) is different from the switched systems investigated in Refs.[50,51]. The switching systems usually switches between two different states. However, the switching of memristor-based nerual network model changes with the change of variable states, and this switching is more complicated. Besides, we consider time-varying delays while Ref.[50,51] considered constant delay, therefore, our model is more general.

2.1.2 | The Slave FOMMNNs

To study the synchronization, the slave delayed FOMMNNs is given by

$$\begin{aligned}\mathcal{D}_t^{\hat{\beta}}\mathfrak{Y}_m(t) &= -\ell_m(\mathfrak{Y}_m(t))\mathfrak{Y}_m(t) + \sum_{n=1}^{\Pi} \upsilon_{mn}(\mathfrak{Y}_m(t))\hbar_n(\mathfrak{Y}_n(t)) \\ &+ \sum_{s=1}^{n}\sum_{\imath=1}^{\mathcal{K}} \varrho_{(\imath)mn}(\mathfrak{Y}_m(t))g_n(\mathfrak{Y}_n(t-\varsigma_\imath(t))) + \mathcal{I}_m(t) + \mu_m(t),\ t \geq 0,\end{aligned} \tag{4}$$

for $m = 1, 2, \dots, \Pi$. Here, $\mathfrak{Y}_m(t)$ represents the state of the m-th neuron of the FOMMNNs (4), $\mu_m(t)$ is an appropriately-designed controller to be determined. The initial value of FOMMNNs (4) is , $\upsilon_{mn}(\mathfrak{Y}_m(t))$ and $\varrho_{(\imath)mn}(\mathfrak{Y}_m(t))$ memristive connection weights, which are expressed by

$$\ell_m(\mathfrak{Y}_m(t)) = \begin{cases} \ell^*_m, |\mathfrak{Y}_m(t)| \leq \Omega_m, \\ \ell^{**}_m, |\mathfrak{Y}_m(t)| > \Omega_m, \end{cases}$$

$$\upsilon_{mn}(\mathfrak{Y}_m(t)) = \begin{cases} \upsilon^*_{mn}, |\mathfrak{Y}_m(t)| \leq \Omega_m, \\ \upsilon^{**}_{mn}, |\mathfrak{Y}_m(t)| > \Omega_m, \end{cases}$$

and

$$\varrho_{(\imath)mn}(\mathfrak{Y}_m(t)) = \begin{cases} \varrho^*_{(\imath)mn}, |\mathfrak{Y}_m(t)| \leq \Omega_m, \\ \varrho^{**}_{(\imath)mn}, |\mathfrak{Y}_m(t)| > \Omega_m, \end{cases}$$

where $m, n = 1, 2, \dots, \Pi, \imath = 1, 2, \dots, \mathcal{K}$, Ω_m indicates the threshold of the switching jumps of the memristors. The other parameters have the same meaning as those defined in FOMMNNs (1).

Similarly, it follows from (4) that

$$\begin{aligned}\mathcal{D}_t^{\hat{\beta}}\mathfrak{Y}_m(t) &\in -\overline{co}[\ell_m(\mathfrak{Y}_m(t))](\mathfrak{Y}_m(t)) + \overline{co}[\upsilon_{mn}(\mathfrak{Y}_m(t))]\hbar_n(\mathfrak{Y}_n(t)) \\ &+ \sum_{n=1}^{\Pi} \overline{co}[\varrho_{(\imath)mn}(\mathfrak{Y}_m(t))]g_n(\mathfrak{Y}_n(t-\varsigma_\imath(t))) + \mathcal{I}_m(t) + \mu_m(t),\end{aligned} \tag{5}$$

for $t \geq 0,\ m = 1, 2, \dots, \Pi$, where

$$\overline{co}[\ell_m(\mathfrak{Y}_m(t))] = \begin{cases} \ell^*_m, & |\mathfrak{Y}_m(t)| < \Omega_m, \\ [\check{\ell}_m, \hat{\ell}_m], & |\mathfrak{Y}_m(t)| = \Omega_m, \\ \ell^{**}_m, & |\mathfrak{Y}_m(t)| > \Omega_m, \end{cases}$$

$$\overline{co}[\upsilon_{mn}(\mathfrak{Y}_m(t))] = \begin{cases} \upsilon^*_{mn}, & |\mathfrak{Y}_m(t)| < \Omega_m, \\ [\check{\upsilon}_{mn}, \hat{\upsilon}_{mn}], & |\mathfrak{Y}_m(t)| = \Omega_m, \\ \upsilon^{**}_{mn}, & |\mathfrak{Y}_m(t)| > \Omega_m, \end{cases}$$

and

$$\overline{co}[\varrho_{(\imath)mn}(\mathfrak{X}_m(t))] = \begin{cases} \varrho^*_{(\imath)mn}, & |\mathfrak{Y}_m(t)| < \Omega_m, \\ [\check{\varrho}_{(\imath)mn}, \hat{\varrho}_{(\imath)mn}], & |\mathfrak{Y}_m(t)| = \Omega_m, \\ \varrho^{**}_{(\imath)mn}, & |\mathfrak{Y}_m(t)| > \Omega_m, \end{cases}$$

where $n = 1, 2, \dots, \Pi$, $\imath = 1, 2, \dots, K$.

Hence, we choose the measurable functions as $\grave{\ell}_m(t) \in \overline{co}[\ell_m(\mathfrak{Y}_m(t))]$, $\grave{\upsilon}_{mn}(t) \in \overline{co}[\upsilon_{mn}(\mathfrak{Y}_m(t))]$ and $\grave{\varrho}_{(\iota)mn}(t) \in \overline{co}[\varrho_{(\iota)mn}(\mathfrak{Y}_m(t))]$ such that

$$\begin{aligned} \mathcal{D}_t^{\hat{\beta}}\mathfrak{Y}_m(t) &= -\grave{\ell}_m(t)\mathfrak{Y}_m(t) + \sum_{n=1}^{\Pi}\grave{\upsilon}_{mn}(t)\hbar_n(\mathfrak{Y}_n(t)) + \sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}}\grave{\varrho}_{(\iota)mn}(t)g_n(\mathfrak{Y}_n(t-\varsigma_\iota(t))) \\ &+ \mathcal{I}_m(t) + \mu_m(t), \qquad t \geq 0, \end{aligned} \tag{6}$$

The synchronization error $\mathfrak{E}_m(t) = \mathfrak{Y}_m(t) - \mathfrak{X}_m(t)$, $(m = 1, 2, \ldots, \Pi)$. Subtracting (3) from (6), we obtain

$$\begin{aligned} \mathcal{D}_t^{\hat{\beta}}\mathfrak{E}_m(t) &= -\grave{\ell}_m(t)\mathfrak{Y}_m(t) + \acute{\ell}_m(t)\mathfrak{X}_m(t) + \sum_{n=1}^{\Pi}\grave{\upsilon}_{mn}(t)\hbar_n(\mathfrak{Y}_n(t)) \\ &- \sum_{n=1}^{\Pi}\acute{\upsilon}_{mn}(t)\hbar_n(\mathfrak{X}_n(t)) + \sum_{s=1}^{n}\sum_{\iota=1}^{\mathcal{K}}\grave{\varrho}_{(\iota)mn}(t)g_n(\mathfrak{Y}_n(t-\varsigma_\iota(t))) \\ &- \sum_{s=1}^{n}\sum_{\iota=1}^{\mathcal{K}}\acute{\varrho}_{(\iota)mn}(t)g_n(\mathfrak{X}_n(t-\varsigma_\iota(t))) + \mu_m(t) \\ &= -\acute{\ell}_m(t)\mathfrak{E}_m(t) + (\acute{\ell}_m(t) - \grave{\ell}_m(t))\mathfrak{Y}_m(t) + \sum_{n=1}^{\Pi}\acute{\upsilon}_{mn}(t)\hbar_n(\mathfrak{E}_n(t)) \\ &+ \sum_{n=1}^{\Pi}(\grave{\upsilon}_{mn}(t) - \acute{\upsilon}_{mn}(t))\hbar_n(\mathfrak{Y}_n(t)) \\ &+ \sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}}(\grave{\varrho}_{(\iota)mn}(t) - \acute{\varrho}_{(\iota)mn}(t))g_n(\mathfrak{Y}_n(t-\varsigma(t))) \\ &+ \sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}}\acute{\varrho}_{(\iota)mn}(t)G_n(\mathfrak{E}_n(t-\varsigma_\iota(t))) + \mu_m(t), \end{aligned} \tag{7}$$

for a.e. $t \geq 0$, $m = 1, 2, \ldots, \Pi$, where $\hbar_n(\mathfrak{E}_n(t)) = \hbar_n(\mathfrak{Y}_n(t)) - \hbar_n(\mathfrak{X}_n(t))$ and $G_n(\mathfrak{E}_n(t-\varsigma_\iota(t))) = g_n(\mathfrak{Y}_n(t-\varsigma_\iota(t))) - g_n(\mathfrak{X}_n(t-\varsigma_\iota(t)))$. The initial condition of the error FOMMNNs system (7) is $\phi(s) = \psi(s) - \pi(s) \in \mathbb{C}([-\varsigma_M, 0], \mathcal{R}^n)$.

Some useful definitions, lemmas and assumptions are given as follows to derive the main theoretical criteria.

Assumption H1. The functions $\hbar_n(\cdot), g_n(\cdot)$ $(n = 1, 2, \ldots, \Pi)$ are assumed to be Lipschitz-continuous. Hence, their exist the known constants $L_n^{\hbar} > 0$, $L_n^g > 0$ which satisfy $|\hbar_n(z_2) - \hbar_n(z_1)| \leq L_n^{\hbar}|z_2 - z_1|$, $|g_n(z_2) - g_n(z_1)| \leq L_n^g|g_n(z_2) - g_n(z_1)|$, $(n = 1, 2, \ldots, \Pi)$, $\forall z_1, z_2 \in \mathcal{R}^n$. Let $L^{\hbar} = \max\limits_{1\leq n\leq\Pi}\left\{L_n^{\hbar}\right\}$, $L^g = \max\limits_{1\leq n\leq\Pi}\left\{L_n^g\right\}$.

Assumption H2. $\hbar_n(\cdot)$, $g_n(\cdot)(n = 1, 2, \ldots, \Pi)$ are assumed to be bounded. Then, there exist known constants $\xi_n^{\hbar} > 0$ and $\xi_n^g > 0$

$$|\hbar_m(\cdot)| \leq \xi_m^{\hbar}, \; |g_m(\cdot)| \leq \xi_m^g, \; m = 1, 2, \ldots, \Pi.$$

Let $\xi^{\hbar} = \max\limits_{1\leq n\leq\Pi}\left\{\xi_n^{\hbar}\right\}$, $\xi^g = \max\limits_{1\leq n\leq\Pi}\left\{\xi_n^g\right\}$ for convenience.

Assumption 3. $\varsigma_\iota(t)$ $(\iota = 1, 2, \ldots, K)$ are bounded differential functions. Let, $0 < \dot{\varsigma}_\iota(t) < \varsigma_D$ and $0 < \varsigma_\iota(t) < \varsigma_M$ hold.

Lemma 1. (Li et al[52]) . If $0 < \hat{\beta} < 1$ and $Q(t) \in C^1[t_0, \infty)$, then

$${}_{t_0}^{C}\mathcal{D}_{t_0,t}^{\hat{\beta}}|Q(t)| \leq sign(Q(t))\mathcal{D}_{t_0,t}^{\hat{\beta}}Q(t), \; t \geq t_0.$$

Lemma 2. Let $\mathcal{O}_m = -sign(\mathfrak{E}_m(t))(\ell_m(\mathfrak{Y}_m(t))\mathfrak{Y}_m(t) - \ell_m(\mathfrak{X}_m(t))\mathfrak{X}_m(t))$, then we can conclude that $\mathcal{O}_m \leq -\breve{\ell}_m|\mathfrak{E}_m(t)| + |\ell_m^{**} - \ell_m^*|\overline{M}_m(m = 1, 2, \ldots, \Pi)$, where $\overline{M}_m = \max\left\{\Omega_m, \mho_m\right\}$.

Proof. According to the characteristics of the memristor, we obtain that

(1) When $\mathfrak{X}_m(t) > \mho_m$ and $\mathfrak{Y}_m(t) > \Omega_m$, we conclude that

$$\begin{aligned}\mathcal{O}_m &= -sign(\mathfrak{E}_m(t))(\ell_m(\mathfrak{Y}_m(t))\mathfrak{Y}_m(t) - \ell_m(\mathfrak{X}_m(t))\mathfrak{X}_m(t))\\ &= sign(\mathfrak{E}_m(t))(\ell_m^{**})(\mathfrak{Y}_m(t) - \mathfrak{X}_m(t))\\ &= -\ell_m^{**}|\mathfrak{E}_m(t)|\\ &\leq -\breve{\ell}_m|\mathfrak{E}_m(t)| \leq -\breve{\ell}_m|\mathfrak{E}_m(t)| + |\ell_m^{**} - \ell_m^{*}|\overline{M}_m.\end{aligned}$$

(2) When $\mathfrak{X}_m(t) \leq \mho_m$ and $\mathfrak{Y}_m(t) \leq \Omega_m$, we conclude that

$$\begin{aligned}\mathcal{O}_m &= -sign(\mathfrak{E}_m(t))(\ell_m(\mathfrak{Y}_m(t))\mathfrak{Y}_m(t) - \ell_m(\mathfrak{X}_m(t))\mathfrak{X}_m(t))\\ &= sign(\mathfrak{E}_m(t))\ell_m^{*}(\mathfrak{Y}_m(t) - \mathfrak{X}_m(t))\\ &= -\ell_m^{*}|\mathfrak{E}_m(t)|\\ &\leq -\breve{\ell}_m|\mathfrak{E}_m(t)| \leq -\breve{\ell}_m|\mathfrak{E}_m(t)| + |\ell_m^{**} - \ell_m^{*}|\overline{M}_m.\end{aligned}$$

(3) When $\mathfrak{X}_m(t) > \mho_m$ and $\mathfrak{Y}_m(t) \leq \Omega_m$, we conclude that

$$\begin{aligned}\mathcal{O}_m &= -sign(\mathfrak{E}_m(t))(\ell_m(\mathfrak{Y}_m(t))\mathfrak{Y}_m(t) - \ell_m(\mathfrak{X}_m(t))\mathfrak{X}_m(t))\\ &= -sign(\mathfrak{E}_m(t))(\ell_m(\mathfrak{X}_m(t)))(\mathfrak{Y}_m(t) - \mathfrak{X}_m(t))\\ &+ (\ell_m(\mathfrak{Y}_m(t))\mathfrak{Y}_m(t) - \ell_m(\mathfrak{X}_m(t))\mathfrak{Y}_m(t))\\ &= -sign(\mathfrak{E}_m(t))(\ell_m^{**}\mathfrak{E}_m(t) + (\ell_m^{**} - \ell_m^{*})\mathfrak{Y}_m(t))\\ &\leq -\breve{\ell}_m|\mathfrak{E}_m(t)| + |\ell_m^{**} - \ell_m^{*}|\Omega_m \leq -\breve{\ell}_m|\mathfrak{E}_m(t)|\\ &\leq -\breve{\ell}_m|\mathfrak{E}_m(t)| + |\ell_m^{**} - \ell_m^{*}|\overline{M}_m.\end{aligned}$$

(4) When $\mathfrak{X}_m(t) \leq \mho_m$ and $\mathfrak{Y}_m(t) > \Omega_m$, we conclude that

$$\begin{aligned}\mathcal{O}_m &- sign(\mathfrak{E}_m(t))(\ell_m(\mathfrak{Y}_m(t))\mathfrak{Y}_m(t) - \ell_m(\mathfrak{X}_m(t))\mathfrak{X}_m(t))\\ &= sign(\mathfrak{E}_m(t))(\ell_m(\mathfrak{Y}_m(t)))(\mathfrak{Y}_m(t) - \mathfrak{X}_m(t))\\ &+ (\ell_m(\mathfrak{Y}_m(t))\mathfrak{X}_m(t) - \ell_m(\mathfrak{X}_m(t))\mathfrak{X}_m(t))\\ &= -sign(\mathfrak{E}_m(t))(\ell_m^{**}\mathfrak{E}_m(t) + (\ell_m^{**} - \ell_m^{*})\mathfrak{X}_m(t))\\ &\leq -\breve{\ell}_m|\mathfrak{E}_m(t)| + |\ell_m^{**} - \ell_m^{*}|\mho_m\\ &\leq -\breve{\ell}_m|\mathfrak{E}_m(t)| + |\ell_m^{**} - \ell_m^{*}|\overline{M}_m.\end{aligned}$$

Hence, we can easily conclude that $\mathcal{O}_m \leq -\breve{\ell}_m|\mathfrak{E}_m(t)| + |\ell_m^{**} - \ell_m^{*}|\overline{M}_m$. This complete the proof. □

Definition 3. The FOMMNNs (1) and (4) are deemed to realize the GAS, if under a suitable designed controller, for any initial value $\phi(s)$ of $\mathfrak{E}(t)$, one always has $\lim_{t\to\infty} \|\mathfrak{E}(t)\|_1 = 0$, where $\mathfrak{E}(t) = (\mathfrak{E}_1(t), \mathfrak{E}_2(t), \ldots, \mathfrak{E}_\Pi(t))^T$. Let $\|\phi(s)\|_1 = \sup_{-\varsigma\leq\theta\leq 0} \|\psi(s) - \pi(s)\|_1$.

Lemma 3. (Wang and Yang[53]) Fractional Barbalat's lemma under the framework of Caputo derivative. If $\int_{t_0}^{t} \mathfrak{B}(s)ds = \mathfrak{A} < +\infty$ when $t \to +\infty$. Besides, if it holds true that ${}_{t_0}^{C}\mathcal{D}_t^{\hat{\beta}}\mathfrak{B}(t)$ is bounded, where $\omega : [0, +\infty) \to \mathcal{R}$, then $\mathfrak{B}(t) \to 0$ as $t \to +\infty$, where $\hat{\beta} \in (0, 1)$.

3 | MAIN RESULTS

In this section, the GAS of the master-slave FOMMNNs are investigated by Adoption two control strategies.

3.1 Linear discontinuous state-feedback controller

Theorem 1. In this section, we adopt the following linear discontinuous state-feedback controller

$$\mu_m(t) = -\zeta_m \mathfrak{E}_m(t) - \eta_m sign(\mathfrak{E}_m(t)), m = 1, 2, \dots, \Pi, \tag{8}$$

where ζ_m, $\eta_m (m = 1, 2, \dots, \Pi)$ are appropriately chosen constants. If ζ_m and η_m satisfy

$$\begin{aligned} \zeta_m &> -\breve{\ell}_m + \xi^{\hbar} \sum_{n=1}^{\Pi} \upsilon_{nm}^{u} + \frac{L^g}{1-\varsigma_D} \sum_{n=1}^{\Pi} \sum_{\iota=1}^{\mathcal{K}} \varrho_{(\iota)ji}^{u}, \\ \eta_m &> |\ell_m^{**} - \ell_m^{*}| \overline{M}_m + \xi^{\hbar} \sum_{n=1}^{\Pi} |\upsilon_{mn}^{*} - \upsilon_{mn}^{**}| + \xi^{g} \sum_{n=1}^{\Pi} \sum_{\iota=1}^{\mathcal{K}} |\varrho_{(\iota)mn}^{*} - \varrho_{(\iota)mn}^{**}|, \end{aligned} \tag{9}$$

where $m = 1, 2, \dots, \Pi$. Then under Assumptions H1-H3, the error FOMMNNs (7) can realize the GAS from Definition 3. Or in other words, the FOMMNNs (4) and (1) can obtain the GAS.

Proof of Theorem 1. Build the Lyapunov functional as follows

$$\mathbb{V}(t, \mathfrak{E}(t)) = \mathbb{V}_1(t, \mathfrak{E}(t)) + \mathbb{V}_2(t, \mathfrak{E}(t)), \tag{10}$$

where $\mathbb{V}_1(t, \mathfrak{E}(t)) = \sum_{m=1}^{\Pi} \mathcal{D}_t^{-(1-\hat{\beta})} |\mathfrak{E}_m(t)|$, and

$$\mathbb{V}_2(t, \mathfrak{E}(t)) = \sum_{m=1}^{\Pi} \sum_{l=1}^{n} \sum_{j=1}^{m} \frac{\varrho_{(\iota)mn}^{u} \xi^g}{1-\varsigma_D} \int_{t-\varsigma_\iota(t)}^{t} |\mathfrak{E}_n(s)| ds.$$

Doing the derivative of $\mathbb{V}(t, \mathfrak{E}(t))$ along the trajectory of FOMMNNs (7) yields

$$\dot{\mathbb{V}} = \dot{\mathbb{V}}_1(t, \mathfrak{E}(t)) + \dot{\mathbb{V}}_2(t) = \frac{d}{dt} \mathcal{D}_t^{\hat{\beta}-1} \left[\sum_{m=1}^{\Pi} |\mathfrak{E}_m(t)| \right] + \dot{\mathbb{V}}_2(t, \mathfrak{E}(t)).$$

Under **Lemma 1**, we conclude that

$$\begin{aligned} \dot{\mathbb{V}}_1(t, \mathfrak{E}(t)) &\leq \sum_{m=1}^{\Pi} sign(\mathfrak{E}_m(t)) \mathcal{D}_t^{\hat{\beta}} \mathfrak{E}_m(t) = \sum_{m=1}^{\Pi} sign(\mathfrak{E}_m(t))(-\grave{\ell}_m(t) \mathfrak{Y}_m(t) \\ &+ \acute{\ell}_m(t) \boldsymbol{\mathfrak{X}}_m(t) + \sum_{n=1}^{\Pi} \grave{\upsilon}_{mn}(t) \hbar_n(\mathfrak{Y}_n(t)) - \sum_{n=1}^{\Pi} \acute{\upsilon}_{mn}(t) \hbar_n(\boldsymbol{\mathfrak{X}}_n(t)) \\ &+ \sum_{n=1}^{\Pi} \sum_{\iota=1}^{\mathcal{K}} \grave{\varrho}_{(\iota)mn}(t) g_n(\mathfrak{Y}_n(t-\varsigma_\iota(t))) - \sum_{n=1}^{\Pi} \sum_{\iota=1}^{\mathcal{K}} \acute{\varrho}_{(\iota)mn}(t) g_n(\boldsymbol{\mathfrak{X}}_n(t-\varsigma_\iota(t))) + \mu_m(t)) \\ &= \sum_{m=1}^{\Pi} sign(\mathfrak{E}_m(t))(-\acute{\ell}_m(t) \mathfrak{E}_m(t) + (\acute{\ell}_m(t) - \grave{\ell}_m(t)) \mathfrak{Y}_m(t) \\ &+ \sum_{n=1}^{\Pi} \acute{\upsilon}_{mn}(t) \hbar_n(\mathfrak{E}_n(t)) + \sum_{n=1}^{\Pi} (\grave{\upsilon}_{mn}(t) - \acute{\upsilon}_{mn}(t)) \hbar_n(\mathfrak{Y}_n(t)) \\ &+ \sum_{n=1}^{\Pi} \sum_{\iota=1}^{\mathcal{K}} (\grave{\varrho}_{(\iota)mn}(t) - \acute{\varrho}_{(\iota)mn}(t)) g_n(\mathfrak{Y}_n(t-\varsigma_\iota(t))) \\ &+ \sum_{n=1}^{\Pi} \sum_{\iota=1}^{\mathcal{K}} \acute{\varrho}_{(\iota)mn}(t)(g_n(\mathfrak{Y}_n(t-\varsigma_\iota(t))) - g_n(\boldsymbol{\mathfrak{X}}_n(t-\varsigma_\iota(t)))) \\ &+ \mu_m(t)). \end{aligned} \tag{11}$$

We can conclude that

$$\begin{aligned}\sum_{m=1}^{\Pi} sign(\mathfrak{E}_m(t))\mu_m(t) &= -\sum_{m=1}^{\Pi} sign(\mathfrak{E}_m(t))(\zeta_m \mathfrak{E}_m(t) + \eta_m sign(\mathfrak{E}_m(t))) \\ &= -\sum_{m=1}^{\Pi} \zeta_m |\mathfrak{E}_m(t)| - \sum_{m=1}^{\Pi} \eta_m \chi_m,\end{aligned} \tag{12}$$

where $\chi_m = |\mathfrak{E}_m(t)| = |\mathfrak{E}_m(t)|^2$.

From **Lemma 2**, we can conclude that

$$\begin{aligned}&\sum_{m=1}^{\Pi} sign(\mathfrak{E}_m(t))(-\grave{\ell}_m(t)\mathfrak{Y}_m(t) + \acute{\ell}_m(t)\mathfrak{X}_m(t)) \\ &\leq \sum_{m=1}^{\Pi} -\breve{\ell}_m |\mathfrak{E}_m(t)| + |\ell_m^{**} - \ell_m^{*}| \overline{M}_m \chi_m.\end{aligned} \tag{13}$$

Similarly, we conclude that

$$\begin{aligned}&\sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi} sign(\mathfrak{E}_m(t))\acute{\upsilon}_{mn}(t)(\hbar_n(\mathfrak{Y}_n(t) - \hbar_n(\mathfrak{X}_n(t))) \\ &\leq \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi} |sign(\mathfrak{E}_m(t))| \upsilon_{mn}^{u} \xi_n^{\hbar} |\mathfrak{E}_n(t)| \\ &\leq \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi} \upsilon_{mn}^{u} \xi^{\hbar} |\mathfrak{E}_n(t)| = \xi^{\hbar} \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi} \upsilon_{nm}^{u} |\mathfrak{E}_m(t)|.\end{aligned} \tag{14}$$

Similarly, we get

$$\begin{aligned}&\sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} sign(\mathfrak{E}_m(t))\acute{\varrho}_{(\iota)mn}(t)(g_n(\mathfrak{Y}_n(t-\varsigma_\iota(t))) - g_n(\mathfrak{X}_n(t-\varsigma_\iota(t))))) \\ &\leq \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} |sign(\mathfrak{E}_m(t))| \varrho_{(\iota)mn}^{u} L_n^{g} |\mathfrak{E}_n(t-\varsigma_\iota(t))| \\ &\leq L^{g} \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} \varrho_{(\iota)mn}^{u} |\mathfrak{E}_n(t-\varsigma_\iota(t))|.\end{aligned} \tag{15}$$

And

$$\begin{aligned}&\sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} sign(\mathfrak{E}_m(t))(\grave{\varrho}_{(\iota)mn}(t) - \acute{\varrho}_{(\iota)mn}(t)) g_n(\mathfrak{Y}_n(t-\varsigma_\iota(t))) \\ &\leq \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} |sign(\mathfrak{E}_m(t))| |\varrho_{(\iota)mn}^{*} - \varrho_{(\iota)mn}^{**}| \xi_n^{g} \\ &\leq \xi^{g} \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} |\varrho_{(\iota)mn}^{*} - \varrho_{(\iota)mn}^{**}| \chi_m,\end{aligned} \tag{16}$$

Similarly, we get

$$\begin{aligned}&\sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi} sign(\mathfrak{E}_m(t))(\grave{\upsilon}_{mn}(t) - \acute{\upsilon}_{mn}(t))\hbar_n(\mathfrak{Y}_n(t)) \\ &\leq \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi} |sign(\mathfrak{E}_m(t))| |\upsilon_{mn}^{*} - \upsilon_{mn}^{**}| \xi_n^{\hbar} \leq \xi^{\hbar} \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi} |\upsilon_{mn}^{*} - \upsilon_{mn}^{**}| \chi_m.\end{aligned} \tag{17}$$

And similarly,

$$\begin{aligned}&\sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\imath=1}^{\mathcal{K}} sign(\mathfrak{E}_m(t))(\grave{\varrho}_{(\imath)mn}(t)-\acute{\varrho}_{(\imath)mn}(t))g_n(\mathfrak{Y}_n(t-\varsigma_\imath(t)))\\ &\leq \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\imath=1}^{\mathcal{K}} |sign(\mathfrak{E}_m(t))||\varrho^{*}_{(\imath)mn}-\varrho^{**}_{(\imath)mn}|\xi^g_n\\ &\leq \xi^g\sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\imath=1}^{\mathcal{K}} |\varrho^{*}_{(\imath)mn}-\varrho^{**}_{(\imath)mn}|\chi_m.\end{aligned}\tag{18}$$

Reckoning the derivatives of $\mathbb{V}_2(t,\mathfrak{E}(t))$ yields

$$\begin{aligned}\dot{\mathbb{V}}_2(t,\mathfrak{E}(t)) &= \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\imath=1}^{\mathcal{K}} \frac{\varrho^u_{(\imath)mn}L^g}{1-\varsigma_D}(|\mathfrak{E}_n(t)|-|\mathfrak{E}_n(t-\varsigma_\imath(t))|(1-\dot{\varsigma}_\imath(t)))\\ &\leq \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\imath=1}^{\mathcal{K}} \frac{\varrho^u_{(\imath)ji}L^g}{1-\varsigma_D}|\mathfrak{E}_m(t)| - \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\imath=1}^{\mathcal{K}} \varrho^u_{(\imath)mn}\xi^g|\mathfrak{E}_n(t-\varsigma_\imath(t))|,\end{aligned}\tag{19}$$

From (12) -(19), we obtain that

$$\begin{aligned}\dot{\mathbb{V}}(t,\mathfrak{E}(t)) &= \dot{\mathbb{V}}_1(t,\mathfrak{E}(t))+\dot{\mathbb{V}}_2(t,\mathfrak{E}(t))\\ &\leq \sum_{m=1}^{\Pi}(-\zeta_m-\breve{\ell}_m+\xi^\hbar\sum_{n=1}^{\Pi}\upsilon^u_{nm}+\sum_{n=1}^{\Pi}\sum_{\imath=1}^{\mathcal{K}}\frac{\varrho^u_{(\imath)ji}L^g}{1-\varsigma_D})|\mathfrak{E}_m(t)|\\ &+\sum_{m=1}^{\Pi}(-\eta_m+|\ell^{**}_m-\ell^{*}_m|\mho_m+\xi^\hbar\sum_{n=1}^{\Pi}|\upsilon^{*}_{mn}-\upsilon^{**}_{mn}|+\xi^g\sum_{n=1}^{\Pi}\sum_{\imath=1}^{\mathcal{K}}|\varrho^{*}_{(\imath)mn}-\varrho^{**}_{(\imath)mn}|)\chi_m\\ &\leq -\sum_{m=1}^{\Pi}\hbar_m|\mathfrak{E}_m(t)|-\sum_{m=1}^{\Pi}\Theta_m\chi_m,\end{aligned}\tag{20}$$

where $\hbar_m=\zeta_m+\breve{\ell}_m-\xi^\hbar\sum_{n=1}^{\Pi}\upsilon^u_{nm}-\sum_{n=1}^{\Pi}\sum_{\imath=1}^{\mathcal{K}}\frac{\varrho^u_{(\imath)ji}\xi^g}{1-\varsigma_D}$, and $\Theta_m=\eta_m-|\ell^{**}_m-\ell^{*}_m|\overline{M}_m+\xi^\hbar\sum_{n=1}^{\Pi}|\upsilon^{*}_{mn}-\upsilon^{**}_{mn}|-\xi^g\sum_{n=1}^{\Pi}\sum_{\imath=1}^{\mathcal{K}}|\varrho^{*}_{(\imath)mn}-\varrho^{**}_{(\imath)mn}|$ for all $m=1,2,\ldots,\Pi$. From (9) and (20), we can conclude that $\hbar_m>0$, $\mho_m>0$. Hence, we can easily get that

$$\dot{\mathbb{V}}(t,\mathfrak{E}(t))\leq -\hbar\sum_{m=1}^{\Pi}|\mathfrak{E}_m(t)| = -\hbar\omega(t),\tag{21}$$

where $\hbar=\min_{1\leq i\leq n}\{\hbar_m\}$, $\omega(t)=\sum_{m=1}^{\Pi}|\mathfrak{E}_m(t)|$. Then we can conclude that $\mathbb{V}(t)\leq -\hbar\int_{t_0}^{t}\omega(s)ds+\mathbb{V}(t_0)$. From (10), we can easily conclude that $\omega(t)$ is non-negatively bounded and $\int_{t_0}^{t}\omega(s)ds$ has a finite limit. Therefore, there exists a positive constant $\mathcal{N}$ which satisfies $|\mathcal{D}^{\hat{\beta}}_t\omega(t)|\leq\mathcal{N}$ when $t\geq t_0$, According to **Lemma 3**, we will further prove that $\omega(t)$ is uniformly continuous. For $t_0\leq t_1<t_2$, when $|t_2-t_1|=\theta(\epsilon)\leq(\frac{\epsilon\Gamma(1+\hat{\beta})}{2\mathcal{N}})^{\frac{1}{\hat{\beta}}}$, we can get

$$\begin{aligned}|\omega_2(t)-\omega_1(t)| &= \mathcal{D}^{-\hat{\beta}}[\mathcal{D}^{\hat{\beta}}\omega_2(t)-\mathcal{D}^{\hat{\beta}}\omega_1(t)]\\ &= \frac{1}{\Gamma(\hat{\beta})}|\int_{t_0}^{t_2}(t_2-s)^{1-\hat{\beta}}[\mathcal{D}^{\hat{\beta}}\omega(s)]ds-\int_{t_0}^{t_1}(t_1-s)^{1-\hat{\beta}}[\mathcal{D}^{\hat{\beta}}\omega(s)]ds|\\ &= \frac{1}{\Gamma(\hat{\beta})}|\int_{t_1}^{t_2}(t_2-s)^{1-\hat{\beta}}[\mathcal{D}^{\hat{\beta}}\omega(s)]ds|+\frac{1}{\Gamma(\hat{\beta})}|\int_{t_0}^{t_1}((t_1-s)^{1-\hat{\beta}})[\mathcal{D}^{\hat{\beta}}\omega(s)]ds|\\ &\leq \frac{\mathcal{N}}{\Gamma(\hat{\beta})}[(t_1-\omega_2)^{\hat{\beta}}-(t_2-t_0)^{\hat{\beta}}+2(t_2-t_1)^{\hat{\beta}}]\\ &\leq \frac{2\mathcal{N}}{\Gamma(\hat{\beta})}(t_2-t_1)^{\hat{\beta}}\leq\epsilon.\end{aligned}$$

From **Lemma 3**, we conclude that $\omega(t)$ is uniformly continuous, and $\lim_{t\to\infty}\omega(t) = 0$. Since $\omega(t) = \sum_{m=1}^{\Pi} |\mathfrak{E}_m(t)|$, it means that $\lim_{t\to\infty} |\mathfrak{E}_m(t)| = 0\,(m = 1, 2, \cdots, \Pi)$. Therefore, The FOMMNNs (4) and (1) can realize the GAS under Definition 3. This completes the proof. □

Corollary 1. For the Master-Slave FOMMNNs (1) and (4), if $\varsigma_1(t) = \varsigma_1$, $\varsigma_2(t) = \varsigma_2$,..., $\varsigma_K(t) = \varsigma_{\mathcal{K}}$, where ς_ι $(\iota = 1, 2, \dots, \mathcal{K})$ are chosen canstants which satisfy $0 < \varsigma_\iota \le \varsigma$, and means that the time delays are time-invariant. Under controller (8), if ζ_m and η_m satisfy the following requirements

$$\zeta_m > -\breve{\ell}_m + \xi^{\hbar} \sum_{n=1}^{\Pi} \upsilon^u_{nm} + \sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} \frac{\varrho^u_{(\iota)ji} L^g}{1-\varsigma_D},$$
$$\eta_m > |\ell^{**}_m - \ell^*_m|\overline{M}_m + \xi^{\hbar} \sum_{n=1}^{\Pi} |\upsilon^*_{mn} - \upsilon^{**}_{mn}| + \xi^g \sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} |\varrho^*_{(\iota)mn} - \varrho^{**}_{(\iota)mn}|,$$

for $m = 1, 2, \dots, \Pi$, then the FOMMNNs (4) are asymptotical synchronized with FOMMNNs (1).

Proof. The proof process is quite similar to the one of the Theorem 3.1. Hence, it is ignored here.

Remark 2. If $\iota = 1$ and $\varsigma_1(t) = \varsigma$ (positive constant), the FOMMNNs are reduced to regular FOMNNs involving constant time delay, which had been discussed in[24,42,54].

3.2 | Adaptive discontinuous state-feedback controller

Usually, the theoretical value of synchronization time is much larger than the required actual time. This is because the amplification technology is used in the theoretical proof process. If we adopt the adaptive strategy, the control strength can adjust itself accordingly, which greatly reduce the control. Hence, in this section, an adaptive discontinuos state-feedback controller is adopted.

The controller is expressed by

$$u_m(t) = -\widehat{\zeta}_m(t)\mathfrak{E}_m(t) - \widehat{\eta}_m(t) sign(\mathfrak{E}_m(t)), \tag{22}$$

where $m = 1, 2, \dots, \Pi$. And the adaptive control law given by

$$\begin{cases} \dot{\widehat{\zeta}}_m(t) = k_m|\mathfrak{E}_m(t)|, & \widehat{\zeta}_m(0) = 0, \\ \dot{\widehat{\eta}}_m(t) = l_m|sign(\mathfrak{E}_m(t))|, & \widehat{\eta}_m(0) = 0, \end{cases}$$

where $k = (k_1, k_2, \dots, k_\Pi)$ and $l = (l_1, l_2, \dots, l_\Pi)$ are two positive vectors, $\widehat{\zeta}_m$ and $\widehat{\eta}_m$ $(m = 1, 2, \dots, \Pi)$ are appropriately chosen control gains.

Theorem 2 (Theorem subhead)**.** Let $0 < \hat{\beta} < 1$ and Assumptions H1-H2 hold, the FOMMNNs (1) will be globally asymptotically synchronized with the FOMMNNs (4) under the adaptive controller (22), if $\widehat{\zeta}_m, \widehat{\eta}_m (m = 1, 2, \dots, \Pi)$ satisfy

$$\begin{aligned} &\widehat{\zeta}_m > -\breve{\ell}_m + L^{\hbar} \sum_{n=1}^{\Pi} \tilde{\upsilon}^u_{nm} + \sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} \frac{\tilde{\varrho}^u_{(\iota)ji} L^g}{1-\varsigma_D}, \\ &\widehat{\eta}_m \ge |\ell^{**}_m - \ell^*_m|\overline{M}_m + \xi^{\hbar} \sum_{n=1}^{\Pi} |\upsilon^*_{mn} - \upsilon^{**}_{mn}| + \xi^g \sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} |\varrho^*_{(\iota)mn} - \varrho^{**}_{(\iota)mn}|. \end{aligned} \tag{23}$$

Proof of Theorem 2. Bulid up the following Lyapunov functional

$$\mathbb{V}(t, \mathfrak{E}(t)) = \mathbb{V}_1(t, \mathfrak{E}(t)) + \mathbb{V}_2(t, \mathfrak{E}(t)) + \mathbb{V}_3(t, \mathfrak{E}(t)), \tag{24}$$

where

$$\mathbb{V}_1(t,\mathfrak{E}(t)) = \sum_{m=1}^{\Pi} \mathcal{D}_t^{-(1-\hat{\beta})}|\mathfrak{E}_m(t)|,$$

$$\mathbb{V}_2(t,\mathfrak{E}(t)) = \sum_{m=1}^{\Pi}\sum_{l=1}^{n}\sum_{j=1}^{K} \frac{\varrho^u_{(\iota)mn}L^g}{1-\varsigma_D} \int_{t-\varsigma_\iota(t)}^{t} |\mathfrak{E}_n(s)|ds$$

$$\mathbb{V}_3(t,\mathfrak{E}(t)) = \sum_{m=1}^{\Pi}\left[\frac{1}{2k_m}(\zeta_m(t)-\zeta_m)^2 + \frac{1}{2l_m}(\eta_m(t)-\eta_m)^2\right],$$

Under **Lemma 1**, Finding the derivation of $\mathbb{V}_1(t)$ along the trajectory of FOMMNNs (7) yields

$$\begin{aligned}
\dot{\mathbb{V}}_1(t) &= \sum_{m=1}^{\Pi} \mathcal{D}_t^{\hat{\beta}}|\mathfrak{E}_m(t)| \le \sum_{m=1}^{\Pi} sign(\mathfrak{E}_m(t))\mathcal{D}_t^{\hat{\beta}}\mathfrak{E}_m(t)\\
&= \sum_{m=1}^{\Pi} sign(\mathfrak{E}_m(t))[-\grave{\ell}_m(t)\mathfrak{Y}_m(t) + \acute{\ell}_m(t)\mathfrak{X}_m(t) + \sum_{n=1}^{\Pi}\acute{\upsilon}_{mn}(t)(\hbar_n(\mathfrak{Y}_n(t)-\hbar_n(\mathfrak{X}_n(t))))\\
&+ \sum_{n=1}^{\Pi}(\grave{\upsilon}_{mn}(t)-\acute{\upsilon}_{mn}(t))\hbar_n(\mathfrak{Y}_n(t))\\
&+ \sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}}(\grave{\varrho}_{(\iota)mn}(t)-\acute{\varrho}_{(\iota)mn}(t))g_n(\mathfrak{Y}_n(t-\varsigma_\iota(t)))\\
&+ \sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}}\acute{\varrho}_{(\iota)mn}(t)(\hbar_n(\mathfrak{Y}_n(t-\varsigma_\iota(t))-\hbar_n(\mathfrak{X}_n(t-\varsigma_\iota(t)\\
&)))) + \mu_m(t)].
\end{aligned} \tag{25}$$

We conclude that

$$\begin{aligned}
&\sum_{m=1}^{\Pi} sign(\mathfrak{E}_m(t))\mu_m(t) = -\sum_{m=1}^{\Pi} sign(\mathfrak{E}_m(t))(\zeta_m(t)\mathfrak{E}_m(t) - \eta_m(t)sign(\mathfrak{E}_m(t)))\\
&= -\sum_{m=1}^{\Pi}\zeta_m(t)|\mathfrak{E}_m(t)| - \sum_{m=1}^{\Pi}\eta_m(t)\chi_m,
\end{aligned} \tag{26}$$

where we denote that $\chi_m = |\mathfrak{E}_m(t)|$.

By **Lemma 2**, it can be derived that

$$\begin{aligned}
&\sum_{m=1}^{\Pi} sign(\mathfrak{E}_m(t))(-\grave{\ell}_m(t)\mathfrak{Y}_m(t) + \acute{\ell}_m(t)\mathfrak{X}_m(t))\\
&\le \sum_{m=1}^{\Pi} -\check{\ell}_m|\mathfrak{E}_m(t)| + |\ell^{**}_m - \ell^{*}_m|\overline{M}_m\chi_m.
\end{aligned} \tag{27}$$

Under Assumption H1, we can conclude that

$$\begin{aligned}
&= \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi} sign(\mathfrak{E}_m(t))\acute{\upsilon}_{mn}(t)\hbar_n(\mathfrak{E}_n(t)) \le \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}|sign(\mathfrak{E}_m(t))|\tilde{\upsilon}^u_{mn}L^\hbar_n|\mathfrak{E}_n(t)|\\
&\le \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\tilde{\upsilon}^u L^\hbar|\mathfrak{E}_n(t)| = L^\hbar\sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\tilde{\upsilon}^u|\mathfrak{E}_m(t)|.
\end{aligned} \tag{28}$$

Similarly, we can conclude that

$$\begin{aligned}
&\sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} sign(\mathfrak{E}_m(t))\acute{\varrho}_{(\iota)mn}(t)(g_n(\mathfrak{Y}_n(t-\varsigma_\iota(t))-g_n(\mathfrak{X}_n(t-\varsigma_\iota(t)))))\\
&\leq \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} |sign(\mathfrak{E}_m(t))|\tilde{\varrho}^{u}_{(\iota)mn}L^g_n|\mathfrak{E}_n(t-\varsigma_\iota(t))| \qquad (29)\\
&\leq L^g\sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} \tilde{\varrho}^{u}_{(\iota)mn}|\mathfrak{E}_n(t-\varsigma_\iota(t))|.
\end{aligned}$$

From Assumption H2, we obtain that

$$\begin{aligned}
&\sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} sign(\mathfrak{E}_m(t))(\grave{\varrho}_{(\iota)mn}(t)-\acute{\varrho}_{(\iota)mn}(t))g_n(\mathfrak{Y}_n(t-\varsigma_\iota(t)))\\
&\leq \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} |sign(\mathfrak{E}_m(t))||\varrho^{*}_{(\iota)mn}-\varrho^{**}_{(\iota)mn}|\xi^g_n \qquad (30)\\
&\leq \xi^g\sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} |\varrho^{*}_{(\iota)mn}-\varrho^{**}_{(\iota)mn}|\chi_m.
\end{aligned}$$

And we have

$$\begin{aligned}
&\sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi} sign(\mathfrak{E}_m(t))(\grave{\upsilon}_{mn}(t)-\acute{\upsilon}_{mn}(t))\hbar_n(\mathfrak{Y}_n(t))\\
&\leq \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi} |sign(\mathfrak{E}_m(t))||\upsilon^{*}_{mn}-\upsilon^{**}_{mn}|\xi^{\hbar}_n \leq \xi^{\hbar}\sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}|\upsilon^{*}_{mn}-\upsilon^{**}_{mn}|\chi_m, \qquad (31)
\end{aligned}$$

Furthermore, we conclude that

$$\begin{aligned}
\dot{\mathbb{V}}_2(t,\mathfrak{E}(t)) &= \sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} \frac{\varrho^{u}_{(\iota)mn}L^g}{1-\varsigma_D}(|\mathfrak{E}_n(t)|-|\mathfrak{E}_n(t-\varsigma_\iota(t))|(1-\dot{\varsigma}_\iota(t)))\\
&\leq \frac{L^g}{1-\varsigma_D}\sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} \varrho^{u}_{(\iota)ji}|\mathfrak{E}_m(t)| - L^g\sum_{m=1}^{\Pi}\sum_{n=1}^{\Pi}\sum_{\iota=1}^{\mathcal{K}} \varrho^{u}_{(\iota)mn}|\mathfrak{E}_n(t-\varsigma_\iota(t))|. \qquad (32)
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\dot{\mathbb{V}}_3(t,\mathfrak{E}(t)) &= \sum_{m=1}^{\Pi}[\frac{1}{2k_m}(\widehat{\zeta}_m(t)-\widehat{\zeta}_m)\dot{\widehat{\zeta}}_m(t)+\frac{1}{2l_m}(\widehat{\zeta}_m(t)-\widehat{\zeta}_m)\dot{\widehat{\zeta}}_m(t)]\\
&= \sum_{m=1}^{\Pi}[(\widehat{\zeta}_m(t)-\widehat{\zeta}_m)|\mathfrak{E}_m(t)|+(\widehat{\eta}_m(t)-\widehat{\eta}_m)\chi_m]. \qquad (33)
\end{aligned}$$

From (25) to (33), we obtain

$$\begin{aligned}\dot{\mathbb{V}}(t,\mathfrak{E}(t)) &= \dot{\mathbb{V}}_1(t,\mathfrak{E}(t)) + \dot{\mathbb{V}}_2(t,\mathfrak{E}(t)) + \dot{\mathbb{V}}_3(t,\mathfrak{E}(t))\\ &\leq \sum_{m=1}^{\Pi}(-\widehat{\zeta}_m - \breve{\ell}_m + L^{\hbar}\sum_{n=1}^{\Pi}\tilde{\upsilon}^u_{nm} + \sum_{n=1}^{\Pi}\sum_{\imath=1}^{\mathcal{K}}\frac{\tilde{\varrho}^u_{(\imath)ji}L^g}{1-\varsigma_D})|\mathfrak{E}_m(t)|\\ &+ \sum_{m=1}^{\Pi}(-\widehat{\eta}_m + |\ell^{**}_m - \ell^{*}_m|\overline{M}_m + \xi^{\hbar}\sum_{n=1}^{\Pi}|\upsilon^{*}_{mn} - \upsilon^{**}_{mn}|\\ &+ \xi^{g}\sum_{n=1}^{\Pi}\sum_{\imath=1}^{\mathcal{K}}|\varrho^{*}_{(\imath)mn} - \varrho^{**}_{(\imath)mn}|)\chi_m\\ &\leq -\widehat{\hbar}_m\sum_{m=1}^{\Pi}|\mathfrak{E}_m(t)| - \sum_{m=1}^{\Pi}\widehat{\Theta}_m\chi_m,\end{aligned} \tag{34}$$

where $\widehat{\hbar}_m = \widehat{\zeta}_m + \breve{\ell}_m - L^{\hbar}\sum_{n=1}^{\Pi}\tilde{\upsilon}^u_{nm} - \sum_{n=1}^{\Pi}\sum_{\imath=1}^{\mathcal{K}}\frac{\tilde{\varrho}^u_{(\varsigma)ji}L^g}{1-\varsigma_D}$, and $\widehat{\Theta}_m = \widehat{\eta}_m - |\ell^{**}_m - \ell^{*}_m|\overline{M}_m + \xi^{\hbar}\sum_{n=1}^{\Pi}|\upsilon^{*}_{mn} - \upsilon^{**}_{mn}| - \xi^{g}\sum_{n=1}^{\Pi}\sum_{\imath=1}^{\mathcal{K}}|\varrho^{*}_{(\imath)mn} - \varrho^{**}_{(\imath)mn}|$, ($m = 1, 2, \dots, \Pi$), From (23) and (34), we conclude that $\widehat{\hbar}_m > 0$, $\widehat{\Theta}_m > 0$. Therefore, it holds true that

$$\dot{\mathbb{V}}(t,\mathfrak{E}(t)) \leq -\widehat{\hbar}\sum_{m=1}^{\Pi}|\mathfrak{E}_m(t)|, \tag{35}$$

where $\widehat{\hbar} = \min_{1\leq i\leq n}\left\{\widehat{\hbar}_m\right\}$, and $\widehat{\omega}(t) = \sum_{m=1}^{\Pi}|\mathfrak{E}_m(t)| \geq 0$. Then, it can be derived that $\mathbb{V}(t,\mathfrak{E}(t)) \leq -\widehat{\hbar}\int_{t_0}^{t}\widehat{\omega}(s)ds + \mathbb{V}(t_0)$ and $\int_{t_0}^{t}\widehat{\omega}(s)ds$ has a finite limitation. Hence, $\widehat{\omega}(t)$ is bounded. Therefore, there exists a given constant $\widetilde{\mathcal{N}_1} > 0$ satisfying $|\mathcal{D}^{\hat{\beta}}_t\widehat{\omega}(t)| \leq \widetilde{\mathcal{N}_1}$ when $t \geq t_0$, from **Lemma 3**, we will further derive that $\widehat{\omega}(t)$ is uniformly continuous. For $t_0 \leq t_1 < t_2$, and when $|t_2 - t_1| = \theta(\epsilon) \leq (\frac{\epsilon\Gamma(1+\hat{\beta})}{2\widetilde{\mathcal{N}}})^{\frac{1}{\hat{\beta}}}$, one can get

$$\begin{aligned}|\widehat{\omega}_2(t) - \widehat{\omega}_1(t)| &= \mathcal{D}^{-\hat{\beta}}[\mathcal{D}^{\hat{\beta}}\widehat{\omega}_2(t) - \mathcal{D}^{\hat{\beta}}\widehat{\omega}_1(t)]\\ &= \frac{1}{\Gamma(\hat{\beta})}|\int_{t_0}^{t_2}(t_2 - x)^{1-\hat{\beta}}[\mathcal{D}^{\hat{\beta}}\widehat{\omega}(x)]dx - \int_{t_0}^{t_1}(t_1 - x)^{1-\hat{\beta}}[\mathcal{D}^{\hat{\beta}}\widehat{\omega}(x)]dx|\\ &= \frac{1}{\Gamma(\hat{\beta})}|\int_{t_1}^{t_2}(t_2 - x)^{1-\hat{\beta}}[\mathcal{D}^{\hat{\beta}}\widehat{\omega}(x)]dx| + \frac{1}{\Gamma(\hat{\beta})}|\int_{t_0}^{t_1}((t_1 - x)^{1-\hat{\beta}})[\mathcal{D}^{\hat{\beta}}\widehat{\omega}(x)]dx|\\ &\leq \frac{\widetilde{\mathcal{N}_1}}{\Gamma(\hat{\beta})}[(t_1 - t_0)^{\hat{\beta}} - (t_2 - t_0)^{\hat{\beta}} + 2(t_2 - t_1)^{\hat{\beta}}]\\ &\leq \frac{2\widetilde{\mathcal{N}_1}}{\Gamma(\hat{\beta})}(t_2 - t_1)^{\hat{\beta}} \leq \epsilon.\end{aligned}$$

Hence, we can obtain that $\widehat{\omega}(t)$ is uniformly continuous. Under **Lemma 3**, we have

$$\lim_{t\to\infty}\widehat{\omega}(t) = 0.$$

Since $\widehat{\omega}(t) = \sum_{m=1}^{\Pi}|\mathfrak{E}_m(t)|$, it means that $\lim_{t\to\infty}|\mathfrak{E}_m(t)| = 0$ ($m = 1, 2, \cdots, \Pi$). Therefore, from Definition 3, we conclude that FOMMNNs systems (4) and (1) can realize the GAS. This complestes the proof. □

4 | CONCLUSIONS

In this paper, we incoporate the fractional calculus into MMNNs, and propose the FOMMNNs for the first time. Based on the Master-Slave concept, this paper investigates the GAS of the Master-Slave FOMMNNs involving time-varying delays. By utilizing the the theories of the set-valued mapping and fractional Filippov inclusions, the discontinuous FOMMNNs are transformed into continuous FOMMNNs. Then, two different strategies are adopted to design the state-feedback controllers. One is a linear and the other one is adaptive. Based on Lyapunov direct functional method, fractional Barbalats lemma and some analysis techniques, several suffient criteria are obtained for ensureing the GAS of delayed FOMMNNs. In this paper, all the proofs procedures are illustrated in numerical forms, thus we will investigate the synchronization and stabilization issue for fractional order memristive BAM neural networks (FOMBAMNNs) by utilizaing the toolbox of LMI later.

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Author contributions

All the authors have equal contributions in the production of the results as well as write-up of the article.

Conflict of interest

The authors declare no potential conflict of interests.

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